Wiener's criterion and obstacle problems for vector valued functions

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1. Introduction

The behaviour at the boundary of solutions of the Dirichlet problem in a set $\Omega\subset\mathbf{R}^n$ is a classical problem in the theory for elliptic boundary value problems. In [13] and [14] Wiener considered the case of Laplace's equation. There he gave a geometrical condition, known as *Wiener's criterion for regular boundary points*, which guarantees that solutions attain the boundary values continuously. The condition was given in terms of a series of capacities, measuring the thickness of the complement of Ω, at the point considered. This was generalized to operators with discontinuous coefficients by Littman, Stampacchia, Weinberger [7], and to quasilinear operators by Maz'ja [9] and Gariepy, Ziemer [3]. See also Hildebrandt, Widman [4].

The pointwise continuity is also of interest in the *regularity theory for solutions of obstacle problems*, that is solutions of variational inequalities where the set of admissible variations is given by an obstacle function ψ. In [1] and [2] Frehse and Mosco studied solutions u in a suitable Sobolev space of the variational inequality: $u(x)\geqq\psi(x)$ for $x\in\Omega$ and $\int_\Omega \nabla u\nabla(v-u)\,dx\geqq 0$ for all v in the same Sobolev space with $v(x)\geqq\psi(x)$ for $x\in\Omega$. With an irregular obstacle function ψ they looked at regularity properties at interior points $x_0\in\Omega$, and one of their results is that solutions are continuous at such points provided a condition of Wiener type is true. Here the condition measures the thickness of certain level sets of ψ at x_0, the meaning of which is precisely described in [1].

The object of this paper is to study *regularity properties of solutions of a class of obstacle problems for vector valued* ($\mathbf{R}^N$-valued, $N\geqq 1$) *functions*, that is when we, instead of one inequality, have a system of inequalities. With a closed and convex set F in $\mathbf{R}^N$, and a closed set E, $E\subset\Omega$, our constraint is of the form $(u-\psi)(x)\in F$ for $x\in E$. Note that in the real case $N=1$, we can for instance choose $F=[0, c]$, $c>0$, and this gives the one-dimensional constraint $\psi(x)\leqq u(x)\leqq\psi(x)+c$ for $x\in E$.

It follows from the regularity theory for the system of differential equations pertaining to our inequality that solutions of our problem are locally Hölder continuous in $\Omega\setminus E$, that is in that part of Ω where we have no constraint, see for instance Hildebrandt and Widman [4]. Our primary concern in this report is the *pointwise continuity* at points which belong to the set E. If $x_0\in E$ and if a Wiener criterion, now measuring the thickness of E at this point, is fulfilled we show that solutions are continuous at x_0. Moreover, in terms of the capacity used in the criterion we give an estimate of the modulus of continuity. In particular if the set E is "sufficiently thick" at x_0 this estimate will give Hölder continuity at this point. The study of this type of regularity was one of the topics in my doctoral thesis [6] presented in April 1983. There the concern was local rather than pointwise regularity and a result on local Hölder continuity was proven. As a last result in this paper we give *an estimate of the modulus of continuity valid locally in* Ω, which in a special case gives local Hölder continuity.

Finally we mention [5], where Hildebrandt and Widman have made an extensive study, concerning regularity and existence of solutions, of the problem where the constraint is of the form $(u-\psi)(x)\in F$ not only for x in E, but for all x in Ω. By introducing the set E we treat a wider class of problems. For instance, the case when E is an $(n-1)$-dimensional manifold, the so called thin obstacle problem is included.

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2. Notations

Let Ω be a bounded and open set in the n-dimensional space $\mathbf{R}^n$, $n\geqq 3$. Put $B_r(x_0)=\{x\in\mathbf{R}^n\colon |x-x_0|<r\}$, $T_r(x_0)=B_r(x_0)\setminus B_{r/2}(x_0)$ and $B_M=\{\xi\in\mathbf{R}^n\colon |\xi|\leqq M\}$. Moreover, let $⨍_S v\,d\mu$ stand for the mean value of v over S with respect to the positive measure μ, that is

$$⨍_S v\,d\mu = \frac{1}{\mu(s)}\int_S v\,d\mu.$$

Denote by $W^{1,p}(\Omega)$, $p\geqq 1$, the Sobolev space of functions η such that

$$\|\eta\|_{W^{1,p}(\Omega)} = \left\{\int_\Omega (|\eta|^p+|\nabla\eta|^p)\,dx\right\}^{1/p} < \infty,$$

and by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the $W^{1,p}(\Omega)$-norm. In the notation for a function space we add the symbol $\mathbf{R}^N$ to denote the corresponding space of $\mathbf{R}^N$-valued functions. For instance, $W^{1,2}(\Omega, \mathbf{R}^N)$ stands for the space of $\mathbf{R}^N$-valued functions with components in $W^{1,2}(\Omega)$. We use the notations $D_\alpha = \frac{\partial}{\partial x_\alpha}$ and $\nabla u = (..., D_\alpha u^i, ...)$, where $1 \leqq \alpha \leqq n$ and $1 \leqq i \leqq N$. Moreover, we use a summation convention such that

$$\int A^{\alpha\beta} D_\alpha u D_\beta (v-u)\, dx \geqq \int f(v-u)\, dx$$

means that

$$\sum_{i=1}^N \sum_{\alpha,\beta=1}^n \int A^{\alpha\beta} D_\alpha u^i D_\beta (v^i - u^i)\, dx \geqq \sum_{i=1}^N \int f^i (v^i - u^i)\, dx,$$

where u^i, v^i and f^i are the components of u, v and f, respectively.

To formulate the conditions on E we need a *notion of capacity*. For any set S in $\mathbf{R}^N$ define

$$C_{1,2}(S) = \inf\left\{\int_{\mathbf{R}^n} \eta^2\, dx \colon \eta \geqq 0 \quad \text{and} \quad G_1 * \eta \geqq 1 \quad \text{on} \quad S\right\},$$

where G_1 is the Bessel kernel defined as the inverse Fourier transform of $\hat{G}_1(\xi) = (1+|\xi|^2)^{-1/2}$. We will also use the notation $\Gamma(r) = r^{2-n} C_{1,2}(T_r(x_0) \cap E)$. Recall that every $v \in W^{1,2}$ has a unique representative $v(x)$ defined capacitary almost everywhere, that is defined pointwise except for a set of capacity zero.

Consequently, when we write $v(x) \in F$ for $x \in E$, where $v \in W^{1,2}(\Omega, \mathbf{R}^N)$, we mean that this relation holds for capacitary almost every $x \in E$. Furthermore, in the notation

$$\omega_r(x_0, v) = \sup_{z, z' \in B_r(x_0)} |v(z) - v(z')|$$

the supremum is taken in the capacitary almost everywhere sense. Finally, different constants appearing in the text will mostly be denoted by the same letter C.

3. Results

We look at solutions u to systems of variational inequalities of the form

(1) $$u \in \mathbf{K} \quad \text{and} \quad \int_\Omega A^{\alpha\beta}(x) D_\alpha u D_\beta (v-u)\, dx \geqq \int_\Omega f(x, u, \nabla u)(v-u)\, dx$$

for all $v \in \mathbf{K}$.

The set $\mathbf{K}$ of admissible variations is a convex set of the form

$$\mathbf{K} = \{v \in W^{1,2}(\Omega, \mathbf{R}^N) \colon (v-\psi)(x) \in F \quad \text{for} \quad x \in E, \quad (v-\psi)(x) \in B_M \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u - \varphi \in W_0^{1,2}(\Omega, \mathbf{R}^N)\},$$

where φ is a prescribed $\mathbf{R}^N$-valued function, E is a closed set, $E \subset \Omega$, and F is a closed and convex set in $\mathbf{R}^N$ such that $0 \in F$. The obstacle function ψ is supposed to be of class $W^{1,2q}(\Omega, \mathbf{R}^N)$, $q>n/2$. Moreover, we suppose that the coefficients $A^{\alpha\beta}$ are in $L^\infty(\Omega)$ and satisfy the following ellipticity condition. There is a positive constant λ such that

$$\lambda|\xi|^2 \leqq A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \quad \text{for all} \quad \xi\in\mathbf{R}^N \quad \text{and} \quad x\in\Omega.$$

The right hand side f is of the form

$$f(x, u, \nabla u) = -D_\alpha g_\alpha(x)+f_0(x, u, \nabla u),$$

where the functions g_α belong to $L^{2q}(\Omega, \mathbf{R}^N)$, $q>n/2$. For the function $f_0=f_0(x, u, \nabla u)$ we assume measurability in Ω if $u\in\mathbf{K}$, and the existence of a number $a\geqq 0$ and a function $b\in L^q(\Omega)$, $q>n/2$, such that

$$|f_0(x, u, p)| \leqq a|p|^2+b \quad \text{for} \quad x\in\Omega,\ p\in\mathbf{R}^{nN} \quad \text{and} \quad u\in\mathbf{K}.$$

Observe that if u is a solution of (1) it is readily seen that $w=u-\psi$ is a solution of a problem of the same kind. The new obstacle function here is identically zero so for the rest of the paper we assume that $\psi\equiv 0$, which means that the constraint is of the form $u(x)\in F$ for $x\in E$. Now assume that *u is a solution of* (1) *and that* $M<\lambda/2a$. The results are formulated in three theorems. The two first deal with the pointwise continuity at points x_0 which belong to E, and the third deals with the local regularity in Ω. Recall that $\Gamma(r)=r^{2-n}C_{1,2}(T_r(x_0)\cap E)$.

Theorem 1. *a) If* $0<r\leqq R\leqq 1/2 \operatorname{dist}(x_0, \partial\Omega)$ *then*

(2) $$\omega^2_{r/4}(x_0, u) \leqq C\Big\{\sum_{i=0}^k \Gamma(R_i)\Big\}^{-1}\int_{B_R(x_0)} |\nabla u|^2|x-x_0|^{2-n}\,dx+CR^\gamma,$$

where $R_i=2^{-i}R$ *and k is such that* $2^{-k-1}R<r\leqq 2^{-k}R$.

b) If $0<R\leqq P/2\leqq 1/2 \operatorname{dist}(x_0, \partial\Omega)$ *then*

(3) $$\int_{B_R(x_0)} |\nabla u|^2|x-x_0|^{2-n}\,dx \leqq e^{-C\sum_{i=0}^l \Gamma(P_i)}\Big\{\int_{B_P(x_0)} |\nabla u|^2|x-x_0|^{2-n}\,dx+CP^\gamma\Big\},$$

where $P_i=2^{-i}P$ *and l is such that* $2^{-l-2}P<R\leqq 2^{-l-1}P$. *The constants γ depend on n and q and the constants C depend on parameters of the problem.*

Theorem 2. *a) If for some* ϱ, $0<\varrho\leqq 1/2 \operatorname{dist}(x_0, \partial\Omega)$,

(4) $$\sum_{i=0}^\infty \Gamma(\varrho_i) = \infty, \quad \varrho_i = 2^{-i}\varrho,$$

u is continuous at x_0.

b) If there is a function B, $B(m)\uparrow\infty$ *when* $m\to\infty$, *such that for every* ϱ, $0<\varrho\leqq\operatorname{dist}(x_0, \partial\Omega)$, *and for every integer* $m>0$,

(5) $$\sum_{i=0}^m \Gamma(\varrho_i) \geqq B(m)$$

then

$$\omega_r(x_0, u) \leqq Ce^{-cB(c\log P/r)} + Cr^{\gamma}$$

for all r, $0 < r \leqq P \leqq \operatorname{dist}(x_0, \partial\Omega)$.

Remark 1. Let $B(m) = B_1 m - B_2$ where B_1 and B_2 are positive constants. Then the estimate in Theorem 2b gives $\omega_r(x_0, u) \leqq C(r^{c^2 B_1} + r^{\gamma})$, and thus the solution u is Hölder continuous at x_0.

Remark 2. Using the subadditivity for the capacity it is not hard to see that, instead of $\Gamma(\varrho_i) = \varrho_i^{2-n} C_{1,2}(T_{\varrho_i}(x_0) \cap E)$, we can have $\varrho_i^{2-n} C_{1,2}(B_{\varrho_i}(x_0) \cap E)$ in (4) and (5) above. Moreover, if we rewrite the condition (4) and (5) in terms of integrals they look like

(4′) $$\int_0^{\varrho} C_{1,2}(B_r(x_0) \cap E) r^{1-n}\, dr = \infty$$

and

(5′) $$\int_{\varrho'}^{\varrho} C_{1,2}(B_r(x_0) \cap E) r^{1-n}\, dr \geqq B'(\log \varrho/\varrho'),$$

where $0 < \varrho' < \varrho$ and B' is a new function of the same type as B.

Theorem 3. *Let y be an arbitrary point in Ω. If the condition in Theorem 2b holds for all $x_0 \in E$ then there is a constant c_0, depending only on parameters of the problem, such that for all r, $0 < 2r \leqq P = \operatorname{dist}(y, \partial\Omega)$,*

$$\int_{B_r(y)} |\nabla u|^2\, dx \leqq Cr^{n-2} e^{-c_0 B(m)} + Cr^{n-2+\gamma},$$

where $P/4r < 2^m \leqq P/2r$. The constants C here depend also on the $W^{1,2}$-norm of u and on $\operatorname{dist}(y, \partial\Omega)$.

The following corollary is a consequence of Theorem 3 and a modified version of the well-known Morrey's lemma, cf. Morrey [11], Theorem 3.5.2.

Corollary 1. *Let $\Omega' \subset\subset \Omega$. Then for all $y \in \Omega'$ and for all r,* $0 < 8r \leqq \operatorname{dist}(\Omega', \partial\Omega)$,

$$\omega_r(y, u) \leqq C\int_{\alpha(r)}^{\infty} e^{-c_0 B(t)/2}\, dt + Cr^{\gamma/2},$$

where $\alpha(r) = \dfrac{1}{\log 2} \log \dfrac{P}{8r}$.

Remark 3. If B is as in Remark 1 then Corollary 1 gives

$$\omega_r(y, u) \leqq Cr^{\gamma'}, \quad \text{where} \quad \gamma' = \min\left(\frac{c_0 B_1}{2\log 2}, \frac{\gamma}{2}\right),$$

and thus the solution u is locally Hölder continuous.

4. Auxiliary lemmata

Lemma 1. *Let u be a solution of* (1). *If $x_0 \in E$ and $0 < r \leqq \operatorname{dist}(x_0, \partial\Omega)$ then for capacitary almost every $z \in B_{r/4}(x_0)$,*

$$|u(z)-\bar{u}|^2+(\lambda-2aM)\int_{B_{r/2}(x_0)} |\nabla u|^2 |x-z|^{2-n}\,dx$$

$$\leqq Cr^{2-n}\int_{T_r(x_0)} |\nabla u|^2\,dx+Cr^{-n}\int_{T_r(x_0)} |u-\bar{u}|^2\,dx+Cr^{\gamma},$$

where $\bar{u}$ is a constant vector in $F \cap B_M$.

Remark 4. The proof of Lemma 1 also gives that $\int_{B_r(x_0)} |\nabla u|^2 |x-x_0|^{2-n}\,dx$ is bounded for all r, $0 < r \leqq \operatorname{dist}(x_0, \partial\Omega)$.

Proof of Lemma 1. Let $\eta \in C_0^\infty(B_r(x_0))$ satisfy $\eta(x)=1$ for $|x-x_0| \leqq 5r/8$, $\eta(x)=0$ for $|x-x_0| \geqq 7r/8$, $|\nabla\eta| \leqq C/r$ and $0 \leqq \eta \leqq 1$. Moreover, with $0<\varrho<r/4$ let $G^\varrho(x, z)$, $z \in B_{r/4}(x_0)$, be the mollification of the Green function G for the elliptic operator $L=-D_\beta(A^{\alpha\beta}D_\alpha)$, that is $G^\varrho(x, z)=⨍_{B_\varrho(z)} G(x, y)\,dy$. Here $A^{\alpha\beta}$ are extended to L^∞-functions defined in an open ball B, $\bar{\Omega} \subset B$, such that the ellipticity condition still holds. As a test function introduce

$$v = u-\varepsilon\eta^2 G^\varrho(\cdot, z)(u-\bar{u}), \quad \text{where} \quad \varepsilon > 0.$$

It is not hard to see that v is an admissible test vector if ε is sufficiently small. If we insert this function in the variational inequality (1) and exploit the technique used by Hildebrandt and Widman in [4], pp. 79 and 80, we obtain the estimate in Lemma 1.

We also need a modified version of a Poincaré inequality of Maz'ja [8]. For a proof we refer to Meyers [10]. As a matter of fact, Corollary 1, p. 117, in [10] together with a homothetic transformation yields:

Lemma 2. *Let E be a closed set in $\mathbf{R}^n$ and $T_r(x_0)$ be such that $T_r(x_0) \cap E \neq \emptyset$. Then there is a positive measure ν with support in $T_r(x_0) \cap E$ such that if $\bar{v}=\int v\,d\nu$ then*

$$\int_{T_r(x_0)} |v-\bar{v}|^2\,dx \leqq Cr^n \{C_{1,2}(T_r(x_0) \cap E)\}^{-1} \int_{T_r(x_0)} |\nabla v|^2\,dx$$

for all $v \in W^{1,2}(T_r(x_0), \mathbf{R}^N)$.

Moreover, one is free to choose the support of ν up to sets of sufficiently small capacity.

Remark. If $\bar{v}=⨍_{T_r(x_0)} v\,dx$ we have the usual Poincaré inequality

$$\int_{T_r(x_0)} |v-\bar{v}|^2\,dx \leqq Cr^2 \int_{T_r(x_0)} |\nabla v|^2\,dx.$$

5. Proofs of the results

Proof of Theorem 1. a) Put $\bar{u}=\int u\,dv$, where v is chosen according to Lemma 2 such that $\bar{u}\in F\cap B_M$. Since z is arbitrary in $B_{r/4}(x_0)$, the estimates in Lemma 1 and Lemma 2 give

$$\omega_{r/4}^2(x_0, u) \leqq C\Gamma(r)^{-1}\int_{T_r(x_0)} |\nabla u|^2 |x-x_0|^{2-n}\,dx + Cr^\gamma.$$

For any R, $0<R\leqq \operatorname{dist}(x_0, \partial\Omega)$ and with $R_i=2^{-i}R$ this yields

$$\Gamma(R_i)\,\omega_{r/4}^2(x_0, u) \leqq C\int_{T_{R_i}(x_0)} |\nabla u|^2 |x-x_0|^{2-n}\,dx + C\Gamma(R_i)R^\gamma$$

for all i, $0\leqq i\leqq k$, where k and r are such that $2^{-k-1}\ R<r\leqq 2^{-k}R$. Observe that this last inequality is trivial for those i where $\Gamma(R_i)=0$. Summing over i we get

$$\omega_{r/4}^2(x_0, u) \leqq C\Big\{\sum_{i=0}^k \Gamma(R_i)\Big\}^{-1}\int_{B_R(x_0)} |\nabla u|^2 |x-x_0|^{2-n}\,dx + CR^\gamma \tag{6}$$

which is the statement in Theorem 1a.

b) With $\bar{u}$ as above, Lemma 1 and Lemma 2 also give

$$\int_{B_{r/2}(x_0)} |\nabla u|^2 |x-x_0|^{2-n}\,dx \leqq C\Gamma(r)^{-1}\int_{T_r(x_0)} |\nabla u|^2 |x-x_0|^{2-n}\,dx + Cr^\gamma.$$

Apply the hole-filling device of Widman [12], that is add $C\Gamma(r)^{-1}\int_{B_{r/2}(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx$ to both sides and divide by $1+C\Gamma(r)^{-1}$ to find

$$I(r/2) \leqq \frac{C}{C+\Gamma(r)}\,I(r) + Cr^\gamma,$$

where $I(r)=\int_{B_r(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx$. Again observe that we have a trivial inequality if $\Gamma(r)=0$.

Since $\Gamma(r)$ is bounded from above we infer

$$I(r/2) \leqq \big(1-c_1\Gamma(r)\big)I(r) + C_2r^\gamma.$$

To eliminate the term C_2r^γ let $C_3=C_2/\big(1-2^{-\gamma}-c_1\Gamma(r)\big)$ and put $J(r)=I(r)+C_3r^\gamma$. Note that it is possible to choose c_1 such that $C_3>0$. In terms of $J(r)$ we have

$$J(r/2) \leqq \big(1-c_1\Gamma(r)\big)J(r).$$

Now fix P, $0<P\leqq \operatorname{dist}(x_0, \partial\Omega)$, put $P_i=2^{-i}P$ and iterate $J(P/2)\leqq\big(1-c_1\Gamma(P)\big)J(P)$ to obtain

$$J(P_{l+1}) \leqq \prod_{i=0}^l \big(1-c_1\Gamma(P_i)\big)J(P) \leqq e^{-c_1\sum_{i=0}^l \Gamma(P_i)}J(P).$$

For any R, $0<R\leqq P/2$, this gives

$$I(R) \leqq e^{-c_1\sum_{i=0}^l \Gamma(P_i)}\big(I(P)+CP^\gamma\big),$$

where l is such that $P2^{-l-2}<R\leqq P2^{-l-1}$, and this completes the proof of Theorem 1b.

Proof of Theorem 2. a) The assumption (4) yields that for any R, $0<R\leqq 1/2\,\mathrm{dist}(x_0, \partial\Omega)$ there is an $r>0$ such that $\{\sum_{i=0}^{k}\Gamma(R_i)\}^{-1}$ becomes arbitrary small. Recall that $R_i=2^{-i}R$ and that k satisfies $2^{-k-1}R<r\leqq 2^{-k}R$. Since $\int_{B_R(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx$ is bounded the continuity follows from the estimate (2) in Theorem 1a.

b) From the assumption (5) it follows that for every R and P, $0<8R\leqq P\leqq \mathrm{dist}(x_0, \partial\Omega)$, $\sum_{i=1}^{l}\Gamma(P_i)\geqq B(l)$, where l is such that $P/4R<2^l\leqq P/2R$. Moreover, if $2^{-k-1}R<r\leqq 2^{-k}R$ we can choose L such that if $R=Lr$ then

$$\sum_{i=0}^{k}\Gamma(R_i)\geqq B(k)\geqq C_4>0.$$

Insert this in the estimates in Theorem 1 to find

$$\omega_{r/4}^2(x_0, u)\leqq CC_4^{-1}\int_{B_R(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx+CL^\gamma r^\gamma$$

and

$$\int_{B_R(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx\leqq Ce^{-cB(c\log P/r)}\Big\{\int_{B_P(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx+CP^\gamma\Big\}$$

for all sufficiently small r, whereupon

$$\omega_{r/4}(x_0, u)\leqq Ce^{-cB(c\log P/r)}+Cr^\gamma$$

and this completes the proof of Theorem 2.

Proof of Theorem 3. Let $x_0\in E$. With $R=r$ and $P=\mathrm{dist}(x_0, \partial\Omega)$ Theorem 1b gives

$$\int_{B_r(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx\leqq e^{-cB(m)}\Big\{\int_{B_P(x_0)}|\nabla u|^2|x-x_0|^{2-n}\,dx+CP^\gamma\Big\},$$

whence

(7) $$\int_{B_r(x_0)}|\nabla u|^2\,dx\leqq Cr^{n-2}e^{-cB(m)}\quad\text{for all}\quad r,\ 0<2r\leqq P.$$

Here $P/4r<2^m\leqq P/2r$. Next we consider the case when $y\in\Omega\setminus E$ and r is such that $B_r(y)\subset\Omega\setminus E$. As in the proof of Lemma 1 we get

$$\int_{B_{r/2}(y)}|\nabla u|^2|x-y|^{2-n}\,dx\leqq Cr^{2-n}\int_{T_r(y)}|\nabla u|^2\,dx+Cr^{-n}\int_{T_r(y)}|u-\bar{u}|^2\,dx+Cr^\gamma,$$

where now $\bar{u}=\fint_{T_r(y)}u\,dx$. If we use Poincaré's inequality to estimate the term $\int_{T_r(y)}|u-\bar{u}|^2\,dx$ we find

$$\int_{B_r(y)}|\nabla u|^2|x-y|^{2-n}\,dx\leqq Cr^{2-n}\int_{T_r(y)}|\nabla u|^2\,dx+Cr^\gamma$$
$$\leqq C\int_{T_r(y)}|\nabla u|^2|x-y|^{2-n}\,dx+Cr^\gamma.$$

As in the proof of Theorem 1b, fill the hole and iterate to arrive at

$$\int_{B_{r/2}(y)} |\nabla u|^2 |x-y|^{2-n}\, dx \leqq C(r/R)^{\gamma_3} \int_{B_r(y)} |\nabla u|^2 |x-y|^{2-n}\, dx + Cr^{\gamma_3}$$

for all r, $0<r\leqq R\leqq \min(\operatorname{dist}(y, \partial\Omega), \operatorname{dist}(y, E))$. It is possible to have the exponent γ_3 here, since we are allowed to take a smaller γ_3 in (7) if necessary. Taking these two last inequalities together we obtain

$$\int_{B_r(y)} |\nabla u|^2\, dx \leqq C(r/R)^{n-2+\gamma_3} \int_{B_R(y)} |\nabla u|^2\, dx + Cr^{n-2+\gamma_3}. \tag{8}$$

Now the estimate (8), dealing with balls $B_r(y)\subset \Omega\setminus E$, is combined with the estimate (7), dealing with balls $B_r(x_0)\subset\Omega$ where $x_0\in E$, whereupon

$$\int_{B_r(y)} |\nabla u|^2\, dx \leqq Cr^{n-2} e^{-c_0 B(m)} + Cr^{n-2+\gamma_3} \tag{9}$$

for all $y\in\Omega$ and all r, $0<2r\leqq \operatorname{dist}(y, \partial\Omega)$. Here m is as in (7). As a matter of fact, the only crucial point is when $y\in\Omega\setminus E$ is near the set E in the sense that $\operatorname{dist}(y, E)\leqq 1/4 \operatorname{dist}(y, \partial\Omega)$. We are left with the following cases: Either $0<r\leqq \operatorname{dist}(y, E)$ or $\operatorname{dist}(y, E)\leqq r\leqq 1/4 \operatorname{dist}(y, \partial\Omega)$. Let x_0 be one of the points in E which is nearest to y. Now, if $0<r\leqq \operatorname{dist}(y, E)=r_0$ then $3r_0\leqq \operatorname{dist}(x_0, \partial\Omega)$ and (8) together with (7) implies that

$$\begin{aligned}\int_{B_r(y)} |\nabla u|^2\, dx &\leqq C(r/r_0)^{n-2+\gamma_3} \int_{B_{3r_0}(x_0)} |\nabla u|^2\, dx + Cr^{n-2+\gamma_3} \\ &\leqq Cr^{n-2+\gamma_3} r_0^{-\gamma_3} e^{-cB(l)} + Cr^{n-2+\gamma_3},\end{aligned}$$

where $P/4\cdot 3r_0<2^l\leqq P/2\cdot 3r_0$. From this we get

$$\int_{B_r(y)} |\nabla u|^2\, dx \leqq Cr^{n-2+\gamma_3} e^{-cB(l)+l\gamma_3 \log 2} = Cr^{n-2+\gamma_3} e^{-c(B(l)-Cl\gamma_3 \log 2)} \tag{10}$$

where $P/4r_0<2^{l+2}\leqq P/2r_0$. According to the definition of Γ there is a constant K such that

$$B(m-1) \geqq B(m)-K \quad \text{for all} \quad m\geqq 1. \tag{11}$$

Due to the possibility of changing the constants involved we can assume that $\gamma_3 C\log 2=K$, and (11) yields

$$B(l)-l\gamma_3 C\log 2 \geqq B(l+1)-(l+1)\gamma_3 C\log 2.$$

Insert an iteration of this in (10) to obtain

$$\int_{B_r(y)} |\nabla u|^2\, dx \leqq Cr^{n-2} e^{-cB(m)} + Cr^{n-2+\gamma_3},$$

where $P/4r<2^m\leqq P/2r$. Moreover, if $\operatorname{dist}(y, E)\leqq r\leqq 1/4 \operatorname{dist}(y, \partial\Omega)$ then $3r\leqq \operatorname{dist}(x_0, \partial\Omega)$ and (7) gives

$$\int_{B_r(y)} |\nabla u|^2\, dx \leqq \int_{B_{3r}(x_0)} |\nabla u|^2\, dx \leqq Cr^{n-2} e^{-cB(l)},$$

where $P/4\cdot 3r<2^l\leqq P/2\cdot 3r$. Again using (11) we find

$$\int_{B_r(y)} |\nabla u|^2\,dx \leqq Cr^{n-2}e^{-cB(m)},$$

where $P/4r<2^m\leqq P/2r$. Thus (9) is established and this completes the proof of Theorem 3.

Proof of Corollary 1. We sketch the proof which is a copy of Morrey's proof in [11]. Without loss of generality we can assume that $u\in C^1$. Fix $y\in\Omega'$ and let $z, z'\in B_r(y)$, $8r\leqq \operatorname{dist}(\Omega', \partial\Omega)$. Put $d=|z-z'|$ and $B=\{x\in\mathbf{R}^n\colon |x-1/2(z+z')|<d\}$. First we estimate the integral $\int_B (u(\xi)-u(x))\,dx$, where ξ is either z or z'. Simple arguments give

$$\left|\int_B (u(\xi)-u(x))\,dx\right| \leqq \frac{3}{2}d\int_B\left\{\int_0^1 |\nabla u(\xi+t(x-\xi))|\,dt\right\}dx.$$

If we interchange the order of integration, put $\eta=\xi+t(x-\xi)$, use Hölder's inequality and the estimate in Theorem 3 we get

$$\left|\int_B (u(\xi)-u(x))\,dx\right| \leqq cd^{n+\gamma/2}+Cd^n\int_0^1 e^{-\frac{c_0}{2}B(m)}\frac{dt}{t}$$

where $P/4td<2^m\leqq P/2td$.

Now,

$$\int_0^1 e^{-\frac{c_0}{2}B(m)}\frac{dt}{t} \leqq \int_0^1 e^{-\frac{c_0}{2}B\left(\frac{1}{\log 2}\log\frac{P}{4td}\right)}\frac{dt}{t}$$

and by a change of variables we see that this last integral equals

$$C\int_{\frac{1}{\log 2}\log\frac{P}{4d}}^{\infty} e^{-\frac{c_0}{2}B(t)}\,dt.$$

Summarizing and using the fact that

$$|u(z)-u(z')| = cd^{-n}\left|\int_B (u(z)-u(z'))\,dx\right|$$

we obtain, via the triangle inequality, that

$$\omega_r(y, u) \leqq Cr^{\gamma/2}+C\int_{\alpha(r)}^{\infty} e^{-\frac{c_0}{2}B(t)}\,dt,$$

where $\alpha(r)=\dfrac{1}{\log 2}\log\dfrac{P}{8r}$. The proof is complete.

References

1. Frehse, J. and Mosco, U., Sur la continuité ponctuelle des solutions locales faibles du problème d'obstacle, *C.R. Acad. Sci. Paris* **295** (10) (1982), 571—574.
2. Frehse, J. and Mosco, U., Wiener obstacles, Collège de France, Seminar V, Pitman 1984.
3. Gariepy, R. and Ziemer, W. P., A regularity condition at the boundary for solutions of quasilinear elliptic equations, *Arch. Rational Mech. Anal.* **67** (1977), 25—39.
4. Hildebrandt, S. and Widman, K.-O., Some regularity results for quasilinear elliptic systems of second order, *Math. Z.* **142** (1975), 67—86.
5. Hildebrandt, S. and Widman, K.-O., Variational inequalities for vectorvalued functions, *J. Reine Angew. Math.* **309** (1979), 191—220.
6. Karlsson, T., Regularity results for solutions of some obstacle problems, Dissertation No. 92-1983, Dept. Math., Linköping University.
7. Littman, W., Stampacchia, G. and Weinberger, H., Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa (3)* **17** (1963), 43—77.
8. Maz'ja, V. G., The Dirichlet problem for elliptic equations of arbitrary order in unbounded regions, *Dokl. Akad. Nauk. SSSR* **150** (1963), 1221—1224.
9. Maz'ja, V. G., On the continuity at a boundary point of solutions of quasilinear elliptic equations, *Vestnik Leningrad. Univ.* **25** (1970), 42—55.
10. Meyers, N. G., Integral inequalities of Poincaré and Wirtinger type, *Arch. Rational Mech. Anal.* **68** (1978), 113—120.
11. Morrey, C. B., *Multiple Integrals in the Calculus of Variations,* Berlin—Heidelberg—New York (1966).
12. Widman, K.-O., Hölder continuity of solutions of elliptic systems, *Manuscripta Math.* **5** (1971), 299—308.
13. Wiener, N., Certain notions in potential theory, *J. Math. and Phys.* **3** (1924), 24—51.
14. Wiener, N., The Dirichlet problem, *J. Math. and Phys.* **3** (1924), 127—146.

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