Extension of a result of Benedek, Calderón and Panzone

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1. Introduction

For X a Banach space and $1 \leqq p \leqq \infty$, L_X^p is the usual Lebesgue space.

The theorem of Benedek, Calderón and Panzone [0] asserts that for $1<p$, $r<\infty$, any operator T: $L_{l^r}^p(\mathbf{R}^n) \to L_{l^r}^p(\mathbf{R}^n)$ of the form $T(f_j)=P.V.\ (K_j * f_j)$ is bounded, the (K_j) being a sequence of convolution kernels K satisfying the conditions

(a) $\|\hat{K}\|_\infty \leqq C$

(b) $|K(x)| \leqq C|x|^{-n}$

(c) $|K(x)-K(x-y)| \leqq C|y|\,|x|^{-n-1}$ for $|y|<\dfrac{|x|}{2}$

and where C is a fixed constant.

Our purpose is to show that this theorem remains true if one replaces l^r by any lattice X with the so-called UMD-property (cf. [2]). Let us recall that a Banach space X is UMD provided for $1<p<\infty$ martingale difference sequences $d=(d_1, d_2, \ldots)$ in $L_X^p[0, 1]$ are unconditional, i.e. $\|\varepsilon_1 d_1+\varepsilon_2 d_2+\ldots\|_p \leqq C_p(X)\|d_1+d_2+\ldots\|_p$ whenever $\varepsilon_1, \varepsilon_2, \ldots$ are numbers in $\{-1, 1\}$. This property is also equivalent to the boundedness of the Hilberttransform on $L_X^p(\mathbf{R})$ (see [3], [1]) and can be characterized geometrically by the existence of a symmetric, biconvex function ζ on $X \times X$ satisfying $\zeta(x, y) \leqq \|x+y\|$ if $\|x\| \leqq 1 \leqq \|y\|$ and $\zeta(0,0)>0$. Let us point out that also for lattices UMD is more restrictive than a condition of r-convexity, s-concavity for some $1<r$, $s<\infty$ (see [9]).

Theorem. *Assume X is a* UMD *space with a normalized unconditional basis (e_j). Then, for $1<p<\infty$, any operator T: $L_X^p(\mathbf{R}^n) \to L_X^p(\mathbf{R}^n)$ defined as*

$$T(\Sigma f_j e_j) = \Sigma T_j(f_j) e_j$$

where the T_j are the singular integral operators considered above, is bounded.

We will use some results on weighted norm inequalities (for a related approach, see [5]).

A positive, locally integrable function ω on $\mathbf{R}^n$ satisfies (A_p) provided, for $1<p<\infty$,

$$\sup_I \left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \omega^{-1/p-1}\right)^{p-1} < \infty,$$

where I runs over all cubes in $\mathbf{R}^n$, for $p=1$,

$$\sup_I \left\{\left(\frac{1}{|I|}\int_I \omega\right) \operatorname*{ess\,sup}_{x\in I} \frac{1}{\omega}\right\} < \infty;$$

for $p=\infty$ (cf. also [10]), there exists $\varepsilon>0$ such that $\int_E \omega \leqq \frac{1}{2}\int_I \omega$ whenever E is a subset of a cube I for which $|E|<\varepsilon|I|$.

The reader is referred to [6], for instance, for the basic theory. We need the following facts

Fact 1 (see [4]). *If ω satisfies (A_∞) and T is a singular integral operator, then*

$$\int |Tf|\omega \leqq C\int f^*\omega \quad \textit{where} \quad f^*(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f|.$$

Fact 2 (see [8]). *If ω is a function on* $[0, 1]$ *satisfying dyadic (A_∞), one has the equivalence*

$$C^{-1}\int S(f)\omega \leqq \int f^*\omega \leqq C\int S(f)\omega$$

for Walsh—Paley series $f=(f_1, f_2, \ldots)$, where

$$f^* = \sup_n |f_n| \quad \textit{and} \quad S(f) = (\Sigma\,|f_n-f_{n-1}|^2+f_0^2)^{1/2}.$$

Of course, there is always uniform dependence between the various involved constants.

2. Proof of the result

Let us first show how to conclude from

Lemma 1. *Under the hypothesis of the theorem, the "maximal operator"*

$$M\colon L^p_X(\mathbf{R}^n) \to L^p_X(\mathbf{R}^n), \quad M(\Sigma f_j e_j) = \Sigma f_j^* e_j$$

is bounded.

Denote (e_j') the dual basis. If X has UMD, also X^* is UMD and Lemma 1 provides a constant $C=C(X)$ such that

$$\|\Sigma f_j^* e_j\|_p \leqq C\|\Sigma f_j e_j\|_p \quad \text{and} \quad \|\Sigma \varphi_j^* e_j'\|_{p'} \leqq C\|\Sigma \varphi_j e_j'\|_{p'}, \quad (p' = p/p-1).$$

In order to show the boundedness of the operator T considered in the theorem, fix norm-1 elements $F=\Sigma f_j e_j$ in $L^p_X(\mathbf{R}^n)$ and $\Phi=\Sigma\varphi_j e'_j$ in $L^{p'}_{X^*}(\mathbf{R}^n)$. Choose $0<\delta<C^{-1}$ and define, for each j, the following function

$$\psi_j = \sum_{k\geqq 0}\delta^k\varphi_j^{(k)}$$

where $\varphi^{(k)}$ is the k-fold maximal function of φ, thus $\varphi^{(k)}=(\varphi^{(k-1)})^*$, $\varphi^{(0)}=|\varphi|$. Clearly $\psi_j^*\leqq\delta^{-1}\psi_j$, so the function ψ_j satisfies (A_1). Hence, for each j,

$$|\langle T_j f_j, \varphi_j\rangle| \leqq \int |T_j f_j|\psi_j \leqq C(\delta)\int f_j^*\psi_j$$

and

$$|\langle T(F), \Phi\rangle| \leqq C(\delta)\|M(F)\|_p \sum_{k\geqq 0}\delta^k\|\Sigma_j\varphi_j^{(k)}e'_j\|_{p'} \leqq \frac{CC(\delta)}{1-\delta C}.$$

We prove lemma 1 in case $n=1$ (the general case is completely similar) and replace for simplicity $\mathbf{R}$ by $[0, 1]$. In what follows, S will be the dyadic square function.

Lemma 2. *A Banach lattice X has* UMD *if and only if* $\|F\|_p\sim\|S(F)\|_p$ *for* $F\in L^p_X$ *(for some or for all* $1<p<\infty$*).*

Proof. The equivalence $\|F\|_p\sim\|S(F)\|_p$ obviously implies unconditionality of Walsh—Paley martingale difference sequences in L^p_X and hence UMD (cf. [2]). Conversely, if X has UMD, then

$$\|F\|_p \sim \int \|\Sigma\varepsilon_n\Delta F_n\|_p d\varepsilon \quad \text{where} \quad \Delta F_n = F_n - F_{n-1}$$

(ε_n being the Rademacher functions) and, by convexity, the latter quantity clearly dominates $\|S(F)\|_p$. Since X is also q-concave for some $p\leqq q<\infty$ (see [2], [9]), we have

$$\int \|\Sigma\varepsilon_n\Delta F_n\|_p\, d\varepsilon \leqq \Big(\int\Big(\int\|\Sigma\varepsilon_n\Delta F_n(\omega)\|^q\, d\varepsilon\Big)^{1/q}d\omega\Big)^{1/p}$$
$$\leqq C_q(X)\Big(\int\Big\|\Big(\int|\Sigma\varepsilon_n\Delta F_n(\omega)|^q\, d\varepsilon\Big)^{1/q}\Big\|^p d\omega\Big)^{1/p} \leqq C\Big(\int\|S(F)(\omega)\|^p\, d\omega\Big)^{1/p}$$

proving the reverse inequality.

Lemma 3. *If ω is a positive, integrable function on* $[0, 1]$ *such that* $S(\omega)\leqq C\omega$ *a.e. then ω is (A_∞) (dyadic) ($C>1$ being some constant).*

Proof. Let I be a dyadic interval, say $|I|=2^{-m}$, and $E\subset I$ with $|E|<\varepsilon|I|$. Considering the normalized measure $2^m dx$ on I, we estimate

$$\frac{1}{|I|}\int_E\omega \leqq \Delta\|\omega\chi_I\|_\Phi\|\chi_E\|_\Psi$$

where Φ, Ψ are the respective Orlicz functions

$$\Phi(t) = |t|(1+\log(1+|t|)),\ \Psi(t) = \exp|t|-1.$$

Denote
$$\omega_I = \frac{1}{|I|} \int_I \omega,$$

$$\omega_I^*(x) = \sup_{x \in J \subset I} \frac{1}{|J|} \int_J \omega \quad (x \in I)$$

$$S_I(\omega) = \omega_I + \left(\sum\nolimits_{n>m} |\Delta \omega_n|^2\right)^{1/2}.$$

Fix $\varrho > 0$. Applying the reverse $L \log L$ result (ω being positive), Davis's result (cf. [7]), it follows from the hypothesis

$$\frac{1}{|I|} \int_I \frac{\omega}{\varrho} \log\left(1+\frac{\omega}{\varrho}\right) \leqq \frac{\omega_I}{\varrho}\left(\log^+ \frac{\omega_I}{\varrho} + K\right) + \frac{K}{\varrho} \frac{1}{|I|} \int_I \omega_I^*$$

$$\leqq \frac{\omega_I}{\varrho}\left(\log^+ \frac{\omega_I}{\varrho} + K\right) + \frac{K'}{\varrho} \frac{1}{|I|} \int_I S_I(\omega) \leqq \frac{\omega_I}{\varrho}\left(\log^+ \frac{\omega_I}{\varrho} + CK''\right)$$

where K, K', K'' are numerical constants. Thus

$$\frac{1}{|I|} \int_I \Phi\left(\frac{\omega}{\varrho}\right) \leqq \frac{\omega_I}{\varrho}\left(\log^+ \frac{\omega_I}{\varrho} + CK'' + 1\right)$$

from which it follows $\|\omega \chi_I\|_\Phi \leqq C \omega_I$.

Also, by hypothesis, $\|\chi_E\|_\Psi \leqq \left(\log \frac{1}{\varepsilon}\right)^{-1}$. Therefore

$$\int_E \omega \leqq \text{const.}\, C (\log \varepsilon^{-1})^{-1} \int_I \omega$$

giving the conclusion for $\varepsilon \to 0$.

Proof of Lemma 1. X and X^* having UMD, Lemma 2 gives

$$\|S(F)\|_p \leqq C\|F\|_p;\ \|S(\Phi)\|_{p'} \leqq C\|\Phi\|_{p'} \quad \text{for} \quad F \in L_X^p[0,1],\ \Phi \in L_{X^*}^{p'}[0,1].$$

Proceeding as above, suppose $F = \Sigma f_j e_j$ and $\Phi = \Sigma \varphi_j e_j'$ norm-1. Fixing $0 < \delta < C^{-1}$, introduce for each j the function

$$\psi_j = |\varphi_j| + \delta S(|\varphi_j|) + \delta^2 S^{(2)}(|\varphi_j|) + \ldots + \delta^k S^{(k)}(|\varphi_j|) + \ldots$$

defining inductively $S^{(k+1)}(|\varphi|) = S(S^{(k)}(|\varphi|))$. One verifies easily that $S(\psi_j) \leqq \delta^{-1} \psi_j$. Thus from Lemma 3 and Fact 2, it follows for each j

$$\left|\int f_j^* \varphi_j\right| \leqq \int f_j^* \psi_j \leqq C(\delta) \int S(|f_j|) \psi_j$$

and therefore

$$|\langle M(F), \Phi\rangle| \leqq C(\delta) \|S(|F|)\|_p \|\Sigma \psi_j e_j'\|_{p'} \leqq \frac{C C(\delta)}{1 - \delta C}.$$

Consequently $\|M(F)\|_p \leqq C\ C(\delta)(1-\delta C)^{-1}$, as required.

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