Singular integrals on product spaces with variable coefficients

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Introduction

The theory of (constant coefficients) singular integral operators, developed by Calderón and Zygmund, has been extended [6] by R. Fefferman and Stein to the product spaces $\mathbf{R}^n \times \mathbf{R}^m$. The point is to deal with kernels $K(x', y')$ that cannot be written in the form $K_1(x') \cdot K_2(y)'$ for otherwise the boundedness of the corresponding operator can be easily obtained by an iteration argument.

In [10], among other things, we proved the boundedness in $L_p(\mathbf{T}\times\mathbf{T})$, $1<p<\infty$, of a variant of the double Hilbert transform H, where the convolution kernel could not be factored out and moreover it was a variable coefficients one. Roughly speaking, such a variant was defined as follows

$$H_* f(x, y) = \int_{(x', y') \in R_y} \frac{1}{x' y'} f(x-x', y-y')\, dx'\, dy' \tag{1}$$

where for every fixed $y \in \mathbf{T}$ and for every integer $k \geqq 0$ the region $R_y \subset \mathbf{T}\times\mathbf{T}$ has the following property: $R_y \cap \{(x', y'): |x'| \sim 2^{-k}\} = \{(x', y'): |x'| \sim 2^{-k}, |y'| \leqq \delta(k, y)\}$, with $0 \leqq \delta(k, y) \leqq 1$ and except for that arbitrarily chosen. This makes the action of H_* on the y' variable closer to the action of the maximal Hilbert transform than to that of the Hilbert transform itself. (The cut off of the domain of integration given by $\chi_{R_y}(x', y')$ was actually smooth.) We also would like to mention that H_* has been introduced to deal with some operator that arise in the study of a problem of almost everywhere convergence of double Fourier series [11]. For this application it is indispensable to study the maximal operator $\tilde{H}_*$ defined by

$$\tilde{H}_* f(x, y) = \sup_{k_0 > 0} \left| \int_{\substack{(x', y') \in R_y \\ |x'| \geqq 2^{-k_0}}} \frac{1}{x' y'} f(x-x', y-y')\, dx'\, dxy' \right|$$

and to prove not only that it is bounded on L_p but also to control it pointwise from above. We did so by proving

$$\tilde{H}_* f(x, y) \leqq c\{M_1 \tilde{H}_2 f(x, y) + M_1(H_* f)(x, y)\} \tag{2}$$

where M_1 denotes the Hardy—Littlewood maximal function acting on x' and $\tilde{H}_2$ denotes the maximal Hilbert transform acting on y'.

In this paper we extend the mentioned results for H_* and $\tilde{H}_*$ to the case in which the product domain is $\mathbf{R}^n \times \mathbf{R}^m$, by introducing variable coefficients operators T defined as in (1) but with $1/x'$ and $1/y'$ replaced by the kernels $K^{(1)}(x') = \Omega_1(x')/\|x'\|^n$ and $K^{(2)}(y') = \Omega_2(y')/\|y'\|^m$ that define the classical singular integral operators, that commute with dilations, in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. It should be stressed that again we have no assumptions of regularity on the dependency of the resulting convolution kernel $K(x', y', y)$ upon y, which is a fairly unusual situation. By a combination of the techniques of [6] and [10] we are able to prove estimates, in certain instances, more refined than those in [6] (compare our Lemma 1 with (2.4) of [6]) by means of which we prove the boundedness on $L_p(\mathbf{R}^n \times \mathbf{R}^m)$ of the operators T. This is done in Part I.

Recently R. Fefferman in [5] gave a simple condition for a bounded operator on $L_2(\mathbf{R}^n \times \mathbf{R}^m)$ to be bounded from $H_p(\mathbf{R}^n \times \mathbf{R}^m)$ to $L_p(\mathbf{R}^n \times R^m)$, $0 < p \leqq 1$, and from $\mathrm{Llg}\, L^{M(n,m)}$ to weak-L_1. He also proved that the operators of the class considered by Journé in [13], which includes to a large extend the class studied in [6], satisfy his condition. In Part II we prove that our operators too (which do not belong to the class studied in [13]) satisfy R. Fefferman's condition.

In Part III we prove a pointwise estimate from above, like (2), for the maximal operator $\tilde{T}$. Let us observe that such an estimate is weaker than the one known for $\tilde{H}$, the standard maximal double Hilbert transform, where the supremum is taken over all possible independent truncations of $|x'|$ and $|y'|$, and which reads as follows

$$\tilde{H} f(x, y) \leqq c\{M_1 \tilde{H}_2 f(x, y) + M_2 \tilde{H}_1 f(x, y) + M_1 M_2 H f(x, y)\}. \tag{3}$$

It should be said though, that it seems unlikely that an estimate like (3) can be proved in our case, due to the potential high irregularity of the dependency of our kernel upon y, that we already pointed out. Moreover, we will show that both T and $\tilde{T}$ satisfy weighted norm inequalities. Finally, we observe that the techniques so developed give similar results for other operators as well: see Theorem 2, 6, 8.

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Part I

On the kernel $K^{(2)}(y)$, $y\in\mathbf{R}^m$, we are going to make the same assumptions satisfied by the kernels that define singular integral operators that commute with dilations [12] and we will have somewhat stronger assumptions on $K^{(1)}(x)$, $x\in\mathbf{R}^n$. Let us suppose that $K^{(1)}(x)=\Omega_1(x)/\|x\|^n$ and $K^{(2)}(y)=\Omega_2(y)/\|y\|^m$ such that Ω_i are homogeneous of degree zero and furthermore satisfy the following conditions

$$(a_1)\qquad \int_{S^{n-1}}\Omega_1(x)\,d\sigma=0;$$

$$(a_2)\qquad \int_{S^{m-1}}\Omega_2(y)\,d\varrho=0$$

where $d\sigma$ and $d\varrho$ are the induced Euclidean measures on the unit spheres S^{n-1} and S^{m-1};

$$(b_1)\qquad \int_0^{2^{-r}}\frac{\omega_1(\delta)}{\delta}\,d\delta\leqq c2^{-r}\quad\text{for every integer}\quad r\geqq 0;$$

$$(b_2)\qquad \int_0^1\frac{\omega_2(\delta)}{\delta}\,d\delta<\infty,\quad\text{where}\quad \omega_1(\delta)=\sup_{\substack{\|x-x'\|\leqq\delta\\ \|x\|=\|x'\|=1}}|\Omega_1(x)-\Omega_1(x')|$$

and ω_2 is similarly defined.

Now let $\varphi(x)$ be a C^∞ function, radial and supported on $\{x\colon 1/2\leqq\|x\|\leqq 2\}$ such that

$$\sum_{k=-\infty}^{+\infty}\varphi(2^kx)=\sum_k\varphi_k(x)=1,\quad x\neq 0.$$

We write

$$K^{(1)}(x)=\sum_{k=-\infty}^{+\infty}K^{(1)}(x)\varphi_k(x)=\sum_k K_k^{(1)}(x).$$

Suppose that to every pair (k,y), $k\in\mathbf{Z}$, $y\in\mathbf{R}^m$ there corresponds $\delta=\delta(k,y)$ with $0\leqq\delta(k,y)\leqq\infty$ and then for g in $L_p(\mathbf{R}^m)$ let

$$T^{(2)}g(y)=\int K^{(2)}(y')g(y-y')\,dy';$$

$$T_\delta^{(2)}g(y)=\int K_\delta^{(2)}(y')g(y-y')\,dy'$$

where $K_\delta^{(2)}(y')=K^{(2)}(y')\chi_{\|y'\|\leqq\delta}(y')$. For every f in $L_p(\mathbf{R}^n\times\mathbf{R}^m)$, $1<p<\infty$, and with $\delta=\delta(k,y)$ we define the variable coefficients operators

$$T_{\varepsilon,N}f(x,y)=\int\sum_{\varepsilon_1\leqq 2^{-k}\leqq N_1}K_k^{(1)}(x')\int_{\varepsilon_2\leqq\|y'\|\leqq N_2}K_\delta^{(2)}(y')f(x-x',y-y')\,dx'\,dy'$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2)$, $0<\varepsilon_i<2$ and $N=(N_1,N_2)$, $N_i>2$. We are going to prove the following

Theorem 1. *In the assumptions listed above there exists a constant A_p depending only upon p, the dimensions n, m and the constants that appear in* (b_1) *and* (b_2) *such that* *for* $1<p<\infty$

(a) $$\|T_{\varepsilon,N}f\|_p \leqq A_p\|f\|_p;$$

(b) $\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} T_{\varepsilon,N}f=Tf$ *exists in* L_p*-norm and* $\|Tf\|_p\leqq A_p\|f\|_p$.

Remark 1. We used sharp truncations to define $K_\delta^{(2)}$ but the theorem holds as well for smooth ones. This was the case in [10] and in the application [11].

Remark 2. Conclusion (a) of the theorem holds for more general operators than $T^{(2)}$, namely for all bounded operators, that we still denote by $T^{(2)}$, for which the following estimate on the maximal operator

$$\tilde{T}^{(2)}g(y) = \sup_{\varepsilon>0}\left|\int_{\|y'\|>\varepsilon} K^{(2)}(y, y')g(y')\,dy'\right|$$

is known: $\tilde{T}^{(2)}g(y)\leqq c\{M_2T^{(2)}g(y)+M_2g(y)\}$. They include the variable coefficients operators defined by "Calderón—Zygmund kernels" following the definition of Coifman and Meyer [2] Chapter IV and the maximal partial sums or Carleson operator

$$Cg(y) = \int_{-\pi}^{\pi} \frac{e^{iN(y)y'}}{y'}\, g(y-y')\,dy'$$

[3], [7], and [9].

We shall make use of the following S-function studied in [6]. Let ψ be a nontrivial, radial, C^∞-function on $\mathbf{R}^n$ supported inside the unit ball, with $\int\psi(x)\,dx=0$. Let $\psi_s(x)=s^{-n}\psi(x/s)$ for $s>0$. Define for $f\in L_p(\mathbf{R}^n)$

$$S_\psi^2 f(x) = \int_0^\infty |\psi_s * f(x)|^2 \frac{ds}{s}$$

then

(4) $$c_p'\|f\|_p \leqq \|S_\psi f\|_p \leqq c_p\|f\|_p,\quad 1<p<\infty.$$

We shall also need the following technical lemma.

Lemma 1. *If* $k, r\in\mathbf{Z}$ *and* $s\sim 2^{-r}$ *(i.e.* $2^{-r-1}\leqq s\leqq 2^{-r}$*) then*

$$|\psi_s * K_k^{(1)} * f(x)| \leqq c_n 2^{-|k-r|} M_1 f(x).$$

Proof. Clearly the convolution $\psi_s * K_k^{(1)}(x)$ is supported inside $\{x\colon \|x\|\leqq 10\, 2^{-r}\}$ if $2^{-r}\geqq 2^{-k}$ and inside $\{x\colon (2^{-k}/10)\leqq\|x\|\leqq 10\, 2^{-k}\}$ if $2^{-k}>2^{-r}$. Now we are going to estimate the values that such a convolution takes on. First assume that $2^{-r}\geqq 10\, 2^{-k}$. It is easy to check that $\|\nabla\psi_S(x)\|\leqq c2^{(r+1)n}$. Therefore using (a_1) we have that

$$|\psi_s * K_k^{(1)}(x)| = \left|\int K_k^{(1)}(u)[\psi_s(x-u)-\psi_s(x)]\,du\right|$$

$$\leqq \int |K_k^{(1)}(u)|\,\|\nabla\psi_s(\tilde{x})\|\,\|u\|\,du \leqq c\,\frac{2^{rn}}{2^{k-r}}.$$

Secondly, assume that $(2^{-k}/10)\leqq 2^{-r}\leqq 10\, 2^{-k}$. Then clearly

$$|\psi_s * K_k^{(1)}(x)| \leqq \int |K_k^{(1)}(u)|\,|\psi_s(x-u)|\,du \leqq c2^{rn} \leqq c' \frac{2^{rn}}{2^{|k-r|}}.$$

Finally, assume $2^{-r}<(2^{-k}/10)$. Then since $\int \psi(u)\,du=0$ we have that

$$|\psi_s * K_k^{(1)}(x)| \leqq \int |K_k^{(1)}(x-u)-K_k^{(1)}(x)|\,|\psi_s(u)|\,du$$

$$\leqq \int \left| \frac{\Omega_1(x-u)}{\|x-u\|^n}\varphi_k(x-u) - \frac{\Omega_1(x)}{\|x-u\|^n}\varphi_k(x-u)\right| |\psi_s(u)|\,du$$

$$+\int \left| \frac{\Omega_1(x)}{\|x-u\|^n}\varphi_k(x-u) - \frac{\Omega_1(x)}{\|x\|^n}\varphi_k(x)\right| |\psi_s(u)|\,du = I_1(x)+I_2(x).$$

We observe that

$$|I_2(x)| \leqq c\int \left| \frac{\varphi_k(x-u)}{\|x-u\|^n} - \frac{\varphi_k(x)}{\|x\|^n}\right| |\psi_s(u)|\,du$$

$$\leqq c_n 2^{k(n+1)} \int |\psi_s(R)|\,R^n\,dR \leqq \frac{c_n 2^{kn}}{2^{r-k}},$$

and

$$I_1(x) = \int \frac{|\varphi_k(x-u)|}{\|x-u\|^n}\,|\Omega_1(x-u)-\Omega_1(x)|\,|\psi_s(u)|\,du$$

$$\leqq \int \frac{|\varphi_k(x-u)|}{\|x-u\|^n}\,\omega_1\left(\frac{\|u\|}{\|x\|}\right) |\psi_s(u)|\,du \leqq c2^{kn}\int_{\delta\leqq(2^{-r}/\|x\|)} \frac{\omega_1(\delta)}{\delta}\,d\delta \leqq c\frac{2^{kn}}{2^{r-k}}.$$

This ends the proof of the lemma.

Now we turn to the

Proof of Theorem 1. Since $\psi*\psi$ has the same properties of ψ, then for y fixed and f in $C_0^\infty(\mathbf{R}^n\times\mathbf{R}^m)$ we have

$$\|S_{\psi*\psi}T_{\varepsilon,N}f(x,y)\|_{L_p(dx)}^p \sim \|T_{\varepsilon,N}f(x,y)\|_{L_p(dx)}^p.$$

So integrating in y we obtain

$$\|S_{\psi*\psi}T_{\varepsilon,N}f\|_{L_p(dx\,dy)}^p \sim \|T_{\varepsilon,N}f\|_{L_p(dx\,dy)}^p.$$

Therefore to prove (a) it is enough to show

$$\|S_{\psi*\psi}T_{\varepsilon,N}f\|_{L_p(dx\,dy)} \leqq A_p\|f\|_p.$$

It is understood that until the end of the proof of (a) we deal with truncated operators $T_{\varepsilon,N}$, but for simplicity of notations we might drop the truncations. We are going to rewrite $S_{\psi*\psi}T_{\varepsilon,N}\,f$ in a suitable way, using the identity $\psi_s*K_\delta^{(2)}*f(x,y)= K_\delta^{(2)}*\psi_s*f(x,y)$, $\delta=\delta(k,y)$, which is true because the two convolution kernels ψ_s

and $K_\delta^{(2)}$, the last one with variable coefficients, act on different variables and so the corresponding operators commute. Let us write

$$S^2_{\psi*\psi}T_{\varepsilon,N}f(x,y)=\sum_{r=-\infty}^{+\infty}\int_{s\sim 2^{-r}}\left|\sum_k \psi_s*K_k^{(1)}*K_\delta^{(2)}*\psi_s*f(x,y)\right|^2\frac{ds}{s}.$$

Now if M_2 denotes the Hardy—Littlewood maximal function acting on y' then [12], p. 67,

$$\tilde{T}^{(2)}g(y)=\sup_{\varepsilon,\varepsilon'}\left|\int_{\varepsilon\leqq\|y'\|\leqq\varepsilon'}K^{(2)}(y')g(y-y')\,dy'\right|\leqq c\{M_2K^{(2)}*g(y)+M_2g(y)\}.$$

By this inequality and Lemma 1 we have that for $s\sim 2^{-r}$ and always $\delta=\delta(k,y)$

$$\left|\sum_k \psi_s*K_k^{(1)}*K_\delta^{(2)}*\psi_s*f(x,y)\right|\leqq\sum_k|\psi_s*K_k^{(1)}*K_\delta^{(2)}*\psi_s*f(x,y)|$$
$$\leqq\sum_k c2^{-|k-r|}M_1K_\delta^{(2)}*\psi_s*f(x,y)\leqq c\{M_1M_2K^{(2)}*\psi_s*f(x,y)+M_1M_2\psi_s*f(x,y)\}.$$

Therefore

$$S^2_{\psi*\psi}T_{\varepsilon,N}f(x,y)\leqq c\left\{\int_{s>0}|M_1M_2K^{(2)}*\psi_s*f(x,y)|^2\frac{ds}{s}+\int_{s>0}|M_1M_2\psi_s*f(x,y)|^2\frac{ds}{s}\right\}.$$

By the maximal theorem of [4] we know that

$$\|S^2_{\psi*\psi}T_{\varepsilon,N}f(x,y)\|_p\leqq\left\|\left(\int_{s>0}|\psi_s*K^{(2)}*f(x,y)|^2\frac{ds}{s}\right)^{1/2}\right\|_p$$
$$+\left\|\left(\int_{s>0}|\psi_s*f(x,y)|^2\frac{ds}{s}\right)^{1/2}\right\|_p.$$

By (4) and the boundedness of $T^{(2)}$ we conclude that this is dominated by $c_p\|f\|_p$. This ends the proof of (a).

Now we shall prove (b). We start by proving it for f in a dense subset of L_p, namely $f\in C_0^\infty$. We can assume $f(x',y')=f_1(x')f_2(y')$, $f_i\in C_0^\infty$. We write $T_{\varepsilon,N}f(x,y)$ as a sum of four terms by breaking up the domain of integration. The first one $T^{(1)}_{\varepsilon,N}$ corresponds to $\|x'\|\leqq 2$, $\|y'\|\leqq 2$. $T^{(1)}_{\varepsilon,N}$ itself can be expressed as a sum of four integrals by writing $f_1(x-x')f_2(y-y')=f_1(x-x')-f_1(x)]\cdot[f_2(y-y')-f_2(y)]+f_1(x)[f_2(y-y')-f_2(y)]+f_2(y)[f_1(x-x')-f_1(x)]+f_1(x)f_2(y)$. The last three integrals are zero, while the first one converges as $\varepsilon\to 0$ and $N\to\infty$ to $T^{(1)}f(x,y)$ for each (x,y), by the dominated convergence theorem. Moreover, $T^{(1)}f(x,y)$ is dominated by a constant and it lives on a fixed compact set. Then we consider $T^{(2)}_{\varepsilon,N}$ which has $\|x'\|\geqq 2$, $\|y'\|\leqq 2$ as domain of integration. In this case we write $f_1(x-x')f_2(y-y')=f_1(x-x')[f_2(y-y')-f_2(y)]+f_1(x-x')f_2(y)$. We insert this in the integral defining $T^{(2)}_{\varepsilon,N}$ and observe that the second integral is zero, while the first one is bounded independently of ε, N and $\{\delta(k,y)\}$ by an L_p-function of x times a bounded function with a bounded support in y. Hence $T^{(2)}_{\varepsilon,N}f(x,y)$ converges for almost every (x,y) as

$\varepsilon\to 0$ and $N\to\infty$. In a similar way one handles the other two terms. So we conclude that for almost every (x, y) the limit $\lim_{\varepsilon\to 0, N\to\infty} T_{\varepsilon,N} f(x, y) = Tf(x, y)$ exists. By the preceding estimates it is clear that $T_{\varepsilon,N} f(x, y) - Tf(x, y)$ is dominated by an L_p-function and so Tf is also the limit of $T_{\varepsilon,N} f$ in L_p-norm. Now by a density argument it is easily shown that the same holds if f belongs to L_p. This ends the proof of Theorem 1.

We also have

Theorem 2. *Let $N(y)$ be any integer-valued, L_∞-function and suppose that* (a_1) *and* (b_1) *are satisfied. Then* (a) *and* (b) *hold for the following operator*

$$Uf(x, y) = \int \sum_k K_k^{(1)}(x') \int_{|y'|<\delta(k,y)} \frac{e^{iN(y)y'}}{y'} f(x-x', y-y')\, dx'\, dy'$$

acting on $L_p(\mathbf{R}^n\times\mathbf{T})$. The norm of the operator U is independent of $N(y)$ and in particular of the L_∞-norm of $N(y)$.

Proof. By Remark 2 we know that (a) holds. Then it is easy to prove that $U_{\varepsilon,N} f$ converges in L_p-norm by writing $U_{\varepsilon,N} f(x, y) = \sum_n U_{\varepsilon,N} f(x, y)\chi_{E_n}(x, y)$ where $E_n = \{(x, y): N(y) = n\}$ and observing that by assumption we are dealing with a finite sum for every fixed $N(y)$. Hence (b) holds.

Part II

In this section we are going to study the boundedness of our operators T from $H_p(\mathbf{R}^n\times\mathbf{R}^m)$ to $L_p(\mathbf{R}^n\times\mathbf{R}^m)$, $0<p\leqq 1$, and from $\mathrm{Llg}\, L^{M(n,m)}(\mathbf{R}^n\times\mathbf{R}^m)$ to weak-L_1 $(\mathbf{R}^n\times\mathbf{R}^m)$. Our proof is based on Theorem 3 that follows. Its proof can be found in [5].

A function $a(x, y)$ defined m $\mathbf{R}^n\times\mathbf{R}^m$ is said to be an H_p-rectangle atom provided a is supported on a rectangle $R = Q_I\times Q_J$, where Q_I is a cube of side I and Q_J is similarly defined,

$\int_{Q_I} a(x, y) x^\alpha dx = 0$ for each multiindex α so that $|\alpha|\leqq N(p)$ and for each $y\in Q_J$,

$\int_{Q_J} a(x, y) y^\beta dy = 0$ for each multiindex β so that $|\beta|\leqq N(p)$ and for each $x\in Q_I$,

and $\|a\|_{L_2(R)} \leqq |R|^{1/2-1/p}$. Here $N(p)$ is a positive integer depending upon p, n and m which becomes large as $p\to 0$ and that can be taken equal to zero if p is sufficiently close to 1 (see [5]). Now we can state

Theorem 3. *Let G be a linear bounded operator on $L_2(\mathbf{R}^n\times\mathbf{R}^m)$. Suppose that for any H_p-rectangle atom a supported on the rectangle R we have*

$$\int_{^cR_\gamma} |G(a)|^p\,dx\,dy \leqq c\gamma^{-1}$$

for every $\gamma\geqq 2$, with R_γ denoting the concentric γ-fold dilation of R. Then G is a bounded operator from $H_p(\mathbf{R}^n\times\mathbf{R}^m)$ to $L_p(\mathbf{R}^n\times\mathbf{R}^m)$, $0<p\leqq 1$.

We will also make use of the following

Lemma 2. *Given $K(x)=\dfrac{\Omega(x)}{\|x\|^n}$, $x\in\mathbf{R}^n$, where Ω is homogeneous of degree zero and satisfies* (a$_1$), *then if φ is as in Part I and $K_0(x)=K(x)\varphi(x)$ we have*

(5) $$|\hat{K}_0(\xi)| \leqq c\|\xi\| \quad \text{if} \quad \|\xi\|<1.$$

If furthermore, Ω is $C^{(1)}$ then

(6) $$|\hat{K}_0(\xi)| \leqq c\|\xi\|^{-1} \quad \text{if} \quad \|\xi\|>1.$$

Proof. (5) follows by a Taylor expansion of $\hat{K}_0(\xi)$ with the origin as a starting point and (6) by one integration by parts.

We shall now require smooth truncations on the y' variable in the definition of our operators T. Suppose that $\varphi_1(x)$ and $\varphi_2(y)$ are C^∞, radial and supported respectively on $\{1/2\leqq\|x\|\leqq 2\}$ and $\{1/2\leqq\|y\|\leqq 2\}$. Define $K_k^{(1)}(x)$ and $K_h^{(2)}(y)$ in the usual way. Then consider the operator

$$Tf(x,y) = \int \sum_k K_k^{(1)}(x-x') \int \sum_{2^{-h}\leqq\delta(k,y)} K_h^{(2)}(y-y')f(x',y')\,dy'\,dx'$$

defined as a limit in the sense of distributions. The existence of the limit can be proved as in the classical case. We are going to show

Theorem 4. *If Ω_1 and Ω_2 satisfy* (a$_1$) *and* (a$_2$) *respectively and furthermore Ω_1 is $C^{(1)}$ and Ω_2 if Lipschitz continuous, then T is bounded from H_p to L_p, $1/2<p\leqq 1$.*

Proof. Let us check that the condition of Theorem 3 is satisfied. We will estimate separately

$$\int_{\substack{\|x\|\sim 2^r|I|\\ \|y\|\sim 2^l\gamma|J|}} |Ta|\,dx\,dy; \quad \int_{\substack{\|x\|\leqq 2|I|\\ \|y\|\sim 2^l\gamma|J|}} |Ta|\,dx\,dy; \quad \int_{\substack{\|x\|\sim 2^r\gamma|I|\\ \|y\|\leqq 2|J|}} |Ta|\,dx\,dy.$$

Our estimates follow closely [5], except in the case $\|x\|\leqq 2|I|$, $\|y\|\sim 2^l\gamma|J|$ which requires a different argument. Assume first $\|x\|\sim 2^r|I|$, $\|y\|\sim 2^l\gamma|J|$. Then in the sums that follow it is enough to add over $2^{-k}\geqq 2^r|I|$, and $2^{-h}\geqq 2^l\gamma|J|$. We have

$$|Ta(x,y)| \leqq \int \sum_k |K_k^{(1)}(x-x')-K_k^{(1)}(x)| \sup_{h_0} \left|\int \sum_{h\geqq h_0} K_h^{(2)}(y-y')a(x',y')\,dy'\right| dx'.$$

Proceeding as in the proof of Lemma 1 and using the fact that Ω_1 is Lipschitz we can prove that $|K_k^{(1)}(x-x')-K_k^{(1)}(x)|\leqq c_n \frac{|I|}{2^{-k(n+1)}}$. So

$$|Ta(x, y)| \leqq \frac{c_n|I|}{(2^r|I|)^{n+1}}\int \sup_{h_0}\left|\int \sum_{h\geqq h_0}(K_h^{(2)}(y-y')-K_h^{(2)}(y))a(x', y')\,dy'\right|dx' \leqq$$

$$\leqq \frac{c_n|I|}{(2^r|I|)^{n+1}}\cdot\frac{|J|}{(2^l\gamma|J|)^{m+1}}\int |a(x', y')|\,dx'\,dy'.$$

Therefore by Schwarz inequality

$$\int_{\substack{\|x\|\sim 2^r|I|\\ \|y\|\sim 2^l\gamma|J|}} |Ta(x, y)|\,dx\,dy \leqq c_n\frac{|R|^{1/2}}{\gamma 2^r 2^l}\|a\|_{L_2(R)}.$$

Now suppose $\|x\|\leqq 2|I|$ and $\|y\|\sim 2^l\gamma|J|$. Fix y. Then by Schwarz inequality and Plancherel formula we have

$$\int_{\|x\|\leqq 2|I|}|Ta(x, y)|\,dx \leqq |I|^{n/2}\left(\int|Ta(x, y)|^2\,dx\right)^{1/2}$$

$$= |I|^{n/2}\left(\int\left|\sum_k \hat{K}_k^{(1)}(\xi)\sum_{2^{-h}\leqq\delta(k,y)}K_h^{(2)}*\hat{a}(\xi, y)\right|^2 d\xi\right)^{1/2}$$

$$\leqq |I|^{n/2}\left(\int\left(\sum_k|\widehat{K_k^{(1)}}(\xi)|\right)^2\sup_{h_0}\left|\sum_{h\geqq h_0}K_h^{(2)}*\hat{a}(\xi, y)\right|^2 d\xi\right)^{1/2}.$$

Since $K_k^{(1)}(x)=2^{kn}K_0^{(1)}(2^k x)$, by Lemma 2 we have $\sum_k|\widehat{K_k^{(1)}}(\xi)|\leqq c$. Therefore integrating both sides of the inequality with respect to y we obtain

$$\int_{\substack{\|x\|\leqq 2|I|\\ \|y\|\sim 2^l\gamma|J|}}|Ta(x, y)|\,dx\,dy \leqq c|I|^{n/2}\int_{\|y\|\sim 2^l\gamma|J|}\left(\sup_{h_0}\left|\sum_{h\geqq h_0}K_h^{(2)}*\hat{a}(\xi, y)\right|^2 d\xi\right)^{1/2} dy$$

$$\leqq c|I|^{n/2}(2^l\gamma|J|)^{m/2}\left(\int_{\|y\|\sim 2^l\gamma|J|}\sup_{h_0}\left|\sum_{h\geqq h_0}K_h^{(2)}*\hat{a}(\xi, y)\right|^2 d\xi\,dy\right)^{1/2}.$$

Since

$$\left|\sum_h K_h^{(2)}*\hat{a}(\xi, y)\right| = \left|\sum_h\int(K_h^{(2)}(y-y')-K_h^{(2)}(y))*\hat{a}(\xi, y')\,dy'\right|$$

$$\leqq \frac{|J|}{(2^l\gamma|J|)^{m+1}}\int|\hat{a}(\xi, y')|\,dy',$$

we have by Plancherel formula that

$$\int_{\substack{\|x\|\leqq 2|I|\\ \|y\|\sim 2^l\gamma|J|}}|Ta(x, y)|\,dx\,dy \leqq \frac{|R|^{1/2}}{2^l\gamma}\|a\|_{L_2(R)}.$$

Similarly one can show that

$$\int_{\substack{\|x\|\sim 2^r\gamma|I|\\ \|y\|\leqq 2|J|}}|Ta(x, y)|\,dx\,dy \leqq \frac{|R|^{1/2}}{2^r\gamma}\|a\|_{L_2(R)}.$$

Then one applies Hölder's inequality and if $\frac{1}{2}<p\leqq 1$ one may add up the preceding estimates and prove the theorem.

Remark 3. For $0<p\leqq\frac{1}{2}$, Theorem 4 can be proved in a similar way assuming more smoothness on the kernels $K^{(1)}$ and $K^{(2)}$ and using the fact that the rectangle atoms have $N(p)$ vanishing moments in each variable separately. See [5].

Also under the same assumption of Theorem 3 as in the H_1 case one can prove (see [5]) that, in case $n=m=1$,

$$m\{|Gf(x, y)| > \alpha, x\in Q_1, y\in Q_2\} \leqq \frac{c}{\alpha} \|f\|_{L\log L}$$

where Q_1 and Q_2 denote the unit square of the two copies of **R**. Therefore under the assumptions of Theorem 4 our operators T map $L\log L$ into weak-L_1, in case $n=m=1$.

Part III

In this section we prove maximal theorems for the operators T and U of Part I, as well as weighted norm inequalities.

Theorem 5. *Let*

$$\tilde{T}f(x, y) = \sup_{k_0>0} \left|\sum_{k\leqq k_0} K_k^{(1)} * K_{\delta(k,y)}^{(2)} * f(x, y)\right|.$$

Then under the same assumptions of Theorem 1 *the following inequality holds*

$$\tilde{T}f(x, y) \leqq c\{M_1\tilde{T}^{(2)}f(x, y)+M_1(Tf)(x, y)\} \tag{7}$$

with c independent of f and $\{\delta(k, y)\}$. *Hence* $\tilde{T}$ *is bounded on* L_p, $1<p<\infty$ *and under the assumptions of Theorem* 4, *from* H_1 *to* weak-L_1.

Proof. We are going to compare $T_{k_0}f(x, y)=\sum_{k\leqq k_0} K_k^{(1)}*K_\delta^{(2)}*f(x, y)$ with $\varphi_{k_0}(x')*Tf(x, y)$ where $\varphi_{k_0}(x')=2^{k_0 n}\varphi(2^{k_0}x')$ and $\varphi(x')$ is chosen to be radial, decreasing, smooth, supported in the unit ball and such that $\int\varphi(x')dx'=1$ and $\varphi(x')\geqq 0$. Let us denote by $F_k(x', y)=K_{\delta(k,y)}^{(2)}*f(x', y)$. Clearly, $|F_k(x', y)|\leqq\tilde{T}^{(2)}f(x', y)$. Now we consider

$$G_{k_0}f(x, y) = \int\left\{\sum_{k\leqq k_0} K_k^{(1)}(x')F_k(x-x', y)-\sum_k \varphi_{k_0}*K_k^{(1)}(x')F_k(x-x', y)\right\}dx'.$$

Observe that

$$\varphi_{k_0}*\left[\sum_k K_k^{(1)}*F_k(x, y)\right] = \sum_k [K_k^{(1)}*\varphi_{k_0}]*F_k(x, y)$$

as one can check by a limiting argument. To estimate $G_{k_0} f(x, y)$ we split the domain of integration into two pieces: $\|x'\| \leqq 10 \cdot 2^{-k_0}$ and $\|x'\| > 10 \cdot 2^{-k_0}$ and so we write $G_{k_0} f = G^{(1)}_{k_0} f + G^{(2)}_{k_0} f$. If $\|x'\| \leqq 10 \cdot 2^{-k_0}$ then

$$\sum_{k \leqq k_0} |K^{(1)}_k(x')| \leqq c 2^{k_0 n} \quad \text{and by} \quad (a_1)$$

$$\sum_k K^{(1)}_k * \varphi_{k_0}(x') = \sum_{k \geqq k_0 - 8} K^{(1)}_k * \varphi_{k_0}(x').$$

Since $K^{(1)}_k(x) = 2^{kn} K^{(1)}_0(2^k x)$, from Lemma 2 it follows that

$$\|K^{(1)}_k * \varphi_{k_0}(x')\|_\infty \leqq \|\check{K}^{(1)}_k \check{\varphi}_{k_0}\|_1 \leqq c_n \frac{2^{k_0}}{2^k} 2^{k_0 n}$$

and so $\sum_{k \geqq k_0 - 8} |K^{(1)}_k * \varphi_{k_0}(x')| \leqq c 2^{k_0 n}$. This implies that $|G^{(1)}_{k_0} f(x, y)| \leqq c M_1 \tilde{T}^{(2)} f(x, y)$. Now suppose $\|x'\| > 10 \cdot 2^{-k_0}$ and observe that

$$\sum_k K^{(1)}_k * \varphi_{k_0}(x') = \sum_{k \leqq k_0} K^{(1)}_k * \varphi_{k_0}(x').$$

Since $\int \varphi(x')\, dx' = 1$ we are led to study

$$\left| \int \sum_{k \leqq k_0} (K^{(1)}_k(x') - K^{(1)}_k(x' - x'')) \varphi_{k_0}(x'')\, dx'' \right|$$

$$\leqq \int \sum_{k \leqq k_0} |K^{(1)}_k(x') - K^{(1)}_k(x' - x'')| \varphi_{k_0}(x'')\, dx'' \leqq \sum_{k \leqq k_0} 2^{k - k_0} 2^{kn} \chi_{\|x'\| < 2 \cdot 2^{-k}}(x'),$$

by computations similar to those of Lemma 1. So $G^{(2)}_{k_0} f(x, y) \leqq c M_1 \tilde{T}^{(2)} f(x, y)$. Hence $T_{k_0} f(x, y) \leqq c \{|\varphi_{k_0} * (Tf)(x, y)| + M_1 \tilde{T}^{(2)} f(x, y)\}$ and from this (5) follows.

In a similar way one proves

Theorem 6. *If*

$$\tilde{U}f(x, y) = \sup_{k_0 > 0} \left| \int \sum_{k \leqq k_0} K^{(1)}_k(x') \int_{|y'| < \delta(k, y)} \frac{e^{iN(y)y'}}{y'} f(x - x', y - y')\, dx'\, dy' \right|,$$

then under the same assumptions of Theorem 2 we have $\tilde{U}f(x, y) \leqq c \{M_1 \tilde{C} f(x, y) + M_1 (Uf)(x, y)\}$ *where* $\tilde{C}$ *denotes the maximal Carleson operator and c is independent of* $N(y)$, $\{\delta(k, y)\}$ *and f.*

Now we are going to prove that weighted norm inequalities hold for the operators T and therefore, by (7), for the $\tilde{T}$'s as well. Following [6] we say that $w(x, y) \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$ if for every $y \in \mathbf{R}^m$ the functions $x \to w(x, y) \in A_p(\mathbf{R}^n)$ and have A_p-norm bounded independently of y and if a similar condition holds for the functions $y \to w(x, y)$ for every $x \in \mathbf{R}$. For the basic facts about $A_p(\mathbf{R}^n)$ weights, see [1] and [8].

Theorem 7. *If T and $\tilde{T}$ are defined as above and* $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$ *then for* $1 < p < \infty$

$$\|Tf\|_{L_p(w)} \leqq c_p \|f\|_{L_p(w)}, \tag{8}$$

$$\|\tilde{T}f\|_{L_p(w)} \leqq c'_p \|f\|_{L_p(w)} \tag{9}$$

with c_p and c'_p depending only upon p, n, m and the constants that appear in (b_1) *and* (b_2).

Proof. The proof of (6) follows the same lines of the proof of Theorem 1, once the following two inequalities have been established [6]:

$$\|S_\psi f\|_{L_p(w)} \leqq c_p \|f\|_{L_p(w)}$$

where $w \in A_p(\mathbf{R}^n)$ and $f \in L_p(w(x)\,dx)$;

$$\left\|\left(\sum (M_1 M_2 f_k)^2\right)^{1/2}\right\|_{L^p(w)} \leqq c'_p \left\|\left(\sum |f_k|^2\right)^{1/2}\right\|_{L_p(w)}$$

where $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$ and $f \in L_p(w(x, y)\,dx\,dy)$.

Finally, since the Carleson operator satisfies weighted norm inequalities [9] we also have

Theorem 8. *If* $w \in A_p(\mathbf{R}^n \times \mathbf{T})$ *and* $1 < p < \infty$ *then*

$$\|Uf\|_{L_p(w)} \leqq c_p \|f\|_{L_p(w)};$$

$$\|\tilde{U}f\|_{L_p(w)} \leqq c'_p \|f\|_{L_p(w)}$$

where c_p and c'_p in particular do not depend upon the function $N(y)$.

Bibliography

1. Coifman, R. R. and Fefferman, C., Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241—250.
2. Coifman, R. R. and Meyer, Y., Au delà des operateurs pseudo-differentiels, *Asterisque* **57** (1978).
3. Fefferman, C., Pointwise convergence of Fourier series, *Ann. of Math.* **98** (1973), 551—572.
4. Fefferman, C. and Stein, E. M., Some maximal inequalities, *Amer. J. Math.* **93** (1971), 107—115.
5. Fefferman, R., Calderón—Zygmund theory for product domains-H^p spaces, *Proc. Nat. Acad. Sci. U.S.A.* (to appear).
6. Fefferman, R. and Stein, E. M., Singular integrals on product spaces, *Advances in Math.* **45** (1982), 117—143.
7. Hunt, R., On the convergence of Fourier series, *Orthogonal expansions and continuous analogues* (Proc. Conf. Edwardsville, Illinois 1967), pp. 235—255, SIU Press 1968.
8. Hunt, R., Muckenhoupt, B. and Wheeden, R., Weighted norm inequalities for conjugate functions and Hilbert transforms, *Trans. Amer. Math. Soc.* **176** (1973), 227—251.
9. Hunt, R. and Young, W. S., A weighted norm inequality for Fourier series, *Bull. Amer. Math. Soc.* **80** (1974), 274—277.
10. Prestini, E., Variants of the maximal double Hilbert transform, *Trans. Amer. Math. Soc.* **290** (1985), 761—771.
11. Prestini, E., Uniform estimates for families of singular integrals and double Fourier series, *J. Austral. Math. Soc.* **41** (1986), 1—12.

12. STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions,* Princeton Univ. Press, Princeton, 1970.
13. JOURNÉ, J. L., Calderón—Zygmund operators on product spaces, *Revista Mat. Ibero americana,* 3 (1985), 55—92.

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