Interaction of progressing waves for semilinear wave equations. II

Richard B. Melrose[1] and Niles Ritter[1]

1. Introduction

This paper is related to the propagation of conormal regularity for solutions to semilinear wave equations, i.e. to the interaction of progressing waves for such an equation. One result of this type is proved here, for the wave operator in three dimensional space-time, concerning propagation of singularities associated to two or more characteristic surfaces, simply tangent along a common line. This special case is analysed in considerable detail for several reasons but principally to check the usefulness of different notions of regularity at such a singular variety.

Three distinct spaces of iterated regularity associated to this geometry are investigated. The first space is the space of Lagrangian distributions associated to the two conormal bundles of the surfaces, which intersect in codimension one because of the simple tangency. This is the type of space which arises in the purely linear case, for example the fundamental solution of the wave operator itself is of this type (microlocally) near the tip of the characteristic cone [MU]. This space would generally be denoted

$$I(\mathbf{R}^3, N^*H_1 \cup N^*H_2), \tag{1.1}$$

where H_1 and H_2 are the two surfaces. However it is not possible to prove propagation results in it for the semilinear equations considered here because it is not multiplicative.

The next space considered is the space of iterated regularity with respect to the vector fields tangent to the surfaces; this is denoted

$$I(\mathbf{R}^3, H_1 \sqcup H_2) \tag{1.2}$$

[1] The first author received partial support from National Science Foundation grant mcs 8306271 and the second was a National Science Foundation Post-Doctoral Fellow during the preparation of this manuscript.

and was defined in [MR]. Propagation for this space was proved in [MR]. One of the principal objectives of this paper is to show that this space is simply not good enough for the general question of the propagation of conormality, a claim made specific below.

The third type of space can also be defined using the notation of [MR], namely as the sum

$$J(\mathbf{R}^3, H_1 \sqcup H_2) = I(\mathbf{R}^3, H_1 \sqcup L) + I(\mathbf{R}^3, H_2 \sqcup L), \quad L = H_1 \cap H_2, \tag{1.3}$$

where L is the line of intersection. A propagation result, of the usual type, is proved for this space, the chief novelty being in the proof that the bounded elements of $J(\mathbf{R}^3, H_1 \sqcup H_2)$ form a (C^∞) ring. This is demonstrated by blowing up the intersection line and defining the space in terms of the properties of the lifts of functions. It is shown below that:

$$I(\mathbf{R}^3, N^*H_1 \sqcup N^*H_2) \subset J(\mathbf{R}^3, H_1 \sqcup H_2) \subset I(\mathbf{R}^3, H_1 \sqcup H_2) \tag{1.4}$$

with both inclusions proper, so the propagation result is finer than that proved in [MR].

To illustrate why the space (1.3) should be preferred to the somewhat larger space (1.2) consider the following rather typical general question of the propagation of singularities. In $\mathbf{R}^3$ with coordinates t, x, y let P be the standard wave operator:

$$P = D_t^2 - D_x^2 - D_y^2. \tag{1.5}$$

Let $\Omega \subset \mathbf{R}^3$ be an open set P-convex with respect to $\Gamma = \Omega \cap \{t=0\}$, i.e. such that the Cauchy problem for $Pu=0$ can be uniquely solved in Ω for arbitrary Cauchy data on Γ. Consider then a real-valued function

$$u \in L^\infty(\Omega), \tag{1.6}$$

meaning that u is *locally* bounded in Ω, which satisfies the semilinear equation

$$Pu = f(.\,, u) \quad \text{in} \quad \Omega, \quad f \in C^\infty(\Omega \times \mathbf{R}). \tag{1.7}$$

It is readily shown that the Cauchy data of u on Γ is then well-defined.

$$\gamma_i u = D_t^i u_{|\Gamma} \in C^{-\infty}(\Gamma), \quad i = 0, 1. \tag{1.8}$$

Suppose that this Cauchy data is conormal with respect to a finite set of points, $L = \{z_1, z_2, \dots, z_N\}$:

$$\gamma_i u \in \sum_{k=1}^N I(\Gamma, T^*_{z_k}\mathbf{R}^2), \quad i = 0, 1. \tag{1.9}$$

The question then is: Where is u singular? This is of course intricately related to *how* it is singular. An example due to M. Beals [Be] shows that if the hypothesis (1.9) is weakened to simply:

$$\operatorname{sing\,supp}(\gamma_i u) \subset L, \tag{1.10}$$

then the singular support of u can fill out all of the solid propagation cones with poles in L, i.e. each of the cones:

(1.11) $$C(\bar{z}) = \{z = (t, x, y) \in \Omega;\ |(x, y)-(\bar{x}, \bar{y})| \leqq |t-\bar{t}|, \bar{z} = (\bar{t}, \bar{x}, \bar{y})\},$$

for $\bar{z} \in L$ (so $\bar{t}=0$). This is in contrast to the linear case, i.e. when f in (1.5) is linear in u, where the singular support of u is confined to the conic surfaces:

(1.12) $$E(\bar{z}) = \{z = (t, x, y) \in \Omega;\ |(x, y)-(\bar{x}, \bar{y})| = |t-\bar{t}|\} = \partial C(\bar{z}),$$

for $\bar{z} \in L$.

Let $E_+(\bar{z})$ be the forward part of $E(\bar{z})$ in (1.12), i.e. the part lying in $\{t \geqq \bar{t}\}$. Define sets $L_0=L, L_1, L_2,$ and so on, successively by:

(1.13)

$$L_{k+1} = L_k \cup \bigcup \{M = E_+(z_1) \cap E_+(z_2) \cap E_+(z_3);\ M \text{ is finite and } z_1, z_2, z_3 \in L_k\},$$

by adding to L_k the sets of triple intersections of the cones based in L_k. The non-empty intersections of three cones either consists of one or two points or else, if the three poles are colinear, part of a line. Thus the finiteness condition just excludes this last case. The general conjecture for the solution of the semilinear equation is:

(1.14) $$(1.5)-(1.9) \Rightarrow \text{sing supp}_s(u) \cap \{t \geqq 0\} \subset \bigcup_{z \in L_k} E_+(z),$$

where the singular support is computed relative to some finite Sobolev space, $H^s(\Omega)$, and k depends on s. The reason for the restriction to finite order singularities is that:

(1.15) $$L_\infty = \bigcup_{k=1}^{\infty} L_k \quad \text{need not be discrete in } \Omega.$$

A simple example illustrating this, with seven initial points is given in §7 below.

Ignoring certain niceties about the rather weak assumption (1.7), *a priori*, on the regularity of u, discussed elsewhere, the conjecture (1.14), is known from the work of Bony, [Bo2], for the case $N \leqq 2$, and from [MR] and [Bo3] in case $N=3$ (and also for certain cases with $N \geqq 4$). In these cases L_∞ is finite and one can take the usual singular support with respect to C^∞ functions in (1.14), with $k=0$ for $N \leqq 2$, $k=1$ for $N=3$:

(1.16) $$(1.5)-(1.9),\ N = 3 \Rightarrow \text{sing supp}(u) \subset \bigcup_{z \in L_1} E_+(z).$$

The case $N=3$ is illustrated in Figure 1 (at the end of the paper). Now $L_1 \setminus L_0$ consists of at most one point (depending on Ω), the triple interaction point, p. Notice the geometry of the four cones after the triple interaction, the new cone $E_+(p)$ meets each of the old cones in a single (half-) line, along which the cones are simply tangent to each other. This is the geometry introduced above and with which this paper is primarily concerned.

In particular, the results of either [MR] or [Bo3] show that (under some additional regularity hypotheses which can be removed) the solution u lies in the space (1.2) near each of these lines of contact between cones, this of course implies in particular the bound on the singular support (1.16). To understand the inadequacy of this regularity, say for the case $N=4$, consider what happens after the first triple interaction. It is readily seen that, if $\Omega=\mathbf{R}^3$, generically (in the placement of the initial points) there must be a further interaction of the wave on two tangent cones with a single cone from a fourth initial point, see Figure 2. According to the conjecture (1.14) this should produce extra singularities only on the cone $E_+(q)$ through the point of intersection of the three cones involved. Since the triple interaction produces singularities in all directions at q it is not reasonable to expect that extra singularities produced by such an interaction should be iteratively more regular than those produced at the triple point p; although it is certainly the case that the Sobolev regularity is higher, hence the finite union in (1.14). Examining the geometry after this interaction, see Figure 3, there are now three cones tangent along a common line, hence simply tangent in pairs. The best iterative regularity one could expect from the results of [MR] or [Bo3] would be that u lies in some ring containing the sum of the three rings (1.2) associated to the three pairs of these hypersurfaces.

Now the difficulty is that, first this sum is not a ring and, secondly, there is no obvious (P-propagative) ring containing it with the property that the wavefront set of its elements is contained in the union of the conormal bundles to all the C^∞ manifolds involved (including the line). This will prevent the proof of (1.14) since such singularities must propagate off the characteristic surfaces on the right side. Of course further interactions will make this problem even worse.

The advantage of the space (1.3) is that the corresponding sum of spaces, when there are three, or more, surfaces (cones in this case) tangent along a line, is a ring. Thus the singularities corresponding to such a space, whilst growing more numerous, are not worse.

It should be emphasised that (1.14) is not proved here. Nor is it shown that after the triple interaction point, p, the solution u to (1.5)—(1.9) actually has the regularity (1.3). It *is* shown that this regularity, if present, persists until further interaction occurs. In fact the same result holds for the space associated to any finite number of tangent cones. Thus if it *can* be shown that the regularity (1.3) is present after triple interaction then it will be relatively straightforward to demonstrate (1.14) in full. We hope to examine this problem elsewhere, in particular to show that the solution u to (1.5)—(1.9) always lies in the space defined by an extension of the definition, by blow up, of (1.3). It is this definition which shows the space to be a ring.

In brief the discussion below proceeds as follows. In § 2 the first blow up of a pair of lines in $\mathbf{R}^2$, simply tangent at a point is considered; in § 3 the second blow up is carried out. This leads to the proof in §4 that the space (1.3) is a ring. In § 5 conditions on such a ring enough to prove the type of propagation theorem mentioned above are given. These are verified in § 6 for the case corresponding to (1.3), i.e. two simply tangent characteristic surfaces, and the case of higher multiplicity also mentioned above. Finally in § 7 the example of seven points for which the set of triple interactions is not discrete is given.

Most of this analysis was carried out while the authors were visiting the Mittag-Leffler Institute. The stimulating atmosphere there was especially appreciated and useful conversations with Jean-Michel Bony and Jeff. Rauch are grateful y acknowledged, as is the warm hospitality and constructive criticism offered by Lars Hörmander. Subsequent conversations with Nils Dencker and Rafe Mazzeo were also helpful.

2. First blow up

For the reasons outlined in the introduction we proceed directly to consider the blow up of two curves, in $\mathbf{R}^2$, simply tangent at the point of blow up. Since we shall only analyse simple examples below we shall not give a coordinate independent definition of normal blow up but simply observe that such a definition does indeed exist, in the C^∞ category, so the operation of introduction of polar coordinates around a submanifold makes more invariant sense than might at first be supposed.

Consider two embedded C^∞ plane curves, K_1, K_2 tangent at the origin, but only simply so, that is having different curvature at 0. Then local coordinates can be introduced, near 0, x, y with respect to which:

$$K_1 = \{y = 0\}, \quad K_2 = \{y = x^2\}. \tag{2.1}$$

Since all considerations here will be local it can be freely assumed that (2.1) holds globally. We wish to consider various operators related to K_1 and K_2 and their behaviour in singular coordinates.

The first set of operators consists of the space of vector fields tangent to K_1 and vanishing at 0. Following the notation of [MR] we introduce the C^∞ variety;

$$\mathscr{K}_1 = K_1 \sqcup \{0\} = \{K_1 \setminus \{0\}, \{0\}\}. \tag{2.2}$$

A C^∞ variety is just a (locally) finite collection of disjoint submanifolds with closed union, the main use being to define the space $\mathscr{V}(\mathscr{K})$, for any C^∞ variety $\mathscr{K}$, of C^∞ vector fields tangent to each element of $\mathscr{K}$. Now, $\mathscr{V}(\mathscr{K})$ is always a C^∞ module and Lie algebra, so we consider a set of generators (as a C^∞ module).

$$\mathscr{V}(\mathscr{K}_1) \text{ is generated by } yD_y, yD_x, xD_x. \tag{2.3}$$

Similarly one finds easily that:

$$\mathscr{V}(\mathscr{K}_2) \text{ is generated by } (y-x^2)D_y, (y-x^2)D_x, x(D_x+2xD_y). \tag{2.4}$$

Next consider the full C^∞ variety defined by the two curves:

(2.5) $$\mathscr{K} = \mathscr{K}_1 \sqcup \mathscr{K}_2 = K_1 \sqcup K_2,$$

where the join of two C^∞ varieties is defined in [MR]. From (2.3) and (2.4),

(2.6) $$\mathscr{V}(\mathscr{K}) \text{ is generated by } 2yD_y+xD_x, (y-x^2)D_x.$$

The spaces considered in [MR] are defined by iterated regularity with respect to such Lie algebras of vector fields. Thus,

(2.7) $$I_k L^2(\mathbf{R}^2, \mathscr{K}) = \{u \in L^2(\mathbf{R}^2);\ \mathscr{V}(\mathscr{K})^j u \subset L^2(\mathbf{R}^2) \forall j \leqq k\}$$

and we also use the compact notation:

(2.8) $$L^\infty I_k L^2(\mathbf{R}^2, \mathscr{K}) = L^\infty(\mathbf{R}^2) \cap I_k L^2(\mathbf{R}^2, \mathscr{K}).$$

Here and below all these are local, i.e. $u \in L^2(\mathbf{R}^2)$ means that

$$\int |\varphi u|^2\, dx < \infty, \quad \forall \varphi \in C_c^\infty(\mathbf{R}^2).$$

As well as the spaces $L^\infty I_k L^2(\mathbf{R}^2, \mathscr{K})$, for $\mathscr{K} = \mathscr{K}_1, \mathscr{K}_2$ or $\mathscr{K}$ we shall also consider the space of Lagrangian distributions associated to the two Lagrangians N^*K_1, N^*K_2. Set

(2.9) $$\Lambda = N^*K_1 \cup N^*K_2$$

and then define $\mathscr{M}(\Lambda) \subset \Psi_p^1(\mathbf{R}^2)$, the space of properly supported pseudodifferential operators on $\mathbf{R}^2$, as the subspace of those operators characteristic on Λ:

(2.10) $$A \in \mathscr{M}(\Lambda) \Leftrightarrow \sigma_1(A) = 0 \quad \text{on} \quad \Lambda.$$

Now, $\mathscr{M}(\Lambda)$ is clearly a Lie algebra, since Λ is a finite union of Lagrangians, and a $\Psi_p^0(\mathbf{R}^2)$-module. As such,

(2.11) $$\mathscr{M}(\Lambda) \text{ is generated by } 2yD_y+xD_x, (y-x^2)D_x, D_x(D_x+2xD_y)\cdot \Psi_p^{-1}(\mathbf{R}^2).$$

Now if we define $I_k L^2(\mathbf{R}^2, \Lambda)$ by replacing $\mathscr{V}(\mathscr{K})$ by $\mathscr{M}(\Lambda)$ in (2.7), then it follows easily that

(2.12) $$u \in I_k L^2(\mathbf{R}^2, \Lambda) \Leftrightarrow A_1^h A_2^i A_3^j u \in L^2(\mathbf{R}^2), \quad h+i+j \leqq k,$$

where A_1, A_2 and A_3 are the first two generators in (2.11) and $A_3 = E\cdot D_x(D_x+2xD_y)$, with $E \in \Psi_p^{-1}(\mathbf{R}^2)$ elliptic. A similar remark applies to the other spaces $I_k L^2(\mathbf{R}^2, \mathscr{K})$.

Now let $B_1: X_1 = [0, \infty) \times S^1 \to \mathbf{R}^2$, be standard polar coordinates:

(2.13) $$B_1(r, \theta) = (r\cos\theta, r\sin\theta),$$

where $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. Writing K for the support of $\mathscr{K}$, i.e. $K = K_1 \cup K_2$, note that:

(2.14) $$B_1^{-1}(K) = F_1 \cup F_2 \cup R_1,$$

where

(2.15) $$F_1 = \{\sin\theta = 0\},\quad F_2 = \{\sin\theta - r\cos^2\theta = 0\},\quad R_1 = \{r=0\}.$$

We divide this up into C^∞ varieties. Let $L_1=\{r=0, \sin\theta=0\}$ be the intersection of each pair of these curves in the manifold with boundary X_1. Then set

(2.16) $$\mathscr{F}_1 = \{F_1\setminus L_1, R_1\setminus L_1, L_1\},\quad \mathscr{F}_2 = \{F_2\setminus L_1, R_1\setminus L_1, L_1\},$$

and

(2.17) $$\mathscr{F} = \{F_1\setminus L_1, F_2\setminus L_1, R_1\setminus L_1, L_1\}.$$

(2.18) **Lemma.** *Under B_1 each element of $\mathscr{V}(\mathscr{K}_1)$, $\mathscr{V}(\mathscr{K}_2)$ and $\mathscr{V}(\mathscr{K})$ lifts to a C^∞ vector field on X_1 and these lifts span, respectively, $\mathscr{V}(\mathscr{F}_1)$, $\mathscr{V}(\mathscr{F}_2)$ and $\mathscr{V}(\mathscr{F})$.*

Proof. Since any C^∞ function lifts under a C^∞ map such as B_1, it suffices to check first that each of the generators in (2.3) and (2.4) lifts. Now, away from $\{r=0\}$,

(2.19) $$D_x = B_{1*}[\cos\theta\cdot D_r - r^{-1}\sin\theta D_\theta],\quad D_y = B_{1*}[\sin\theta D_r + r^{-1}\cos\theta D_\theta]$$

so the vector fields in (2.3) lift to

(2.20)
$$\sin^2\theta\cdot rD_r+\sin\theta\cos\theta D_\theta,\quad \sin\theta\cos\theta\cdot rD_r-\sin^2\theta D_\theta,\quad \cos^2\theta\cdot rD_r-\sin\theta\cos\theta D_\theta.$$

These vector fields generate the same $C^\infty(X_1)$ module as

(2.21) $$rD_r \quad\text{and}\quad \sin\theta D_\theta,$$

which is just the Lie algebra $\mathscr{V}(\mathscr{F}_1)$. A similar argument applies to the lift of $\mathscr{V}(\mathscr{K}_2)$.

Direct computation of the lifts of the vector fields in (2.6) gives

(2.22) $$(1+\sin^2\theta)rD_r+\sin\theta\cos\theta D_\theta,\quad (\sin\theta-r\cos^2\theta)(\cos\theta\cdot rD_r-\sin\theta D_\theta).$$

Near $r=0$, i.e. R_1, but away from $\sin\theta=0$, these span the same C^∞ module as rD_r and D_θ. Near $\sin\theta=0$, introduce as new C^∞ coordinates in X_1,

(2.23) $$t = \sin\theta/\cos\theta = y/x,\quad\text{and}\quad x.$$

Then the vector fields in (2.22) locally span the same $C^\infty(X_1)$-module as

(2.24) $$xD_x+tD_t,\quad (t-x)[xD_x-tD_t].$$

These two vector fields are easily seen to generate the Lie algebra $\mathscr{V}(\mathscr{F})$, proving the lemma.

Let $\omega_1=r\,dr\,d\theta$ be the polar measure on X_1, and denote by L^2_B the space of locally square-integrable functions with respect to this measure. Thus:

(2.25) $$B_1^*\colon L^2(\mathbf{R}^2) \leftrightarrow L^2_B(X_1),$$

is an isomorphism. From Lemma 2.18 we have directly:

(2.26) **Corollary.** *B_1^* is an isomorphism, for all $k\in\mathbf{N}$, on the spaces:*

(2.27)
$$B_1^*\colon I_kL^2(\mathbf{R}^2,\mathscr{H})\leftrightarrow I_kL_B^2(X_1,\mathscr{P}),\quad\text{for}\quad(\mathscr{H},\mathscr{P})=(\mathscr{K},\mathscr{F}),(\mathscr{K}_i,\mathscr{F}_i),\quad i=1,2.$$

Next we wish to examine the lift to X_1 of the space $I_kL^2(\mathbf{R}^2,\Lambda)$. Since this is a subspace of $I_kL^2(\mathbf{R}^2,\mathscr{K})$ it follows from Corollary 2.26 that

$$B_1^*(I_kL^2(\mathbf{R}^2,\Lambda))\subset I_kL_B^2(X_1,\mathscr{F}). \tag{2.28}$$

This gives good control of the lift away from the intersecting curves, i.e. away from L_1. In view of this we consider simpler local coordinates on X_1 near L_1 given in (2.23). In terms of these coordinates consider the lift of the differential operator Q, involved in (2.11):

$$Q=D_x(D_x+2xD_y). \tag{2.29}$$

Thus if Q_1 is defined by $B_1^*(Qv)=Q_1(B_1^*v)$, for all $v\in C^\infty(\mathbf{R}^2)$ then

$$Q_1=x^{-1}\cdot(xD_x-tD_t)\cdot x^{-1}\cdot(xD_x-tD_t+xD_t). \tag{2.30}$$

In [Me] certain Sobolev spaces, $H_b^m(X_1)$, associated to the boundary of a manifold with boundary, such as X_1, are discussed. In particular in the coordinates x, t, where the boundary is defined by $x=0$, the space of negative integral order can be defined simply:

$$v\in H_b^{-p}(X_1)\text{ near }L_1\Leftrightarrow v=v_0+\textstyle\sum_{i=1}^p(xD_x)^iD_t^{p-i}v_i,\quad v_i\in L_b^2(X_1), \tag{2.31}$$

where $L_b^2(X_1)$ is the space of square-integrable functions with respect to the measure $x^{-1}\,dx\,dt$. Thus,

$$L_b^2(X_1)=x^{1/2}L^2(X_1)=xL_B^2(X_1),\text{ near }L_1. \tag{2.32}$$

In terms of the desingularized form of Q:

$$Q_1'=(xD_x-tD_t)(xD_x-tD_t+xD_t) \tag{2.33}$$

we can bound the behaviour of the lift of $I_kL^2(\mathbf{R}^2,\Lambda)$.

(2.34) **Lemma.** *If $\varphi\in C_c^\infty(X_1)$ has support in the coordinate neighbourhood of x and t in (2.23) then any $u\in I_kL^2(\mathbf{R}^2,\Lambda)$ satisfies the regularity conditions:*

$$(Q_1')^pW_1^iW_2^j\varphi(B_1^*u)\in\textstyle\sum_{l=1}^p x^{l-1}H_b^{-l}(X_1),\quad\textit{provided}\quad p+i+j\leqq k. \tag{2.35}$$

where W_1 and W_2 are the lifts to X_1 of the first two vector fields in (2.11).

Proof. The original definition (2.12) of $I_kL^2(\mathbf{R}^2,\Lambda)$ can be written in terms of Q and the first two vector fields, V_1 and V_2 in (2.11):

$$Q^pV_1^iV_2^ju\in H^{-p}(\mathbf{R}^2),\quad\text{provided}\quad p+i+j\leqq k. \tag{2.36}$$

Now the Sobolev space of negative order can be expressed in terms of differential operators as:

(2.37) $$v \in H^{-p}(\mathbf{R}^2) \Leftrightarrow w = \sum_{|\alpha| \leq p} D^\alpha v_\alpha, \quad v_\alpha \in L^2(\mathbf{R}^2).$$

Clearly then from (2.32),

$$v \in H^{-p}(\mathbf{R}^2) \Rightarrow x^p \varphi \cdot B_1^* v \in x^{-1} H_b^{-p}(X_1).$$

Applying this to (2.36) it follows that $x^{2p}(Q_1)^p W_1^i W_2^j \varphi(B_1^* u)$ is in the space on the right in (2.35). To get (2.35) from this is only a matter of a simple commutation argument to pass from $x^{2p} Q_1^p$ to $(Q_1')^p$.

3. Second blow up

After one blowing up the tangency between the curves in (2.1) has been removed. However there is still multiplicity in the sense that three lines pass through each of the two points of L_1. It is therefore convenient to blow up, again, around these points, i.e. polar coordinates will be introduced there. The doubly blown up manifold, X_2 can then be visualized as the exterior of four circles, each of radius 2, centred at the points $(\pm 1, \pm 1)$. The new blow down map will be written:

(3.1) $$B_2\colon X_2 \to X_1, \quad B = B_1 \cdot B_2\colon X_2 \to \mathbf{R}^2.$$

In X_2 there are four distinguished (non-connected) curves. First there are the original curves lifted:

(3.2) $$G_i = \mathrm{cl}\,[B^{-1}(K_i \setminus L)] = \mathrm{cl}\,[B_2^{-1}(F_i \setminus L_1)], \quad i = 1, 2.$$

Next there are the two radial lines, the new one and the old one lifted to the new space.

(3.3) $$R_3 = B_2^{-1}(L_1), \quad R_2 = \mathrm{cl}\,[B_2^{-1}(R_1 \setminus L_1)].$$

From these curves we form three C^∞ varieties, the lifts to X_2 of $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}$, although the concept of the lift of a C^∞ variety will not be defined in general here:

(3.4) $$\mathscr{G}_i = G_i \sqcup R_2 \sqcup R_3, \quad \mathscr{G} = G_1 \sqcup G_2 \sqcup R_2 \sqcup R_3 = \mathscr{G}_1 \sqcup \mathscr{G}_2, \quad i = 1, 2,$$

i.e. as the collections consisting of each of the appropriate curves, minus the intersection with the others together with these intersections. Let ω_2 be the lift to X_2 of ω_1, i.e. the lift under B of the original Lebesgue measure, and again denote by $L_B^2(X_2)$ the space of functions locally square integrable with respect to it:

(3.5) $$B^*\colon L^2(\mathbf{R}^2) \leftrightarrow L_B^2(X_2).$$

The first result on lifting to X_2 is a special case of a general result on the lifting under blow down maps of the conormal functions associated to a variety with normal crossings: the variety stays normal and the functions stay conormal.

(3.6) **Lemma.** *Under the double blow down map:*

$$B^*: I_k L^2(\mathbf{R}^2, \mathscr{K}_i) \leftrightarrow I_k L^2_B(X_2, \mathscr{G}_i). \tag{3.7}$$

Proof. As in Corollary 2.26 this is just a matter of examining the analogue of Lemma 2.18, i.e. checking that the vector fields in (2.21) lift under B_2 to span $\mathscr{V}(\mathscr{G}_1)$ as a $C^\infty(X_2)$ module. Away from L_1, X_2 is diffeomorphic to X_1 so we can consider generators of $\mathscr{V}(\mathscr{F}_1)$ in the simpler form arising in the coordinates (2.23), i.e.

$$xD_x \quad \text{and} \quad tD_t, \quad \text{in} \quad x \geqq 0,\ t\in\mathbf{R}. \tag{3.8}$$

Under the polar coordinates which implement B_2, nearby

$$B_2(\varrho, \psi) = (\varrho\cos\psi, \varrho\sin\psi) = (x, t), \quad \varrho \geqq 0, \quad \psi\in[-1/2\pi, 1/2\pi], \tag{3.9}$$

these lift to

$$\cos^2\psi\cdot\varrho D_\varrho - \cos\psi\sin\psi D_\psi \quad \text{and} \quad \sin^2\psi\cdot\varrho D_\varrho + \cos\psi\sin\psi D_\psi. \tag{3.10}$$

It is readily seen that these two vector fields span $\mathscr{V}(\mathscr{G}_1)$ locally. This completes the proof of the Lemma, since the demonstration for $\mathscr{G}_2$ is completely parallel.

Now consider the space analogous to that in (1.3) for the C^∞ variety composed of the two simply tangent lines K_1 and K_2:

$$J_k L^2(\mathbf{R}^2, \mathscr{K}) = I_k L^2(\mathbf{R}^2, \mathscr{K}_1) + I_k L^2(\mathbf{R}^2, \mathscr{K}_2), \tag{3.11}$$

and then the subspace of bounded elements:

$$L^\infty J_k L^2(\mathbf{R}^2, \mathscr{K}) = L^\infty(\mathbf{R}^2) \cap J_k L^2(\mathbf{R}^2, \mathscr{K}). \tag{3.12}$$

We shall not attempt here to explain this notation in a wider context, but simply note the important consequence of Lemma 3.6.

(3.13) **Proposition.** *Each space $L^\infty J_k L^2(\mathbf{R}^2, \mathscr{K})$ in (3.12), where $\mathscr{K}$ is the C^∞ variety composed of two curves in the plane simply tangent at a point, is a C^∞ algebra, i.e. with $\mathscr{R} = L^\infty J_k L^2(\mathbf{R}^2, \mathscr{K})$,*

$$F\in C^\infty(\mathbf{R}^2\times\mathbf{R}^n), \quad u_i\in\mathscr{R} \quad i = 1, \ldots, n \Rightarrow F(., u_1, \ldots, u_n)\in\mathscr{R}. \tag{3.14}$$

Proof. It is shown in [MR] that the space $L^\infty(X_2)\cap I_k L^2_B(X_2, \mathscr{G})$ is a C^∞ algebra. Thus to prove the Proposition it suffices to show that:

$$B^*: J_k L^2(\mathbf{R}^2, \mathscr{K}) \leftrightarrow I_k L^2_B(X_2, \mathscr{G}), \tag{3.15}$$

since $B^*: L^\infty(\mathbf{R}^2) \leftrightarrow L^\infty(X_2)$. According to Lemma 3.6, and the definition (3.11) of $J_k L^2(\mathbf{R}^2, \mathscr{K})$, (3.15) is equivalent to

$$I_k L^2_B(X_2, \mathscr{G}) = I_k L^2_B(X_2, \mathscr{G}_1) + I_k L^2_B(X_2, \mathscr{G}_2). \tag{3.16}$$

It is only necessary to verify (3.16) locally near each point of X_2, since all the spaces are local. Near any given point of X_2 either $\mathscr{G}=\mathscr{G}_1$ or $\mathscr{G}=\mathscr{G}_2$ locally, in the sense that their component manifolds coincide locally. Corresponding to the first case we have:

(3.17) $$\mathscr{V}(\mathscr{G})=\mathscr{V}(\mathscr{G}_1) \text{ and } \mathscr{V}(\mathscr{G}_1)\subset\mathscr{V}(\mathscr{G}_2) \text{ locally.}$$

Clearly (3.17) implies the converse inclusion for the iterated spaces, i.e.

(3.18) $$I_k L^2_B(X_2,\mathscr{G})=I_k L^2_B(X_2,\mathscr{G}_1),\quad I_k L^2_B(X_2,\mathscr{G}_2)\subset I_k L^2_B(X_2,\mathscr{G}_1) \text{ locally.}$$

This of course implies (3.16) near such points. The other case in (3.17) has the same consequence so (3.16) holds everywhere and the Proposition is proved.

This is the main result needed from this section for the proof of the propagation theorem described in the Introduction. We shall however proceed to analyse the other spaces introduced above. In particular we shall prove the inclusions analogous to (1.4) (and essentially implying those) in $\mathbf{R}^2$, i.e.

(3.19) $$I_k L^2(\mathbf{R}^2,\Lambda)\subset J_k L^2(\mathbf{R}^2,\mathscr{K})\subset I_k L^2(\mathbf{R}^2,\mathscr{K})$$

defined in (2.12), (3.11) and (2.8) respectively. Then we shall discuss the deficiencies of the last space in (3.19), or rather its superfluity.

Consider then (3.19). The second inclusion is trivial from the definition (3.11) and the obvious inclusions

(3.20) $$\mathscr{V}(\mathscr{K})\subset\mathscr{V}(\mathscr{K}_1),\quad \mathscr{V}(\mathscr{K})\subset\mathscr{V}(\mathscr{K}_2).$$

Similarly it follows directly from the definitions of the spaces that the first is contained in the last. The first inclusion, amounting to a splitting of the first space, is not quite so obvious so it will be considered in detail.

(3.21) **Lemma.** *Under B, the double polar blow down map* (3.3), *the space defined in* (2.12) *lifts into that in* (3.11)*:*

(3.22) $$B^*\colon I_k L^2(\mathbf{R}^2,\Lambda)\to I_k L^2_B(X_2,\mathscr{G}).$$

Proof. First recall (2.28). Using this away from L_1 we certainly have (3.22) away from the new radial line R_3, since B_2 is a diffeomorphism there. Thus it suffices to consider the lift to X_2 of functions on X_1 of the form $w=\varphi B_1^* u$, where $u\in I_k L^2(\mathbf{R}^2,\Lambda)$ and φ has support near L_1. Lemma 2.34 gives information on the regularity of such functions, w, and whilst (2.35) is strictly weaker than the condition $u\in I_k L^2(\mathbf{R}^2,\Lambda)$ we shall only use the estimates (2.35) to prove (3.22).

So suppose that w, with support sufficiently near L_1, satisfies these estimates,

(3.23) $$(Q_1')^p W_1^i W_2^j w\in \textstyle\sum_{m=0}^p x^{m-1} H_b^{-m}(X_1), \quad\text{for}\quad p+i+j\leqq k.$$

Then to analyse $B_2^* w$ we need to consider the lifts to X_2 of the operators Q_1', W_1 and

W_2 under (3.9), since near the other component of L_1 the same argument can be applied. Instead of the coordinates ϱ and ψ of (3.9) it is convenient, as in § 2, to use projective coordinates in more restricted subsets of X_2. Thus, the desired conclusion:

(3.24) $$B_2^* w \in I_k L_B^2(X_2, \mathscr{G})$$

is again local on X_2.

We first examine (3.24) near $R_2 \cap R_3 = L_3$, away from the curves G_1 and G_2. Here the obvious coordinates are:

(3.25) $$t \text{ and } s = x/t$$

in terms of the coordinates (2.23) on X_1. Writing $B_2^*(Q_1' w) = Q_2(B_2^* w)$, it is then easily seen that:

(3.26) $$Q_2 = (2sD_s - tD_t)(2sD_s - tD_t + stD_t - s^2 D_s), \quad B_{2*}(tD_t) = W_1.$$

and no use will be made of W_2 here. Choosing $\mu \in C_c^\infty(X_2)$ to have small support near $s=t=0$, the inclusions (3.23) imply, for $p+i \leq k$,

(3.27) $$Q_2^p (tD_t)^i (w_1) = \sum_{j+l=0}^{p+i} (tD_t)^j (sD_s)^l e_{jl}, \; e_{jl} \in L_B^2(X_2), \; w_1 = \mu B_2^*(w).$$

Now we claim that this implies in turn,

(3.28) $$(tD_t)^i (sD_s)^j w_1 \in L_B^2(X_2), \quad \text{for} \quad i+j \leq k.$$

Certainly this implies (3.24) locally near $R_2 \cap R_3$. Now deducing (3.28) from (3.27) is really a matter of elliptic regularity estimates, and can be easily accomplished in a calculus of pseudodifferential operators on a manifold with corner, generalizing that in [Me]; since this is not available in the literature we give an elementary derivation. Decompose Q_2 as:

(3.29) $$Q_2 = Q_2' + s(2sD_s - tD_t)tD_t + E, \quad Q_2' = (1 - 1/2s)(2sD_s - tD_t)^2,$$

where E is a first order operator in sD_s and tD_t. A simple commutation argument shows first that (3.27) remains true if Q_2 is replaced by $Q_2 - E$. Now it is also the case that (3.27) remains valid with Q_2 replaced by Q_2', even though the difference is not of first order. This is simply because the difference has a factor tD_t in it, so can be controlled inductively by the estimates (3.27). Thus

(3.30) $$(2sD_s - tD_t)^{2p} (tD_t)^i w_1 = \sum_{j+l=0}^{p+i} (sD_s)^j (tD_t)^l e_{jl}, \; e_{jl} \in L_B^2(X_2), \quad p+i \leq k.$$

Now we can also assume that w_1 and all the e_{jl} have supports in $[0, 1] \times [0, 1]$. These conditions can be reduced to constant coefficient estimates by introducing logarithmic coordinates, $S = \log(s)$, $T = \log(t)$, since then:

$$D_T = tD_t, \quad D_S = sD_s.$$

In the local coordinates s, t the measure ω_2 is equivalent to $s^2 t\, ds\, dt$, which becomes exponential in S and T. Thus (3.30) becomes:

$$(3.31)\qquad (2D_S - D_T)^{2p} D_T^i(\tilde{w}_1) \in H^{-p}(\mathbf{R}^2), \quad p+i \leqq k,$$

where $\tilde{w}_1(S, T) = e^{S+(1/2)T} w_1(s, t)$. Here, $H^{-p}(\mathbf{R}^2)$ is the usual global Sobolev space. The combination of operators in (3.31) is elliptic, so this implies

$$(3.32)\qquad \tilde{w}_1 \in H^k(\mathbf{R}^2).$$

Translating back to the coordinates s and t this is just (3.28).

This has taken care of one small part of (3.24), so that now we can cut $B_2^* w$ off to have support away from $R_2 \cap R_3$. In particular we shall now use the other projective coordinates

$$(3.33)\qquad x \text{ and } z = t/x,$$

in place of (3.25). Then assume that $v \in C_c^\infty(X_2)$ has support in the corresponding coordinate patch and consider the regularity of

$$w_2 = v B_2^* w.$$

Now, we need to translate (3.23) to regularity for w_2, in particular to examine the operators Q_1', W_1 and W_2 in the coordinates (3.33). Once again we can ignore W_2 and simply note that Q_1' and W_1 lift to

$$(3.34)\qquad Q_3 = [xD_x - 2zD_z][xD_x + 2(1-z)D_z], \; xD_x.$$

Thus we can assume that w_2 satisfies the iterated regularity conditions:

$$(3.35)\qquad Q_3^p (xD_x)^i w_2 \in x^{-3/2} H_b^{-p}([0, \infty)\times \mathbf{R}) \quad i+p \leqq k.$$

To deduce (3.24) from this we could proceed in an elementary fashion as in the discussion of w_1 above, but since there is only a simple boundary present the analysis can be shortened by using the calculus of [Me]. In particular setting $i=k$ in (3.35) gives the regularity:

$$(3.36)\qquad (xD_x)^k w_2 \in x^{-3/2} L_b^2(X_2)\; [= L_B^2(X_2) \text{ locally}],$$

so we need only consider microlocal regularity, with respect to totally characteristic operators, in the region where xD_x is not elliptic. Consider the first factor in Q_3, in (3.34). Away from $G_1 = \{z=0\}$, this is elliptic where xD_x is not, so a simple commutation argument using (3.35), with $i=0$, implies that

$$(3.37)\qquad (xD_x + 2(1-z)D_z)^p w_2 \in x^{-3/2} L_b^2(X_2), \quad p \leqq k, \quad \text{in} \quad |z| > \varepsilon.$$

The corresponding estimates intermediate between (3.37) and (3.36) can be obtained in the the same way, or follow in any case by interpolation. Together, xD_x and $(1-z)D_z$ generate, locally $\mathscr{V}(\mathscr{G})$, so (3.24) holds away from G_1. Of course, using the other factor in Q_3 the same conclusion holds near G_1 so this completes the proof of the proposition.

We remark again that the inclusion (1.4) is easily obtained from this proposition and the discussion above. This result also indicates strongly that the space in the centre in (1.4) is the smallest "reasonable" space, J, which contains the ("very linear") space of Lagrangian functions and yet has the property that its bounded elements (i.e. of J) form a ring. To emphasize this point we shall conclude this section by examining the largest space $I_k L^2(\mathbf{R}^2, \mathscr{K})$ further, in terms of the blow up, and show why it does not deserve to be thought of as consisting of the "conormal" functions associated to the C^∞ variety $\mathscr{K}$.

(3.38) **Proposition.** *Let $\mathscr{W}$ be the Lie algebra of all C^∞ vector fields on X_2 which are tangent to $\mathscr{G}$ and vanish at the part R_3 of the boundary, then*

(3.39) $$B^*\colon I_k L^2(\mathbf{R}^2, \mathscr{K}) \leftrightarrow I_k L^2_B(X_2, \mathscr{W}).$$

Proof. In view of Lemma 2.18, to prove this result it is only necessary to show that:

(3.40) $$\mathscr{W} \text{ is spanned by the } B_2\text{-lifts of } \mathscr{V}(\mathscr{F}).$$

We leave the details to the interested reader, except to note the following simple, but crucial lemma which shows the effect of the multiplicity in the variety $\mathscr{F}$, the three lines passing through a point.

(3.41) **Lemma.** *Let U be a real vector space of dimension two. If L_1, L_2 and L_3 are three distinct lines in U then the space of linear vector fields tangent to all three is one dimensional.*

Proof. Choose a basis of U such that $L_1=\{x=0\}, L_2=\{y=0\}$ in the induced coordinates, so $L_3=\{ax=by\}$, $a, b\neq 0$. Changing variable from x to ax/b reduces L_3 to $\{x=y\}$. Now it is obvious that the only linear vector field tangent to all three lines is xD_x+yD_y, the radial vector field. This is true of course in any basis.

(3.42) *Remark.* Proposition 3.38 shows that the elements of $I_k L^2(\mathbf{R}^2, \mathscr{K})$ lifted to X_2 only have iterated regularity with respect to the Lie algebra $\mathscr{W}$. To make clear how weak this is note that any harmonic function near the part of the boundary, R_3, where $\mathscr{W}$ differs from $\mathscr{V}(\mathscr{G})$, which is in L^2 actually has such iterated regularity. Thus an element of $IL^2(\mathbf{R}^2, \mathscr{K})$ can "include" an arbitrary function in some finite Sobolev space on a line, the corresponding Dirichlet data. This is not what one expects of a conormal function. More particularly, although these "hidden" singularities do not affect the wavefront set, which is contained in the conormal variety, they can be expected to eventually release a shower of singularities on further interaction in the types of hyperbolic problems of basic interest here. The J-type spaces do not have these complicating features.

4. Tangent surfaces

Suppose that H_1 and H_2 are two C^∞ surfaces in $\mathbf{R}^3$ which are simply tangent along a common line:

(4.1) $$H_1 \cap H_2 = L \quad \text{is a } C^\infty \text{ line}$$

and

(4.2) in any plane transversal to L, H_1 and H_2 are simply tangent curves.

Using the results of §§ 2 & 3 we can easily analyse the three spaces associated to such a pair of surfaces in the Introduction, and prove the inclusions (1.4). In fact the C^∞ varieties formed from H_1 and H_2 are all products of those examined above.

(4.3) **Lemma.** *If H_1 and H_2 satisfy* (4.1) *and* (4.2) *near $p \in L$ then there are local coordinates x, y, z with p as the origin with respect to which:*

(4.4) $$H_1 = \{y = 0\}, \quad H_2 = \{y = x^2\}.$$

Proof. By the implicit function theorem H_1 can be brought to the normal form in (4.4). Since it is a curve in H_1, L can simultaneosly be brought to the appropriate normal form $\{x = y = 0\}$. Then H_2 must have as a defining function

$$h_2 = y - ax^2, \quad a = a(x, z) \neq 0,$$

changing the sign of y if necessary it can be assumed that $a > 0$, then replacing y by $y/\sqrt{a}$ gives the desired form.

Now let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}$ be the three C^∞ varieties defined from these two surfaces by:

(4.5) $$\mathscr{H}_1 = \{H_1 \setminus L, L\}, \quad \mathscr{H}_2 = \{H_2 \setminus L, L\}, \quad \mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2.$$

Using the definition (2.7), with $L^2(\mathbf{R}^2)$ replaced by $L^2(\mathbf{R}^3)$ etc., this allows us to associate with $\mathscr{H}$ the spaces of iterated regularity $I_k L^2(\mathbf{R}^3, \mathscr{H})$ and

(4.6) $$J_k L^2(\mathbf{R}^3, \mathscr{H}) = I_k L^2(\mathbf{R}^3, \mathscr{H}_1) + I_k L^2(\mathbf{R}^3, \mathscr{H}_2).$$

Similarly set $\Lambda = N^*H_1 \cup N^*H_2$, define $\mathscr{M}(\Lambda) \subset \Psi^1_P(\mathbf{R}^3)$ by (2.10) and $I_k L^2(\mathbf{R}^3, \Lambda)$ by using the same definition with $\mathscr{M}(\Lambda)$ as Lie algebra of operators.

(4.7) **Proposition.** *If H_1 and H_2 are two C^∞ surfaces in $\mathbf{R}^3$ satisfying* (4.1) *and* (4.2) *then for any $k \in \mathbf{N}$,*

(4.8) $$I_k L^2(\mathbf{R}^3, \Lambda) \subset J_k L^2(\mathbf{R}^3, \mathscr{H}) \subset I_k L^2(\mathbf{R}^3, \mathscr{H}), \quad \Lambda = N^*H_1 \cup N^*H_2.$$

Moreover,

(4.9) $$L^\infty J_k L^2(\mathbf{R}^3, \mathscr{H}) = L^\infty(\mathbf{R}^3) \cap J_k L^2(\mathbf{R}^3, \mathscr{H}) = L^\infty I_k(\mathbf{R}^3, \mathscr{H}_1) + L^\infty I_k L^2(\mathbf{R}^3, \mathscr{H}_2)$$

is a C^∞ algebra.

Proof. Most of this proposition follows, essentially trivially, from the discussion above of the case of two tangent curves. All the spaces are obviously local and coordinate independent, so one can work with the model case (4.4). The extra variable z is then a parameter in all the blowing up operations, for example a basis of $\mathcal{V}(\mathcal{H}_1)$ is given by (2.3) and D_z. The results proved above on the lifts of these space carry through essentially unchanged, so (4.8) follows easily from the analogue of Lemma 3.21 and the characterization of the lift of $J_k L^2(\mathbf{R}^3, \mathcal{H})$ to $\mathbf{R}\times X_2$, i.e. the analogue of (3.15). In fact the remainder of the proposition also follows from this discussion, since not only is $L^\infty J_k L^2(\mathbf{R}^3, \mathcal{H})$ shown to be a ring, as in Proposition 3.13, but the characterization of the lift under the double blow up implies the equality in (4.9).

5. P-propagative algebras

Despite some danger of confusion from excessive terminology we shall consider here the notion of a P-propagative algebra, including in this way natural properties which give iterative regularity for solutions of (1.7). Thus throughout we shall suppose that P is a strictly hyperbolic operator of second order in some open set Ω, in $\mathbf{R}^n$ or a manifold, which is P-convex with respect to the subsets (in the past)

$$\Omega_\delta = \{p\in\Omega;\ t(\varrho) < \delta\}, \quad \text{for some time function } t \text{ and } \quad 0 < \delta < \bar{\delta}.$$

Thus we assume that the continuation form of the Cauchy problem is well-posed in Ω:

(5.1) if $f\in C^{-\infty}(\Omega)$ and $f=0$ in Ω_ϱ, there exists a unique $u\in C^{-\infty}(\Omega)$ with $u=0$ in Ω_δ and $Pu=f$ in Ω.

Now, let $J_k(\Omega)$ be a filtered space of locally square-integrable functions on Ω:

$$L^2(\Omega) = J_0(\Omega) \supset J_1(\Omega) \supset \ldots \supset J_k(\Omega) \supset \ldots \supset J(\Omega) = \bigcap_{k=0}^{\infty} J_k(\Omega). \tag{5.2}$$

We require that solving the linear wave equation increase the iterative regularity of these spaces, in the sense that:

$$Pu = f\in J_{k-1}(\Omega),\quad u\in J_{k-1}(\Omega),\quad u=f=0 \text{ in } \Omega_\delta \Rightarrow u\in J_k(\Omega). \tag{5.3}$$

We shall also require each of the subspaces of locally bounded functions to be a ring, or more precisely:

$$L^\infty J_k(\Omega) = L^\infty(\Omega)\cap J_k(\Omega) \quad \text{is a } C^\infty \text{ algebra,} \tag{5.4}$$

i.e. each of these satisfies (3.14) in place of $\mathcal{R}$. It is also convenient to be able to localize with cutoff functions, so all these spaces should be $C^\infty(\Omega)$-modules

$$C^\infty(\Omega)\cdot J_k(\Omega) \subset J_k(\Omega)\,\forall\, k\in\mathbf{N}. \tag{5.5}$$

(5.6) *Definition.* A filtered space of functions $J_k(\Omega)$ as in (5.2) which satisfies (5.3), (5.4) and (5.5) will be called a P-propagative algebra.

Of course the main reason to introduce this definition is just to have explicit conditions on $J_k(\Omega)$ which imply iterative regularity for solution to (1.7).

(5.7) **Proposition.** *If* $u \in L^\infty(\Omega)$ *satisfies*

(5.8) $$Pu = f(., u) + g, \quad in\ \Omega,$$

with P a strictly hyperbolic operator of second order, Ω P-convex with respect to Ω_δ as above, then

(5.9) $$u = 0 \quad in \quad \Omega_\delta, \quad g \in J_{k-1}(\Omega) \Rightarrow u \in J_k(\Omega).$$

Proof. This is immediate for $k=0$, since $J_0(\Omega) = L^2(\Omega)$. Then (5.3) and (5.4) allow an inductive argument. Indeed, if $u \in J_{p-1}(\Omega)$ then (5.4) shows that

(5.10) $$f(\,.\,, u) + g \in J_{p-1}(\Omega), \quad p \leqq k,$$

so (5.3), and the fact that the function in (5.10) vanishes in Ω_δ implies that $u \in J_p(\Omega)$. This completes the proof of the proposition.

Many examples of P-propagative algebras were given in [MR]. We recall some of the terminology from there and add a little more. The rings (hence algebras) considered in [MR] are of the form $I_k L^2(\Omega, \mathcal{V})$ for some Lie algebra of C^∞ vector fields on Ω.

(5.11) *Definition.* A Lie algebra and C^∞ module of vector fields, $\mathcal{V}$, is said to be P-complete if:

(5.12) $$[P, \mathcal{V}] \subset \Psi^0 \cdot P + \Psi^1 \cdot \mathcal{V} + \Psi^1,$$

where $\Psi^m = \Psi^m_P(\Omega)$ is the space of properly supported pseudodifferential operators of order m on Ω.

(5.13) **Proposition.** *If $\mathcal{V}$ is a P-complete, locally finitely generated C^∞ module and Lie algebra of C^∞ vector fields on Ω, P-convex with respect to Ω_δ, then*

(5.14) $$J_k(\Omega) = I_k L^2(\Omega, \mathcal{V})$$

defines a P-propagative algebra.

Proof. See [MR].

In [MR] some sufficient conditions are given on the Lie algebra $\mathcal{V}(\mathcal{K})$ arising from tangency to a C^∞ variety, $\mathcal{K}$, in order that it be P-complete. The conormal variety of any C^∞ variety is the set of common zeros of all the vector fields tangent to it:

(5.15) $$N^*\mathcal{K} = \{\varrho \in T^*\Omega;\ \sigma_1(V) = 0 \text{ at } \varrho,\ \forall V \in \mathcal{V}(\mathcal{K})\}.$$

Then $\mathscr{H}$ is said to be *characteristic* for P if it satisfies an intersection condition with the characteristic variety, $\sum=\{p=0\}$, of P:

(5.16) $$N^*\mathscr{H}\cap\sum=\bigcup_{j=1}^N \Lambda_j,\ \Lambda_j \text{ closed conic Lagrangian in } T^*\Omega.$$

It is said to be *characteristically complete* if in addition:

(5.17) $\mathscr{N}(\mathscr{H})=\{a\in C^\infty(\sum);\ a=0 \text{ on } N^*\mathscr{H}\}$ is generated by $\sigma_1(\mathscr{V}(\mathscr{H}))$, as a $C^\infty(\sum)$ module.

(5.18) **Proposition.** *If $\mathscr{H}$ is a C^∞ variety of finite type, i.e. such that $\mathscr{V}(\mathscr{H})$ is locally finitely generated, and is characteristically complete for P then $\mathscr{V}(\mathscr{H})$ is P-complete.*

Proof. See [MR].

Now the difference between some of the spaces considered here and those considered in [MR] is that they involve sums of spaces to which Proposition 5.18 applies. For these we use the following simple observation.

(5.19) **Proposition.** *If $J_k^{(i)}(\Omega)$ $i=1, \dots, N$, define P-propagative algebras on a fixed domain Ω satisfying* (5.1) *and the sum*

(5.20) $$J_k(\Omega)=\sum_{i=1}^N J_k^{(i)}(\Omega)$$

is such that:

(5.21) $$L^\infty J_k(\Omega)=L^\infty(\Omega)\cap J_k(\Omega)=\sum_{i=1}^N L^\infty(\Omega)\cap J_k^{(i)}(\Omega)$$

is a C^∞ algebra, then $J_k(\Omega)$ is a P-propagative algebra.

Proof. Obviously $J_k(\Omega)$ satisfies (5.5), and by assumption (5.4) also holds, so it is only necessary to check the P-propagative condition (5.3). Thus, suppose that $u\in J_{k-1}(\Omega)$ vanishes in Ω_δ, for some $0<\delta<\delta_0$ and has $Pu=f\in J_{k-1}(\Omega)$. From the definition, (5.20) of the $J_k(\Omega)$,

(5.22) $$f=\sum_{i=1}^N f_i,\ f_i\in J_{k-1}^{(i)}(\Omega).$$

Now in (5.22) the f_i do not necessarily vanish in Ω_δ. However we can always choose a function $e(t)$ which is C^∞, is identically 1 on supp (f), and vanishes in some $\Omega_{\delta'}$, with $0<\delta'<\delta$. Multiplying through (5.22) by this function, and using the assumption that all the spaces are C^∞ modules ensures that the f_i do vanish in the past. Then, using the P-convexity of Ω, solve each equation $Pu_i=f_i$, with $u_i=0$ in $\Omega_{\delta'}$. By assumption then $u_i\in J_k^{(i)}(\Omega)$, and by the uniqueness of the solution to the Cauchy problem,

$$u=\sum_{i=1}^N u_i\in J_k(\Omega).$$

6. Tangent characteristic surfaces

We now proceed to apply the analysis, by blowing up, of conormal spaces in §§2—4 and the propagation results of §6 to deduce a result on the propagation of conormality.

Let $\Omega \subset \mathbf{R}^3$ be an open set and suppose that

$$H_1, H_2, \ldots, H_N \subset \Omega \tag{6.1}$$

are closed embedded C^∞ surfaces. Let $L \subset \Omega$ be an embedded C^∞ line and suppose that:

$$H_i \cap H_j = L \quad \text{for} \quad i, j \in \{1, \ldots, N\},\ i \neq j. \tag{6.2}$$

We shall further assume that (4.2) holds locally near L for each pair:

$$H_i \text{ and } H_j \text{ are simply tangent along } L,\ i \neq j. \tag{6.3}$$

Then it is natural, following the ideas of §4, to associate to the C^∞ variety:

$$\mathscr{H} = \{L, H_i \setminus L,\ i = 1, \ldots, N\} \tag{6.4}$$

the space of functions:

$$J_k L^2(\Omega, \mathscr{H}) = \sum_{i=1}^N I_k L^2(\Omega, \mathscr{H}_i), \tag{6.5}$$

where $\mathscr{H}_i = \{L, H_i \setminus L\}$ are the individual C^∞ varieties. In particular

$$\mathscr{H} = \bigcup_{i=1}^N \mathscr{H}_i. \tag{6.6}$$

(6.7) **Theorem.** *If P is a strictly hyperbolic linear partial differential operator of second order with C^∞ coefficients in an open set $\Omega \subset \mathbf{R}^3$, which is P-convex with respect to Ω_δ in the sense that* (5.1) *holds, and $H_1, \ldots, H_N$ are closed embedded characteristic surfaces as in* (6.1)—(6.3) *and $f \in C^\infty(\Omega, \mathbf{R})$ then if $J_k(\Omega) = J_k L^2(\Omega, \mathscr{H})$ is defined by* (6.5)*:*

(6.8)

$$Pu = f(., u) + g \quad in \quad \Omega,\ u \in L^\infty(\Omega),\ g \in J_k(\Omega),\ u = 0 \quad in \quad \Omega_\delta \Rightarrow u \in J_{k+1}(\Omega),\ \forall k \in \mathbf{N}.$$

Proof. According to Proposition 5.7 it is enough to show that the $J_k(\Omega)$ form a P-propagative algebra in the sense of Definition 5.6. Applying Proposition 5.19 this in turn follows once it is demonstrated that $L^\infty J_k(\Omega)$ is a C^∞ ring with a decomposition (5.21), where $J_k^{(i)}(\Omega) = I_k L^2(\Omega, \mathscr{H}_i)$, and that each of these spaces is P-propagative. The proof therefore follows directly from the next two propositions, once it is noted that a line such as L along which two characteristic surfaces are tangent must be a bicharacteristic.

(6.9) **Proposition.** *Let $H_1, H_2, \ldots, H_N$ be a finite collection of closed embedded C^∞ surfaces in $\Omega\subset\mathbf{R}^3$ satisfying* (6.1)—(6.3). *Then, in terms of* (6.5), $L^\infty J_k(\Omega)=L^\infty(\Omega)\cap J_k(\Omega)$ *is a C^∞ ring and*

$$L^\infty J_k(\Omega)=\sum_{i=1}^N L^\infty(\Omega)\cap I_k L^2(\Omega,\mathscr{H}_i). \tag{6.10}$$

Proof. This is a generalization of Proposition 4.7, in particular (4.9), which gives the case $N=2$. Now (6.10) is certainly true if it is true locally near each point of L. As in the proof of Proposition 4.7 local coordinates can be introduced so that the analogue of (4.4):

$$H_i=\{y,0\},\quad H_j=\{y=x^2\} \tag{6.11}$$

holds for any prearranged pair, $i\neq j$, of indices. Thus if F is the double blow down, i.e. polar coordinate, map discussed in §§ 2, 3, then:

$$F^* I_k L^2(\Omega,\mathscr{H}_r)=I_k L^2_B(X_2,\mathscr{G}_r),\quad r=i,j \tag{6.12}$$

where $\mathscr{G}_r$ is the lifted variety in X_2.

Of course one cannot expect to arrange a simple form such as (4.4) to hold simultaneously for all the H_r. Observe, however, that to reduce any other pair to normal form, starting from (6.11), only a diffeomorphism fixing each point of L is required (see the proof of Lemma 4.3). Such a diffeomorphism lifts under F to a diffeomorphism of X_2. This can be seen directly, or alternatively it follows from the observation that any C^∞ vector field which vanishes at L lifts to a C^∞ vector field on X_2 tangent to the fibres of R_1 and R_2. Since a diffeomorphism fixing L can be obtained by integration of a time-dependent vector field of this type the lifting follows by functoriality. Thus, (6.12) actually holds with a fixed F for all $r=1,\ldots,N$ simultaneously. Since the lifted surfaces G_r are all disjoint the argument of Proposition 4.7, which only relies on this disjointness, gives (6.10) and shows that $L^\infty J_k(\Omega)$ is a C^∞ ring as desired.

(6.13) **Proposition.** *Let P be a strictly hyperbolic operator of second order in $\Omega\subset\mathbf{R}^3$, H a characteristic surface for P and $L\subset H$ a bicharacteristic line, then the space $I_k L^2(\Omega,\mathscr{K})$, where $\mathscr{K}=\{L, H\setminus L\}$, is a P-propagative algebra.*

Proof. This is a result of the type discussed in more detail in [MR]. In suitable local coordinates,

$$H=\{y=0\},\ L=\{x=y=0\}, \tag{6.13}$$

so $\mathscr{V}(\mathscr{K})$ has three local generators:

$$yD_y,\ xD_x,\ yD_x \text{ span } \mathscr{V}(\mathscr{K})\quad\text{locally.} \tag{6.14}$$

In particular it follows readily that $\mathscr{K}$ is complete, i.e. $\sigma_1(\mathscr{V}(\mathscr{K}))$ spans, as a $C^\infty(T^*\Omega)$-module, the space of functions on $T^*\Omega$ vanishing on $N^*\mathscr{K}$, see (5.15).

Now the characteristic variety, $\sum$, of a wave operator such as P, is a strictly convex cone in each fibre. As N^*L is a plane in each fibre, this and the assumption that L is a bicharacteristic for P means that the Hamilton vector field of the symbol of P is tangent to N^*L at $\sum$, so N^*L is tangent to $\sum$, hence is of dimension two. Now, the Hamilton vector field is also tangent to N^*H and is non-radial, from which it follows that:

$$N^*L \cap \sum = N^*H \cap N^*L. \tag{6.15}$$

Thus the C^∞ variety $\mathscr{K}$ is characteristic for P in the sense of (5.16). Since N^*L meets N^*H cleanly, in codimension one, it also follows easily from the completeness, remarked on above, that (5.17) holds, i.e. $\mathscr{K}$ is characteristically complete with respect to P. With (6.14) this fulfills the hypotheses of Proposition 5.18, so $\mathscr{V}(\mathscr{K})$ is a P-complete Lie algebra, as in (5.12). Finally then Proposition 5.13 can be applied to show that the space $I_kL^2(\Omega, \mathscr{V})$ is P-propagative, proving the proposition.

(6.16) *Remark*. Theorem 6.7 shows that propagation of conormality, in the sense of these J_k-spaces, can be maintained even in the presence of arbitrarily high multiplicity. As noted briefly in the Introduction, such high multiplicity will occur generically even in the simplest case, of conormal initial data at a finite number of point in the plane. Of course the importance of Theorem 6.7 is limited unless, and until, it can be shown that this type of conormality arises immediately after triple (and higher order) interactions. This will necessitate the introduction of related spaces at the triple point, an analysis which will be taken up elsewhere. Note that no propagation result of the type of Theorem 6.7 has been proved for the bigger space $I_kL^2(\Omega, \mathscr{H})$ and for the reasons given in §4, no such result is likely to be true.

7. Geometric degradation

In this section we give an example to show that the set L_∞ in (1.15) of iteratively defined triple interaction points need not be discrete. This means that in a proof of (1.14) some degree of smoothing must be shown for the anomalous singularities as compared to the incoming singularities. Of course under sufficiently strong hypotheses on the regularity of the solution this is known from the work of Bony [Bo1], see also [BR], extending that of Rauch and Reed [RR1] in one space dimension.

To justify, on geometric grounds, the definition of the varieties in (1.14), recall that we can expect the smallest possible variety of this type to be characteristic for P, in the sense of (5.15). Then, following a similar lemma in [MR]:

(7.1) **Lemma.** *Any C^∞ variety $\mathscr{H}$ containing the N points $\{z_1\}, \{z_2\}, \dots, \{z_N\}$ in $\{t=0\}$ and characteristic for P contains the characteristic cone $E_+(q)$ with pole at any point q lying on three or more cones in $\mathscr{H}$ not all tangent at q.*

Proof. If three or more cones pass through q and not all of them are tangent then either there are three meeting transversally at q or else two or more are tangent along a line $L\subset H$ but meeting a third transversally. In either case $\{q\}\in\mathscr{H}$. Thus $T_q^*\mathbf{R}^3\subset N^*\mathscr{H}$. The only Lagrangian submanifold of $T^*\mathbf{R}^3$ contained in the characteristic variety of P and containing $T_q^*\mathbf{R}^3\cap\sum$ is the closure of $N^*(E_+(q)\setminus\{q\})$. Thus it follows that $E_+(q)\subset H$ (see [MR]), proving the lemma.

(7.2) **Lemma.** *If there exists a C^∞ variety $\mathscr{H}'$ characteristic for P and containing the initial points $\{z_1\}, \{z_2\}, \dots, \{z_N\}$ then the minimal characteristic variety containing these points is:*

(7.3) $$\mathscr{H}=\bigcup\{E_+(q);\, q\in L_k\} \text{ for sufficiently large } k \text{ (in } t>0).$$

Proof. By definition a C^∞ variety is a locally finite collection of manifolds so the existence of $\mathscr{H}'$, together with Lemma 7.1, shows that the iterative definition of the L_k given in the Introduction will be locally finite. Obviously this gives the minimal C^∞ variety.

The condition that there be a characteristic C^∞ variety for P containing the initial points is therefore equivalent to the discreteness of the set L_∞ in (1.15). We now give an example to show that this is not always the case. For convenience let us use the notation $p=(t, z)\in\mathbf{R}\times\mathbf{R}^2$ for a point in $\mathbf{R}^3$.

(7.4) **Proposition.** *Let $p_0=0$, $p_i=(0, z_i)$, $i=1, \dots, 6$, be the centre and six vertices of a hexagon with unit sidelength in the plane, then the set L^∞ defined by* (1.15) *is not discrete and in particular* $(1+\sqrt{3}, 0)$ *is a point of accumulation.*

Proof. At time $t=r=1/\sqrt{3}$, the cones $E_+(p_1)$, $E_+(p_2)$ and $E_+(p_0)$ meet transversally at $q_1=(r, b_1)$. By the symmetry of the configuration six points $q_i=(r, b_i)$ are in L_1, forming a hexagon at time $t=r$, of sidelength r and centred at 0.

At time $t=2r$ the six cones $E_+(q_j)$ meet transversally at the origin in space, so $(2r, 0)\in L_2$. Notice too that the cone $E_+(p_i)$ meets the cone $E_+(q_{i+2})$ with its tangent cone $E_+(p_{i+2})$ non-tangentially at $(2r, b_{i+1})$, and hence $(2r, b_{i+1})\in L_2$, cyclically for $i=1, \dots, 6$. Thus inside L_2 is a new hexagon, with its centre, at $t=2r$ and with sidelength r. Proceeding inductively a sequence of such hexagons appears, at successive time steps $2r^k$, with sidelength r^k. Hence a point of accumulation must appear in $\mathscr{H}$ by the time:

$$T=\sum_{j=1}^{\infty} 2r^j = 1+\sqrt{3}.$$

This completes the proof of the proposition.

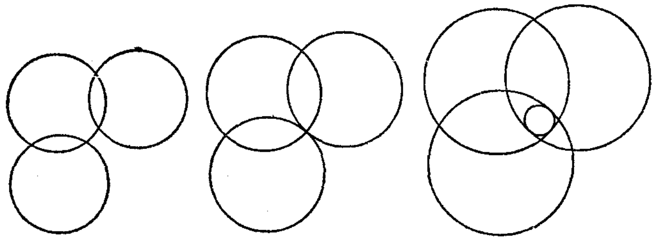

Figure 1

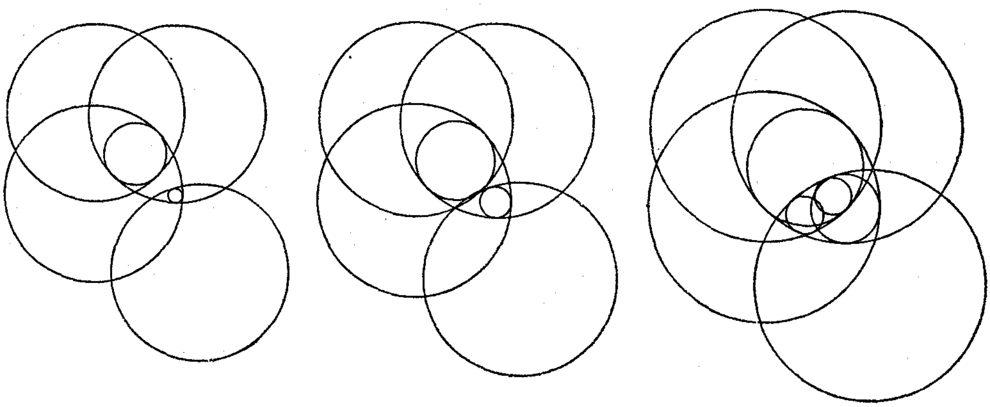

Figure 2

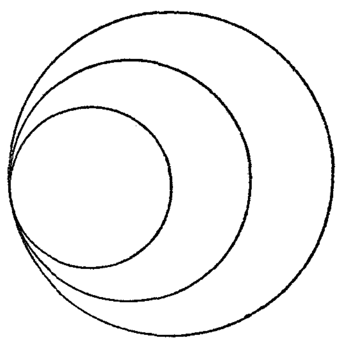

Figure 3

References

Be. Beals, M., Nonlinear wave equations with data singular at one point (preprint).

BR. Beals, M. and Reed, M., Propagation of singularities for hyperbolic pseudo differential operators with non-smooth coefficients, *Comm. Pure Appl. Math.* **35** (1982), 169—184.

Bo1. Bony, J.-M., Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles nonlinéares, *Ann. Sci. École Norm. Sup.* **14** (1981), 209—246.

Bo2. Bony, J.-M., Interaction des singularités pour les équations aux dérivées partielles non linéares, *Sem. Goulaouic—Schwartz,* 1979—1980, #22 and *Sem. Goulaouic—Meyer—Schwartz* 1981—1982, #2.

Bo3. Bony, J.-M., Second microlocalization and propagation of singularities for semi-linear hyperbolic equations. (preprint).

Me. Melrose, R. B., Transformation of boundary problems, *Acta Math.* **147** (1981), 149—236.

MR. Melrose, R. B. and Ritter, N., Interaction of nonlinear progressing waves for semilinear wave equations, *Ann. of Math.* **121** (1985), 187—213.

MU. Melrose, R. B. and Uhlmann, G., Lagrangian intersection and the Cauchy problem, *Comm. Pure Appl. Math.* (1979), 483—519.

RR1. Rauch, J. and Reed, M., Propagation of singularities for semilinear hyperbolic equations in one space variable, *Ann. of Math.* **111** (1980), 531—552.

RR2. Rauch, J. and Reed, M., Singularities produced by the nonlinear interaction of three progressing waves; examples, *Comm. Partial Differential Equations* **7** (1982), 1117—1133.

RR3. Rauch, J. and Reed, M., Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension, *Duke Math. J.* **49** (1982), 397—475.

Received November 30, 1985

Richard B. Melrose
MIT
Cambridge, MA 02139
U. S. A.

Niles Ritter
University of California
Los Angeles, California 90024
U. S. A.