Local solvability of second order differential operators on nilpotent Lie groups

Linda Preiss Rothschild

1. Introduction

The main objective of this work is to establish sufficient conditions for the local solvability of certain left invariant differential operators on a nilpotent Lie group G. The operators to be considered are of the form

$$L = \sum_{j,k} a_{jk} X_j X_k + \sum_{p,q} C_{p,q}[X_p, X_q] \tag{1.1}$$

where $\{X_j\}$ is a set of generators for the Lie algebra $\mathfrak{G}$ of G, (a_{jk}) is a positive definite quadratic form, and each C_{pq} is a complex constant. If the C_{pq} are all real, Hörmander's criterion [16] implies that L is hypoelliptic and locally solvable. However, if the C_{pq} are imaginary both hypoellipticity and local solvability may fail as happens for instance when G is the Heisenberg group. Nevertheless, we will show that for many interesting classes of groups, all operators of the form (1.1) are locally solvable, even when not hypoelliptic.

This investigation has its origin in the author's attempt to understand the significance of the criterion for solvability of the Lewy equation, as well as the associated boundary Laplacian equation, given by Greiner, Kohn, and Stein [7]. (Similar results had previously been obtained in a different context by Sato, Kawai, and Kashiwara [30].) In [7], $\mathfrak{G}$ is the Heisenberg algebra, say of dimension three, and $L=X_1^2+X_2^2+i[X_1, X_2]$. Among other results it is proved that the equation $Lu=f$, f smooth, has a local smooth solution u at x_0 if and only if the orthogonal projection of f onto the L^2 kernel of L is real analytic near x_0. This result suggests a close relationship between the existence of a nontrivial global L^2 kernel for L^t and the local nonsolvability of L (see [2]).

Any unitary irreducible representation π of G acting on a Hilbert space $\mathscr{H}$ determines a corresponding representation, again denoted π of $\mathfrak{G}$ on $\mathscr{H}$; hence $\pi(L)$ is also defined as an operator on $\mathscr{H}$. For the Heisenberg group, the existence of a nontrivial L^2 kernel for L is equivalent to $\pi(L)$ having a zero eigenvalue for any infinite dimensional irreducible representation π. For many other classes of nilpotent Lie algebras, such as the "free" ones of step two with more than two generators the situation is different. For such it may happen that $\pi(L)$ has zero eigenvalues for many values of π, but not for all π in an open set of the parametrizing space for the representations. Thus we are led to a more careful study of the eigenvalues of $\pi(L)$ as π varies.

Our general approach to proving local solvability for operators of the form (1.1) may be described roughly as follows. Given $f \in C_0^\infty(G)$, decompose $\pi(f) = \int f(g)\pi(g)dg$ into its action on the eigenspaces of $\pi(L)$. As π varies over most representations the eigenvalues of $\pi(L)$ are almost algebraic functions of the parametrization of the representations. If f is regarded as a distribution, one may hope to divide each component of f in the above decomposition by the corresponding eigenvalue of $\pi(f)$, using the division of distributions of Hörmander [14] and Lojasiewicz [21]. This process is accomplished by making estimates using the Plancherel formula for G.

In [5] Folland and Stein proved, for operators L of the form (1.1) on the Heisenberg group, that the injectivity of $\pi(L)$ for all non-trivial irreducible representations π implies hypoellipticity and local solvability of L. Later Rockland [25] generalized this result to left invariant differential operators on the Heisenberg group homogeneous under automorphic dilations. His methods involve use of the explicit Plancherel formula. Rockland conjectured that for a general graded nilpotent Lie group G and a homogeneous left invariant L on G, injectivity of $\pi(L)$ for all non-trivial irreducible representations implies that L is hypoelliptic. This conjecture was recently proved by Helffer and Nourrigat [10].

The idea of relating the injectivity of transformed differential operators to the hypoellipticity of the operators goes back to Grušin [8]. In this work a notion of homogeneity is defined for a class of partial differential operators with polynomial coefficients which are elliptic away from a submanifold. A partial Fourier transform is taken in certain variables (see § 3) and the original operator is proved to be hypoelliptic if and only if all the resulting transformed operators are injective on L^2.

In discussing local solvability on Lie groups, one should note that there are very beautiful, general results for operators which are both left and right invariant, i.e., those which come from the center of the universal enveloping algebra. The first such result was obtained by Rais [24], who proved that any bi-invariant differential operator on a nilpotent group is locally solvable. The same result was then proved for semi-simple groups by Helgason [12], and for solvable groups by Duflo—Rais [4] and Rouvière [29]. Then Duflo [3] gave a general proof for any Lie group. One of our main results (see § 13) depends on a very special case of Rais' Theorem. An excellent survey of local solvability of bi-invariant differential operators and related questions is given in Helgason [11].

This paper is the third revision of a manuscript first circulated in 1978 and revised in 1979 and 1980. The first two versions contained several serious mathematical errors. After the appearance of the earlier versions some of these results, as well as related ones, were obtained more simply by Helffer [9], Lévy-Bruhl [18], [19], [20] as well as the author and Tartakoff [28].

I wish to thank Peter Greiner and E. M. Stein for early discussions on local solvability, and Lawrence Corwin for help with the second version, especially section 5. Also, the idea for the counter-example in § 15 is due to Schmuel Friedland. Finally, I am indebted to the referee, who pointed out a large number of minor mistakes, as well as a small number of major ones.

2. Notation and main results

In what follows $\mathfrak{G}$ will always denote a two step nilpotent Lie algebra and G its corresponding simply connected Lie group. We shall assume that $\mathfrak{G}$ decomposes as a vector space $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2$ with $[\mathfrak{G}_1,\mathfrak{G}_1]=\mathfrak{G}_2$, and $[\mathfrak{G}_1,\mathfrak{G}_2]=[\mathfrak{G}_2,\mathfrak{G}_2]=(0)$. $\mathfrak{G}$ carries a natural family of automorphic dilations given by

$$\delta_s(X) = sX, \quad \text{if} \quad X\in\mathfrak{G}_1$$

and

$$\delta_s(T) = s^2 T \quad \text{if} \quad T\in\mathfrak{G}_2.$$

These dilations extend in a natural way to $\mathcal{U}(\mathfrak{G})$, the universal enveloping algebra of $\mathfrak{G}$, which may be identified with the set of all left invariant differential operator on G.

A left invariant differential operator D on G is *homogeneous* of degree d if $\delta_s(D)=s^d D$. By an appropriate choice of the basis $\{X_1, \dots, X_n\}$ of $\mathfrak{G}_1$, any operator of the form (1.1) may be written

$$L = \sum_{j=1}^n X_j^2 + \sum_{q=1}^p C_q T_q, \tag{2.1}$$

where $\{T_q,\ 1\leq q\leq p\}$ is a basis of $\mathfrak{G}_2$. Then L is homogeneous of degree 2, and the term $\sum_{q=1}^p C_q T_q$ may well affect the existence and regularity of solutions. Necessary and sufficient conditions for the hypoellipticity for most operators of the form (2.1) in terms of the C_q were given in [27]. A differential operator D is *locally solvable* at a point x_0 if there exist neighborhoods $V\subset U$ of x_0 such that for every f smooth in U i.e. $f\in C^\infty(U)$, there exists $u\in C^\infty(V)$ such that $Du=f$ is valid in V. For operators of the form (2.1) it can be shown that hypoellipticity implies local solvability (see [25]), but we shall prove here that L is locally solvable in many cases even when hypoellipticity fails.

The main results will be described in terms of classes of groups. $\mathfrak{G}$ is a *free* algebra on n generators if $\dim \mathfrak{G}_1 = n$ and $\dim \mathfrak{G}_2$ is as large as possible, i.e. $\dim \mathfrak{G}_2 = n(n-1)$. This means that the only linear relation among the commutators is skew symmetry $[X_j, X_k] = -[X_k, X_j]$. At the other extreme, $\mathfrak{G}$ is a Heisenberg algebra if $\dim \mathfrak{G}_2 = 1$ and for any non-zero linear functional λ on $\mathfrak{G}_2$, $\det(\lambda([X_j, X_k])) \neq 0$.

Our main results on the local solvability of operators of the form (2.1) may be summarised as follows.

(2.2) Theorem. *Let $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ be a two step graded nilpotent Lie algebra and L the second order left invariant differential operator on G defined by*

$$L = \sum_{j=1}^{n} X_j^2 + \sum_{q=1}^{p} C_q T_q$$

with $\{X_j\}$, $\{T_q\}$ basis of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ respectively. Then L is locally solvable in the following cases.

(i) *$\mathfrak{G}$ is free on n generators, $n > 2$.*

(ii) *For all linear functionals λ on $\mathfrak{G}_2$, $\det \lambda([X_j, X_k]) = 0$; e.g. in particular if $\dim \mathfrak{G}_1$ is odd.*

3. Outline of the proofs

To illustrate one of the main techniques without the machinery of group representations, we first consider the following example. Suppose

$$D = \frac{\partial^2}{\partial x_1^2} + x_1^2 \frac{\partial^2}{\partial t^2} + i\beta \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x_2^2}$$

for some $\beta \in \mathbf{R}$, D acting on $\mathbf{R}^3$. D is homogeneous in the sense of Grušin [8]. If λ, ξ are dual variables to t, x_2, respectively, then the partial Fourier transform D with respect to t and x_2 is

$$D\hat{\ } = \frac{\partial^2}{\partial x_1^2} - x_1^2 \lambda^2 - \beta\lambda - \xi^2.$$

After the change of variables $u_\lambda = |\lambda|^{1/2} x_1$, for $\lambda \neq 0$, $D\hat{\ }$ becomes

$$D\hat{\ } = |\lambda| \left(\frac{d^2}{du_\lambda^2} - u_1^2 \right) - \lambda - \xi^2. \tag{3.1}$$

The operator $\dfrac{d^2}{du^2} - u_\lambda^2$ has as eigenfunctions the Hermite functions h_α with eigenvalues $-(2\alpha+1)$ $\alpha = 0, 1, 2, \ldots$. Hence zero is an eigenvalue of $D\hat{\ }$ when $\lambda = -1$, $\xi = 0$ and $\alpha = 0$. By Grušin's criterion D is not hypoelliptic near $x_1 = 0$.

Nevertheless, (3.1) may be used to prove that D is locally solvable. Indeed, to solve

$$Du = f, \quad f \in C_0^\infty$$

locally, it would suffice to solve

$$Du = \frac{\partial^m}{\partial t^m} f_1 \quad f_1 \in C_0^\infty, \tag{3.2}$$

$m > 0$, since $\frac{\partial^m}{\partial t^m} f_1 = f$ has a local solution $f_1 \in C_0^\infty$.

To solve (3.2) we begin by expanding the partial Fourier transform of $\frac{\partial^m}{\partial t^m} f_1$ in terms of eigenfunctions of $D^\wedge$:

$$\left(\frac{\partial^m}{\partial t^m} f_1\right)^\wedge = i^m \lambda^m \hat{f_1} \sim i^m \lambda^m \textstyle\sum_\alpha (f^\wedge, h_\alpha) h_\alpha. \tag{3.3}$$

Now to solve (3.2), it suffices to divide each term in (3.3) by the eigenvalue $m_\alpha(\lambda, \xi)$ of $D^\wedge$ on h_α i.e.

$$m_\alpha(\lambda, \xi) = -|\lambda|(2\alpha+1) - \beta\lambda - \xi^2$$

when $2\alpha+1 > |\beta|$, $\lambda^m / m_\alpha(\lambda, \xi)$ is locally bounded, and so the division make sense in the context of L^2. For the finitely many α for which $2\alpha+1 \leqq |\beta|$, the division may still be performed in the sense of distributions. Indeed,

$$m_\alpha(\lambda, \xi) = \frac{(-\lambda(2\alpha+1) - \beta\lambda - \xi^2)(\lambda(2\alpha+1 - \beta\lambda - \xi^2)}{|\lambda|(2\alpha+1) - \beta\lambda - \xi^2}.$$

Thus, division of the function (f, h_α) by $m_\alpha(\lambda, \xi)$ is equivalent to multiplication by $|\lambda|(2\alpha+1) - \beta\lambda - \xi^2$ and division by the numerator of $m_\alpha(\lambda, \xi)$, which is a polynomial, q_α. The multiplication is possible because the power λ^m may be made so large that the singularity of $|\lambda|$ at $\lambda = 0$ is effectively killed. Finally, division of the resulting distribution by the polynomial q_α is possible by the results of Hörmander [14] and Lojasiewicz [21]. Since the resulting distribution is tempered, the distribution may be pulled back to t, x_2, giving a solution of (3.2).

For the operator L of the form (2.1) we use the group Fourier transform rather than the Euclidean one. We recall some basic facts about harmonic analysis on a nilpotent group G. For every irreducible unitary representation π of G on a Hilbert space $\mathscr{H}$ and any $f \in C_0^\infty$ there is an operator $\pi(f)$ on $\mathscr{H}$ defined by

$$\pi(f) = \int f(g) \pi(g) \, dg,$$

where dg is the usual Euclidean measure on G (which agrees with the Haar measure). The Plancherel theorem then states that there is a measure $d\mu(\pi)$ on the set of all irreducible unitary representations of G such that

$$\|f\|^2 = \int \operatorname{tr}(\pi(f)\pi(f)^*) \, d\mu(\pi).$$

Here * denotes the adjoint and tr denotes the trace. (The operator $\pi(f)$ is always of trace class.)

Hence $f \in L^2(G)$ may be identified with the distribution

$$\psi \to \int \mathrm{tr}\,(\pi(\bar{\psi})^* \pi(f))\, d\mu(\pi),$$

where $\bar{\psi}$ is the complex conjugate of ψ. Here $\mathscr{H} = L^2(\mathbf{R}^d)$ for some d. Now consider the operator $\pi(L)$, where π is identified also with a representation of the Lie algebra $\mathfrak{G}$. The eigenfunctions of $\pi(L)$ on $L^2(\mathbf{R}^d)$ are the Hermite functions $h_\alpha = h_{\alpha_1}(y_1) \dots h_{\alpha_d}(y_d)$ $y_i \in \mathbf{R}$, with eigenvalues $m_\alpha(\pi)$. Let P_α be the orthogonal projection onto the subspace of $L^2(\mathbf{R}^2)$ spanned by h_α. One may then decompose $\pi(f)$ as

$$\pi(f) \sim \textstyle\sum_\alpha \pi(f) P_\alpha \tag{3.4}$$

and attempt to divide each term by the function $m_\alpha(\pi)$ as before. However, there are considerably more technical problems in this case. For example, the parameter space (ξ, λ) for $D^\wedge$ has a singularity only at $\lambda = 0$, i.e. (3.4) is valid whenever $\lambda \neq 0$. The parametrizing space for the L^2 decomposition (3.4) has more complicated singularities. Furthermore, the eigenvalues $m_\alpha(\pi)$ involve not only absolute values of the main parameters but also eigenvalues of matrices with entries in the main parameter space.

One of the main methods of dealing with the singularities involved is to solve the equation

$$L\sigma = Zf,$$

where Z is a left invariant differential operator which is itself locally solvable and which has the property that $\pi(Z)$ is a polynomial in the parameter space which vanishes to a high power on the singularities. This is a generalization of an idea used in [25] for the Heisenberg group.

4. Unitary representations of G

We shall calculate $\pi(L)$ for almost all representations π of G. To do this, we follow the orbit method of Kirillov [17], for those representations needed for the Plancherel formula [22].

Let $\mathfrak{G}^*$ be the linear dual of $\mathfrak{G}$. The orbits of $\mathfrak{G}^*$ are the sets of the form

$$\{g \cdot \ell;\ g \in G\}$$

for $\ell \in \mathfrak{G}^*$. Here the coadjoint action $g \cdot \ell = \mathrm{Ad}^*\, g \circ \ell$ is defined by $g \circ \ell(X) = \ell(g^{-1} \cdot X)$, where $g^{-1} \cdot X = \mathrm{Ad}\,(g^{-1}) X$, Ad denoting the adjoint representation. By the Kirillov theory, the set of all irreducible unitary representations of G is one-one correspondence with the set of orbits in $\mathfrak{G}^*$. We shall not discuss the general method of assigning a representation to an orbit in $\mathfrak{G}^*$, but shall restrict the discussion to those representations occurring in the Plancherel formula.

If $\ell\in\mathfrak{G}^*$, the *radical* of ℓ is defined as

$$\operatorname{Rad}\ell = \{Y_1\in\mathfrak{G}\colon\ \ell([Y_1, Y]) = 0 \text{ for all } Y\in\mathfrak{G}\}.$$

If $\ell=(\xi,\lambda)$ with $\xi\in\mathfrak{G}_1^*$ and $\lambda\in\mathfrak{G}_2^*$, then $\operatorname{Rad}\ell=\operatorname{Rad}(0,\lambda)\supset\mathfrak{G}_2$. By renumbering the X_j's if necessary, we may assume that $X_1, X_2, \ldots, X_{2d}$ is a basis for the complement of $\operatorname{Rad}\ell$ on a Zariski open subset $\mathcal{O}\subset\mathfrak{G}^*$. Thus $\det\big(\ell([X_j, X_k])\big)_{1\leqq j,k\leqq 2d}\neq 0$ if $(0,\lambda)\in\mathcal{O}$. We identify λ with $(0,\lambda)$ and put $\lambda_q=\lambda(T_q)$, $1\leqq q\leqq p$.

Now if $S(\lambda)=(\lambda[X_j, X_k])$, ℓ will be called regular if $\lambda\in\mathcal{O}$ and $S(\lambda)$ has the maximal number of distinct eigenvalues among $\{S(\lambda'), \lambda'\in\mathfrak{G}_2^*\}$. If λ is regular there is an orthogonal change of basis

$$\{X_1, \ldots, X_n\}\to\{U_1^\lambda, \ldots, U_d^\lambda, V_1^\lambda, \ldots, V_d^\lambda, W_1^\lambda, \ldots, W_{n-2d}^\lambda\}$$

such that

$$\lambda([U_j^\lambda, V_k^\lambda]) = \delta_{jk}\varrho_j,$$

where $\{\pm i\varrho_j\}_{j=1,2\ldots d}$ is the set of non-zero eigenvalues of the matrix $S(\lambda)$, $\varrho_j>0$, and

$$\lambda([U_j^\lambda, U_k^\lambda]) = \lambda([V_j^\lambda, V_k^\lambda]) = \lambda([W_s^\lambda, U_k^\lambda]) = \lambda(W_s^\lambda, V_k^\lambda]) = 0.$$

If $\xi=(\xi_1, \xi_2', \ldots, \xi_{n-2d})$, $\xi_j\in\mathbf{R}$, define the linear functional $\ell=(\xi,\lambda)$ by

(4.1i) $$\ell(T_q) = i\lambda_q$$

(4.1ii) $$\ell(W_s^\lambda) = i\xi_s$$

(4.1iii) $$\ell(U_j^\lambda) = \ell(V_k^\lambda) = 0 \quad \text{all } j, k.$$

It then follows from the general theory that a set of representations associated to $\{(\xi,\lambda)\colon \lambda\in\mathcal{O}\}$ is sufficient for the Plancherel measure [22].

We now describe the infinitesimal representation π_ℓ of $\mathfrak{G}$ associated to the linear functional ℓ. $\pi_\ell(\mathfrak{G})$ acting on the Hilbert space $L^2(\mathbf{R}^d)$ is in given as follows,

(4.2i) $$\pi_\ell(U_j^\lambda) = |\varrho_j|^{1/2}\frac{\partial}{\partial y},$$

(4.2ii) $$\pi_\ell(V_j^\lambda) = i\varrho_j^{1/2}y_j, \quad 1\leqq j\leqq d$$

(4.2iii) $$\pi_\ell(W_k^\lambda) = i\xi_k,$$

(4.2iv) $$\pi_\ell(T_q) = i\lambda_q, \quad 1\leqq q\leqq p.$$

It will be useful to know π_ℓ on G, also. A linear function $\ell=(\lambda,\xi)$ is called regular if λ is regular. Now let ℓ regular be fixed and give G the coordinates (u, v, w, t) determined by

(4.3) $$(u, v, w, t)\to\exp(u\cdot U+v\cdot V+w\cdot W+t\cdot T)$$

where $u \cdot U = \sum u_j U_j$, etc. Then

$$\pi_\ell(w, u, v, t)\, h(y) = e^{i(\xi \cdot w + \lambda \cdot t + \Sigma \varrho_j^{1/2} y_j v_j)}\, h(y + \varrho^{1/2} \cdot u), \tag{4.4}$$

where $\varrho^{1/2} \cdot u = \sum \varrho_j^{1/2} u_j$, $h \in L^2(\mathbf{R}^d)$.

Now let $h_{\alpha_j}(y_j)$ be the j^{th} Hermite function in the variable y_j, and put

$$h_\alpha = h_{\alpha_1}(y_1) h_{\alpha_2}(y_2) \dots h_{\alpha_d}(y_d)$$

for any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, with each α_j a non-negative integer.

(4.5) Proposition. *If* $\ell = (\xi, \lambda)$, $\lambda \in \mathcal{O}$, *then*

$$\pi_\ell(L) = -\sum_{k=1}^{n-2d} \xi_k^2 - \sum_{j=1}^{d} \varrho_j \left(\frac{\partial^2}{\partial y_j^2} - y_j^2 \right) + i \sum_{t=1}^{p} C_t \lambda_t. \tag{4.6}$$

The eigenfunctions for $\pi_\ell(L)$ *are* $\{h_\alpha = h_{\alpha_1}(y_1) h_{\alpha_2}(y_2) \dots h_{\alpha_d}(y_d)\}$ *with eigenvalues*

$$m_\alpha(\ell) = -\sum_{k=1}^{n-2d} \xi_k^2 - \sum_{j=1}^{d} \varrho_j (2\alpha_j + 1) + i \sum C_t \lambda_t \tag{4.7}$$

for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$.

Proof. Since the U_j^λ, V_j^λ, W_k^λ are obtained from the X_s by an orthogonal change of basis,

$$\sum_{s=1}^{n} X_s^2 = \sum_{j=1}^{d} (U_j^{\lambda 2} + V_j^{\lambda 2}) + \sum_{k=1}^{n-2d} W_k^{\lambda 2}.$$

The proposition is then immediate from (4.2).

5. A cross-section of generic representations

Each $\ell = (\lambda, \xi) \in \mathfrak{G}^*$ determines an equivalence class of representations. We shall make a choice of a representation from each class. Let $\sqrt{-1} r_1, \sqrt{-1} r_2, \dots, \sqrt{-1} r_m$, be the distinct non-zero eigenvalues with positive imaginary part of $S(\lambda) = (\lambda [X_j, X_k])$ for λ regular. Then $m = 2d$, and $r_j = \pm \varrho_k$, some k. A function of λ will be called *rational-radical* if it is obtained from the r_j and the coordinate functions λ_k by a finite sequence of successive operations of forming rational functions and taking square roots. A function of $\ell = (\lambda, \xi)$ is *rational-radical* if it is a polynomial in ξ with coefficients which are rational-radical functions of λ.

(5.1) Theorem. *For each* $\ell = (\lambda, \xi)$, $\lambda \in \mathcal{O}$ *there is a choice of an irreducible representation* π_ℓ *of* $\mathfrak{G}$ *satisfying the following.*

(i) *For each* X_j, $\ell \to \pi_\ell(X_j)$ *is an operator with rational-radical coefficients.*

(ii) $\pi_\ell(\sum X_j^2) = -\sum \xi_j^2 + \sum_{j=1}^{d} \varrho_j \left(\frac{\partial^2}{\partial y_j^2} - y_j^2 \right)$, *where* $\pm \sqrt{-1}\ \varrho_j$ *are the (not necessarily distinct) non-zero eigenvalues of* $S(\lambda)$.

Proof. By Section 4, it suffices to find a basis $X_1'(\lambda), X_2'(\lambda), \ldots, X_n'(\lambda)$ such that

(5.2) $$X_j'(\lambda) = \sum a_{jk}(\lambda) X_k,$$

where the a_{jk} are rational-radical functions,

(5.3) $$\lambda([X_j'(\lambda), X_k'(\lambda)] = \operatorname{diag}\left(\begin{bmatrix} 0 & \varrho_1 \\ -\varrho_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \varrho_2 \\ -\varrho_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \varrho_d \\ -\varrho_\lambda & 0 \end{bmatrix}, 0 \ldots 0\right),$$

where diag is the $n \times n$ matrix with the indicated 2×2 blocks down the main diagonal, and

(5.4) $$\sum X_j'(\lambda)^2 = \sum X_j^2.$$

For suppose (5.2) and (5.3) are satisfied. Then we may define the representation π_ℓ by taking

$$U_j^\lambda = X_{2j-1}'(\lambda), \quad j = 1, 2, \ldots, d$$

$$V_j^\lambda = X_{2j}'(\lambda), \quad j = 1, 2, \ldots, d,$$

and

$$W_k^\lambda = X_{k+2d}'(\lambda) \quad k = 1, 2, \ldots, n-2d,$$

and defining π_ℓ by (4.2i—iv).

We shall need some preliminaries before defining $X_j'(\lambda)$. A family ϑ_λ of vector spaces parametrized by $\lambda \in \mathfrak{G}_2^*$ will be called *orthonormal rational-radical* if each space in ϑ_λ has a basis of orthonormal vectors $Y_1(\lambda), Y_2(\lambda), \ldots, Y_n(\lambda)$, where $Y_k = \sum b_{ks}(\lambda) X_s$ with $b_{ks}(\lambda)$ rational-radical.

(5.5) Lemma. *Let ϑ_λ be a family of orthonormal rational-radical vector spaces and ω_λ an orthonormal rational-radical subfamily preserved by linear operators S_λ on the spaces in ϑ_λ. Then the following subfamilies are again orthonormal rational-radical:*

(1) $\omega_\lambda^\perp$

(2) $\ker S_\lambda$

(3) $\ker (S_\lambda^2 + r_k^2)$.

Proof. For (1), extend the given orthonormal rational-radical basis of ω_λ to one for all of ϑ_λ. Now the use of Cramer's rule shows there is an orthonormal rational-radical basis dual to the given one, from which one can extract a basis of $\omega_\lambda^\perp$. Next, $\ker S_\lambda = (\text{range } S_\lambda^t)^\perp$, where t denotes transpose. An orthonormal rational-radical basis for range S_λ^t is obtained by applying S_λ^t to the given basis. Now the proof follows from that of (1). The argument for $\ker (S_\lambda^2 + r_k^2)$ is the same.

The idea for the above lemma is essentially contained in a similar result by Corwin—Greenleaf [1]. We may now complete the proof of Theorem 5.1 by defining $X_j'(\lambda)$. Let S_λ be the transformation defined by $S(\lambda)$.

By Lemma 5.5 choose an orthonormal rational-radical basis for $\ker S_\lambda$ and $\ker(S_\lambda^2+r_k^2)$ for all k. Since $S(\lambda)$ is skew symmetric, the spaces are all mutually orthogonal. If $\dim\ker(S_\lambda^2+r_k^2)>2$ for any k we shall choose a particular basis. For this, let $Y_1(\lambda)\in\ker(S_\lambda^2+r_k^2)$ be an arbitrary element of the basis. Then let ω_1^λ be the subfamily spanned by $Z_1(\lambda)=Y_1(\lambda)$ and $Z_2(\lambda)=S_\lambda Y_1(\lambda)/r_k$. Note that since S_λ is skew symmetric

$$(Z_1(\lambda), Z_2(\lambda)) = (Y_1(\lambda), S_\lambda Y_1(\lambda)/r_k) = -\left(\frac{S_\lambda Y_1(\lambda)}{r_k}, Y_1(\lambda)\right)$$
$$= -(Z_1(\lambda), Z_2(\lambda)).$$

Therefore $(Z_1(\lambda), Z_2(\lambda))=0$. Also,

$$\|S_\lambda Y_1(\lambda)\|^2 = (S_\lambda Y_1(\lambda), S_\lambda(\lambda)) = (-S_\lambda^2 Y_1(\lambda), Y_1(\lambda)) = r_k^2,$$

which shows that the set $\{Z_1(\lambda), Z_2(\lambda)\}$ is orthonormal. Now S_λ preserves $\ker(S_\lambda^2+r_k^2)\cap W_1^\lambda$ and the above procedure may be repeated. With respect to the resulting basis S_λ restricted to $\ker(S_\lambda^2+r_k^2)$ has the matrix

$$\operatorname{diag}\left(\begin{pmatrix} 0 & r_k \\ -r_k & 0\end{pmatrix}, \begin{pmatrix} 0 & r_k \\ -r_k & 0\end{pmatrix}, \ldots, \begin{pmatrix} 0 & r_k \\ -r_k & 0\end{pmatrix}\right).$$

Now the $X_j'(\lambda)$ may be taken to be the bases of $\ker(S_\lambda^2+r_k^2)$ and $\ker S_\lambda$. This completes the proof of Theorem 5.1.

For fixed $\lambda\in\mathcal{O}$ it will be useful to define coordinates corresponding to the $X'(\lambda)$. The following is a consequence of the construction in the proof of Theorem 5.1.

(5.6) Proposition. *Let* (x, t) *and* (u, v, w, t) *be coordinates for G defined by*

$$(x, t) \leftrightarrow \exp(x\cdot X+t\cdot T)$$

and

$$(u, v, w, t) \leftrightarrow \exp(u\cdot U+v\cdot V+w\cdot W+t\cdot T),$$

where $x\cdot X=\sum x_j X_j$, *etc, and* $\lambda\in\mathcal{O}$ *is fixed with* $U=U^\lambda$, $V=V^\lambda$ *and* $W=W^\lambda$ *defined by* (5.4). *Then*

$$x = R(\lambda)\begin{pmatrix} u \\ v \\ w\end{pmatrix}, \tag{5.7}$$

and

$$\begin{pmatrix} u \\ v \\ w\end{pmatrix} = R^{-1}(\lambda)x, \tag{5.8}$$

where $R(\lambda)$ *and* $R^{-1}(\lambda)$ *are matrices with rational-radical coefficients.*

A modification of Lemma 5.5 will be needed in § 13.

(5.9) Proposition. *One of the* W_i^λ, *say* W_1^λ, *may be chosen so that*

$$W_1^\lambda = \sum q_k(\lambda) X_k / (\sum q_k(\lambda)^2)^{1/2},$$

where $q_k(\lambda)$ *is rational in* λ.

Proof. Let S_λ be the linear transformation defined by $S(\lambda)$. Since S_λ^t has polynomial coefficients in λ, a basis of $(S_\lambda^t)^\perp$ may be chosen with rational coefficients. (This basis will not be orthonormal, in general.)

6. The Plancherel measure of G

Let $\mathcal{O} \subset \mathfrak{G}$ be the set of all regular elements of $\mathfrak{G}$ as defined in § 4. Let $Q(\ell)$ be the polynomial defined by $Q(\ell) = \det(\ell[X_j, X_k]_{1 \leq j, k \leq 2d})$. $Q(\ell) = Q(\lambda)$ is a non-vanishing Ad G^*-invariant polynomial on $\mathcal{O}$.

(6.1) Proposition. *The Plancherel measure on* $\mathcal{O}/\mathrm{Ad}^* G$, *the set of all orbits in* $\mathcal{O}$ *is given by*

$$d\mu(\ell) = r(\lambda)\, d\xi\, d\lambda,$$

where $r(\lambda)$ *is a rational-radical function, as defined in* § 5.

Proof. We write $d\mu(\ell) = r(\lambda, \xi) d\xi d\lambda$ and compute $r(\lambda, \xi)$ by a theorem of Pukanszky [23, Proposition 3]. For this we identify W_k^λ with the dual of ξ_k^λ i.e. $W_k^\lambda = d\xi_k^\lambda$, $k = 1, 2, \ldots, n-2d$. Similarly, we identify T_p with $d\lambda_p$, and X_k with $d\varphi_k$, where $\{\varphi_k\}$ is a set of linear functionals dual to $\{X_k\}_{1 \leq k \leq n}$. Then by Pukanszky's Theorem, $r(\lambda, \xi)$ is determined by

$$Q(\lambda)^{1/2} \prod_{g=1}^{p} \wedge T_q \wedge \prod_{j=1}^{n} \wedge X_j = r(\lambda, \xi) \prod_{k=1}^{n-2d} \wedge W_k^\ell \wedge \prod_{q=1}^{p} \wedge T_q \wedge \prod_{j=1}^{2d} X_j.$$

By Theorem 5.1, we may write $W_k^\ell = X'_{k+2d}(\lambda) = \sum_{s=1}^{n} a_{k+2d, s}(\lambda) X_s$, where each $a_{q,s}$ is rational-radical. Hence $r(\lambda, \xi) = r(\lambda)$ is also rational-radical.

(6.2) Corollary. *There is a polynomial* $q(\lambda)$ *and an integer* N_0 *such that if* $N \geqq N_0$ *the measure*

$$(1 + |\xi| + |\lambda|)^{-N} q(\lambda)\, d\mu(\ell)$$

is finite.

7. Estimates for $D_\ell^\beta \|\pi_\ell(\psi) h_\alpha\|^2$

We denote by $\|\ \|$ the usual Hilbert space norm in $L^2(\mathbf{R}^d)$. We study here the growth of $D_\ell^\beta\|\pi_\ell(\psi)h_\alpha\|^2$ for ℓ regular in terms of the Schwartz norms of ψ. For this we write

$$\|\varphi\|_{k,N} = \sup_{\substack{|\beta|\leq k \\ (x,t)\in G}} (1+|x|^2+|t^2|)^{N/2}|D_{x,t}^\beta\varphi(x,t)|$$

for non-negative integers k, N and $\varphi\in C_0^\infty(G)$. Our main result of this section is the following, the present formulation of which was suggested by the referee.

(7.1) Proposition. *For every multi-index β and positive integer N there is a rational function $q_\beta(\lambda)$ depending only on β, and integers k' and N' such that for any $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_d)$ there exists $C_\alpha>0$ satisfying*

$$(1+|\ell|^2)^{N/2}\left|D_\ell^\beta\|\pi_\ell(\psi)h_\alpha\|^2\right| \leq C_\alpha|q_\beta(\lambda)|\,\|\psi\|_{k',N'}^2 \tag{7.2}$$

for all $\psi\in C_0^\infty(G)$. Furthermore, C_α may be chosen, depending on β and N so that

$$\textstyle\sum_{\alpha\in\mathbf{N}^d} C_\alpha < \infty. \tag{7.3}$$

The main part of the proof of Proposition 7.1 is the following.

(7.4) Lemma. *For every multi-index β there is a finite set $\mathscr{D}$ of differential operators on G and rational-radical functions $r_{\beta,i}(\lambda)$, $D_i\in\mathscr{D}$, such that for any α,*

$$D_\lambda^\beta\|\pi_\ell(\psi)h_\alpha\|^2 = \mathscr{D}\textstyle\sum_i r_{\beta,i}(\lambda)\|\pi_\ell(D_i\psi)h_\alpha\|^2 \tag{7.5}$$

for all regular λ and all $\psi\in C_0^\infty(G)$.

Proof. By 4.4 we have for $h\in L^2(\mathbf{R}^d)$

$$\pi_\ell(\psi)h = \int \psi(x(u,v,w),t)e^{i(\xi\cdot w+\lambda\cdot t+\Sigma\varrho_j^{1/2}y_jv_j)}h(y+\varrho^{1/2}\cdot u)\,du\,dv\,dw\,dt.$$

(Note that $dudvdw=dx$, since $\{U, V, W\}$ are obtained from the $\{X\}$ by a unitary transformation.) Now the proof of Lemma 7.4 is straightforward, but tedious. By dominated convergence we may differentiate under the integrals and for differentiation in λ, it suffices to study the effects of the following differentiations:

$$\frac{\partial}{\partial\lambda_j}e^{i\lambda\cdot t} = t_je^{i\lambda\cdot t}, \tag{7.6}$$

$$\frac{\partial}{\partial\lambda_j}e^{i\varrho_k^{1/2}y_kv_k} = i\left(\frac{\partial v_k}{\partial\lambda_j}y_k\varrho_k^{1/2}+\frac{\partial\varrho_k^{1/2}}{\partial\lambda_j}v_ky_k\right)e^{i\varrho_k^{1/2}y_kv_k} \tag{7.7}$$

$$\frac{\partial}{\partial\lambda_j}h(y+\varrho^{1/2}\cdot u) = \textstyle\sum_k\left(\frac{\partial\varrho_k^{1/2}}{\partial\lambda_j}u_k\frac{\partial h}{\partial y_k}(y+\varrho^{1/2}\cdot u)\right) \tag{7.8}$$

$$\frac{\partial}{\partial\lambda}\psi(x(u,v,w),t) = \sum\frac{\partial x_i}{\partial\lambda_j}(u,v,w)\frac{\partial\psi}{\partial x_i}(x(u,v,w),t). \tag{7.9}$$

$$\frac{\partial}{\partial\lambda_j}e^{i\xi_kw_k} = \xi_k\frac{\partial w_k}{\partial\lambda_j}e^{i\xi_kw_k}. \tag{7.10}$$

The contribution of each term above may be absorbed in the right hand side of (7.5) as follows. The contribution of (7.6) is absorbed in the coefficients of the D_i's. For (7.7), note that $\frac{\partial \varrho^{1/2}}{\partial \lambda_j}$ is rational-radical and $v_j=\sum S_{jk}(\lambda)X_k$, with $S_{jk}(\lambda)$ rational-radical, by (5.8). To deal with the factor y_k, write

$$y_k e^{i\varrho_k^{1/2} y_k v_k} = \frac{1}{i\varrho_k}\frac{\partial}{\partial v_k}(e^{i\varrho_k^{1/2} y_k v_k}),$$

and use integration by parts to put the differentiation on $\psi(x(u, v, w), t)$. Then use (5.7) and (5.8) again to write $\frac{\partial}{\partial v_k}$ in terms of the $\frac{\partial}{\partial x_j}$. The term on the right in (7.8) is handled similarly, using

$$\frac{\partial h}{\partial y_k}(y+\varrho^{1/2}\cdot u) = \varrho_k^{-1/2}\frac{\partial}{\partial u_k}(h(y+\varrho^{1/2}\cdot u)),$$

integration by parts, and (5.7) and (5.8).

For (7.10) we may handle by (5.7) and (5.8) as before. For the term ξ_k, we use

$$i\xi_k = \frac{\partial}{\partial w_k}(e^{i\xi\cdot w})$$

and integration by parts, then (5.7) and (5.8).

Finally, differentiation in ξ occurs only in

$$\frac{\partial}{\partial \xi_k}(e^{i\xi\cdot w}) = iw_k e^{i\xi\cdot w}$$

and the factor w_k is handled like u_k or v_k. This completes our sketch of the proof of Lemma 7.4.

(7.11) Lemma. *For any even integer N_1 there exists C_{N_1} such that*

$$(1+|\ell|^2)^{N_1/2}\|\pi_\ell(\varphi)h_\alpha\|^2 \leqq C_{N_1}\left\|\pi_\ell((\textstyle\sum X_j^2)^{N_1}\varphi)h_\alpha\right\|^2$$

for all $\varphi\in C_0^\infty(G)$.

Proof. $\pi_\ell((\sum X_j^2)^{N_1}\varphi)h_\alpha=\pi_\ell(\varphi)\pi_\ell(\sum X_j^2)^{N_1}h_\alpha=\pi_\ell(\varphi)(-\sum\varrho_j(2\alpha_j+1)-\sum\xi_k^2)^{N_1}h_\alpha$. Since $\sup_j \varrho_j\geqq C|\lambda|$, the lemma follows.

(7.12) Lemma. *There exists N_2 such that*

$$\|\pi_\ell(\varphi)h_\alpha\| \leqq C_\alpha(\ell)\left\|\pi_\ell((\textstyle\sum X_j^2)^{N_2}\varphi)h_\alpha\right\|$$

all $\varphi\in C_0^\infty(G)$, with

$$C_\alpha(\ell) \leqq q_1(\lambda)\frac{1}{\prod_j(2\alpha_j+1)^{N_2/d}}, \tag{7.13}$$

where $q_1(\lambda)$ is rational and independent of α.

Proof. As in the proof of Lemma 7.11,

$$\|\pi_\ell(\psi)h_\alpha\| = \left\|\frac{\pi_\ell(\psi)}{(\sum \varrho_j(2\alpha_j+1)-\sum \xi_k^2)^{N_2}}\pi_\ell(\sum X_j^2)^{N_2}h_\alpha\right\|$$
$$\leqq \frac{|q_1(\lambda)|}{(\prod_j(2\alpha_j+1))^{N_2/d}}\left\|\pi_\ell((\sum X_j^2)^{N_2}\psi)h_\alpha\right\|$$

with $q_1(\lambda)=(\prod_{j=1}\varrho_j)^{N_2/d}$, which is a polynomial if N_2/d is an even integer.

(7.14) Lemma. *If* $N_3\geqq 0$ *is sufficiently large, then*

(7.15) $$\|\pi_\ell(\varphi)h_\alpha\| \leqq C_{N_3}\|\varphi\|_{0,N_3},$$

with C_{N_3} *independent of* α.

Proof. It is well known that

$$\|\pi_\ell(\varphi)h\| \leqq \|\varphi\|_{L^1(G)}\|h\|_{L^2(\mathbf{R}^d)}.$$

The Lemma follows since $\|h_\alpha\|_{L^2}=1$ and $\|\varphi\|_{L^1(G)}$ is bounded by the right hand side of (7.15).

We may now prove proposition 7.1. By Lemma 7.4, given N, there exist N_1, a rational-radical function r_β, and a finite set $\mathscr{D}$ of differential operators with polynomial coefficients such that

(7.16) $$(1+|\ell|^2)^N\|D_\ell^\beta\pi_\lambda(\varphi)h_\alpha\|^2 \leqq |r_\beta(\lambda)|(1+|\ell|^2)^{N_1}\textstyle\sum_i\|\pi_\ell(D_i\varphi)h_\alpha\|^2$$

Now let $\varphi_i=D_i\varphi$. By Lemma 7.11 the right hand side of (7.16) is bounded by

(7.17) $$C_{N_1}|r_\beta(\lambda)|\textstyle\sum_i\left\|\pi_\ell((\sum X_j^2)^{N_1}\varphi_i)h_\alpha\right\|^2$$
$$\leqq CC_\alpha(\ell)|r_\beta(\lambda)|\textstyle\sum_i\left\|\pi_\ell((\sum X_j^2)^{N_2}\varphi_i)h_\alpha\right\|^2,$$

for some N_2, with $C_\alpha(\ell)$ satisfying (7.13). Now apply Lemma 7.14 to obtain from (7.16) and (7.17)

$$(1+|\ell|^2)^N\|D_\ell^\beta\pi_\ell(\varphi)h_\alpha\|^2 \leqq CC_\alpha(\lambda)|q_2(\lambda)|\|\varphi\|_{k,N_3}$$

for some k and some N_3, where $q_2(\lambda)$ is chosen so that

$$|q_2(\lambda)| \geqq \max_i|q_1(\lambda)r_{\beta,i}(\lambda)|.$$

Now the proposition is proved provided that N_2 is chosen large enough that

$$\sum_\alpha\frac{1}{(\prod_{j=1}^d(2\alpha_j+1))^{N_2/d}}<\infty.$$

8. Rapid decrease of $\ell\to\|\pi_\ell(Z\psi)h_\alpha\|^2$

The main result of this section is the following easy consequence of Proposition 7.1.

Proposition 8.1. *Let k and N be fixed. Then there exists a polynomial $z(\lambda)$ and integer k', N' such that*

$$\sup_{\substack{\beta\leqq k\\ \ell}}\{(1+|\ell|^2)^N D_\ell^\beta(\|z(\lambda)\pi_\ell(\psi)h_\alpha\|^2)\}^{1/2}\leqq C_\alpha\|\psi\|_{B,N'},\tag{8.1}$$

$$\sum C_\alpha<\infty.\tag{8.2}$$

Proof. Choose $q_\beta(\lambda)$ rational as in Proposition 7.1 depending on β, and put $q(\lambda)=\sum_{|\beta|\leqq k}q_\beta(\lambda)^2+1$. Then $q(\lambda)=p_1(\lambda)/p_2(\lambda)$, p_1, p_2 polynomials of degrees s_1 and s_2, respectively. Put $z(\lambda)=p_2(\lambda)^{k+1}$. Then $(1+|\ell|^2)^N D_\ell^\beta\|z(\lambda)\pi_\ell(\psi)h_\alpha\|^2$ is a finite sum of terms of the form $(1+|\ell|^2)^N(p_2(\lambda))^{k+1-j}D_\ell^\gamma\|\pi_\ell(\psi)h_\alpha\|^2$, where $|\gamma|=|\beta|-j$. Now apply Proposition 7.1 to k, N_2, where $N_2\geqq N+kS_2/2+S_1/2$. Then

$$\left|(1+|\ell|^2)^N|\varrho_2(\lambda)|^{k+1-j}\right|\leqq C\frac{(1+|\ell|^2)^{N_2}}{|p_1(\lambda)|}|p_2(\lambda)|.$$

Since $\dfrac{(1+|\ell|^2)^{N_2}}{p_1(\lambda)}p_2(\lambda)D_\ell^\gamma\|\pi_\ell(\psi)h_\alpha\|^2\leqq C_\alpha\|\psi\|_{k',N'}$ for some k', N' by Proposition 7.1, Proposition 8.1 is proved.

9. Solvability of $L\sigma_\alpha=Zf_\alpha$

Recall that if $f\in L^2(G)$, f has the L^2 decomposition

$$f\sim\sum_\alpha f_\alpha$$

where α runs over all multi-indices $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)$, $\alpha_i\geqq 0$, and f_α is uniquely determined by the condition

$$\pi_\ell(f_\alpha)=\pi_\ell(f)P_\alpha$$

for every regular ℓ. Here P_α is the orthogonal projection onto the subspace of $L^2(\mathbf{R}^d)$ spanned by the Hermite function $h_\alpha=h_{\alpha_1}(y_1)\dots h_{\alpha_d}(y_d)$. By Proposition 4.3, $\pi_\ell(L)h_\alpha=m_\alpha(\ell)h_\alpha$ where $m_\alpha(\ell)=-\sum_k\xi_k^2-\sum_j\varrho_j(2\alpha_j+1)+i\sum_q C_q\lambda_q$. We would like to prove that $L\sigma_\alpha=Zf_\alpha$ has a distribution solution σ_α for any $f\in C_0^\infty(G)$ for some non-zero $Z\in\mathscr{U}(\mathscr{J}_2)$ provided that $m_\alpha(\ell)$ does not vanish identically on an open set.

(9.1) Example. Suppose $\mathscr{J}$ is the three-dimensional Heisenberg algebra, and $L=X_1^2+X_2^2+i[X_1,X_2]$. Then $m_0=-|\lambda|-\lambda$ vanishes for all $\lambda<0$, and $L\sigma_0=Zf_0$ is not solvable for "most" f, if $Z\neq 0$. (See [7].)

Recall that the *regular* set is the open subset of all $(\xi, \lambda)=\ell$ for which $\det \lambda([X_j, X_k])$, $1 \leqq j, k \leqq 2d$, does not vanish and for which the number of distinct eigenvalues is maximal. The regular set consists of a finite number of connected components. We now prove the key theorem in solving $L\sigma=f$.

(9.2) Theorem. *There exists* $Z \in \mathcal{U}(\mathcal{J}_2)$ *such that the equation*

$$L\sigma_\alpha = Zf_\alpha \tag{9.3}$$

has a global distribution solution σ_α *whenever* $f \in C_0^\infty(G)$, *and*

(9.4) $m_\alpha(\ell)$ *does not vanish on any component of the regular set.*

Proof. Regard Zf_α as a tempered distribution, and note that $(Zf)_\alpha = Zf_\alpha$. By the Plancherel Theorem for G, if $\psi \in C_0^\infty(G)$,

$$\begin{aligned} Zf_\alpha(\psi) &= \int \operatorname{tr}\left(\pi_\ell(\bar{\psi})^* \pi_\ell(Zf_\alpha)\right) d\mu(\ell) \\ &= \int \operatorname{tr}\left(\pi_\ell(\bar{\psi})^* \pi_\ell(Zf) P_\alpha\right) d\mu(\ell) \\ &= \int \left(\pi_\ell(Zf) h_\alpha, \pi_\ell(\bar{\psi}) h_\alpha\right) d\mu(\ell), \end{aligned}$$

where $(\ ,\ )$ denotes the inner product in $L^2(\mathbf{R}^d)$. If $z(\lambda)$ is the polynomial satisfying

$$z(\lambda) = \pi_\ell(Z^\tau),$$

then

$$Zf_\alpha(\psi) = \int z(\lambda)\left(\pi_\ell(f) h_\alpha, \pi_\ell(\bar{\psi}) h_\alpha\right) d\mu(\ell). \tag{9.5}$$

Now if σ_α satisfies (9.3), then

$$\sigma_\alpha(L^\tau \psi) = L\sigma_\alpha(\psi) = Zf_\alpha(\psi). \tag{9.6}$$

Hence σ_α is determined on the subspace $\{L^\tau \psi : \psi \in C_0^\infty(G)\}$ by

$$\sigma_\alpha(L^\tau \psi) = \int z(\lambda)\left(\pi_\ell(f) h_\alpha, \pi_\ell(\bar{\psi}) h_\alpha\right) d\mu(\ell).$$

To prove that σ_α extends to a distribution, by the Hahn—Banach Theorem it suffices to prove the crucial estimate

$$|\sigma_\alpha(L^\tau \psi)| \leqq C_{\alpha, f} \|L^t \psi\|_{k, N} \tag{9.7}$$

for some constant $C_{\alpha, f}$ and some k, $N \geqq 0$.

We shall compare $Zf_\alpha(L^\tau \psi)$ with $Z' f_\alpha(\psi)$. First observe that

$$\pi_\ell(L^\tau \psi) h_\alpha = \pi_\ell(\psi) \pi_\ell(L) h_\alpha = \pi_\ell(\psi) m_\alpha(\ell) h_\alpha. \tag{9.8}$$

By (9.5) and (9.8),

$$Zf_\alpha(\psi) = \int z(\lambda)\left(\pi_\ell(f) h_\alpha, \pi(\bar{\psi}) h_\alpha\right) d\mu(\ell). \tag{9.9}$$

Now suppose $z=z_1z_2z_3$ with

$$z_j = \pi_\ell(Z_j^\tau), \quad j = 1, 2, 3. \tag{9.10}$$

Applying Schwartz' inequality to (9.9) we obtain

$$|Zf_\alpha(\psi)| \leqq \left\{\int \|\pi_\ell(f)\, h_\alpha\|^2\, d\mu(\ell)\right\}^{1/2} \tag{9.11}$$

$$\cdot\left\{\int |z_1(\lambda)|^2\, |z_2(\lambda)|^2\, |z_3(\lambda)|^2\, \|\pi_\ell(\psi)\, h_\alpha\|^2\, d\mu(\ell)\right\}^{1/2}.$$

The first factor on the right hand side is just $\|f_\alpha\|$, where $\|\ \|$ denotes the L^2 norm in G. In order to estimate the second factor we need the following.

(9.12) Lemma. *For any N', there exist N, k and polynomials $z_3(\lambda)$ and $z_3'(\lambda)$ such that*

$$\sup_\ell\,(1+|\ell|)^{N'}\, \|z_1(\lambda)\, z_3(\lambda)\, \pi_\ell(\psi)\, h_\alpha\|$$

$$\leqq C_\alpha \sup_{\substack{\ell \\ |\beta|\leqq k}} \{(1+|\ell|)^{2N} \left|D_\ell^\beta(\|m_\alpha(\ell)\, z_1(\lambda)\, z_3'(\lambda)\, \pi_\ell(\psi)\, h_\alpha\|^2)\right|\}^{1/2}.$$

The proof of this lemma, which requires some estimates of Hörmander and Lojasiewicz, will be given in § 10. Assuming Lemma 9.12, we now complete the proof of Theorem 9.2. Write $Z=Z_1Z_2Z_3$, with Z_j defined by (9.10). Choose a polynomial $z_1(\lambda)$ as in Proposition 8.1. Next choose N' and $z_2(\lambda)$ so that

$$\int (1+|\ell|^2)^{N'}\, |z_2(\lambda)|^2\, d\mu(\lambda) < \infty, \tag{9.13}$$

which is possible by Proposition 5.6. Then for any choice of $z_3(\lambda)$, by (9.13),

$$|\sigma_\alpha(L^\tau\psi)| \leqq C'_{\alpha,f} \sup_\ell \left\{(1+|\ell|^2)^{N'}\, \|z_1(\lambda) z_3(\lambda) \pi_\ell(\psi) h_\alpha\|^2 \right. \tag{9.14}$$

$$\left. \cdot\int (1+|\ell|^2)^{-N'}\, |z_2(\lambda)|^2\, d\mu(\ell)\right\}^{1/2}$$

$$\leqq C''_{\alpha,f} \sup_\ell\, \{(1+|\ell|^2)^{N'}\, \|z_1(\lambda)\, z_3(\lambda)\, \pi_\ell(\psi)\, h_\alpha\|^2\}^{1/2}.$$

Finally, choose $z_3(\lambda)$ to satisfy Lemma 9.12. Then (9.14) gives

$$|\sigma_\alpha(L^\tau\psi)| \leqq C''_{\alpha,f}\, C'_\alpha \sup\left\{(1+|\ell|^2)^N \left|D_\ell^\beta \|z_1(\lambda)\, m_\alpha(\ell)\, z_3'(\lambda) \pi_\ell(\psi)\, h_\alpha\|^2\right|\right\}^{1/2} \tag{9.15}$$

$$\leqq C''_{\alpha,f}\, C_\alpha \sup\, \{(1+|\ell|)^{2N}\, N|D_\ell^\beta \|z_1(\lambda) \pi_\ell(Z_3'^\tau L^\tau\psi)\, h_\alpha\|^2|\}^{1/2}$$

since

$$m_\alpha(\ell)\, z_3'(\lambda)\, \pi_\ell(\psi)\, h_\alpha = \pi_\ell(Z_3'^\tau L^\tau\psi)\, h_\alpha.$$

(Here $z_3'=\pi_\ell(Z_3')$.) Now apply Proposition 8.1 to the right hand side of (9.15). Then the proof of theorem 9.5 is complete, modulo Lemma 9.12.

10. Application of the estimates of Hörmander—Lojasiewicz

In order to apply the estimates of Hörmander [14] and Lojasiewicz [21] involving the division of distributions by polynomials for the proof of Lemma 9.12 we must replace $m_\alpha(\ell)$ by a polynomial. This is accomplished by the following.

(10.1) Lemma. *Suppose $m_\alpha(\ell)$ does not vanish identically on any component $\mathscr{C}_j$ of the regular set. Then there is a non-zero polynomial $q_\alpha^j(\ell)$ such that*

$$m_\alpha(\ell) = q_\alpha^j(\ell)/s_\alpha^j(\ell) \quad on\ \mathscr{C}_j,$$

where s_α^j is a function with the following property: for each k, there is a polynomial $z_4(\lambda)$ and an integer $N''>0$ such that

$$\sup_{|\beta|\leqq k} \left|D_\ell^\beta (z_4(\lambda) s_\alpha^j(\ell))\right| \leqq C(1+|\ell|)^{N''} \tag{10.2}$$

for all ℓ.

Proof. Let $ir_1(\lambda), ir_2(\lambda), \ldots, ir_m(\lambda)$ be the distinct eigenvalues of the matrix $S(\lambda)$ on $\mathscr{C}^j$. By the implicit function theorem it is not hard to show that the $r_k(\lambda)$ are analytic functions on $\mathscr{C}^j$. Then there are constants $C_k(\alpha)$ such that

$$m_\alpha(\ell) = -\sum_{j=1}^{n-2d} \xi_j^2 - \sum_{k=1}^m |r_k(\lambda)|\, C_k(\alpha) + i \sum_{q=1}^P C_q \lambda_q. \tag{10.3}$$

Now let $P(m)$ be the set of all permutations on $\{1, 2, \ldots, m\}$. If $\tau\in P(m)$, then $\tau=(\tau_1, \tau_2, \ldots, \tau_m)$ with $1\leqq\tau_i\leqq m$, and we put

$$m_\alpha^\tau(\ell) = -\sum_{j=1}^{n-2d} \xi_j^2 - \sum_{k=1}^m r_{\tau_k}(\lambda)\, C_k(\alpha) + i \sum_{q=1}^P C_q \lambda_q.$$

Then each m_α^τ is an analytic function on $\mathscr{C}_j$, and one may assume the $C_k(\alpha)$ have been chosen so that

$$m_\alpha(\ell) = m_\alpha^{\tau_0}(\ell) \quad \text{on } \mathscr{C}_j$$

for some τ_0. (For this one must take the coefficient in (10.3) to be zero for one of each pair of conjugate roots.) Now let K_j be the cardinality of $P_0^j=\{\tau\in P(m)\colon m_\alpha^\tau(\ell)$ does not vanish identically on $\mathscr{C}_j\}$, i.e. $K_j=|P_0^j|$. Define a symmetric polynomial q_α^j in the roots of $S(\lambda)$ by

$$q_\alpha^j = \sum_{\substack{|P^j|=K_j \\ P^j\subset P(m)}} \prod_{\tau\in P^j} m_\alpha^\tau(\ell). \tag{10.4}$$

Hence $q_\alpha^j(\ell)$ is actually a polynomial in ℓ. Furthermore,

$$q_\alpha^j = \prod_{\tau\in P_0^j} m_\alpha^\tau(\ell) \quad \text{on} \quad \mathscr{C}_j,$$

since all the other terms must vanish on $\mathscr{C}_j$, by definition of K^j. Furthermore, q_α^j does not vanish identically on $\mathscr{C}_j$ since each m_α^τ is analytic and not identically zero if $\tau\in P_0^j$. Furthermore, since, by assumption, $m_\alpha^{\tau_0}$ does not vanish identically on $\mathscr{C}_j$,

$\tau_0 \in P_0^j$, and

$$m_\alpha(\ell) = q_\alpha^j(\ell) / \prod_{\substack{\tau \in P_0^j \\ \tau \neq \tau_0}} m_\alpha^\tau(\ell).$$

If $z_4(\lambda)$ is a sufficiently high power of a polynomial vanishing off the regular set, it is clear that

$$s_\alpha^j = \prod_{\substack{\tau \in P_0^j \\ \tau \neq \tau_0}} m_\alpha^\tau(\ell)$$

satisfies (10.2) for some C, N''.

We now apply Lemma 10.1 and the estimates of [14] and [21] to prove the following.

(10.5) Lemma. *There are polynomials $z_5(\lambda)$ and $z_5'(\lambda)$, vanishing off the regular set, and positive integers k, N', for any given N, such that if $\chi(\ell) \in C^k$,*

$$\sup_{\ell \in \mathscr{C}_j} (1+|\ell|)^{2N} |z_5(\lambda)\chi(\ell)| \leqq C_\alpha \sup_{\substack{\ell \in \mathscr{C}_j \\ |\beta| \leqq k}} (1+|\ell|)^{2N'} \left| D_\ell^\beta \left(m_\alpha^2(\ell) z_5'^2(\lambda) \chi(\ell) \right) \right|$$

for any component $\mathscr{C}_j$.

Proof. For any j, put

$$\chi_j(\ell) = \begin{cases} \chi(\ell) & \ell \in \mathscr{C}_j \\ 0 & \ell \notin \mathscr{C}_j. \end{cases}$$

Then one may choose $z_5^2(\lambda)$, vanishing of order at least k off the regular set, so that $z_5(\lambda)\chi_j(\ell) \in C^k$. If $q_\alpha^j(\ell)$ is the polynomial of Lemma 10.1, then by [14, formula (4.3)], applied to the polynomial $(q_\alpha^j(\ell) z_4(\lambda))^2$,

$$\sup_\ell (1+|\ell|)^{2N} |z_5^2(\lambda)\chi_j(\ell)| \tag{10.6}$$

$$\leqq C_\alpha' \sup_{\substack{\ell \\ |\beta| \leqq k}} (1+|\ell|)^{2N'''} \left| D_\lambda^\beta D_\xi^\gamma \left((q_\alpha^j(\ell))^2 (z_4(\lambda))^2 z_5^2(\lambda) \chi_j(\lambda) \right) \right|$$

for some $N''' > 0$, where z_4 is the polynomial of Lemma 10.1. Now

$$(q_\alpha^j(\ell))^2 z_4^2(\lambda) z_5^2(\lambda) \chi_j(\ell) = m_\alpha(\ell)^2 (s_\alpha^j(\ell) z_4(\ell))^2 z_5^2(\lambda) \chi_j(\ell) \tag{10.7}$$

since $q_\alpha^j(\ell) = m_\alpha(\ell) s_\alpha^j(\ell)$ on $\mathscr{C}_j$, and both sides vanish off $\mathscr{C}_j$. Hence the lemma follows by 10.2, with $z' = q_\alpha^j(\ell) z_4(\lambda) z_5(\lambda)$.

We may now complete the proof of Lemma 9.12. First choose any polynomial $z_6(\lambda)$ as in Proposition 8.1 so that $\|z_6(\lambda)\pi_\ell(\psi)h_\alpha\|^2$ has k continuous derivatives in ℓ, and put

$$\chi(\ell) = \|z_6(\lambda)\pi_\ell(\psi)h_\alpha\|^2.$$

By Lemma 10.5,

$$\sup_\ell (1+|\ell|)^{2N} \|z_5(\lambda) z_6(\lambda) \pi_\ell(\psi) h_\alpha\|^2$$

$$\leqq C_\alpha \sup_{\substack{\ell \\ |\beta| \leqq k}} (1+|\ell|)^{2N'} \left| D_\ell^\beta \left(\|m_\alpha(\ell) z_5'(\lambda) z_6(\lambda) \pi_\ell(\psi) h_\alpha\|^2 \right) \right|$$

from which Lemma 9.12 follows for $z_3 = z_5' z_6$ by taking square roots.

11. Polar coordinates in $so(n)$, the set of all skew symmetric matrices

In order to prove our main result, the case where $\mathfrak{G}$ is the free Lie algebra of step 2, we introduce a change of coordinates in $so(n)$, $n=2d$, which may be identified with $\mathfrak{G}_2^*$ by

$$\lambda\in\mathfrak{G}_2^* \leftrightarrow (\lambda[X_j, X_k])\in so(n),$$

which is $1-1$ and onto since $\mathfrak{G}$ is free. It will be convenient to note that $so(n)$ has a natural inner product, given by

$$(A, B) = \operatorname{tr}(AB^*), \tag{11.1}$$

where tr denotes trace. Now for any $\lambda_0\in\mathfrak{G}_2^*$ there is a matrix

$$A(\varrho) = \operatorname{diag}\left(\begin{pmatrix} 0 & \varrho_1 \\ -\varrho_1 & 0\end{pmatrix}, \begin{pmatrix} 0 & \varrho_2 \\ -\varrho_2 & 0\end{pmatrix}, \dots, \begin{pmatrix} 0 & \varrho_d \\ -\varrho_d & 0\end{pmatrix}\right) \tag{11.2}$$

which is conjugate to λ_0 via a unitary matrix. Let $\mathfrak{U}$ be the subalgebra of $so(n)$ consisting of all matrices of the form (11.2) and $\mathfrak{N}$ the orthogonal complement of $\mathfrak{U}$ in $so(n)$ under the pairing (11.1). Then

$$\dim\mathfrak{N} = \dim so(n)-\dim\mathfrak{U} = n(n-1)/2-n/2 = n^2/2-n. \tag{11.3}$$

Choose a basis $\{I_j\}$ of $\mathfrak{N}$. If $\omega^0\in\mathfrak{N}$ there is a mapping of a neighborhood of ω^0 into $SO(n)$, the connected group corresponding to $so(n)$, given by

$$\omega \to K(\omega) = \prod_{j=1}^{n^2/2-n} \operatorname{Exp}\omega_j I^j s$$

if $\omega=\sum\omega_j I^j$. The tangent space to the image may be identified with $\mathfrak{N}$. Now define Φ: $\mathfrak{U}\times\mathfrak{N}\to so(n)$ by

$$\Phi(\varrho, \omega) = K(\omega)A(\varrho)K(\omega)^{-1}.$$

(11.4) Theorem. *Φ is a local isomorphism in a neighborhood of (ω^0, ϱ^0) if $\Phi(\varrho^0, \omega^0)$ is regular. More precisely,* $\det d\Phi$ *is a symmetric polynomial in the ϱ_j which vanishes only if the eigenvalues of $A(\varrho^0)$ are not all distinct. Hence*

$$\det d\Phi = q(\lambda), \tag{11.5}$$

where $q(\lambda)$ is polynomial which does not vanish on the regular set.

Proof. Since the ϱ_j are the roots of the characteristic polynomial of $S(\lambda)$, the last statement of the theorem will follow from the rest. The theorem will be proved by calculating $\det d\Phi$.

We follow a similar calculation by Helgason [11, Chapter VII, Proposition 3.1]. Let M be the connected component of the centralizer of $\mathfrak{U}$ in $SO(n)$ i.e. the set of matrices y for which $yAy^{-1}=A$ for all $A\in\mathfrak{U}$. Let $t(y)$: $SO(n)/M\to SO(n)/M$ be defined by

$$t(y)(xM) = yxM.$$

For any $y \in SO(n)$, $Y \in so(n)$, let $\mathrm{Ad}\, y \cdot Y = yYy^{-1}$. Now since the map K: $\mathfrak{N} \to SO(n)$ is a local isomorphism of a neighborhood of 0 in $\mathfrak{N}$ onto an open neighborhood in $SO(n)/M$, it suffices to consider Φ as the mapping

$$\Phi\colon\ SO(n)/M \times \mathfrak{A} \to so(n)$$

given by $\Phi(yM, A) \to yAy^{-1}$. (Since M centralizes A this is well defined.)

Now suppose $(y_0 M, A(\varrho_0)) \in SO(n)/M \times \mathfrak{U}$ with $A_0 = A(\varrho_0)$ regular. If B runs through $\mathfrak{N}$ and A runs through $\mathfrak{U}$,

$$\left(dt(y_0)B, A\right)$$

runs through the tangent space of $SO(n)/M \times \mathfrak{U}$ at $(y_0 M, A_0)$. By definition,

$$\Phi(y_0 M, A_0 + sA) = \mathrm{Ad}\, y_0(A_0 + sA), \quad s \in \mathbf{R}.$$

Hence

$$d\Phi_{(y_0 M, A_0)}\left(dt(y_0)B, A\right) = \mathrm{Ad}\, y_0([B, A_0] + A).$$

Using the given basis for $\mathfrak{N}$ we may calculate $\det \mathrm{Ad}\, y_0([B, A_0]+A) = \det([B, A_0]+A)$ explicitly. First, since $[\mathfrak{N}, A_0] \subset \mathfrak{N}$ and $\mathfrak{U} + \mathfrak{N}$ is a direct sum, $\det d\Phi$ is the product of the determinant of the mappings, $dt(y_0)\mathfrak{N} \to \mathfrak{N}$ given by $dt(y_0)B \to [B, A_0]$ and $\mathfrak{U} \to \mathfrak{U}$ given by the identity. The determinant of the second mapping is one, while the determinant of the first is an easy calculation. This completes the proof of Theorem (11.4).

12. Distribution solutions of $L\sigma = Zf$ on free groups

By Theorem 9.2, for a given multi-index α, there is a global distribution σ_α with

$$L\sigma_\alpha = Zf_\alpha$$

provided that m_α is not identically zero on any open set. In specific cases, where more information is known about the functions $m_\alpha(\ell)$, we shall be able to solve the equation $L\sigma = Zf$.

(12.1) Theorem. *Let $\mathfrak{G}$ be free on n generators, $X_1, X_2, \dots, X_n$, with n even, $n > 2$ and put $L = \sum X_j^2 + i \sum_{1 \leq j,k \leq n} a_{jk}[X_j, X_k]$ where the a_{jk} are real constants. Then there exists $Z \in U(\mathfrak{G}_2)$ such that*

$$L\sigma = Zf \tag{12.2}$$

has a global distribution solution for any $f \in C_0^\infty(G)$.

Remark. The restriction $n > 2$ cannot be removed, since the theorem is false if $\mathfrak{G}$ is the Heisenberg algebra. The conditions n even and a_{jk} real are not essential, but are merely reductions to the crucial case. (See Theorems (13.1) and (13.8)).

The proof of Theorem 12.1 will be given in two parts. First we show that (9.4) is satisfied for all multi-indices α.

(12.3) Lemma. *If $\mathfrak{G}$ is free on $n>2$ generators, $m_\alpha(\ell)$ does not vanish on any component of the regular set.*

Proof. Let $\ell_0=(0, \lambda_0)$ where λ_0 is the linear functional determined by

$$\lambda_0([X_1, X_2]) = \lambda_1 \neq 0$$
$$\lambda_0([X_1, X_3]) = \lambda_2 \neq 0$$
$$\lambda_0([X_2, X_3]) = 0$$

and $\lambda_0([X_j, X_k])=0$ if j or k is greater than 3. Now if $\mathscr{C}$ is any component of the regular set, then there is a point of the form λ_0 in its closure, where λ_1, and λ_2 are so chosen that

$$m_\alpha(\ell_0) = (2\alpha_1+1)\sqrt{\lambda_1^2+\lambda_2^2}-2a_{12}\lambda_1-2a_{13}\lambda_2 \neq 0.$$

(12.4) Corollary. *For every multi-index α, the estimates of Lemma 9.12 hold for some polynomials $z_3(\lambda)$ and $z_3'(\lambda)$.*

We now come to the main part of the proof of Theorem 12.1. In order to prove the existence of a solution σ for (12.2), it suffices to prove that Lemma 9.12 holds with C_α, $z_3(\lambda)$ and $z_3'(\lambda)$ all independent of α. We divide the multi-indices α into two groups. Let

$$\mathscr{A}_1 = \left\{\alpha\colon \sup_{1\leqq i\leqq n} (2\alpha_i+1) \leqq n \textstyle\sum_{j,k}|\alpha_{jk}|\right\},$$

and let $\mathscr{A}_2$ be the complement of $\mathscr{A}_1$. Since $\mathscr{A}_1$ is a finite set, Lemma 9.12 is true with C_α, N and k all independent of α. By applying Proposition 7.1 as in Sections 10 and 11 and summing over α (which is possible by (7.3)) it suffices to prove

(12.5) Proposition. *There exist polynomials $z(\lambda)$ and $z'(\lambda)$ and an integer N' such that*

$$\sup_\lambda |z^2(\lambda)\chi(\lambda)| \leqq C \sup_{\substack{\lambda \\ |\beta|\leqq 2}} (1+|\ell|^2)^{N'} D_\lambda^\beta\big(m_\alpha^2(\lambda)(z'(\lambda))^2\chi(\lambda)\big), \tag{12.6}$$

for all $\chi\in C^2(so(n))$, the space of twice differentiable functions, and all multi-indices $\alpha\in\mathscr{A}_2$.

(Cf. Lemma 10.5.) Note that in the case of a free algebra, there is no ξ-component if ℓ is regular and hence $\ell=\lambda$.

The proof is roughly based on the fact that if $\alpha\in\mathscr{A}_2$, then m_α is locally a linear function in one of its variables. Then we shall show that it is possible to divide by the square of this function. For this, we shall follow the arguments in Schwartz

[31, Chapitre IV, § 5]. Complications arise in this case because of our need to change variables to obtain a linear function.

The division will be based on the following elementary estimate.

(12.7) Lemma. *Let $\delta>0$ be given. Then there exists a constant $C_\delta>0$ such that for all functions $g\in C^k(\mathbf{R})$ and any real constants a and b with $|a|\geqq\delta$,*

$$\sup_{t\in\mathbf{R}}|g(t)| \leqq C_\delta \sup_{\substack{k\leqq 2\\ t\in\mathbf{R}}}\left|\frac{d^k}{dt^k}(at+b)^2 g(t)\right| \tag{12.8}$$

Proof. An elementary argument using Taylor's formula for $x^2h(x)$ shows that

$$\sup_{y\in\mathbf{R}}|h(y)| \leqq \sup_{k\leqq 2}\left|\frac{d^k}{dx^k}(x^2h(x))\right| \tag{12.9}$$

all $h\in C^2(\mathbf{R})$.

Now (12.8) follows from (12.9) by (using the change of variables $y=at+b$.

We may now prove Proposition 12.5. Suppose $\alpha\in\mathscr{A}_2$ is fixed. Then $2\alpha_i+1> n\sum_{j,k}|a_{jk}|$ for some i, and we may as well assume $i=1$. Let Φ be the polar coordinates introduced in § 11. Then

$$m_\alpha(\Phi(\varrho,\omega)) = -\sum\varrho_j(2\alpha_j+1)+\sum\tau_k(\omega)\varrho_k \tag{12.10}$$

for some real-valued functions τ_k. Now fix $\omega, \varrho_2, \varrho_3, \dots, \varrho_d$ and let ϱ_1 vary. Since

$$m_\alpha(\Phi(\varrho,\omega)) = -\varrho_1[(2\alpha_1+1)-\tau_1]-\textstyle\sum_{j\geqq 2}\varrho_j[(2\alpha_j+1)-\tau_j],$$

$$m_\alpha(\Phi(\varrho,\omega)) = a\varrho_1+b, \tag{12.11}$$

where $a=(2\alpha_1+1)+\tau_1$ and $b=-\sum_{j\geqq 2}\varrho_j(2\alpha_j+1-\tau_j)$. Furthermore, since $\alpha\in\mathscr{A}_2$, $|\alpha|\geqq\delta>0$, δ constant. Hence we may apply (12.9) to the function $h_1(\varrho_1)=h(\varrho_1,\varrho_2,\dots,\varrho_d,\omega)$.

Finally, suppose $\chi\in C^k(so(n))$. Then $\tilde\chi=\chi\circ\Phi\in C^k(\mathfrak{U}\times\mathfrak{N})$. Let $p(\lambda)$ be the symmetric polynomial defined by

$$p(\lambda) = \Pi_{j=1}^d\varrho_j^2(\lambda).$$

Since $p(\lambda)$ is a symmetric polynomial in λ which vanishes at any $\varrho_j=0$, $p(\lambda)$ vanishes to order at least 2 on the set where any $\varrho_j=0$. Now for $\varphi\in C^k(so(n))$, let $\tilde\varphi\in C^k(\mathfrak{U}\times\mathfrak{N})$ defined by $\tilde\varphi(\varrho,\omega)=\varphi(\Phi(\varrho,\omega))$ and $\tilde\varphi_1\colon\mathbf{R}\to\mathbf{C}$ defined by

$$\tilde\varphi_1(t)=\begin{cases}\tilde\varphi(t,\varrho_2,\dots,\varrho_d,\omega) & \text{if}\quad t\geqq 0\\ 0 & \text{otherwise.}\end{cases}$$

Clearly

$$\sup_{t\in\mathbf{R}}|\tilde\varphi_1(t)| = \sup_{\substack{\lambda\in so(n)\\ \lambda=\Phi(\varrho,\omega)\\ \varrho_2,\varrho_3,\dots,\varrho_d,\omega\ \text{fixed}}}|\varphi(\lambda)|, \tag{12.12}$$

since any λ can be written as $\lambda=\Phi(\varrho_1, \varrho_2, \ldots, \varrho_d, \omega)$ with $\varrho_1 \geqq 0$. We shall show that there exists an integer N' and a polynomial $w(\lambda)$ such that for all $\varphi\in C^2(so(n))$,

$$(12.13)\qquad \sup_{\substack{t\in\mathbf{R}\\ k\leqq 2}}\left|\frac{d^k}{dt^k}(w^2\varphi)_1^{\sim}(t)\right| \leqq C' \sup_{\substack{\beta\leqq 2\\ \lambda=\Phi(\varrho,\omega)\\ \varrho_2,\varrho_3,\ldots,\varrho_d,\,\omega \text{ fixed}}} (1+|\lambda|^2)|^{N'}\left|D_\lambda^\beta(\varphi)(\lambda))\right|$$

Now suppose (12.13) is proved. We shall obtain (12.6). For this, choose s even and large enough so that $p^s\varrho_{k_1}$ and $p^s\varrho_{k_1}\varrho_{k_2}$ are in $C^2(so(n))$ for all k_1, k_2 and satisfy

$$(12.14)\qquad \sup_{\substack{k\leqq 2\\ |\beta|\leqq k}}\left|D_\lambda^\beta(p^s q_1(\varrho))\right| \leqq C_{q_1}(1+|\lambda|^2)^{N_1}$$

for any polynomial $q_1(\varrho)$ of degree less than or equal to two. Then let $z(\lambda)=p^s(\lambda)\omega(\lambda)$. By (12.8) applied to the function $g(t)=(\tilde{z}_1^2\tilde{\chi}_1)(t)$,

$$(12.15)\qquad \sup_{t\in\mathbf{R}}|\tilde{z}_1^2\tilde{\chi}_1(t)| \leqq C_\delta \sup_{\substack{k\leqq 2\\ t\in\mathbf{R}}}\left|\frac{d^k}{dt^k}(m_\alpha^2 z^2\chi)_1^{\sim}(t)\right|$$

for all $\chi\in C^2(so(n))$. By (12.15) and (12.12) we have

$$(12.16)\qquad \sup_{\substack{\lambda\in so(n)\\ \lambda=\Phi(\varrho,\omega)\\ \varrho_2,\varrho_3,\ldots,\varrho_d,\,\omega\text{ fixed}}} |z^2\chi(\lambda)| \doteq \sup_{t\in\mathbf{R}}|\tilde{z}_1^2\tilde{\chi}_1(t)| \leqq C_\delta \sup_{\substack{k\leqq 2\\ t\in\mathbf{R}}}\left|\frac{d^k}{dt^k}(m_\alpha^2 z^2\chi)_1^{\sim}(t)\right|$$

Now put $\varphi=p^s(\lambda)m_\alpha^2(\lambda)\chi(\lambda)$, and apply (12.13) to the right hand side of (12.16). Then

$$(12.17)\qquad \sup_{\substack{t\in\mathbf{R}\\ k\leqq 2}}\left|\frac{d^k}{dt^k}(p^s w^2(\lambda)m_\alpha^2\tilde{\chi})_1^{\sim}(t)\right| \leqq C' \sup_{\substack{|\beta|\leqq 2\\ \lambda=\Phi(\varrho,\omega)\\ \varrho_2,\varrho_3,\ldots,\varrho_d,\,\omega\text{ fixed}}} \left|(1+|\lambda|^2)^{N'}D_\lambda^\beta(p^s m_\alpha^2\chi)(\lambda)\right|,$$

Hence (12.6) will follow immediately from (12.16) & (12.17), since $p^s m_\alpha^2\in C^2(so(n))$.

We still must prove (12.13). By direct calculation, for $t>0$, in any local coordinate system

$$(12.18)\qquad \left|\frac{d^2}{dt^2}(w\varphi)_1^{\sim}\right| = \left|\frac{\partial^2}{\partial\varrho_1^2}\tilde{w}\tilde{\varphi}\right| = \left|\left(\sum\frac{\partial\lambda_k}{\partial\varrho_1}\frac{\partial}{\partial\lambda_k}\right)^2(w\varphi)(\lambda)\right|$$

$$\leqq \sum\nolimits_{j,k}\left|\frac{\partial\lambda_k}{\partial\varrho_1}\right|\left|\frac{\partial\lambda_j}{\partial\varrho_1}\right|\left|\frac{\partial^2}{\partial\lambda_k\partial\lambda_j}(w\varphi)\right|$$

$$+\sum\nolimits_{j,k}\left|\frac{\partial\lambda_k}{\partial\varrho_1}\right|\left|\frac{\partial}{\partial\lambda_k}\left(w\left(\frac{\partial\lambda_j}{\partial\varrho_1}\right)\right)\right|\left|\frac{\partial}{\partial\lambda_j}(\psi)\right|.$$

By Theorem 11.4, there exists C_1 so that

$$\left|\frac{\partial\lambda_k}{\partial\varrho_1}\right| \leqq C_1,$$

all k, uniformly in λ. Also, $d\Phi^{-1}=(d\Phi)^{-1}$ is a polynomial in the coefficients of $d\Phi$ divided by $\det d\Phi$. Hence one may choose $w(\lambda)$, divisible by a power of the polynomial $q(\lambda)=\det d\Phi$ as well as by a power of $p(\lambda)$, so that $\left|\frac{\partial}{\partial\lambda_k}\left(w\left(\frac{\partial\lambda_j}{\partial\varrho_1}\right)\right)\right|$ is bounded by a constant plus a power of $|\lambda|$. Then (12.13) follows from (12.18).

Now the proof of Proposition 12.5, and hence of Theorem 12.1, is complete.

13. Solvability of $L\sigma=Zf$ on other nilpotent groups

In other situations in which one has good control over $m_\alpha(\ell)$, it may be possible to prove solvability.

(13.1) Theorem. *If* $\dim \mathfrak{G}$ *is odd, or, more generally, if* $\det \lambda([X_j, X_k])=0$ *for all linear functions* λ *on* $\mathfrak{G}_2$, *then there exists* $Z\in\mathscr{Z}(\mathfrak{G})$, *the center of* $\mathscr{U}(\mathfrak{G})$, *such that*

$$L\sigma = Zf$$

has a distribution solution σ *for all* $f\in C_0^\infty(G)$, *where* $L=\sum_{j=1}^n X_j^2+i\sum_{q=1}^p a_q T_q$.

The hypotheses of the theorem imply for all ℓ regular,

$$m_\alpha(\ell) = -\sum_{k=1}^{n-2d}\xi_k^2-\sum|\varrho_j|(2\alpha_j+1)-\sum a_q\lambda_q$$

with $n-2d>0$. We will proceed by proving an analogue of Lemma 13.7, with Z determined by the following.

(13.2) Proposition. *There exists* $Z\in\mathscr{Z}(\mathfrak{G})$ *the center of* $\mathfrak{G}$, *and polynomials* $p(\lambda)$ *and* $q(\lambda)$, *with* $q(\lambda)\geqq 0$, *such that for all regular* $\ell=(\lambda, \xi)$

$$\pi_\ell(Z_1) = iq(\lambda)^{1/2}p(\lambda)\,\xi_1. \tag{13.3}$$

Proof. By Proposition 5.6, we may choose W_1^λ so that there are polynomials $p(\lambda)$ and $q(\lambda)$ so that

$$p(\lambda)W_1^\lambda = \left(\sum_{k=1}^n p_k(\lambda)X_k\right)/(q(\lambda))^{1/2}, \tag{13.4}$$

where $p_k(\lambda)$ is a polynomial for all k. Now put

$$Z_1 = \sum p_k(-iT)X_k$$

$T=(T_1, T_2, \ldots, T_p)$. Then

$$\pi_\ell(Z_1) = \sum_{k=1}^n \pi_\ell(p_k(-iT))\pi_\ell(X_k) = \sum_{k=1}^n p_k(\lambda)\,\pi_\ell(X_k).$$

Now by (13.4)

$$p(\lambda)\,i\xi_1 = \left(\sum_{k=1}^n p_k(\lambda)\,\pi_\ell(X_k)\right)/q(\lambda)^{1/2}.$$

Hence

$$\pi_\ell(Z_1) = iq(\lambda)^{1/2}p(\lambda)\,\xi_1.$$

which proves (13.3). Finally since $\pi(Z_1)$ is a scalar operator for almost all irreducible unitary representations π of G, it follows that $Z_1 \in \mathscr{Z}(\mathfrak{G})$. (Of course this could also be proved directly by showing that $\pi_\ell([Z_1, X_k])=0$ for all k.) This completes the proof of Proposition 13.2. The analogue of Proposition 12.5 is the following.

(13.5) Lemma. *Let* Z_1 *be as in Proposition* 13.2. *Then there exists an integer* N *such that*

$$\sup_\ell |\pi_\ell(Z_1^4)\chi(\ell)| \leqq C \sup_{\substack{0\leqq j\leqq 2\\ \ell}} (1+|\xi|+|\lambda|)^N \left|\frac{\partial^j}{\partial \xi_1^j}(m_\alpha(\ell)^2\chi(\ell))\right|,$$

for all $\chi \in C^2$.

Proof. Write $m_\alpha(\ell)$ as

$$m_\alpha(\ell) = -\xi_1^2 - k,$$

where

$$k = k(\alpha, \xi', \lambda) = -\sum_{i=2}^{n-2d} \xi_i^2 - \sum_{j=1}^{d} |\varrho_j|(2\alpha_j+1) - \sum a_q\lambda_q,$$

with $\xi'=(\xi_2, \xi_3, \ldots, \xi_{n-2d})$. Consider first the case where $k>0$. Then

$$|\pi_\ell(Z_1^4)\chi(\ell)| = |q^2(\lambda)p^4(\lambda)\xi_1^4\chi(\ell)| \leqq C_1(1+|\lambda|)^N \left||-\xi_1^2-k|^2\chi(\ell)\right|,$$

with C_1 independent of ℓ, if N is sufficiently large. Next suppose $k<0$ and put $\gamma=\sqrt{-k}$. Then

$$m_\alpha(\ell) = -(\xi_1+\gamma)(\xi_1-\gamma).$$

Suppose first that $\xi_1>0$. Then $|\xi_1|\leqq|-\xi_1-\gamma|$, so that

$$(13.6)\qquad |\pi_\ell(Z_1^4)\chi(\ell)| = |q^2(\lambda)p^4(\lambda)\xi_1^4\chi(\ell)| \leqq C_1(1+|\lambda|)^N|(-\xi_1-\gamma)^2\xi_1^2\chi(\ell)|.$$

Suppose now that one can prove that there exists $C_2>0$, independent of γ, such that

$$(13.7)\qquad \sup_\ell |g(\ell)| \leqq C_2 \sup_\ell \left|\frac{\partial^2}{\partial \xi_1^2}((-\xi_1+\gamma)^2 g(\ell))\right|$$

for all $g\in C^2$. Now let $g(\ell)=(1+|\lambda|)^N|(-\xi_1-\gamma)^2\xi_1^2\chi(\ell)|$. Then by (13.6 and (13.7),

$$(13.8)\qquad \sup_{\substack{\ell\\ \xi_1\geqq 0}} |\pi_\ell(Z_1^4)\chi(\ell)| \leqq C_3 \sup_{\substack{\ell\\ \xi_1\geqq 0}} \left|\frac{\partial^2}{\partial \xi_1^2}((1+|\lambda|)^N(-\xi_1-\gamma)^2\xi_1^2(-\xi_1+\gamma)^2\chi(\ell)\right|$$

if $\xi_1>0$. This proves (13.5) for $\xi_1>0$, modulo (13.7). The case $\xi_1\leqq 0$ is similarly proved, since $|\xi_1|\leqq|-\xi_1+\gamma|$ in that case, with (13.7) replaced by

$$(13.7')\qquad \sup_\ell |g(\ell)| \leqq C_2 \sup_\ell \left|\frac{\partial^2}{\partial \xi_1^2}((-\xi_1-\gamma)^2 g(\ell))\right|$$

for all $g\in C^2$, from which we may derive (13.8), but for $\xi_1\leqq 0$. This will prove (13.5).

It remains to show (13.7) and (13.7′). (13.7) follows from (12.8). The proof of (13.7′) is similar. This proves Lemma 13.4.

The remainder of the proof of Theorem 13.1 is exactly analogous to that of Theorem 12.1.

Finally, for the case where $L=\sum_{j=1}^{n} X_j^2+\sum_{q=1}^{p} C_q T_q$ with C_q not pure imaginary, we refer to the following result of P. Lévy-Bruhl [18, Théorème 6.1].

(13.8) Theorem. *If* $L=\sum_{j=1}^{n} X_j^2+\sum_{q=1}^{p} C_q T_q$, *where* $\operatorname{Re} C_q \neq 0$ *for at least one index* q, *then* L *and* L^* *are both locally solvable.*

Now Theorem 2.2 will follow from Theorems 12.1, 13.1 and 13.8 provided that one can show that if there exists $Z \subset \mathscr{Z}(\mathfrak{G})$ such that $L\sigma = Zf$ has a distribution solution for all $f \in C_0^\infty(G)$, then L is locally solvable.

14. Existence of local smooth solutions

We prove here that the results of § 12 and § 13 imply local solvability. This will complete the proof of Theorem 2.2. The methods of this section are completely standard.

(14.1) Proposition. *Suppose there exists* $Z \in \mathscr{Z}(\mathfrak{G})$, *the center of* $\mathscr{U}(\mathfrak{G})$ *such that*

$$L\sigma = Zf \tag{14.2}$$

has a distribution solution σ *for any* $f \in C_0^\infty$. *Then for any* $f_1 \in C_0^\infty(G)$ *and any open* $U \subset G$ *with compact closure, there is a distribution* σ_1 *satisfying*

$$L\sigma_1 = f_1 \quad \text{in } U. \tag{14.3}$$

Proof. Since Z is bi-invariant, it is locally solvable by Rais' theorem [24]. More precisely, given $f_1 \in C_0(G)$ there exists $f_2 \in C^\infty(G)$ such that $Zf_2 = f_1$. If $f = \varphi f_2$, where $\varphi \in C_0^\infty(G)$ and $\varphi \equiv 1$ in a neighborhood of U, then the solution σ of (14.2) also satisfies (14.3).

The proof of Theorem 2.2 will be completed by a general result which is, no doubt, known.

(14.4) Theorem. *The following are equivalent for a left invariant differential operator* D *on a Lie group* G.

(i) *There exist neighborhoods* $U \supset V$ *of* 0 *in* G *such that*

$$D\sigma = f \quad \text{in } V \tag{14.5}$$

has a distribution solution σ *for every* $f \in C^\infty(U)$.

(ii) *There exist neighborhoods* $U' \supset V'$ *of* 0 *in* G *such that* (14.5) *has a smooth solution for every* $f \in C^\infty(U')$.

(iii) *There exists a neighborhood* V'' *of* 0 *and a distribution* τ *such that*

$$D\tau = \delta \quad \text{in } V'',$$

i.e. τ *is a local fundamental solution for* D.

Proof. We follow the method of Rouvière [29], with some modifications. One has the obvious implications (iii) implies (ii) implies (i) (by convolution with τ), so it suffices to prove (i) implies (iii). First if (14.5) is solvable for all $f \in C_0^\infty(U)$ and the closure of U is compact, then it is also solvable for all f with k continuous derivatives in U, for some k. Indeed the solvability of (14.5) implies that the bilinear form

$$\langle f, v \rangle = \int fv \, dg$$

is separately continuous on $C^\infty(U) \times C_0^\infty(\bar{V})$. Here one takes the usual topology for $C^\infty(U)$ and the least fine topology which makes the mapping $v \to D^\tau v$ continuous from $C_0^\infty(U)$ to $C^\infty(U)$. By the Banach—Steinhaus theorem since U has compact closure,

$$\langle f, v \rangle \leqq C \sup_{|\alpha| \leqq k} |D^\alpha f| \sup_{|\beta| \leqq k'} |D^\beta (D^\tau v)| \tag{14.6}$$

for some k, k'. The inequality (14.6) then extends to all $f \in C^k(U)$ and by the Hahn—Banach theorem there exists a distribution σ on U such that $D\sigma = f$.

Next, by Sobolev's lemma, there exists $K > 0$ such that $L_K^2(U) \subset C^k(U)$, where L_K^2 is the space all functions on U with all derivatives up to order K in $L^2(U)$. Now if $R_1, R_2, \ldots, R_s$ is a basis for the right invariant vector field on G, then $E = \sum_{j=1}^s R_j^2$ is an elliptic operator which commutes with all left invariant differential operators on G and hence with D. If J is sufficiently large E^J has a fundamental solution $f_K \in L_K^2(U'')$ for some neighborhood U'' of 0, i.e.

$$E^J f_K = \delta \quad \text{in } U''.$$

By the previous remarks, there is a distribution σ such that

$$D\sigma = f_K \quad \text{in } V''$$

for some neighborhood $V'' \subset U''$ of 0. Then

$$D(E^J \sigma) = E^J D\sigma = \delta \quad \text{in } V'',$$

so $\tau = E^J \sigma$ satisfies (iii).

15. An example of an unsolvable L

In earlier versions of this paper we conjectured that the condition (9.4) is always satisfied for all multi-indices α unless $\mathfrak{G}$ is the quotient of a direct sum of Heisenberg algebras by a subspace of $\mathfrak{G}_2$. (It is easy to check that for such $\mathfrak{G}$ there is an unsolvable L of the form (2.1).) The conjecture is false, as is shown by the following example, the idea for which was given me by Schmuel Friedland.

First, any 2 step Lie algebra $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2\mathfrak{G}_2=[\mathfrak{G}_1, \mathfrak{G}_2]$ may be constructed as follows. Let V be a subspace of $so(n)$. Then $\mathfrak{G}=\mathfrak{G}_V$ has basis $X_1, X_2, \dots, X_n$ with the relations $\sum_{i<j} a_{ij}[X_i, X_j]=0$ for every skew symmetric matrix $(a_{ij})\in V^{\perp}$, where $\perp$ is false with respect to the inner product $A\cdot B=\frac{1}{2}$ tr (AB). Then any $S\in V$ defines a linear functional $\lambda_s\in\mathfrak{G}_2^*$ by $\lambda_s([X_i, X_j])=s_{ij}$, which is well defined by the definition of the linear relations in $\mathfrak{G}_2$. Conversely, all of $\mathfrak{G}_2^*$ is obtained in this way.

(15.1) Proposition. *Let $\mathfrak{G}_V$ be the 2-step Lie algebra constructed as above with $n=4$ and V spanned by* $A=\left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 2 \\ \multicolumn{2}{c|}{0} & -2 & 0 \end{array}\right)$ *and* $B=\left(\begin{array}{cc|cc} & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \\ \hline -1 & 0 & & \\ 0 & -1 & \multicolumn{2}{c}{0} \end{array}\right)$. *Then the operator $L=X_1^2+X_2^2+X_3^2+X_4^2-i[X_1, X_2]-i[X_3, X_4]$ does not satisfy (9.4) for $\alpha=(1, 1)$ and is hence unsolvable. However, $\mathfrak{G}$ is not the quotient of a product algebra $\mathfrak{G}'=\mathfrak{G}_1'+\mathfrak{G}_2'$ Heisenberg algebras by a subspace of $\mathfrak{G}_2'$.*

Proof. Let $\lambda=\lambda_{\lambda, A+\lambda_2 B}$. Then a direct calculation shows that the matrix $(\lambda([X_i, X_j]))=\lambda_1 A+\lambda_2 B$ has eigenvalues

(16.2)
$$\pm i\varrho_1=(\pm 3i\lambda_1-\sqrt{-7\lambda_1^2-\lambda_2^2})/2$$
$$\pm i\varrho_2=(\pm 3i\lambda_1+\sqrt{-7\lambda_1^2-\lambda_2^2})/2$$

in the open set where $|\lambda_2|<\sqrt{2}\lambda_1$. Then in that set $-(\varrho_1+\varrho_2)=-3\lambda$, and $-i\lambda(i[X_1, X_2]+i[X_3, X_4])=s_{12}+s_{34}=3\lambda_1$. Hence (9.4) is not satisfied for $\alpha=(1, 1)$. By [2, Theorem 5], L is unsolvable.

Now if $\mathfrak{G}$ were the quotient of a direct sum of Hersenberg algebras, it would have to be the quotient of a product of two three-dimensional Heisenberg algebras. Since $\dim\mathfrak{G}_2=2$, $\mathfrak{G}$ would itself be a direct sum of these algebras. In this case the eigenvalues of the matrix $(\lambda[X_i, X_j])$ would be linear functions in λ_1 and λ_2, which they are not.

16. Open problems

The local solvability results proved here are unfortunately incomplete even for operators of the form (1.1) on two-step nilpotent Lie groups. In view of more recent results ([18], [19], [20], [27]), it is likely that these can be solved by simpler methods than those employed here. One reasonable conjecture is the following:

Conjecture: *An operator of the form* (1.1) *is locally solvable on a two-step nilpotent Lie group if and only if $m_\alpha(\ell)$ does not vanish identically on any component of the regular set, for every multi-index α.* (See Theorem 9.2.)

The necessity of the condition on the $m_\alpha(\ell)$ has recently been proved by L. Corwin and the author [2].

It would also be interesting to extend these results to more general operators, more general groups, or even operators constructed from more arbitrary vector fields.

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Linda Preiss Rothschild
Mathematics Department
University of Wisconsin — Madison
Van Vleck Hall
480 Lincoln Drive
Madison, Wisconsin 53706
U.S.A.