On analytic functions which are in H^p for some positive p

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1. Introduction. The main results

We shall say that a domain G in the complex plane $\mathbf{C}$ is a H-domain (where H stands for Hardy) if every function analytic in the unit disk U with values in G belongs to H^p for some $p>0$. If $E=\complement G$ is the complement of G, we define

$$\gamma(r) = \operatorname{cap}(E \cap \{|w| \leqq r\}),$$
$$h(r) = (\log(5r/\gamma(r)))^{-1},$$
$$H(r) = \int_1^r h(s)\, ds/s.$$

We shall prove

Theorem 1. *Suppose that*

(1) $$\liminf_{r\to\infty} H(r)/\log r > 0.$$

Then G is a H*-domain.*

Theorem 2. *Let $a>0$ be given and suppose that for all sufficiently large r, we have the inequality $\gamma(r)<ar$. Then, if a is small enough, G is a* H*-domain if and only if* (1) *holds.*

Remark. From the proof, we shall see that we can take $a=34^{-8}$. If $\limsup_{r\to\infty} \gamma(r)/r=1$, G can be a H-domain also when condition (1) does not hold. An example is given after the proof of Lemma 1 in Section 2. Some extra condition will be needed in situations which are not covered by Theorem 2.

To introduce the condition which will be used in the present paper, let us consider an open subset M of $\{|w|=R\}$ and let $\omega_R(w, M)$ be the harmonic measure of M with respect to $\{|w|<R\}\setminus E$. We shall say that the complement E of G is not too large if there exists a positive number A with the following property: for

each large circle $\{|w|=R\}$ which does not intersect E and for each open subset M of $\{|w|=R\}$ of angular measure π, we define

$$\Omega(r, R) = \{e^{i\varphi}\colon \omega_R(re^{i\varphi}, M) \geqq (r/R)^A\}.$$

Then, if $m(\cdot)$ denotes Lebesgue measure on the unit circle, we have

$$m\Omega(r, R) \geqq \pi, \quad 0 < r \leqq R. \tag{2}$$

Theorem 3. *Assume that E is not too large. Then G is a* H*-domain if and only if* (1) *holds.*

Remark. In the definition above, we might just as well have worked with an open subset M of angular measure πs, where s is a small given positive number, and required that we would have $m\Omega(r, R) \geqq \pi s$. The price to pay would have been certain technical complications in Lemma 2. For simplicity, we have restricted ourselves to the situation discussed above.

A similar question has been discussed by Hansen and Hayman [1]. They prove (in our terminology) that if the area $A(r)$ of the set $G \cap \{|w|<r\}$ satisfies certain conditions, G will be a H-domain.

Many of the proofs in the present paper use techniques which are due to Hayman and Pommerenke [3].

2. The main lemma

Without loss of generality, we can assume that $0 \in G$. Let $\omega = \omega_R$ be the harmonic measure of $\{|w|=R\}$ with respect to G, i.e., the function which is harmonic in G, 1 a.e. on $\{|w|=R\} \setminus E$ and 0 on $\partial G \cap \{|w|<R\}$, except possibly on a set of capacity zero. We extend ω_R to a function subharmonic in the disk $\{|w|<R\}$ by defining it to be zero elsewhere in the disk.

Lemma 1. *G is a* H*-domain if and only if for some positive constants p_0 and C, we have*

$$\omega_R(0, G) \leqq CR^{-p_0}, \quad R \geqq 1. \tag{3}$$

Proof. In the proof that condition (3) is sufficient, we need a result of Hayman and Weitsman [4]. Suppose that f is analytic in U, that $f(U) \subset G$ and that $f(0)=0$. Let $G(R)$ be that component of $G \cap \{|w|<R\}$ which contains the origin. Let $2\pi L(r, R)$ be the total length of the arcs on $\{|z|=r\}$ where $|f(z)|>R$. Then we have (cf. Theorem 4 in [4]) that

$$L(r, R) \leqq \omega_R(0). \tag{4}$$

We also note that (cf. [4], p. 135)

$$(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta = -\int_0^{\infty} R^p\,dL(r,R) = p\int_0^{M(r)} L(r,R)R^{p-1}\,dR,$$

where $M(r)=\sup_\theta|f(re^{i\theta})|$. Thus, if (3) holds, we see that $f\in H^p$ for $0<p<p_0$.

Conversely, assume that G is a H-domain. Let F map U onto the infinite covering surface R_G over G in such a way that $F(0)=0$. Since G is a H-domain, there exists $p>0$ such that

$$\int_0^{2\pi}|F(re^{i\theta})|^p\,d\theta \leqq \text{Const.} < \infty, \quad 0<r<1. \tag{5}$$

Let G_1 be the component of $\{|w|<R\}\setminus E$ which contains the origin and let f_1 map U onto the infinite covering surface R_{G_1} over G_1 in such a way that $f_1(0)=0$. As in Hayman and Pommerenke [3], we apply a result of Nevanlinna (cf. [5], p. 33) which says that $\omega_R(0)$ is exactly the proportion of the circle ∂U which is mapped by f_1 onto points of $\{|w|=R\}$. Thus we have

$$\omega_R(0)R^p \leqq (2\pi)^{-1}\int_0^{2\pi}|f_1(e^{i\theta})|^p\,d\theta \leqq \liminf_{r\to 1-}(2\pi)^{-1}\int_0^{2\pi}|f_1(re^{i\theta})|^p\,d\theta$$
$$\leqq \liminf_{r\to 1-}(2\pi)^{-1}\int_0^{2\pi}|F(re^{i\theta})|^p\,d\theta \leqq \text{Const.}$$

We have used Fatou's lemma, subordination (cf. [2], pp. 74, 76) and (5). Thus (3) holds with $p_0=p$ and the lemma is proved.

Remark. We can now construct a H-domain for which condition (1) does not hold. Let $R_n=\exp(n^2)$, $n=1,2,\ldots$. Let

$$E=\bigcup_1^\infty\{w\colon |w|=R_n,\ |\arg w|\geqq 2\pi/R_{n+1}\},$$

and let G be the complement of E. We see that

$$\omega_R(0,G)\leqq (R_{n+1})^{-1}<R^{-1}, \quad R_n\leqq R\leqq R_{n+1}.$$

Consequently, all analytic functions in U with values in G are in H^p, $0<p<1$. A disk of radius r has capacity r. Thus, if $R_n\leqq R\leqq R_{n+1}$, we have

$$H(R)\leqq \sum_1^{n-1}\int_{R_k}^{R_{k+1}}(\log(5s/R_k))^{-1}\,ds/s \leqq n\log n\,(1+o(1)), \quad n\to\infty.$$

It follows that $\liminf_{R\to\infty} H(R)/\log R=0$, i.e., condition (1) does not hold.

3. Proof of Theorem 1

We can assume that $0\in G$. If R is a large positive number, we define $\omega=\omega_R$ as in Section 2 and put $B(r)=\sup_{|w|=r}\omega_R(w)$, $0\leqq r\leqq R$. Let $r>0$ be given such that $5r<R$ and choose $w_0\in C$, $|w_0|=r$. The capacity of $E_0=E\cap\{|w-w_0|<2r\}$ is at least $\gamma(r)$. Let ω_0 be the harmonic measure of E_0 with respect to $\{|w-w_0|<4r\}\setminus E_0$. From a lemma of Hayman and Pommerenke (cf. [3], p. 220), we have the estimate

(6) $$\omega_0(w_0)\geqq \log(5/4)/\log(5r/\gamma(r)),\quad |w_0|=r.$$

Let us now consider the function $h(w)=\omega_R(w)-B(5r)(1-\omega_0(w))$ which is harmonic in $G_0=G\cap\{|w-w_0|<4r\}$. On the outer circle with radius $4r$, we have $\omega_0(w)=0$ and $\omega_R(w)\leqq B(5r)$, and thus h is non-positive on this part of ∂G_0. But h is non-positive also on that part of ∂G_0 which is inside this circle, since $\omega_R(w)=0$ at all those points except possibly a set of capacity zero. Thus the maximum principle shows that we have $h(w_0)\leqq 0$, and we obtain from (6) that

$$B(r)=\sup_{|w_0|=r}\omega(w_0)\leqq\sup_{|w_0|=r}B(5r)(1-\omega_0(w_0))\leqq B(5r)(1-h(r)\log(5/4)).$$

Thus, if $1\leqq r<5$, we have

$$\log B(r)\leqq-\left(\textstyle\sum_{k=1}^{n}h(5^{k-1}r)\right)(\log 5/4),$$

where n is the largest integer such that $5^{n+1}\leqq R$. It follows that

$$\log\omega(0)\leqq \text{Const.}\int_1^5\log B(r)\,dr/r\leqq-\text{Const.}\int_1^R h(s)\,ds/s.$$

Thus, if (1) holds, there exists a positive number c such that

$$\omega_R(0)=\omega(0)\leqq\exp(-c\log R)=R^{-c},\quad R\geqq 1.$$

Applying Lemma 1, we see that G is a H-domain, and Theorem 1 is proved.

4. A lemma on harmonic measures

The following result will be needed in the proofs of Theorems 2 and 3.

Lemma 2. *Let E be a compact subset of $\{|w|\leqq R\}$ such that if $\gamma=\operatorname{cap}E$, we have*

(7) $$r_0=2R(\gamma/2R)^{1/8}<R/17.$$

Let M be an open subset of $\{|w|=R\}$ of angular measure at least π and such that $\overline{M}\cap E=\emptyset$. Let $\omega(\cdot,M)$ be the harmonic measure of M with respect to $\{|w|<R\}\setminus E$.

Then the measure of the set $F=F(r)=\{\varphi:\ \omega(re^{i\varphi}, M)\geqq 1/16\}$ *is at least* π *when* $r_0\leqq r\leqq R/17$.

Remark. When applying the lemma, we can take $\gamma\leqq R\,34^{-8}$.

Proof. Let $N=\{|w|=R\}\setminus(M\cup E)$. Let $\omega(\cdot, N)$ be the harmonic measure of N with respect to $\{|w|<R\}\setminus E$. As before, we extend $\omega(\cdot, M)$ and $\omega(\cdot, N)$ to subharmonic functions in the disk $\{|w|<R\}$ by defining them to be zero at points in the interior where they are not defined. Finally, we have a superharmonic function $\omega(\cdot, E)$ which is the harmonic measure of E with respect to $\{|w|<R\}\setminus E$ in $\{|w|<R\}\setminus E$ and 1 on E. We clearly have

$$(8)\qquad \omega(w, M)+\omega(w, N)+\omega(w, E) = 1, \quad |w| < R.$$

We also define $I_X(r)=(2\pi)^{-1}\int_0^{2\pi}\omega(re^{i\varphi}, X)\,d\varphi$, where X can be either one of the sets M, N and E.

We first note that

$$I_E(r) \leqq \big(\log(2R/\gamma)\big)^{-1}\log(2R/r), \quad 0<r<R.$$

The case $r=R/2$ can be found in Hayman and Pommerenke (cf. [3], p. 221). The general case is handled in exactly the same way. Thus, if there exists $r\in[0, R)$ such that $I_E(r)>1/8$, we must have $r<r_0$. Consequently, we obtain

$$(9)\qquad I_E(r) \leqq 1/8, \quad r_0 \leqq r < R.$$

If $I_E(0)\leqq 1/8$, (9) is clearly also true since $I_E(r)$ is nonincreasing on $(0, R)$.

Let $|X|$ denote the angular measure of the set X divided by 2π. From Poisson's formula for the unit disk, we see that

$$\omega(w, M) \leqq 9|M|/8, \quad \omega(w, N) \leqq 9|N|/8 \leqq 9(1-|M|/8), \quad |w| \leqq R/17.$$

From (8) and (9), it follows that $I_M(r)\geqq(9|M|-2)/8$, $r_0\leqq r\leqq R/17$. On the other hand, we have also the estimate

$$I_M(r) \leqq (1-|F|)/16+9|M||F|)/8, \quad 0 \leqq r \leqq R/17.$$

Combining these two estimates, we obtain Lemma 2.

5. Proof of Theorem 2

We shall construct a decreasing sequence of harmonic minorants of the harmonic measure $\omega_R(\cdot, G)$ for a large given R.

Let E be a compact subset of a circular ring $\{R_1\leqq|w|\leqq R_2\}$ with the property that $E\cap\{|w|=r\}\neq\emptyset$ for all $r\in[R_1, R_2]$. Let E_0 be the circular projection of E

onto the positive real axis, i.e., E_0 is the interval $[R_1, R_2]$. Then we know that

$$\text{cap}\, E \geqq \text{cap}\, E_0 = (R_2 - R_1)/4. \tag{10}$$

We have assumed that $\gamma(r) \leqq ar$ for all sufficiently large r, where $a>0$ is given. It follows from (10) that if $a \in (0, 1/20)$, say, there is at least one R in each interval of logarithmic length at least $5a$ such that

$$E \cap \{|w| = R\} = \emptyset. \tag{10}$$

We shall estimate $\omega_R(0)$ for numbers R such that (10) holds. Since $\omega_R(0)$ is a non-increasing function of R, this will give us control of $\omega_R(0)$ for all large R.

Let now R be given such that (10) holds. To construct a decreasing sequence $\{r_n\}$, let us take $r_1 = R$ and assume that $r_2, \ldots, r_n$ are known. We choose r_{n+1} in such a way that (10) holds with $R = r_{n+1}$ and that we have

$$2r_n(\gamma(r_n)/2r_n)^{1/8} \leqq r_{n+1} \leqq (1+6a)2r_n(\gamma(r_n)/2r_n)^{1/8}. \tag{11}$$

This is possible since the logarithmic length of the interval defined in (11) is larger than $5a$. We now use Lemma 2 to define a sequence $\{M_n\}$ of subsets of the circles $\{\{|w| = r_n\}\}$. If $M_1 = \{|w| = R\}$ and $M_2, \ldots, M_n$ are defined, we apply Lemma 2 to the disk $\{|w| < r_n\}$ with $E = E \cap \{|w| \leqq r_n\}$. Consequently, there exists a subset M_{n+1} of $\{|w| = r_{n+1}\}$ such that

$$\omega_n(r_{n+1}e^{i\varphi}, M_n) \geqq 1/16, \quad r_{n+1}e^{i\varphi} \in M_{n+1},$$

where $\omega_n(\cdot, M) = \omega_r(\cdot, M_n)$, the angular measure of M_{n+1} is at least π and M_{n+1} has a positive distance to E. From the maximum principle, we deduce

$$\omega_R(w) \geqq \omega_2(w, M_2)/16, \quad |w| \leqq r_2, \tag{12}$$

$$\omega_n(w, M_n) \geqq \omega_{n+1}(w, M_{n+1})/16, \quad |w| \leqq r_{n+1}, \quad n = 2, 3, \ldots. \tag{13}$$

Let q be the last index such that we can choose r_{q+1} according to (11). From (12) and (13), we deduce that

$$\sup_{\varphi} \omega_R(r_{q+1}e^{i\varphi}) \geqq 16^{-q}. \tag{14}$$

Since the domain G is connected and the inequality $\gamma(r) \leqq ar$ holds for all sufficiently large r, there exists a constant $C = C(G)$ only depending on G such that

$$\sup_{\varphi} \omega(r_{q+1}e^{i\varphi}) \leqq C\omega_R(0). \tag{15}$$

Since G is a H-domain, it follows from Lemma 1 and (14) that there exists $p>0$ such that

$$16^{-q} \leqq \text{Const.}\, R^{-p}. \tag{16}$$

We need an estimate of q. From (11), we see that

$$\log(R/r_{q+1}) = \sum_1^q \log(r_k/r_{k+1}) \geqq \text{Const.} \sum_1^q \log(2r_k/\gamma(r_k)).$$

It follows from Schwarz inequality that we have

$$q^2 \leqq \left(\sum_1^q \log(2r_k/\gamma(r_k))\right)\left(\sum_1^q (\log(2r_k/\gamma(r_k)))^{-1}\right),$$

and we obtain

$$\log(R/r_{q+1}) \geqq \text{Const.}\, q^2 \Big/ \int_1^R (\log(2s/\gamma(s)))^{-1}\, ds/s.$$

Combining this inequality with (16), we see that for all large R, we have

$$(\log R)^{-1} \int_1^R (\log(2s/\gamma(s)))^{-1}\, ds/s \geqq \text{Const.} > 0.$$

Thus G is a H-domain only if (1) holds. The sufficiency was proved in Theorem 1. This concludes the proof of Theorem 2.

6. Proof of Theorem 3

Let us assume that condition (1) does not hold. We shall prove that $\limsup_{R\to\infty} \log \omega_R(0)/\log R = 0$, i.e., that G can not be a H-domain since we have Lemma 1.

Let $\delta > 0$ be given and define $T = T(\delta) = \{r \geqq 1 : \gamma(r) \geqq r\delta\}$ and $T_1 = \{r \geqq 1 : s \leqq r \leqq s/\delta$ for some $s \in T\}$.

Since γ is continuous to the right, $\complement T_1 \cap (1, \infty)$ is an open set. If $r \in T_1$ and $r \geqq 1/\delta$, there exists a closed interval containing r, contained in T_1 and of logarithmic length $\log(1/\delta)$. Thus $(1, \infty)$ is a union of intervals contained in $\complement T_1$ or T_1. Essentially all intervals contained in T_1 have logarithmic length at least $\log(1/\delta)$. We also note that we have

$$\text{(17)} \qquad \gamma(r) \geqq r\delta^2, \quad r \in T_1, \quad \gamma(r) < r\delta, \quad r \in \complement T_1.$$

The boundary points of these intervals form an increasing sequence with only a finite number of points in each compact interval. Since (1) does not hold, there exists a sequence $\{R_n\}$ increasing to infinity such that $H(R_n)/\log R_n \to 0$, $n \to \infty$. Without loss of generality, we can assume that $\{|w| = R_n\} \cap E = \emptyset$, $n = 1, 2, \ldots$ (cf. the discussion at the beginning of Section 5).

We assume in the sequel that δ is so small that we can use Lemma 2. If $R \in \{R_n\}$, we shall succesively choose a decreasing sequence of real numbers. Let $r_1 = R$, and assume that $r_2, \ldots, r_n$ have been chosen in such a way that $\{|w| = r_k\} \cap E = \emptyset$, $k = 1, 2, \ldots, n$.

i) Suppose that $\gamma(r_n) \geqq \delta r_n$ which means that we are in T_1. Let $\alpha = \sup_{r \leqq r_n} r$, $r \in \complement T_1$. Let (β, α) be the associated interval in $\complement T_1$. If $2\beta < \alpha$, there exists $\varrho \in (\beta, \alpha)$ such that $\{|w| = \varrho\} \cap E = \emptyset$ and we choose $r_{n+1} = \varrho$. This is correct since we have $\gamma(r) < \delta r$ in $\complement T_1$ and δ is small. Otherwise, let $\alpha_1 = \sup_{r \leqq \beta} r$, $r \in \complement T_1$, and let (β_1, α_1) be the next associated interval in $\complement T_1$. If $2\beta_1 < \alpha_1$, we find ϱ as above in the interval (β_1, α_1) and define $r_{n+1} = \varrho$. After a finite number of steps, we have either found r_{n+1} or got down to 1, since the logarithmic length of an interval in T_1 is at least $\log(1/\delta)$.

ii) Suppose $\gamma(r_n) < \delta r_n$. Then, we find ϱ in the interval $\left(2r_n(\gamma(r_n)/2r_n)^{1/8}, r_n/17\right)$ such that $\{|w| = \varrho\} \cap E = \emptyset$ and define $r_{n+1} = \varrho$ (cf. (7) in Lemma 2).

We have now found the finite, decreasing sequence $\{r_n\}$ and can define the harmonic minorants of the harmonic measure ω_R (cf. Section 2). We shall define a sequence $\{M_n\}$ of subsets of $\left\{\{|w| = r_n\}\right\}$, all of angular measure at least π. If $M_1 = \{|w| = R\}$ and $M_2, \ldots, M_n$ are defined, there are two possibilities.

i) If $\gamma(r_n) \geqq \delta r_n$, we define

$$M_{n+1} = \{r_{n+1}e^{i\varphi} \colon \omega_n(r_{n+1}e^{i\varphi}, M_n) \geqq (r_{n+1}/r_n)^A\}.$$

From our assumption that E is not too large, it is clear that the angular measure of M_{n+1} is at least π. (As before, we write ω_n instead of ω_{r_n}.)

ii) If $\gamma(r_n) < \delta r_n$, we define

$$M_{n+1} = \{r_{n+1}e^{i\varphi} \colon \omega_n(r_{n+1}e^{i\varphi}, M_n) \geqq 1/16\}.$$

According to Lemma 2, the angular measure of M_{n+1} is at least π. We can now define the decreasing sequence of harmonic minorants. In the two possible cases, we have the following alternatives:

(i) $\omega_R(w) \geqq \omega_2(w, M_2)(r_2/R)^A, \quad |w| \leqq r_2.$

(ii) $\omega_R(w) \geqq \omega_2(w, M_2)/16, \quad |w| \leqq r_2.$

(i) $\omega_n(w) \geqq \omega_{n+1}(w, M_{n+1})(r_{n+1}/r_n)^A, \quad |w| \leqq r_{n+1},$

(ii) $\omega_n(w) \geqq \omega_{n+1}(w, M_{n+1})/16, \quad |w| \leqq r_{n+1},$

$n = 2, 3, \ldots$.

Arguing in the same way as in the proof of Theorem 2, we see that

$$\omega_R(0) \geqq \text{Const. } 16^{-q} \prod_{n \in J} (r_{n+1}/r_n)^A, \tag{18}$$

where J is the index set which occurs in case (i) and $q = q(R)$ is the number of steps from case (ii). The number of intervals in $T_1(R) = T_1 \cap [1, R]$ is approximately at most $\log R/(\log(1/\delta))$, and we have

$$\sum_{n \in J} \log(r_n/r_{n+1}) \leqq m_l T_1(R) + (\log 2)(\log R)/\log(1/\delta),$$

where m_l stands for logarithmic length.

The same reasoning as in the proof of Theorem 2 shows that

$$q^2 \leqq \text{Const.} \left(\sum_{n\in JJ} \log (r/r_{n+1})\right)\left(\sum_{n\in JJ} (\log (2r_n/\gamma(r_n)))^{-1}\right),$$

where JJ is the index set which occurs in case (ii). Consequently, we have

$$q^2 \leqq \text{Const.} \log R \left(\int_1^R (\log (2s/\gamma(s)))^{-1} ds/s.\right.$$

When $R\in\{R_n\}$, we deduce that $q(R_n)/\log R_n \to 0$, $n\to\infty$. Since (17) holds, we must also have that $m_l T_1(R_n)/\log R_n \to 0$, $n\to\infty$. It follows from (18) that

$$\limsup \log \omega_R(0)/\log R \geqq -(\log 2)/(\log (1/\delta)).$$

Since $\delta>0$ is arbitrary, the upper limit must in fact be zero, and Theorem 3 is proved.

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