Area integral estimates for elliptic differential operators with non-smooth coefficients

Björn E. J. Dahlberg, David S. Jerison* and Carlos E. Kenig*

In this note we shall prove inequalities comparing the area integral and nontangential maximal functions for solutions to second order elliptic equations in a domain in $\mathbf{R}^n$, in which both the coefficients of the equation and the domain satisfy very weak regularity conditions to be formulated later (cf. [7]). Such inequalities have been proved by many authors in increasing generality. (See [1, 6, 9, 11], where further references can be found.) The most general setting up to now is that of harmonic functions in Lipschitz domains [6]. There the key additional point is the fact that harmonic measure for the standard Laplace operator satisfies A_∞ (a scale-invariant form of mutual absolute continuity) with respect to surface measure on the boundary of the domain. By contrast, surface measure need not exist in our more general context. Moreover, even if the domain is smooth (and hence has a surface measure), L-harmonic measure and surface measure are mutually singular for some choices of the elliptic operator L [2]. This is the main new difficulty.

We shall first state and prove the theorem in a special case that retains the main difficulty. Recall that a bounded domain $D\subset\mathbf{R}^n$ is called a Lipschitz domain if ∂D can be covered by finitely many open right circular cylinders whose bases have positive distance from ∂D and corresponding to each cylinder I there is a coordinate system (x, y) with $x\in\mathbf{R}^{n-1}$, $y\in\mathbf{R}$ with y axis parallel to the axis of I, and a function $\varphi\colon \mathbf{R}^{n-1}\to\mathbf{R}$ satisfying a Lipschitz condition $\big(|\varphi(x)-\varphi(z)|\leqq M|x-z|\big)$ such that $I\cap D=\{(x,y)\colon y>\varphi(x)\}\cap I$ and $I\cap\partial D=\{(x,y)\colon y=\varphi(x)\}\cap I$. Denote by $L=\sum_{i,j=1}^n \frac{\partial}{\partial X_i} a_{ij}(X)\frac{\partial}{\partial X_j}$ a uniformly elliptic operator with bounded, measurable coefficients, that is, $a_{ij}\in L^\infty(\mathbf{R}^n)$ and for some $c>0$, $\sum_{i,j=1}^n a_{ij}(X)\xi_i\xi_j\geqq c|\xi|^2$ for

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all $\xi\in\mathbf{R}^n$. We say that u is L-harmonic in D or $Lu=0$ in D if $\frac{\partial u}{\partial X_i}$ $u\in L^2_{\mathrm{loc}}(D)$ and

$$\int \sum_{i,j=1}^n \left(\frac{\partial u}{\partial X_j}(X)\right) a_{ij}(X) \frac{\partial \varphi}{\partial X_i} \varphi(X)\, dX = 0$$

for all $\varphi\in C_0^\infty(D)$. The theorem of de Giorgi and Nash [14] says that u has a Hölder continuous representative (which we also denote by u). Let $D\subset\mathbf{R}^n$ be a (connected) Lipschitz domain. Fix for all time a point $X_0\in D$. For each continuous function f on ∂D there is a unique L-harmonic function u in D continuous in $\bar{D}$ such that $u=f$ on ∂D. The L-harmonic measure at X_0 is the representing measure of the linear functional $f\mapsto u(X_0)\equiv \int_{\partial D} f(P)d\omega(P)$.

For $P\in\partial D$, $\alpha>0$, a nontangential approach region to P in D is given by $\Gamma(\alpha,P,D)=\{X\in D\colon |X-P|<(1+\alpha)\operatorname{dist}(X,\partial D)\}$. The area integral is defined for $P\in\partial D$ by

$$A(u,\alpha,P) = \left(\int_{\Gamma(\alpha,P,D)} |X-P|^{2-n} |\nabla u(X)|^2\, dX\right)^{1/2}.$$

The nontangential maximal function is defined by

$$N(u,\alpha,P) = \sup\{|u(X)|\colon X\in\Gamma(\alpha,P,D)\}.$$

Denote $B(X,r)=\{Y\in\mathbf{R}^n\colon |X-Y|<r\}$ and

$$\Delta(P,r) = B(P,r)\cap\partial D \quad \text{for} \quad P\in\partial D.$$

A positive measure μ on ∂D satisfies A_∞ with respect to ω if there exist positive constants θ, c_1, c_2 such that for any $\Delta=\Delta(P,r)$ and any $E\subset\Delta$,

$$c_1\left(\frac{\omega(E)}{\omega(\Delta)}\right)^{1/\theta} \leqq \frac{\mu(E)}{\mu(\Delta)} \leqq c_2\left(\frac{\omega(E)}{\omega(\Delta)}\right)^{\theta}.$$

Theorem. *Let $\alpha_1, \alpha_2, \alpha_3$ be positive real numbers. Let D be a bounded Lipschitz domain in $\mathbf{R}^n$, $n\geqq 2$. Let μ be a positive measure satisfying A_∞ with respect to ω. Let $\Phi\colon [0,\infty)\to[0,\infty)$ be an unbounded, nondecreasing, continuous function satisfying $\Phi(0)=0$ and $\Phi(2t)\leqq C\Phi(t)$. There are positive constants c_1 and c_2 such that if u is an L-harmonic function in D and u vanishes at X_0, then*

$$c_1\int \Phi(N(\alpha_1,u,.))\,d\mu \leqq \int \Phi(A(\alpha_2,u,.))\,d\mu \leqq c_2\int \Phi(N(\alpha_3,u,.))\,d\mu.$$

The constants c_1 and c_2 are independent of u, but depend on $\alpha_1, \alpha_2, \alpha_3, D, \Phi$, and X_0.

We shall begin the proof by recalling some well-known facts about harmonic measure. We shall use the notation $\omega_D(F,Z)$ for the L-harmonic function of Z in D with boundary values 1 on F and 0 elsewhere on ∂D. Thus $\omega(F)=\omega_D(F,X_0)$ for $F\subset\partial D$. Let $Z\in D$ and choose $P\in\partial D$ so that $|Z-P|=s=\operatorname{dist}(Z,\partial D)$. Then

(1) $$1 \geqq \omega_D(\Delta(P,s),Z) > c > 0 \quad \text{(cf. [3])}.$$

The constant c depends only on the Lipschitz constant of D, provided s is sufficiently small that Z and $\Delta(P, s)$ are contained in a single coordinate cylinder.

Throughout the remainder of the proof we shall use the notation $A_1 \lesssim A_2$ to mean that there is a constant $c>0$ such that $cA_1 \leqq A_2$, and the constant c depends only on the Lipschitz constant of D and X_0. We shall assume that any sets on which A_1 and A_2 depend are in a single coordinate cylinder of D. $A_1 \simeq A_2$ means $A_1 \lesssim A_2$ and $A_2 \lesssim A_1$. In this notation (1) can be written

$$(1') \qquad \omega_D(\Delta(P, s), Z) \simeq 1.$$

Another important property of harmonic measure is the doubling property [3]

$$(2) \qquad \omega(\Delta(P, 2s)) \lesssim \omega(\Delta(P, s)).$$

For $P \in \partial D$ and s suitably small, the Lipschitz character of D implies that there exists $Y \in D$ such that $\operatorname{dist}(Y, \partial D) \simeq |Y-P| \simeq s$. The lemma of Carleson, Hunt and Wheeden, and Caffarelli et al. [3] says that for every $F \subset \Delta(P, s)$,

$$(3) \qquad \omega_D(F, Y) \simeq \omega_D(F, X_0)/\omega_D(\Delta(P, s), X_0).$$

For $P_0 \in \partial D$, $r>0$, and a closed set $E \subset \Delta(P_0, r)$ it is well-known that one can construct a "sawtooth" region $\Omega = \Omega(E, P_0, r)$ over E. The properties of Ω are that it is a Lipschitz domain satisfying

(i) For suitable $\alpha_1, \alpha_2, c_1, c_2$

$$\bigcup\{\Gamma(\alpha_1, P, D) \cap B(P, c_1 r)\colon P \in E\} \subset \Omega \subset \bigcup\{\Gamma(\alpha_2, P, D) \cap B(P, c_2 r)\colon P \in E\}.$$

(ii) $(\partial\Omega) \cap (\partial D) = E$.

(iii) There exists $X \in \Omega$ such that $\operatorname{dist}(X, \partial\Omega) \simeq r$.

(iv) The Lipschitz constants of Ω depend only on D. (This somewhat cumbersome description is designed to work equally well when Lipschitz domains are replaced by NTA domains, defined below.)

The crucial step in the proof is to compare the L-harmonic measure of Ω with that of D:

Main lemma. *Let ν denote L-harmonic measure for Ω at X. There exists $\theta>0$ such that*

$$\omega(E)/\omega(\Delta) \leqq \nu(E)^\theta, \quad \text{for} \quad E \subset \Delta = \Delta(P_0, r).$$

Here θ depends on the Lipschitz constant of D, but not on E or Δ.

Proof. Let $\{Q_j\}$ denote a Whitney decomposition of $\mathbf{R}^n \setminus E$, that is, a disjoint family of cubes Q_j whose union in $\mathbf{R}^n \setminus E$ and whose distances to E are comparable to their respective diameters. Let $\tilde{Q}_j$ be a cube centered at a point of $\partial\Omega$ whose diameter and whose distance from ∂D are comparable to the diameter of Q_j. Let $\tilde{\tilde{Q}}_j$ be a cube centered at a point of ∂D whose distance from Q_j and whose diameter are comparable to the diameter of Q_j. Conditions (i) and (iv) imply that for every cube Q_j of diameter less than $10r$ the cubes $\tilde{Q}_j$ and $\tilde{\tilde{Q}}_j$ exist and there is a cube Q_j^* concentric with Q_j with diameter a fixed multiple of the diameter of Q_j such that Q_j^* contains $\tilde{Q}\cup\tilde{\tilde{Q}}_j$. Define a measure $\bar{\nu}$ on $\Delta(P_0, 2r)$ by

$$\bar{\nu}(F) = \nu(F\cap E)+\sum_j \frac{\omega(F\cap Q_j)}{\omega(Q_j^*\cap\partial D)}\nu(Q_j^*\cap\partial\Omega).$$

We want to prove the inequalities

For $s \leqq r,\quad |P-P_0| \leqq r,\quad \Delta' = \Delta(P, s),\quad F\subset\Delta',$

$$\bar{\nu}(F)/\bar{\nu}(\Delta') \leqq \omega(F)/\omega(\Delta'). \tag{4}$$

$$\bar{\nu}(\Delta) \simeq 1. \tag{5}$$

The real variable lemma of Coifman and C. Fefferman [4], says that (4) implies a kind of converse to itself: *There exists* $\theta>0$ *such that if* $\Delta'\subset\Delta$,

$$\omega(F)/\omega(\Delta') \lesssim \left(\bar{\nu}(F)/\bar{\nu}(\Delta')\right)^{\theta}.$$

Using (5), we see that our main lemma is the special case $F=E$, $\Delta'=\Delta$.

For the proof of (4), consider two cases.

Case (a). The cubes Q_j *intersecting* Δ' *all have diameter less than* $100s$.

We claim that in this case

$$\bar{\nu}(\Delta')\simeq \nu(\Delta''),\quad \text{where}\quad \Delta'' = B(P, cs)\cap\partial\Omega. \tag{6}$$

The inequality $\bar{\nu}(\Delta')\lesssim\nu(\Delta'')$ is easy to see for suitably large c. For the converse, observe that if $Q_j\cap\Delta(P, s/2)\neq\emptyset$, then $Q_j\cap\Delta'$ is a large part of $Q_j\cap\partial D$. Hence, by the doubling property (2) $\omega(Q_j^*\cap\partial D)\lesssim\omega(Q_j\cap\Delta')$. Therefore,

$$\bar{\nu}(\Delta') \gtrsim \nu(\Delta'\cap E)+\sum_{Q_j\cap\Delta(P,\frac{1}{2}s)\neq\emptyset}\nu(Q_j^*\cap\partial\Omega) \gtrsim \nu(\Delta'')$$

by the doubling condition for ν. This proves (6).

Choose $Y\in\Omega$ such that $\operatorname{dist}(Y,\partial\Omega)\simeq\operatorname{dist}(Y,\partial D)\simeq|Y-P|\simeq s$. Property (3) implies that

$$\omega(F\cap Q_j)/\omega(Q_j^*\cap\partial D) \simeq \omega_D(F\cap Q_j, Y)/\omega_D(Q_j^*\cap\partial D, Y).$$

Also, the analogue of (3) for ν says that for any $G\subset\Delta''$,

$$\omega_\Omega(G, Y) \simeq \nu(G)/\nu(\Delta'').$$

These two facts and (6) imply that

$$\frac{\bar{\nu}(F)}{\bar{\nu}(\Delta')} \simeq \frac{1}{\nu(\Delta'')}\left(\nu(F\cap E)+\sum_j \frac{\omega(F\cap Q_j)}{\omega(Q_j^*\cap\partial D)}\nu(Q_j^*\cap\partial\Omega)\right)$$

$$\simeq \omega_\Omega(F\cap E, Y)+\sum_j \frac{\omega_D(F\cap Q_j, Y)}{\omega_D(Q_j^*\cap\partial D, Y)}\omega_\Omega(Q_j^*\cap\partial\Omega, Y).$$

Next, we claim that $\omega_\Omega(Q_j^*\cap\partial\Omega, Y)\lesssim\omega_D(Q_j^*\cap\partial D, Y)$. Indeed the doubling property (2) implies $\omega_\Omega(Q_j^*\cap\partial\Omega, Y)\lesssim\omega_\Omega(\tilde{Q}_j\cap\partial\Omega, Y)$. (Recall that $\tilde{Q}_j$ was defined at the beginning of the proof of the main lemma.) Moreover, (1) implies that $\omega_D(Q_j^*\cap\partial D, Z)\simeq 1$ for $Z\in\tilde{Q}_j$. Hence, by the maximum principle, $\omega_\Omega(\tilde{Q}_j\cap\partial\Omega, Y)\lesssim \omega_D(Q_j^*\cap\partial D, Y)$, and the claim follows. Applying the claim to the sum over j and the maximum principle to the first term, we find that

$$\frac{\bar{\nu}(F)}{\bar{\nu}(\Delta')} \lesssim \omega_D(E\cap F, Y)+\sum_j \omega_D(F\cap Q_j, Y) = \omega_D(F, Y).$$

But by (3), $\omega_D(F, Y)\simeq\omega(F)/\omega(\Delta')$, concluding the proof in case (a).

Case (b). Some cube Q_j intersecting Δ' has diameter greater than $100s$.

In this case Δ' is contained in a union of cubes Q_j of comparable diameter whose distance apart is at most the diameter. Therefore, for any fixed l such that $Q_l\cap\Delta'\neq\emptyset$ and any $F\subset\Delta'$,

$$\bar{\nu}(F) = \sum_j \frac{\omega(F\cap Q_j)}{\omega(Q_j^*\cap\partial D)}\nu(Q_j^*\cap\partial D)$$

$$\simeq \frac{\nu(Q_l^*\cap\partial D)}{\omega(Q_l^*\cap\partial D)}\sum_j \omega(F\cap Q_j) = c_l\,\omega(F),$$

with $c_l=\nu(Q_l^*\cap\partial D)/\omega(Q_l^*\cap\partial D)$. Inequality (4) follows immediately.

We conclude the main lemma with the proof of (5). Assuming that E is nonempty, the case $\Delta'=\Delta$ falls under case (a). Therefore, by (6) and (1),

$$\bar{\nu}(\Delta) \simeq \nu(B(P_0, r)\cap\partial\Omega) = \omega_\Omega(B(P_0, r)\cap\Omega, X) \simeq 1.$$

Having proved the main lemma, we can now follow the lines of the proof given in [6]. We shall first recall some facts from earlier work. Let $g(X, Y)$ be the Green function for a Lipschitz domain Ω. If $Lu=0$ in Ω and $u(X)=0$, then [12]

$$(7)\qquad 2\int_\Omega \sum_{i,j=1}^n a_{ij}(Y)\frac{\partial u}{\partial Y_i}(Y)\frac{\partial u}{\partial Y_j}(Y)g(X, Y)\,dY = \int_{\partial\Omega} u(P)^2\,d\nu(P)$$

where ν is the harmonic measure at X for Ω. The comparison lemma [3, 5] says that if $P\in\partial D$, $Y\in\Omega$, $\operatorname{dist}(Y, \partial\Omega)=s=|Y-P|$ and $|X-P|>2s$, then

$$(8)\qquad g(X, Y) \simeq \nu(B(P, s)\cap\partial\Omega)s^{2-n}.$$

Denote $B=B(Z,t)$, $B^*=B(Z,(1+\eta)t)$ for some $\eta>0$. If $Lu=0$ in B^*, then for all $Z', Z''\in B$,

$$|u(Z')-u(Z'')| \lesssim (|Z'-Z''|/t)^{\alpha}\left(\frac{t^2}{|B^*|}\int_{B^*}|u-b|^2\right)^{1/2} \tag{9}$$

$$\lesssim (|Z'-Z''|/t)^{\alpha}\left(\frac{t^2}{|B^*|}\int_{B^*}|\nabla u|^2\right)^{1/2},$$

where $b=\frac{1}{|B|}\int_B u$. (Here, and elsewhere $|\cdot|$ denotes Eulidean volume in $\mathbf{R}^n$. The first inequality is due to de Giorgi, Nash, and Moser. The second is Poincaré's inequality, cf. [14].)

$$\int_B |\nabla u|^2 \lesssim t^{-2}\int_{B^*} u^2 \qquad [14] \tag{10}$$

There exists $p>1$ such that

$$\left(\frac{1}{|B|}\int_B |\nabla u|^{2p}\right)^{1/2p} \lesssim \left(\frac{1}{|B^*|}\int_{B^*}|\nabla u|^2\right)^{1/2} \qquad [13] \tag{11}$$

Let $0<\alpha'<\alpha''$ and abbreviate $A'(P)=A(y,\alpha',P)$; $N''(P)=N(u,\alpha'',P)$.

Lemma 1. *Suppose that* $A'(P_1)\leqq\lambda$ *for some* $P_1\in\Delta=\Delta(P_0,r)$. *Given* $\alpha>1$ *and* $\beta>1$, *there exists* $\varepsilon>0$ *depending only on the Lipschitz constant of* D, α', α'' α, β, *and* μ *such that*

$$\alpha\mu\{P\in\Gamma\colon\ A'(P)\geqq\beta\lambda,\ N''(P)\leqq\varepsilon\lambda\}\leqq\mu(\Delta).$$

Proof. Denote $E=\{P\in\Delta\colon A'(P)\geqq\beta\lambda,\ N''(P)\leqq\varepsilon\lambda\}$. Let

$$A_t(P)=\left(\int_{\Gamma(\alpha',P,D)\cap B(P,t)}|u(Y)|^2|Y-P|^{2-n}\,dY\right)^{1/2}.$$

We claim that for any $\tau>0$ there exists $\varepsilon>0$ such that for $P\in E$, $A_{\tau r}(P)>\frac{\beta-1}{2}\lambda$. Denote $U=\Gamma(\alpha',P,D)\setminus B(P,\tau r)$. Then $U=U_1\cup U_2\cup U_3$ where $U_1=U\cap B(P,tr)$, $U_2=(U\cap\Gamma(\alpha',P_1,D))\setminus B(P,tr)$ $U_3=U\setminus\Gamma(\alpha',P_1,D)$ and t will be chosen later. To prove the claim it suffices to show that for any $\eta>0$ we can choose t and $\varepsilon>0$ so that

$$\int_{U_1\cup U_3}|\nabla u(Y)|^2|Y-P|^2\,dY<\eta\lambda^2$$

and

$$\int_{U_2}|\nabla u(Y)|^2|Y-P|^{2-n}\,dY<(1+\eta)\lambda^2.$$

The second inequality follows from the fact that $A'(P_1)\leqq\lambda$ and the observation that t sufficiently large $|Y-P|^{2-n}\leqq(1+\eta)|Y-P_1|^{2-n}$ for all $Y\in U_2$. Now fix t sufficiently large. Denote $R_j=B(P,2^j\tau r)\setminus B(P,2^{j-1}\tau r)$, $j=1,2,\ldots$. Notice that $R_j\cap U\subset Q_j$, where Q_j is a finite union of balls of radius comparable to $2^j r$, whose distance to P and to ∂D is also comparable to $2^j r$. Thus

$$\int_{U_1} |\nabla u(Y)|^2 |Y-P|^{2-n}\, dY \lesssim \sum_{2^j\tau \leqq 2t} \int_{Q_j} |\nabla u(Y)|^2 (2^j r)^{2-n}\, dY.$$

Let Q_j^* denote the union of slightly larger balls than those of Q_j, but still satisfying $Q_j^* \subset \Gamma(\alpha'', P, D)$. This is possible because $\alpha'' < \alpha'$. Hence, from (10),

$$\int_{U_1} |\nabla u(Y)|^2 (2^j r)^{2-n}\, dY \lesssim (2^j r)^{-n} \int_{Q_j^*} |u(Y)|^2\, dY \lesssim \varepsilon^2 \lambda^2, \quad \text{and}$$

$$\int_{U_1} |\nabla u(Y)|^2 |Y-P|^{2-n}\, dY \lesssim \varepsilon^2 \lambda^2 \log t/\tau < \eta \lambda^2$$

for ε sufficiently small depending on t. Finally, for U_3, observe that $|R_j \cap U_3| \lesssim r(2^j r)^{n-1}$. Applying Hölder's inequality, (11) and then (10) we find that for some $p>1$,

$$\begin{aligned}
\int_{U_3} |\nabla u(Y)|^2 |Y-P|^{2-n}\, dY &\simeq \sum_{j=1}^{\infty} \int_{R_j \cap U_3} |\nabla u|^2 (2^j r)^{2-n} \\
&\leqq \sum_j (2^j r)^{2-n} |R_j \cap U_3|^{1-1/p} \Big(\int_{R_j \cap U_3} |\nabla u|^{2p} \Big)^{1/p} \\
&\leqq \sum_j (2^j r)^{2-n} |R_j \cap U_3|^{1-1/p} \left(\frac{1}{|Q_j|} \int_{Q_j} |\nabla u|^{2p} \right)^{1/p} |Q_j|^{1/p} \\
&\lesssim \sum_j (2^j r)^{2-n} |R_j \cap U_3|^{1-1/p} \left(\frac{1}{|Q_j|} \int_{Q_j^*} |\nabla u|^2 \right) |Q_j|^{1/p} \\
&\simeq \sum_j (2^j r)^{2-n} 2^{j(1/p-1)} \int_{Q_j^*} |\nabla u|^2 \leqq \sum 2^{j(1/p-1)} \varepsilon^2 \lambda^2 \lesssim \varepsilon^2 \lambda^2.
\end{aligned}$$

This concludes the proof of the claim.

Let $\Omega = \Omega(E, P_0, r)$ be a sawtooth region with the properties (i) to (iv) above, and let $X \in \Omega$ be given by (iii). Suppose that $\alpha' < \alpha_1 < \alpha_2 < \alpha''$ and that $\tau < c_1/2$, in the notation of (i). As before, let ν denote the L-harmonic measure for Ω at X. Then as in [6, Lemma 3] we can use (8) and (7) to obtain

$$\left(\frac{p-1}{2} \right)^2 \lambda^2 \nu(E) \leqq \int_E A_{\tau r}(P)^2\, d\nu(P) \lesssim \int_{\partial\Omega} u(P)^2\, d\nu(P) \lesssim \varepsilon^2 \lambda^2.$$

The main lemma now implies $\omega(E)/\omega(\Delta) \lesssim (\varepsilon^2/(\beta-1))^{1/\theta}$. Hence, since μ belongs to A_∞ with respect to ω, $\alpha\mu(E) < \mu(\Delta)$ for sufficiently small ε, concluding Lemma 1.

Reversing the roles of A and N, let

$$A''(P) = A(u, \alpha'', P), \quad N'(P) = N(u, \alpha', P), \quad \alpha'' > \alpha' > 0.$$

Lemma 2. *Let* $q(t, P) = \sup \{\mu\{P' \in \Delta\colon A''(P') > t\}/\mu(\Delta)\colon \Delta = \Delta(P, r), r > 0\}$. *Suppose that* $N'(P_1) \leqq \lambda$ *for some* $P_1 \in \Delta = \Delta(P_0, r)$. *Suppose also that* α'' *is a sufficiently large multiple of* α'. *Given* $\alpha > 1$ *and* $\beta > 1$, *there exist* $\varepsilon > 0$ *and* $0 < \delta < 1/2$

with the same dependence as in Lemma 1 *such that*

$$\alpha\mu\{P\in\Delta:\ N'(P)\geqq\beta\lambda,\ q(\varepsilon\lambda, P)\leqq\delta\}\leqq\mu(\Delta).$$

Proof. Denote $E=\{P\in\Delta:\ N'(P)\geqq\beta\lambda, q(\varepsilon\lambda, P)\leqq\delta\}$ and $F=\{P\in\Delta:\ A''(P)\leqq\varepsilon\lambda\}$. Clearly, $E\subset F$. Let $\Omega=\Omega(E, P_0, r)$ be a sawtooth region above E and let $X\in\Omega$ be as in (iii) above, $\alpha'<\alpha_1<\alpha_2<\alpha''$ where α_1 and α_2 are as in (i) above. There exist α_3 and $\tau>0$ such that for $P\in E$

$$\Gamma(\alpha_2, P, \Omega)\supset B(P, \tau r)\cap\Gamma(\alpha', P, D).$$

Provided α'' is a suitably large multiple of α', for any $Y\in\Gamma(\alpha', P, D)\setminus B(P, \tau r)$, there exists a sequence $Y_1, \dots, Y_N=Y$ such that $Y_1\in\Gamma(\alpha', P_1, D), B(Y_j, \eta r)\subset\Gamma(\alpha'', P, D)$ and $|Y_j-Y_{j+1}|<1/2\eta r$. (Here η and N are independent of r.) Hence, by (9) for $Z', Z''\in B(Y_j, 1/2\eta r)$

$$|u(Z')-u(Z'')|\leqq C\left(r^{2-n}\int_{B(Y_j,\eta r)}|\nabla u|^2\right)^{1/2}\leqq CA''(P)\leqq C\varepsilon\lambda.$$

Therefore $|u(Y_1)-u(Y)|\leqq CN\varepsilon\lambda$, and for sufficiently small ε, $|u(Y)|\leqq|u(Y_1)|+ +C\varepsilon N\lambda\leqq(1+C\varepsilon N)\lambda<\beta\lambda$. The same argument applied to X instead of Y yields

$$|u(X)|\leqq\left(1+\frac{\beta-1}{2}\right)\lambda.$$

Suppose that $P\in E$. The foregoing estimate implies that there exists $Y\in\Gamma(\alpha', P, D)\cap B(P, \tau r)$ such that $|u(Y)\geqq\beta\lambda$. Hence, $|u(Y)-u(X)|\geqq\frac{\beta-1}{2}$. Denote $\tilde{N}(P)=\sup\{|u(Y)-u(X)|:\ Y\in\Gamma(\alpha_3, P, \Omega)\}$. We have just shown that $E\subset\left\{P:\ \tilde{N}(P)\geqq\frac{\beta-1}{2}\lambda\right\}$.

Next, denote $F(Y)=\{P\in F:\ Y\in\Gamma(\alpha'', P, D)\}$. Let $l(Y)=\operatorname{dist}(Y, \partial D)$ and $f(Y)=\int_{F(Y)}d\omega_D(., X)$. Then by Fubini's theorem

$$\varepsilon^2\lambda^2\geqq\int_F A''(P)^2\,d\omega_D(., X)\geqq\int_\Omega f(Y)|\nabla u(Y)|^2 l(Y)^{2-n}\,dY.$$

For each $Y\in\Omega$ there exists $\pi Y\in E$ such that $Y\in\Gamma(\alpha_2, \pi Y, D)$ and $|Y-\pi Y|\cong l(Y)$. Let $\Delta'=\Delta\left(\pi Y, \frac{1}{10}\alpha'' l(Y)\right)$. Clearly, $\Delta'\cap F\subset F(Y)$, provided α'' is a suitably large multiple of α_2. Therefore,

$$f(Y)\geqq\omega_D(\Delta'\cap F, X)=\frac{\omega_D(\Delta'\cap F, X)}{\omega_D(\Delta', X)}\omega_D(\Delta', X).$$

By (3), $\omega_D(\Delta'\cap F, X)/\omega_D(\Delta', X)$ is comparable to $\omega_D(\Delta'\cap F, X_0)/\omega_D(\Delta', X_0)$. By (8), if $|Y-X|>\eta r/2$. Then $\omega_D(\Delta', X)$ is comparable to $g_D(X, Y)l(Y)^{n-2}$. Recall that because $\pi Y \in E$, $\mu(\Delta' \setminus F)/\mu(\Delta') \leqq \delta$. Because μ satisfies A_∞ with respect to $\omega_D(.,X_0)$, for sufficiently small δ,

$$\omega_D(\Delta' \cap F, X_0)/\omega_D(\Delta', X_0) > (1 - C\delta^\theta) > \tfrac{1}{2}.$$

In all,

$$C\varepsilon^2\lambda^2 \geqq \int_{\Omega \setminus B(X,r/2)} g_D(Y_0, Y)\,|\Delta u(Y)|^2\,dY.$$

Next, for η small enough that $B(X, 4\eta r) \subset \Omega$,

$$\int_{B(X,2\eta r)} |\Delta u(Y)|^2 l(Y)^{2-n}\,dY \leqq \varepsilon^2\lambda^2.$$

Hence, by (9)

$$\sup\{|u(Z)-u(X)|:\ Z \in \bar{B}(X, \eta r)\} \leqq C\varepsilon\lambda.$$

Thus by (7)

$$\int_{B(X,\eta r)} |\Delta u(Y)|^2 g_{B(X,\eta r)}(X, Y)\,dY \leqq (C\varepsilon\lambda)^2.$$

It is easy to check using the maximum principle and interior inequalities for the Green function in a region containing D [7, Theorem 4] that for $Y \in B(X, \eta r/2)$,

$$g_D(X, Y) \leqq C g_{B(X,\eta r)}(X, Y).$$

Hence,

$$\int_\Omega g_D(X, Y)\,|\Delta u(Y)|^2\,dY \leqq C\varepsilon^2\lambda^2.$$

By the maximum principle, $g_\Omega(X, Y) \leqq g_D(X, Y)$. Finally, using

(7) $$\int_{\partial\Omega} |u(P)-u(X)|^2\,d\omega_\Omega(.,X) \leqq C\varepsilon^2\lambda^2.$$

The weak L^2 estimate for the nontangential maximal function $\tilde{N}$ says [3]

$$\omega_\Omega\left(\left\{P \in \partial\Omega:\ \tilde{N}(P) \geqq \frac{\beta-1}{2}\lambda\right\}, X\right) \leqq C\varepsilon^2.$$

Because $E \subset \left\{P \in \partial\Omega:\ N(P) \geqq \frac{\beta-1}{2}\lambda\right\}$ and because of the main lemma and the A_∞ property for μ,

$$\frac{\mu(E)}{\mu(\Delta)} \leqq \left(\frac{\omega_D(E, X_0)}{\omega_D(\Delta, X_0)}\right)^\theta \leqq (C\varepsilon^2)^{\theta^2}.$$

This concludes Lemma 2.

The remainder of the proof of the theorem is well-known.

Except for the problem of interchanging α' and α'' it involves only a real variable argument due to Burkholder and Gundy [1]. The only difference is that the supremum of $|\nabla u(Y)|$ need not exist in our case. However, with the help of (9) it is possible to replace the supremum of $|\nabla u(Y)|$ by a suitable difference quotient. For instance, the auxiliary function of [1]

$$D(P) = \sup\{|\nabla u(Y)||Y-P|: Y\in\Gamma(\alpha, P, D)\}$$

may be replaced by

$$\tilde{D}(P) = \sup\{|u(Z')-u(Z'')|: Z', Z''\in B(z, t)\subset\Gamma(\alpha, P, D)\}.$$

Notice that $\tilde{D}(P)\leqq A'(P)$ for $\alpha'>\alpha$ because of (9).

Two generalizations

A) The theorem and its proof are valid with no change when the domain D is nontangentially accessible (NTA) rather than Lipschitz. Following [12], we call a domain D NTA if

(a) There exist constants $A>1$, $r_0>0$ such that for every r, $0<r<r_0$, and every $P\in\partial D$, there exists $X\in D$ such that $|X-P|<Ar$ and $B(X, r/A)\subset D$.

(b) Property (a) holds for the complement of D.

(c) For every $C>0$, there exists N such that if $0<\varepsilon<r_0$, $X, Y\in D$, $\operatorname{dist}(X,\partial D)>\varepsilon$, $\operatorname{dist}(Y,\partial D)>\varepsilon$ and $|X-Y|<C\varepsilon$, then there is a sequence of N points of D $X=X_1, X_2, \ldots, X_N=Y$ such that $|X_j-X_{j+1}|<\varepsilon/A$ and $B(X_j, 2\varepsilon/A)\subset D$. The analogues of (2), (3), (7), and (8) and the construction of Ω for NTA domains can be found in [12].

B) The operator L need not be uniformly elliptic. It can satisfy a non-uniform ellipticity condition

$$C^{-1}m(X)|\xi|^2 \leqq \sum_{i,j=1}^n a_{ij}(X)\xi_i\xi_j \leqq cm(X)|\xi|^2$$

where $m(X)$ satisfies either Muckenhoupt's condition

$$(A_2) \qquad \sup_{B=B(X,t)} \left(\frac{1}{|B|}\int_B m\right)\left(\frac{1}{|B|}\int_B m^{-1}\right) < \infty$$

or $m(X)=|f'(X)|^{1-2/n}$, where f is a global quasiconformal mapping of $\mathbf{R}^n$ and $|f'(X)|$ denotes the modulus of the Jacobian determinant of f.

Denote $\theta(X, r)=r^2/m(B(X, r))$, where $m(B)=\int_B m$. The area integral is defined by

$$\left(\int_{\Gamma(\alpha,P,D)}\sum_{i,j=1}^n a_{ij}(X)\frac{\partial u}{\partial X_i}(X)\frac{\partial u}{\partial X_j}(X)\theta(X, |X-P|)\,dX\right)^{1/2}$$
$$\simeq \left(\int_{\Gamma(\alpha,P,D)}|\nabla u(X)|^2 m(X)\theta(X, |X-P|)\,dX\right)^{1/2}.$$

The nontangential maximal function is defined as before. Once, again, the proof of the theorem is the same. The statements of (2), (3) and (7) are the same. Inequalities (8) to (11) are replaced by

(8′) $$g(X, Y) \simeq \nu(B(P, s)\cap\partial\Omega)\theta(P, s).$$

(9′) $$|u(Z')-u(Z'')| \lesssim (|Z'-Z''|/t)^{\alpha}\left(\theta(Z, t)\int_{B^*}|\Delta u|^2 m\right)^{1/2}.$$

(10′) $$\int_B |\nabla u|^2 m \leqq t^{-2}\int_{B^*} u^2 m.$$

(11′) $$\left(\frac{1}{m(B)}\int_B |\Delta u|^{2p} m\right)^{1/2p} \leqq \left(\frac{1}{m(B^*)}\int_{B^*} |\Delta u|^2 m\right)^{1/2}.$$

For the proofs of these results, except for (11′), see [7] and [8]. In order to prove (11′), observe that if $B\subset\subset B'\subset\subset B^*$, then for some $q<2$ and some constant c depending on u, [8]

$$\left(\frac{1}{m(B)}\int_B |\nabla u|^2 m\right)^{1/2} \leqq \operatorname{diam} B\left(\frac{1}{m(B)}\int_{B'} |u-c|^2 m\right)^{1/2}$$
$$\leqq \left(\frac{1}{m(B)}\int_{B^*} |\nabla u|^q m\right)^{1/q}.$$

Inequality (11′) now follows from the variant of Gehring's lemma [10, Prop. 5.1] that is applicable to the case in which the ball B^* on the right hand side is the double of the ball on the left, rather than the same ball.

References

1. Burkholder, D. and Gundy, R., Distribution function inequalities for the area integral, *Studia Math.*, **44** (1972), 527—544.
2. Caffarelli, L., Fabes, E. and Kenig, C., Completely singular elliptic-harmonic measures, *Indiana Univ. Math. J.*, **30** (1981), 917—924.
3. Caffarelli, L., Fabes, E., Mortola, S. and Salsa, S., Boundary behavior of nonnegative solutions of elliptic operators in divergence form, *Indiana Univ. Math. J.*, **30** (1981), 621—640.
4. Coifman, R. and Fefferman, C., Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.*, **51** (1974), 241—250.
5. Dahlberg, B., On estimates of harmonic measure, *Arch. Rational Mech. Anal.*, **65** (1977), 272—288.
6. Dahlberg, B., Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for functions harmonic in a Lipschitz domain, *Studia Math.*, **67** (1980), 297—314.
7. Fabes, E., Jerison, D. and Kenig, C., Boundary behavior of solutions to degenerate elliptic equations, *Conf. on harmonic analysis in honor of A. Zygmund*, Vol. 2, Wadsworth. Math. Series, Belmont, Calif. 1983, pp. 577—589.

8. Fabes, E., Kenig, C. and Serapioni, R., The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations,* **7** (1) (1982), 77—116.
9. Fefferman, C. and Stein, E., H^p spaces of several variables, *Acta Math.*, **129** (1972), 137—193.
10. Giaquinta, M. and Modica, G., Regularity results for some classes of higher order nonlinear elliptic systems, *J. Reine Angew. Math.*, **311—312** (1979), 145—169.
11. Gundy, R. and Wheeden, R., Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh—Paley series, *Studia Math.*, **49** (1974), 107—124.
12. Jerison, D. and Kenig, C., Boundary behavior of harmonic functions in non-tangentially accessible domains, *Adv. in Math.*, **46** (1982), 80—147.
13. Meyers, N., An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat.* (3) **17** (1963), 189—206.
14. Moser, J., On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.*, **14** (1961), 577—591.

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Björn E. J. Dahlberg
Uppsala University
Uppsala, Sweden

David S. Jerison
M.I.T. Cambridge, MA 02 139

Carlos E. Kenig
University of Minnesota
Minneapolis, MN 55 455