Boundedness of the shift operator related to positive definite forms: An application to moment problems

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Introduction

1. Positive definite forms. Suppose S is an involution semigroup and E is a complex linear space. Let $\omega\colon S\times E\times E\to\mathbf{C}$ be a map such that for every $s\in S$ $\omega(s,\cdot,-)$ is a (hermitian) bilinear form. We call ω simply *a form* (over (S, E)) although it is in fact a family of forms on E, indexed by S. We will see a little while later that we are not far from being precise at this point.

We say that a form ω is *positive definite* (in short: PD) if for all finite sequences $s_1, \dots, s_n\in S$ and $f_1, \dots, f_n\in E$

$$\sum_{ij} \omega(s_i^* s_j, f_j, f_i) \geqq 0.$$

Such forms appear in many circumstances. Let us describe some of them:

1⁰ Suppose $\{\mu_n\}_{n=0}^{\infty}$ is a sequence of real numbers like in the classical moment problem. Then

$$\omega(n, \xi, \eta) = \mu_n \xi\bar{\eta}$$

is a form over $(\mathbf{N}, \mathbf{C})$. Here $\mathbf{N}$ is understood as an additive semigroup of nonnegative integers with involution being just the identity mapping.

2⁰ Let $\varphi\colon S\to B(H)$ ($B(H)$ stands for the algebra of all *bounded* linear operators in a Hilbert space H) be a PD map arising from the Sz.-Nagy dilation theory [14]. It leads to a PD form

$$\omega(s,f,g) = \langle\varphi(s)f, g\rangle,\ f, g\in H,\ s\in S.$$

3⁰ The next sort of examples comes from *unbounded* operators in a Hilbert space. It is commonly known that in this case forms (in their usual meaning) rather than operators themselves are more appropriate to deal with. So as to have a concrete example (of a form in our sense) in mind take an unbounded symmetric operator A, denote by $C^\infty(A)$ the set of all f's such that all the powers $A^n f$ are well defined and define

$$\omega(n, f, g) = \langle A^n f, g\rangle, \quad f, g\in C^\infty(A).$$

We get a PD form over $(\mathbf{N}, C^\infty(A))$.

4^0 Another kind of forms comes from operator valued stochastic processes. The covariance kernel, generally depending on two separated variables s and t, may depend, and in many cases does, on the product s^*t. If this happens we get our form.

2. The Schwarz inequality. Let $\mathscr{F}(S, E)$ denote the complex linear space of all functions from S to E which are zero but a finite number of s. For $h, k\in\mathscr{F}(S, E)$ define

$$\Omega(h, k) = \sum_{s,t}\omega(t^*s, h(s), k(t)).$$

We get in this way a hermitian bilinear form on $\mathscr{F}(S, H)$ corresponding to ω. This correspondence goes back. Indeed, take $s\in S$ and $f\in E$ and define $\delta_{sf}\in\mathscr{F}(S, E)$ as $\delta_{sf}(s)=f$ and $=0$ otherwise. Then

$$\omega(t^*s, f, g) = \Omega(\delta_{sf}, \delta_{tg}).$$

This is why we have called ω just a form. It is easily seen that Ω is PD (i.e. $\Omega(h, h)\geqq 0$) if and only if so is ω.

Positive definiteness of ω implies immediately (for example via Ω) the following Schwarz inequality

$$(1)\qquad \Big|\sum_{i,k}\omega(t_k^*s_i, f_i, g_k)\Big|^2 \leqq \sum_{i,j}\omega(s_i^*s_j, f_j, f_i)\sum_{kl}\omega(t_k^*t_l, g_l, g_k)$$

for $s_1, \dots, s_m, t_1, \dots, t_n\in S$ and $f_1, \dots, f_m, g_1, \dots, g_n\in E$. Moreover we have the following *symmetry* relation

$$\omega(t^*s, f, g) = \omega(s^*t, g, f).$$

3. Factorization. We can apply to Ω the well known procedure (following Aronszajn—Kolmogorov) giving us the factorization (in terms of ω)

$$(2)\qquad \omega(s^*t, f, g) = \langle F(t)f, F(s)g\rangle$$

where, for every $s\in S$, $F(s)$ is a linear operator from E to some Hilbert space H_ω. Moreover the linear span of $F(S)E$, call it H_ω^0, is dense in H_ω. This minimality condition determines F and H_ω up to unitary equivalence. As the most appropriate reference in this matter we recommend [8].

The shift operator

4. Definition of the shift operator. Take $u \in S$. Since an arbitrary element of H_ω^0 is $\sum F(s_i)f_i$ with some $s_1, \ldots, s_n \in S$ and $f_1, \ldots, f_n \in E$, we can try to define $\varphi(u)$, called *the shift operator*, in the following way

$$\varphi(u) \textstyle\sum_i F(s_i)f_i = \sum_i F_i(us_i)f_i. \tag{3}$$

It is easily seen, via (2), that $\varphi(u)$ is well defined if the following implication holds:

$$\textstyle\sum_{ij} \omega(s_i^* s_j, f_j, f_i) = 0 \Rightarrow \sum_{ij} \omega(s_i^* u^* u s_j, f_j, f_i) = 0 \tag{4}$$

Proposition. *$\varphi(u)$ is the well defined linear operator with the domain $D(\varphi(u)) = H_\omega^0$. The adjoint $\varphi(u)^*$ always exists and*

$$\varphi(u^*) \subset \varphi(u)^*, \quad \varphi(u)^*|_{H_\omega^0} = \varphi(u^*). \tag{5}$$

Thus $\varphi(u)$ is closable. Moreover the mapping $u \to \varphi(u)$ is multiplicative.

Proof. Use the Schwarz inequality (1) with $t_i = u^* u s_i$, $g_i = f_i$. Then we get

$$\left|\textstyle\sum_{ij} \omega(s_i^* u^* u s_j, f_j, f_i)\right|^2 \leqq \sum_{ij} \omega(s_i^* s_j, f_j, f_i) \sum_{ij} \omega(s_i^* (u^* u)^2 s_j, f_j, f_i)$$

and this shows the implication (4). Linearity of $\varphi(u)$ follows also from (4). To see (5) write, using (2),

$$\langle \varphi(u) \textstyle\sum_i F(s_i) f_i, \sum_j F(t_j) g_j \rangle = \sum_{ij} \omega(t_j^* u s_i, f_i, g_j)$$
$$= \omega((u^* t_j)^* s_i, f_i, g_j) = \langle \textstyle\sum_i F(s_i) f_i, \varphi(u^*) \sum_j F(t_j) f_j \rangle.$$

Since $\varphi(u) = \varphi((u^*)^*)$, It follows from (5) that $\varphi(u)$ is closable. Multiplicativity of φ follows just from its definition.

Now we can explicitly write (2) using $\varphi(u)$

$$\omega(t^* us, f, g) = \langle \varphi(u) F(s) f, F(t) g \rangle$$

or, if the semigroup has a unit e,

$$\omega(u, f, g) = \langle \varphi(u) Vf, Vg \rangle \tag{6}$$

with $V = F(e)$. Furthermore

$$\|Vf\|^2 = \omega(e, f, f) \tag{7}$$

5. Main result. We deduce from (1) the following simple

Lemma. *Let $v \in S$ be such that $v^* = v$. Then*

$$\left|\textstyle\sum_{ij} \omega(s_i^* v s_j, f_j, f_i)\right| \leqq \left[\sum_{ij} \omega(s_i^* s_j, f_j, f_i)\right]^{1-2^{-k}} \times \left[\sum_{ij} \omega(s_i^* v^{2k} s_j, f_j, f_i)\right]^{2^{-k}} \tag{8}$$

for $k = 1, 2, \ldots$.

Proof. Use (1) with $t_i = vs_i$ and $g_i = f_i$. We have

$$\left|\sum_{ij} \omega(s_i^* v s_j, f_j, f_i)\right|^2 \leqq \sum_{ij} \omega(s_i^* s_j, f_j, f_i) \sum_{ij} \omega(s_i^* v^2 s_j, f_j, f_i).$$

Denote by $p(v) = \sum_{ij} \omega(s_i^* v s_j, f_j, f_i)$ and $a = \sum_{ij} \omega(s_i^* s_j, f_j, f_i)$. Then the above can be rewritten as follows

$$|p(v)|^2 \leqq a p(v^2). \tag{9}$$

This implies

$$|p(v)|^{2^k} \leqq a^{2^k - 1} p(v^{2^k}). \tag{10}$$

Indeed, suppose

$$|p(v)|^{2^{k-1}} \leqq a^{2^{k-1}-1} p(v^{2^{k-1}}).$$

Then, by (9)

$$|p(v)|^{2^k} = \left(|p(v)|^{2^{k-1}}\right)^2 \leqq \left(a^{2^{k-1}-1} p(v^{2^{k-1}})\right)^2$$
$$\leqq a^{2^k - 2} p(v^{2^{k-1}})^2 \leqq a^{2^k - 1} p(v^{2^k}).$$

This gives (10) and, after taking the 2^k-th root, implies (8).

We are interested in condition that would guarantee that the operator $\varphi(u)$ is bounded on H_ω^0 and consequently extends to a bounded operator on H_ω. A look at definition of $\varphi(u)$ as well as the factorization formula enables us to state that $\varphi(u)$ is bounded if and only if the following condition is satisfied

$$\sum_{ij} \omega(s_i^* u^* u s_j, f_j, f_i) \leqq c_1(u) \sum_{ij} \omega(s_i^* s_j, f_j, f_i) \tag{BC$_1$}$$

where $c_1(u)$ is independent of s_i and f_i.

Besides (BC_1) consider two more conditions

$$\omega(s^* u^* u s, f, f) \leqq c_2(u)\, \omega(s^* s, f, f) \tag{BC$_2$}$$

$$\liminf_{k\to\infty} \left(\sum_{ij} \omega(s_i^* (u^* u)^{2^k} s_j, f_j, f_i)\right)^{2^{-k}} \leqq c_3(u). \tag{BC$_3$}$$

We show, in the same way as we did in [16] (see also [11], [12], [13] and [9, Complement 4, pp. 509—510]) for forms discussed in the case 2° of the first section, that these conditions are equivalent. Our lemma provides us at once the following

Proof. (i) implies (ii) trivially. To show that (BC_2)→(BC_3) observe first that the repeated use of (BC_2) gives

$$\omega\big(s^*(u^*u)^{2^k}s, f, f\big) \leqq c(u)_2^{2^k-1} c(u^*)_2^{2^k-1} \omega(s^*s, f, f).$$

Now we can write

$$\textstyle\sum_{ij}\omega\big(s_i^*(u^*u)^{2^k}s_j, f_j, f_i\big) \leqq \sum_{ij}\big|\omega\big(s_i^*(u^*u)^{2^k}s_j, f_j, f_i\big)\big|$$
$$\leqq \textstyle\sum_{ij}\big[\omega\big(s_i^*(u^*u)^{2^k}s_i, f_i, f_i\big)\big]^{1/2}\big[\omega\big(s_j^*(u^*u)^{2^k}s_j, f_j, f_j\big)\big]^{1/2}$$
$$= \big[\textstyle\sum_i\big(\omega\big(s_i^*(u^*u)^{2^k}s_i, f_i, f_i\big)\big)^{1/2}\big]^2$$
$$\leqq c_2(u)^{2^k-1} c_2(u^*)^{2^k-1}\big[\textstyle\sum_i\big(\omega(s_i^*s_i, f_i, f_i)^2\big]^{1/2}.$$

To obtain the second inequality we have used the Schwarz inequality with $s_i^*(u^*u)^{2^k}s_j = \big(s_i^*(u^*u)^{2^{k-1}}\big)\big((u^*u)^{2^{k-1}}s_j\big)$, applying it to each ingredient of the sum separately. Consequently

$$\liminf_{k\to\infty}\big(\textstyle\sum \omega\big(s_i^*(u^*u)^{2^k}s_j, f_j, f_i\big)\big)^{2^{-k}} \leqq c_2(u)^{1/2} c_2(u^*)^{1/2}.$$

The implication (iii)→(i) is a matter of Lemma. If we choose all constants to be minimal, then we can check that they are related as has been indicated in theorem.

Corollary 1. *The shift operator $\varphi(u)$ is bounded if and only if any of the equivalent statements of Theorem* 1 *holds true. The norm of $\varphi(u)$ is $\|\varphi(u)\| \leqq c_1(u)$ and, when $c_1(u)$ is minimal in* (BC_1), $\|\varphi(u)\| = c_1(u)$.

Remark 2. In the case when S is *commutative* we can simplify (BC_3) in the following way: Lemma and the Schwarz inequality give us

$$\omega(s^*u^*us, f, f) \leqq \big(\omega(s^*s, f, f)\big)^{1-2^{-k}}\big(\omega(s^*su^*u)^{2^k}, f, f)\big)^{2^{-k}}$$
$$\leqq \big(\omega(s^*s, f, f)\big)^{1-2^{-k}}\big(\omega((u^*u)^{2^k}s^*s, f, f)\big)^{2^{-k}}$$
$$\leqq \big(\omega(s^*s, f, f)\big)^{1-2^{-k}}\big(\omega((u^*u)^{2^{k+1}}, f, f)\big)^{2^{-k-1}}\big(\omega((s^*s)^2, f, f)\big)^{1/2}.$$

Thus the following condition

(BC_3') $$\liminf_{k\to\infty}\big(\omega((u^*u), f, f)\big)^{2^{-k}} \leqq c_3'(u)$$

forces (BC_2) with $c_2(u) \leqq c_3'(u)$. If S has a *unit*, (BC_3) implies trivially (BC_3') with $c_3'(u) \leqq c_3(u)$. Consequently $c_1(u) = c_2(u) = c_3(u) = c_3'(u)$. This will help us to find the constants $c_i(u)$ involved in Theorem and consequently to determine precisely the norm of $\varphi(u)$.

Applications

6. One-parameter moment problem. Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Call it a moment sequence (on **R**) if there exists a non-negative measure μ such that

$$\mu_n = \int_{-\infty}^{+\infty} \lambda^n \mu(d\lambda).$$

This is the classical result of Hamburger which says that $\{\mu_n\}$ is a moment sequence (on **R**) if and only if

$$\sum_{m,n=1}^{p} \mu_{m+n} \xi_m \bar{\xi}_n \geqq 0 \tag{12}$$

for all finite sequences $\xi_1, \ldots, \xi_p$. In other words the form $\mu(m, \xi, \eta) = \mu_m \xi \bar{\eta}$ is PD. Our Theorem characterizes those moment sequences for which the measure μ is concentrated on the interval $[-a, a]$. Call such a sequence $\{\mu_n\}$ a moment sequence on $[-a, a]$.

Theorem 2. *$\{\mu_n\}$ is a moment sequence on $[-a, a]$ if and only if it satisfies* (12) *and*

$$\mu_{2m+2} \leqq a^2 \mu_{2m} \quad m = 0, 1, \ldots . \tag{13}$$

Then

$$a^2 = \liminf_{k \to \infty} \mu_{2^k}^{2^{-k}} \tag{14}$$

and the measure μ is uniquely determined.

Proof. The operator $\varphi(1)$ is a bounded selfadjoint operator with the norm equal to a. This follows from Theorem 1, both Remarks and Proposition (cf. (5)). Let E be the spectral measure of $\varphi(1)$. Then we have

$$\mu_n = \langle \varphi(1)^n V1, V1 \rangle = \int_{-a}^{a} \lambda^n \langle E(d\lambda) V1, V1 \rangle$$

where V is given as in (7). We see what the measure μ is.

This theorem, especially (15), gives a necessary and sufficient condition for the Jacobi matrix corresponding to the moment sequence $\{\mu_n\}$ to be bounded (cf. [2, p. 7] and also [4]). Condition (13) essentially simplifies what is given there.

Using (14) we get a simple corollary of Theorem 1

Corollary 2. *$\{\mu_n\}$ is a moment sequence on $[-1, 1]$ if and only if it is* PD *and bounded.*

7. Two-parameter moment problem. Going in the same way as in the preceding section we can get the following

Theorem 3. *A necessary and sufficient condition in order that a sequence* $\{\mu_{mn}\}_{m,n=0}^{\infty}$ *is a moment sequence on the rectangle* $[-a, a]\times[-b, b]$, *i.e.*

$$\mu_{mn}=\int_{-a}^{a}\int_{-b}^{b}\lambda_1^m\lambda_2^n\mu(d(\lambda_1,\lambda_2)),$$

is that $\{\mu_{mn}\}$ *is* PD *which means*

$$\sum_{i,j}\mu_{m_i+m_j,n_i+n_j}\xi_i\bar{\xi}_j\geqq 0,$$

and

$$\mu_{2m+2,n}\leqq a^2\mu_{2m,2n}$$

$$\mu_{2m,2n+2}\leqq b^2\mu_{2m,2n}.$$

The measure μ *is uniquely determined and*

$$a^2=\liminf_{k\to\infty}\mu_{2^k,0}^{2-k},\quad b^2=\liminf_{k\to\infty}\mu_{0,2^k}^{2-k}.$$

The proof needs the same arguments as that before. The semigroup in this case is just $\mathbf{N}\times\mathbf{N}$ with $(m,n)(p,q)=(m+p,n+q)$ and $(m,n)^*=(m,n)$. It is generated by two elements $(1,0)$ and $(0,1)$. The operators $\varphi(1,0)$ and $\varphi(0,1)$ are selfadjoint, bounded and commuting (because $(1,0)$ and $(0,1)$ commute).

We can state an analogue of Corollary 2 in this case too.

Theorem 3 improves result of [3].

8. Complex moment problem. Here we consider the same semigroup as before with the involution defined in another way. Let $S=\mathbf{N}\times\mathbf{N}$ and $(m,n)(p,q)=(m+p,n+q)$ and $(m,n)^*=(n,m)$. This semigroup is generated by one element $(1,0)$. The operator $\varphi(1,0)$, if it is bounded, becomes normal. This follows easily from Proposition. Thus we have the following

Theorem 4. *A necessary and sufficient condition for the sequence of complex numbers* $\{\mu_{m,n}\}_{m,n=0}^{\infty}$ *to be a moment problem on the circle* $|\lambda|\leqq a$ *that is to be of the form*

$$\mu_{m,n}=\int_{|\lambda|\leqq a}\lambda^m\lambda^{-n}\mu(d\lambda)$$

is that $\{\mu_{mn}\}$ *is* PD:

$$\sum\mu_{m_j+n_i,m_i+n_j}\xi_i\bar{\xi}_j\geqq 0,$$

and

$$\mu_{k+1,k+1}\leqq a^2\mu_{k,k}.$$

In this case

$$a^2=\liminf_{k\to\infty}\mu_{2^k,2^k}^{-2^k}$$

and the nonnegative measure is uniquely determined.

This contributes to what is in [4] and [1]. Also an analogue of Corollary 2 is easy to formulate.

9. Operator moment problem. Suppose $A_0, A_1, \ldots$ is a sequence of (possible unbounded) operators with the same dense domain D in some Hilbert space H. Moreover suppose

$$\sum_{ij} \langle A_{i+j} f_j, f_i \rangle \geqq 0 \tag{15}$$

for all finite sequences $f_1, \ldots, f_n$ in D. Such moment sequences have been considered in [14] and later in [6] and [7]. First of all notice that, by the Schwarz inequality, if A_0 is a bounded operator so are all $A_1, A_2, \ldots$ but the converse is not true. Then A_0 is bounded if and only if so is V involved in (7). If $\varphi(1)$ is a bounded operator (here again $S=\mathbf{N}$), then we have its spectral measure E and we get

$$(A_n f, g) = \int_{-a}^{a} \lambda^n \langle F(d\lambda) f, g \rangle \tag{16}$$

where

$$\langle F(\cdot) f, g \rangle = \langle E(\cdot) V f, V g \rangle, \quad f, g \in D. \tag{17}$$

and this does not depend on whether V is bounded or not. Anyhow, the values of the measure $F(\cdot)$ are (possible unbounded) positive operators. We get the following

Theorem 5. *The sequence $\{A_n\}$ is of the form* (16) *with F factoring as in* (17) *if and only if it satisfies* (16) *and*

$$\langle A_{2n+2} f, f \rangle \leqq a^2 \langle A_{2n} f, f \rangle$$

for all $n=0, 1, \ldots$. *Then*

$$a^2 = \liminf_{k \to \infty} \langle A_{2^k} f, f \rangle^{2^{-k}}.$$

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