The extension problem for certain function spaces involving fractional orders of differentiability

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1. Introduction

The purpose of this paper is to study the question of extendability to the whole space of functions defined on sub-domains of $\mathbf{R}^n$ and satisfying certain smoothness conditions. The usual Sobolev spaces of integral order are defined by

$$L_k^p(\Omega) = \{f \in L^1_{\mathrm{loc}}(\Omega) \colon D^\beta f \in L^p(\Omega), \quad \text{for all} \quad |\beta| \leqq k\},$$

when Ω is connected, $1 \leqq p \leqq \infty$ and $k \in Z^+$; the derivatives are assumed to exist in the sense of distributions on Ω. $\|f\|_{L_k^p(\Omega)}$ is defined to be

$$\textstyle\sum_{0 \leqq |\beta| \leqq k} \|D^\beta f\|_{L^p(\Omega)}.$$

By an extension operator for $L_k^p(\Omega)$ we will mean a bounded linear operator $\Lambda\colon L_k^p(\Omega) \to L_k^p(\mathbf{R}^n)$, such that $\Lambda(f) \equiv f$ on Ω. Ω will be called an *extension domain* for L_k^p if such an extension operator exists.

Calderon [4] showed that if $\partial\Omega$ is locally the graph of a Lipschitz function, then Ω is an extension domain for L_k^p, for all $1 < p < \infty$ and $k \in Z^+$. Stein [14] extended this result to include the endpoints $p = 1, \infty$ and moreover constructed an extension operator completely independent of k (as well as p). The class of known extension domains was enlarged by Jones [10], who showed that (ε, δ) domains (defined below) are also extension domains for L_k^p, $1 \leqq p \leqq \infty$ and $k \in Z^+$. Furthermore, (ε, ∞) domains are extension domains for the Dirichlet space of functions (modulo constants) with gradients in $L^n(\mathbf{R}^n)$ and for BMO [9]. This class of domains is relatively sharp: if $\Omega \subset \mathbf{R}^2$ is a bounded finitely connected extension domain for L_1^2, then Ω is an (ε, ∞) domain.

Ω is an (ε, δ) domain if there are constants $\varepsilon \in (0, \infty)$ and $\delta \in (0, \infty)$ such that for any $x, y \in \Omega$ with $|x-y|<\delta$, there exists a rectifiable path $\gamma \subset \Omega$ such that

(1.1) $l(\gamma) \leqq \varepsilon^{-1}|x-y|$

(1.2) $d(z, \partial\Omega) \geqq \varepsilon \cdot \inf(|z-x|, |z-y|)$ if $z \in \gamma$,

where $d(z, \partial\Omega)$ is the distance from z to $\partial\Omega$, and $l(\gamma)$ is the length of γ. In $\mathbf{R}^2$, (ε, δ) domains are intimately connected with the theory of quasiconformal mapping:

Theorem A [1, 11]: *If $\Gamma \subset \mathbf{R}^2$ is a Jordan curve, the following are equivalent:*

(1.3) *One or both of the regions bounded by Γ are (ε, ∞) domains for some $\varepsilon>0$.*

(1.4) *Γ is a quasicircle.*

(1.5) *There is a constant $M<\infty$ such that for any $x, y \in \Gamma$, at least one of the two subarcs of Γ with endpoints x and y contains no z such that $|x-z| \geqq M \cdot |x-y|$.*

A Jordan curve $\Gamma \subset \mathbf{R}^2$ is called a quasicircle if it is the image of the unit circle under a globally quasiconformal mapping of $\mathbf{R}^2$. The equivalence of (1.4) and (1.5) is due to Ahlfors [1]; the equivalence of (1.3) and (1.4) was shown by Martio and Sarvas [11] and Jones (unpublished). Examples of (ε, δ) domains include domains whose boundaries are given locally as graphs of functions in the Zygmund class Λ_1, or of functions with gradient in BMO [8], and the classical snowflake domain of conformal mapping theory.

In this paper we investigate the extension problem for the same class of domains, but for more general function spaces than the L_k^p. By means of certain maximal operators N_α, we define (see (2.4)) for arbitrary open Ω function spaces $\mathfrak{N}_\alpha^p(\Omega)$, for all $\alpha>0$ and $1<p<\infty$. These maximal operators have been considered previously in [3] and [5], for instance. When α is a positive integer $\mathfrak{N}_\alpha^p(\mathbf{R}^n)$ coincides with $L_\alpha^p(\mathbf{R}^n)$, but when α is not an integer then $\mathscr{L}_\alpha^p(\mathbf{R}^n) \subsetneqq \mathfrak{N}_\alpha^p(\mathbf{R}^n) \subsetneqq \mathscr{L}_{\alpha-\varepsilon}^p(\mathbf{R}^n)$, for all $\varepsilon>0$; furthermore $\mathfrak{N}_\alpha^p(\Omega)$ does not coincide with the space of restrictions to Ω of functions in $\mathscr{L}_\alpha^p(\mathbf{R}^n)$, for any sub-domain Ω of $\mathbf{R}^n$. Here $\mathscr{L}_\alpha^p$ is the usual potential space as defined for instance in Stein [14].

Our principal result is

Theorem 1.1. *If $\Omega \subset \mathbf{R}^n$ is an open connected (ε, δ) domain, then Ω is an extension domain for $\mathfrak{N}_\alpha^p$, for all $1<p<\infty$ and $\alpha>0$. More precisely, for any $N>0$ there exists an extension operator Λ_N such that*

$$\|\Lambda_N f\|_{\mathfrak{N}_\alpha^p(\mathbf{R}^n)} \leqq C_{p,\alpha} \|f\|_{\mathfrak{N}_\alpha^p(\Omega)},$$

for all $1<p<\infty$ and all $0<\alpha<N$.

The proof is based on ideas of P. W. Jones. This theorem unifies his extendability results for BMO and for the Sobolev spaces; that there should exist such a unification is not surprising since the maximal operators N_α which characterize $\mathfrak{N}_\alpha^p$

reduce to the sharp function, which characterizes BMO, when $\alpha=0$. A minor improvement on the main result of [10] even when $\alpha\in Z^+$ is that the extension operator is independent of α, for α in any bounded range $(0, N)$.

There is also a partial converse to Theorem 1.1, generalizing a result of Gol'dsthein, Latfullin and Vodop'yanov (see also [10]):

Theorem 1.2. *Suppose that* $\Omega\subset\mathbf{R}^2$ *is finitely connected. Suppose* $0<\alpha\leqq 1$ *and* $p\cdot\alpha=2$. *If* Ω *is an extension domain for* $\mathfrak{N}_\alpha^p$, *then* Ω *is an* (ε, δ) *domain.*

The (ε, ∞) (or (ε, δ)) condition is not necessary for $n\neq 2$, or for $n=2$ if $p\cdot\alpha\neq 1$. The proof also yields some insight into the cases $n\neq 2$ or $p\cdot\alpha\neq 2$.

The paper is organized as follows. Section 2 states, mostly without proof, the geometric properties of (ε, δ) domains needed later. The reader is referred to [9] and [10] for details. We also define N_α and $\mathfrak{N}_\alpha^p$, describe a method of approximating functions by polynomials, and derive some basic properties of such approximations. Theorem 1.1 is proved in the third section. The final section is devoted to studying the necessity of the (ε, δ) condition for extendability in $\mathbf{R}^2$.

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2. Notation and preliminaries

$\Omega\subset\mathbf{R}^2$ will be open and connected, and Ω^c will denote the complement of *the closure of* Ω. Q will always denote a closed cube in $\mathbf{R}^n$, and $l(Q)$ is its edgelength. $r\cdot Q$ is the cube concentric with Q with $l(r\cdot Q)=r\cdot l(Q)$. $M(f)$ is the Hardy—Littlewood maximal function of f. $\alpha\in\mathbf{R}$ will be positive, and $m=m(\alpha)$ *is the greatest integer strictly less than* α. $\chi_S(x)$ denotes the characteristic function of S. No two occurrences of C need denote the same constant.

$\mathfrak{W}(\Omega)$ denotes a fixed Whitney decomposition of Ω. Thus $\mathfrak{W}(\Omega)=\{Q_k\}$ where $\bigcup Q_k=\Omega$ and

(2.1) Q_j and Q_k have disjoint interiors if $j\neq k$

(2.2) $c_1 l(Q_k)\leqq d(Q_k, \partial\Omega)\leqq c_2 l(Q_k)$

(2.3) $\sum_k \chi_{c_3\cdot Q_k}(x)\leqq c_4$.

We may take c_1 and c_3 to be as large as desired. $d(Q_k, \partial\Omega)$ denotes the distance between Q_k and $\partial\Omega$. A Whitney chain Γ is a subset $\Gamma=\{Q_0, \dots, Q_k\}\subset\mathfrak{W}(\Omega)$ such that $Q_j\cap Q_{j+1}\neq\emptyset$. The length of Γ is k, and Γ is said to connect Q_0 and Q_k.

The fundamental maximal operator, for $f\in L^1_{\text{loc}}(\Omega)$ and $x\in\Omega$, is

$$N_\alpha f(x) = \inf_P \sup_{x\in Q\subset\Omega} l(Q)^{-n-\alpha}\int_Q |f(y)-P(y)|\,dy, \tag{2.4}$$

where P runs over all polynomials of degree less than or equal to m. If there exists P for which the supremum is finite, then P is unique and is denoted P_x. Information concerning N_α may be found in [3] and [5].

Definition 2.1. $\mathfrak{N}^p_\alpha(\Omega)=\{f\in L^p(\Omega)\colon N_\alpha f\in L^p(\Omega)\}$, *for* $1<p<\infty$. $\|f\|_{\mathfrak{N}^p_\alpha(\Omega)}=\|f\|_{L^p(\Omega)}+\|Nf\|_{L^p(\Omega)}$.

Remark. When $p=\infty$, the methods of this paper apply equally well. Suppose that Ω is an (ε,δ) domain. If $\alpha\notin Z$, then the set of functions f on Ω with $N_\alpha f\in L^\infty(\Omega)$ coincides with the set of restrictions to Ω of functions $f\in\Lambda_\alpha(\mathbf{R}^n)$. When $\alpha=k\in Z$, Λ_α is replaced by $L^\infty_k(\mathbf{R}^n)$.

The following lemma is almost completely proved in Calderón [3, Theorem 4].

Lemma 2.2. *Suppose* $k\in Z^+$, $1<p\leqq\infty$ *and* Ω *is open and connected. Then for any* $f\in L^1_{\text{loc}}(\Omega)$,

$$N_k f\in L^p(\Omega) \Leftrightarrow D^\beta f\in L^p(\Omega) \quad \text{for all} \quad |\beta|=k,$$

and

$$\|N_\alpha f\|_{L^p(\Omega)} \sim \textstyle\sum_{|\beta|=k}\|D^\beta f\|_{L^p(\Omega)}.$$

Proof. Calderón has shown that $N_k f\in L^p(\Omega)$ implies $D^\beta f\in L^p(\Omega)$, for all $|\beta|=k$. Conversely, if $f\in L^p_k(\Omega)$, then given $x\in Q\subset\Omega$ we can approximate f (in L^p_k if $p<\infty$) in Q by smooth functions. If M_ω denotes the maximal function along line segments in direction ω for each $\omega\in S^{n-1}$, then Taylor's theorem yields

$$N_k f(x) \leqq C\textstyle\sum_{|\beta|=k}\int_{S^{n-1}} M_\omega(D^\beta f)(x)\,d\omega.$$

It will be convenient to work with an equivalent variant of N_α. Define

$$\tilde N_\alpha f(x) = \sup_{x\in Q\subset\Omega}\inf_P l(Q)^{-n-\alpha}\int_Q |f(y)-P(y)|\,dy, \tag{2.5}$$

where again P runs over all polynomials of degree $\leqq m$. Certainly $\tilde N_\alpha f(x)\leqq N_\alpha f(x)$, for all x.

Lemma 2.3. $N_\alpha f(x)\leqq C\tilde N_\alpha f(x)$ *for all* x, *where* $C=C(n,\alpha)$ *is independent of* x *and* Ω.

Proof. Suppose $\tilde N_\alpha f(x)<\infty$; we may suppose that $x=0$ and $\tilde N_\alpha f(0)=1$. If $\sqrt{n}2^{-k}<d(0,\partial\Omega)$ choose a polynomial P_k of degree $\leqq m$ such that

$$2^{k(n+\alpha)}\int_{Q_k}|f(y)-P_k(y)|\,dy \leqq 2,$$

where Q_k has center 0 and side length 2^{-k}. Now

$$\int_{Q_0} |P_{k+1}(2^{-k-1}x) - P_k(2^{-k-1}x)|\, dx = 2^{n(k+1)} \int_{Q_{k+1}} |P_{k+1}(x) - P_k(x)|\, dx$$
$$\leqq 2^{n(k+1)} \Big(\int_{Q_{k+1}} |P_{k+1}(x) - f(x)|\, dx + \int_{Q_k} |P_k(x) - f(x)|\, dx \Big)$$
$$\leqq 4 \cdot 2^{n(k+1)} \cdot 2^{-k(n+\alpha)} = 2^{n+2} \cdot 2^{-k\alpha}.$$

Since the $L^1(Q_0)$ and $L^\infty(Q_0)$ norms are equivalent on the space of polynomials of degree $\leqq m$,

$$|P_{k+1}(x) - P_k(x)| \leqq C \cdot 2^{-k\alpha}, \quad \text{for all} \quad x \in Q_k.$$

Moreover, if

$$P_k(x) = \sum_{|\beta| \leqq m} a_{k,\beta} x^\beta$$

then

$$|a_{k,\beta} - a_{k+1,\beta}| \leqq C \cdot 2^{-k(\alpha - |\beta|)}.$$

Hence there exists a_β such that $a_{k,\beta} \to a_\beta$ as $k \to \infty$, and we define $P(x) = \sum_{|\beta| \leqq m} a_\beta x^\beta$. Then

$$\int_{Q_k} |f(x) - P(x)|\, dx \leqq \int_{Q_k} |f(x) - P_k(x)|\, dx + \int_{Q_k} |P(x) - P_k(x)|\, dx.$$

The second term is easily estimated, since for $x \in Q_k$,

$$|P(x) - P_k(x)| \leqq C \sum_{|\beta| \leqq m} 2^{-k|\beta|} |a_\beta - a_{k,\beta}| \leqq C 2^{-k\alpha}.$$

Hence

$$l(Q_k)^{-n-\alpha} \int_{Q_k} |f(x) - P(x)|\, dx \leqq C(n).$$

It follows at once that the same estimate holds with a larger value of $C(n)$ if Q_k is replaced by any cube in Ω centered at 0. Then a similar argument handles arbitrary Q.

Lemma 2.4. *If $x_0, x_1, y \in Q \subset \Omega$ and $N_\alpha f(x_i) < \infty$, then for all $|\beta| \leqq m$,*

$$|D_y^\beta P_{x_0}(y) - D_y^\beta P_{x_1}(y)| \leqq C \cdot l(Q)^{\alpha - |\beta|} \cdot (N_\alpha f(x_0) + N_\alpha f(x_1)).$$

Proof. By dilation it suffices to assume that $l(Q) = 1$. Then

$$\|D^\beta P_{x_0} - D^\beta P_{x_1}\|_{L^\infty(Q)} \leqq C \|P_{x_0} - P_{x_1}\|_{L^1(Q)} \leqq C\Big(\int_Q |f - P_{x_0}| + \int_Q |f - P_{x_1}|\Big)$$
$$\leqq C(N_\alpha f(x_0) + N_\alpha f(x_1)).$$

In order to construct extension operators which are more or less independent of α, we utilize the following approximation scheme:

Proposition 2.5. (See [2] and also [6].) *Let Q_0 be the unit cube. For each fixed $N \in Z^+$, there is a linear projection operator Π: $L^1(Q_0) \to \{$polynomials of degree less than $N\}$ such that for any integer $M \leqq N$,*

$$\|D^\beta (f - \Pi f)\|_{L^p(r \cdot Q_0)} \leqq C(r) \sum_{|\gamma| = M} \|D^\gamma f\|_{L^p(r \cdot Q_0)}, \tag{2.6}$$

for $1 \leqq p \leqq \infty$ *and* $|\beta| < M$. *Furthermore,*

$$\|D^\beta(\Pi f)\|_{L^p(Q_0)} \leqq C \sum_{|\gamma|=|\beta|} \|D^\gamma f\|_{L^p(Q_0)}, \tag{2.7}$$

for all $|\beta| < N$.

(Π is given by an integral operator of the form

$$\Pi f(x) = \int_{x+y \in Q_0} f(x+y) \cdot (A^* h)(x+y)\, dy,$$

where A^* is the formal adjoint of a differential operator A with polynomial coefficients (in y), acting in the y-variable, such that $A(P) \equiv P(0)$ if P is any polynomial of degree $< N$. h is any function in $C_0^\infty(Q_0)$ with $\int h = 1$.) The techniques of this paper do not require the full strength of this proposition; we shall use only the fact that Π is a projection onto the space of polynomials of a certain degree, the estimate

$$\|\Pi f\|_{L^1(Q_0)} \leqq C \|f\|_{L^1(Q_0)},$$

and the same estimate with Q_0 replaced by a fixed dilate. Thus a simpler approximation method would suffice.

Given an arbitrary Q and $f \in L^1(Q)$, we associate to f and Q a polynomial P by translating and dilating Q so that it is identified with Q_0, applying Π, and then reversing the dilation and translation. It will always be assumed that the integer N of Proposition 2.5 is larger than any value of α under consideration.

Next we review some properties of (ε, δ) domains; proofs may be found in [9] and [10]. In the remainder of this section Ω will be an (ε, δ) domain.

Lemma 2.6 [10]. *Suppose Ω is an (ε, δ) domain. There exists $C(\varepsilon, \delta) > 0$ such that if $Q \in \mathfrak{B}(\Omega^C)$ and $l(Q) \leqq C(\varepsilon, \delta)$, then there exists $Q^* \in \mathfrak{B}(\Omega)$ such that*

$$l(Q^*) \sim l(Q) \tag{2.8}$$

and

$$d(Q^*, Q) \leqq C \cdot l(Q). \tag{2.9}$$

Let $W = \{Q \in \mathfrak{B}(\Omega^C)\colon l(Q) \leqq C(\varepsilon, \delta)\}$. For each Q W make a fixed choice of $Q^* \in \mathfrak{B}(\Omega)$ satisfying (2.8) and (2.9). Q^* will be called the reflection of Q. The next lemma is another straightforward consequence of the definitions.

Lemma 2.7 [10]. *Suppose that $Q_0, Q_1 \in W$ and Q_0 meets Q_1. Then there is a Whitney chain $\Gamma_{0,1} \subset \mathfrak{B}(\Omega)$ of length at most $C(\varepsilon)$, connecting Q_0^* to Q_1^*. Moreover, if we choose a fixed such $\Gamma_{j,k}$ for each intersecting pair $Q_j, Q_k \in W$, then*

$$\sum_{\substack{Q_j, Q_k \in W \\ Q_j \cap Q_k \neq \emptyset}} \sum_{R_i \in \Gamma_{j,k}} \chi_{10\sqrt{n} \cdot R_i}(x) \in L^\infty \tag{2.10}$$

and

$$\sum_{Q \in W} \chi_{Q^*}(x) \in L^\infty. \tag{2.11}$$

For the remainder of this section and the next, we make a fixed choice of the $\Gamma_{j,k}$ as above. Then let $\tilde{\Gamma}_{j,k}$ denote $\bigcup_{R_i \in \Gamma_{j,k}} (10\sqrt{n} \cdot R_i)$. A key geometric property of (ε, δ) domains is.

Lemma 2.8 [9]. *Suppose Ω is an (ε, δ) domain. Then there exists $R = R(\varepsilon, \delta) < \infty$ such that any dyadic cube Q of length at most $C(\varepsilon, \delta)$ intersects some $Q^1 \in \mathfrak{B}(\Omega) \cup \mathfrak{B}(\Omega^C)$ with $l(Q^1) \geqq 2^{-R} \cdot l(Q)$.*

Together with the next lemma this provides the foundation for our estimates.

Lemma 2.9. *Suppose that $\Omega \subset \mathbf{R}^n$ is an (ε, δ) domain, and $f \in \mathfrak{N}_\alpha^p(\Omega)$. Suppose $Q_0, Q_1 \in W$ and $Q_0 \cap Q_1 \neq \emptyset$. Let P_i be the polynomial associated to f on Q_i^* by Π. Then*

$$\|D^\beta (P_0 - P_1)\|_{L^\infty(Q_1)} \leqq C l(Q_1)^{\alpha - |\beta| - n} \int_{\tilde{\Gamma}_{0,1}} N_\alpha f(x)\, dx,$$

for all $|\beta| \leqq m$.

Proof. Consider the quantity

$$\|P_0 - P_1\|_{L^\infty(Q_1)} \leqq C l(Q_1)^{-n} \|P_0 - P_1\|_{L^1(Q_1)}$$
$$\leqq C l(Q_1)^{-n} \|P_0 - P_1\|_{L^1(Q_1^*)}.$$

Let $\Gamma_{0,1}$ be the Whitney chain chosen above. $\Gamma_{0,1} = \{R_0, \dots, R_k\}$, where $R_0 = Q_0^*$ and $R_k = Q_1^*$.

$$\|P_0 - P_1\|_{L^1(Q_1^*)} \leqq \sum_{j=0}^k \|P_j - P_{j+1}\|_{L^1(Q_1^*)} \leqq C \sum_{j=0}^k \|P_j - P_{j+1}\|_{L^1(R_j)},$$

where P_j is the polynomial associated to f on R_j by Π. (We use repeatedly the equivalence of all norms on the finite-dimensional space of all polynomials of degree less than N.) Finally,

$$\|P_j - P_{j+1}\|_{L^1(R_j)} \leqq \int_{R_j} |f - P_j| + \int_{R_j} |f - P_{j+1}|$$
$$\leqq \int_{R_j} |f - P_j| + \int_{C(n) \cdot R_{j+1}} |f - P_{j+1}|.$$

To estimate the first integral, choose a polynomial q of degree $\leqq m$ so that

$$\int_{R_j} |f - q| \leqq 2 \operatorname*{Inf}_P \int_{R_j} |f - P| \leqq 2 l(R_j)^{n+\alpha} \cdot \operatorname*{Inf}_{x \in R_j} N_\alpha f(x),$$

where the infimum is taken over all polynomials P of degree $\leqq m$. Since $f - P_j = f - \Pi(f) = (f - q) - \Pi(f - q)$, by (2.6)

$$\int_{R_j} |f - P_j| \leqq C \int_{R_j} |f - q| \leqq C l(R_j)^{n+\alpha} \inf_{x \in R_j} N_\alpha f(x)$$
$$\leqq C l(R_j)^\alpha \int_{R_j} N_\alpha f(x)\, dx \leqq C l(Q_1)^\alpha \int_{R_j} N_\alpha f(x)\, dx.$$

The second integral is treated in the same way, completing the proof in the case $|\beta|=0$. The general case follows from homogeneity and the fact that

$$\|D^\beta(P_0-P_1)\|_{L^\infty(Q)} \leqq C\|P_0-P_1\|_{L^\infty(Q)}$$

when $l(Q)=1$.

Note that if $d(Q_0, Q_1)\leqq Cl(Q_0)$ then by the triangle inequality the same conclusion holds, with $\tilde{\Gamma}_{0,1}$ replaced by a union of finitely many $\tilde{\Gamma}_{i,j}$'s. Fix a smooth partition of unity $\{\varphi_j\}$ such that

$$\sum \varphi_j \equiv 1 \quad \text{on} \quad \Omega^C \tag{2.12}$$

$$\operatorname{supp}(\varphi_j) \subset \frac{17}{16} Q_j \quad \text{for} \quad Q_j \in \mathfrak{W}(\Omega^C) \quad \text{and} \tag{2.13}$$

$$\|D^\beta \varphi_j\|_\infty \leqq Cl(Q_j)^{-|\beta|}. \tag{2.14}$$

Finally, there is

Lemma 2.10 [10]. *If Ω is an (ε, δ) domain then $\partial\Omega$ has measure zero.*

This is an immediate consequence of Lemma 2.8.

3. Estimates for the extension operator

Suppose $f\in L^1_{\text{loc}}(\Omega)$. For each $Q_j\in W$, let P_j be the polynomial associated to f on Q_j via the projection generator Π of Proposition 2.5. The extension operator $\Lambda\colon L^1_{\text{loc}}(\Omega)\to L^1_{\text{loc}}(\Omega\cup\Omega^C)$ is defined by

$$\Lambda f(x) = \begin{cases} \sum_{Q_j\in W} \varphi_j(x) P_j(x) & \text{if} \quad x\in\Omega^C \\ f(x) & \text{if} \quad x\in\Omega. \end{cases}$$

Observe that $\|\Lambda f\|_{L^p}\leqq C_p\|f\|_{L^p(\Omega)}$, $1\leqq p\leqq\infty$. For since $\partial\Omega$ has measure zero, it suffices to estimate $\|\Lambda f\|$ on Ω^c. φ_j is supported in $\frac{17}{16}Q_j$, and the $\frac{17}{16}Q_j$ have bounded overlap. Hence

$$\int_{\Omega^c} |\Lambda f(x)|^p\,dx \leqq C\sum_{Q_j\in W}\int_{\frac{17}{16}Q_j} |P_j(x)|^p\,dx$$

$$\leqq C\sum_{Q_j\in W}\int_{Q_j^*} |P_j(x)|^p\,dx \leqq C\sum_{Q_j\in W}\int_{Q_j^*}|f(x)|^p\,dx,$$

by construction of the projection Π. By the finiteness condition (2.11), this is dominated by $C\|f\|^p_{L^p(\Omega)}$.

Consider the auxiliary functions $\mathfrak{M}$ and $\mathfrak{M}'$ defined as follows:

$$\mathfrak{M}(x)=\begin{cases} N_\alpha f(x) & \text{if } x\in\Omega \\ l(Q_j)^{-n}\int_{C_0\cdot Q_j^*} N_\alpha f(y)\,dy & \text{if } x\in Q_j\in W \\ 0 & \text{otherwise,}\end{cases} \tag{3.2}$$

where C_0 is large enough that for any $Q_j\in W$, $\bigcup_i \tilde{\Gamma}_{ij}\subset C_0\cdot Q_j^*$.

$$\mathfrak{M}'(x)=\begin{cases} f(x) & \text{if } x\in\Omega \\ \sum_{Q_j\in W}\|P_j\|_{L^\infty(Q_j)}\cdot\chi_{2\cdot Q_j}(x) & \text{if } x\in\Omega^c.\end{cases} \tag{3.3}$$

Essentially the same argument as given above for $\|\Lambda f\|_{L^p}$ shows that $\|\mathfrak{M}'\|_{L^p}\leqq C\|f\|_{L^p(\Omega)}$ and $\|\mathfrak{M}\|_{L^p}\leqq C\|N_\alpha f\|_{L^p(\Omega)}$.

This section is devoted to the proof of the pointwise inequality

Theorem 3.1. *Suppose that Ω is an (ε,δ) domain, $f\in L^1_{\text{loc}}(\Omega)$ and Λf, $\mathfrak{M}$ and $\mathfrak{M}'$ are defined as above. Then for all $x\in\mathbf{R}^n$,*

$$\tilde{N}_\alpha(\Lambda f)(x)\leqq C\cdot M\big(M(\mathfrak{M})\big)(x)+C\cdot M(\mathfrak{M}')(x).$$

Theorem 1.1 follows at once, by Lemma 2.3 and the definition of $\mathfrak{N}^p_\alpha$.

Let S be any cube in $\mathbf{R}^n$. $\text{Inf}_P\, l(S)^{-n-\alpha}\int_S|\Lambda f-P(y)|dy$ will be estimated according to several cases. $C(\varepsilon,\delta)$ denotes the constant of Lemma 2.6.

Case 1. $l(S)\geqq\dfrac{1}{200\sqrt{n}}\cdot C(\varepsilon,\delta)$. Then

$$l(S)^{-n-\alpha}\int_S|\Lambda f(y)|\,dy\leqq C\cdot l(S)^{-n}\int_S|\Lambda f(y)|\,dy\leqq C\cdot\inf_{x\in S}M(\mathfrak{M}')(x).$$

Case 2. S meets some $Q\in\mathfrak{B}(\Omega^c)\setminus W$, and $l(S)\leqq\dfrac{1}{100\sqrt{n}}\cdot C(\varepsilon,\delta)$. Then

$$\inf_P l(S)^{-n-\alpha}\int_S|\Lambda f(y)-P(y)|\,dy\leqq C\sum_{\substack{|\beta|+|\gamma|\leqq m+1\\ S\cap(17/16)Q_j\neq\emptyset\\ Q_j\in W}}\|D^\beta\varphi_j\|\cdot\|D^\gamma P_j\|,$$

the norms being sup norms over $\dfrac{17}{16}\cdot Q_j$. All $Q_j\in W$ for which S meets $\dfrac{17}{16}\cdot Q_j$ have length comparable to $C(\varepsilon,\delta)$, so $\|D^\beta\varphi_j\|\leqq C$ and

$$\|D^\gamma P_j\|_{L^\infty\left(\frac{17}{16}\cdot Q_j\right)}\leqq C\|P_j\|_{L^\infty\left(\frac{17}{16}\cdot Q_j\right)}.$$

Thus

$$\inf_P l(S)^{-n-\alpha}\int_S|\Lambda f(y)-P(y)|\,dy\leqq C\sum_{S\cap\operatorname{supp}(\varphi_j)\neq\emptyset}\|P_j\|_{L^\infty(Q_j)}\leqq\mathfrak{M}'(x)$$

for any $x\in S$, since $S\cap\operatorname{supp}(\varphi_j)\neq\emptyset$ implies $S\subset 2\cdot Q_j$ when $l(S)\leqq\dfrac{1}{100\sqrt{n}}C(\varepsilon,\delta)$.

Case 3. $S \subset \Omega$. This requires no comment.

Case 4. S intersects some $Q_0 \in W$ with $l(S) \leqq \frac{1}{100} l(Q_0)$, and S does not belong to Case 2.

Fix $x_0 \in S$. For each j for which supp (φ_j) intersects S let $q_j(y)$ be the Taylor polynomial of degree m for $\varphi_j(y) \cdot (P_j - P_0)$ at x_0, let $q(y)$ be the Taylor polynomial of degree m for $P_0(y)$ at x_0, and let $P(y) = \sum_j q_j(y) + q(y)$. (Recall that m is the largest integer strictly less than α.)

$$\int_S |\Lambda f(y) - P(y)|\, dy \leqq \sum_j \int_S |\varphi_j(y)(P_j - P_0)(y) - q_j(y)|\, dy$$

$$+ \int_S |P_0(y) - q(y)|\, dy \leqq Cl(S)^{m+n+1} \sum_{|\beta|=m+1} \|D^\beta(\varphi_j(P_j - P_0))\|_{L^\infty(Q_0)}$$

$$+ Cl(S)^{m+n+1} \sum_{|\beta|=m+1} \|D^\beta P_0\|_{L^\infty(Q_0)}.$$

If $|\gamma| + |\tau| = m+1$, then

$$|D^\gamma \varphi_j| \cdot |D^\tau (P_j - P_0)| \leqq C \cdot l(Q_0)^{-|\gamma|} \cdot l(Q_0)^{\alpha - |\tau| - n} \int_{\tilde{\Gamma}_{0,j}} N_\alpha f(x)\, dx$$

on Q_0, by Lemma 2.9. To estimate $\|D^\beta P_0\|_{L^\infty(Q_0)}$, choose a polynomial $p(x)$ of degree at most m such that

$$\int_{Q_0^*} |f(y) - p(y)|\, dy \leqq 2 \cdot l(Q_0^*)^{n+\alpha} \inf_{x \in Q_0} \tilde{N}_\alpha f(x).$$

Then if $|\beta| = m+1$

$$\|D^\beta P_0\|_{L^\infty(Q_0)} = \|D^\beta(P_0 - p)\|_{L^\infty(Q_0)} \leqq Cl(Q_0)^{-m-1} \|P_0 - p\|_{L^\infty(Q_0^*)}$$

$$\leqq Cl(Q_0)^{-m-n-1} \int_{Q_0^*} |f(y) - p(y)|\, dy,$$

since $P_0 - p = \Pi(f-p)$.

Altogether

$$l(S)^{-n-\alpha} \int_S |\Lambda f(y) - P(y)|\, dy \leqq C \left[\frac{l(S)}{l(Q_0)}\right]^{m+1-\alpha} \cdot l(Q_0)^{-n} \int_{\bigcup_j \tilde{\Gamma}_{0,j}} N_\alpha f(x)\, dx$$

$$\leqq C \cdot l(Q_0)^{-n} \int_{C_0 \cdot Q_0^*} N_\alpha f(x)\, dx \leqq C \cdot \mathfrak{M}(x_0),$$

for any $x_0 \in S$. In the second-to-last inequality the bounded overlap property (2.10) of the $\tilde{\Gamma}_{0,1}$'s has been invoked.

Case 5. S is dyadic, $l(S)$ is no larger than the constant $C(\varepsilon, \delta)$ of Lemma 2.8, S meets no cube in $\mathfrak{B}(\Omega^c) \setminus W$, and S meets some $Q \in W$ with $l(S) > \frac{1}{100} \cdot l(Q)$. This, the main case, includes precisely those dyadic cubes not covered by the previous cases. The following argument is adapted from that given by Jones [9] for BMO.

Let R be the integer of Lemma 2.8. Divide the dyadic cube S into dyadic cubes $\{Q_j^{(1)}\}$ of lengths $2^{-R}l(S)$. Let $F_1=\{Q_j^{(1)}: Q_j^{(1)}$ is contained in some $Q\in\mathfrak{B}(\Omega)\cup\mathfrak{B}(\Omega^c)\}$. By Lemma 2.8,

$$\Big|\bigcup_{Q_j^{(1)}\in F_1} Q_j^{(1)}\Big| \geqq c|S| \quad (c>0).$$

Subdivide each $Q_j^{(1)}\notin F_1$ into dyadic cubes $Q_k^{(2)}$ of length $2^{-R}l(Q_j^{(1)})$, and let $F_2=\{Q_k^{(2)}: Q_k^{(2)}$ is contained in some $Q\in\mathfrak{B}(\Omega)\cup\mathfrak{B}(\Omega^c)\}$. Continue this process inductively, constructing $F_k=\{Q_j^{(k)}\}$ for each $k\geqq 1$, such that

$$\text{(3.4)} \qquad \Big|S\setminus \bigcup_{K\leqq N}\ \bigcup_{Q_j^{(k)}\in F_k} Q_j^{(k)}\Big| \leqq (1-c)^N\cdot |S|$$

(3.5) $Q_j^{(k)}$ and $Q_i^{(l)}$ have disjoint interiors unless $(j,k)=(i,l)$.

(3.6) Each $Q_j^{(k)}$ is contained in some $Q\in\mathfrak{B}(\Omega)\cup\mathfrak{B}(\Omega^c)$ with $l(Q)\sim l(Q_j^{(k)})$.

The proof of (3.4) is by induction using Lemma 2.8, and it follows that

$$\Big|S\setminus\bigcup_k \bigcup_{Q_j^k\in F_k} Q_j^{(k)}\Big| = 0.$$

To each $Q=Q_j^{(k)}$ associate polynomials $\bar{P}_Q$ and P_Q as follows: Let $\tilde{Q}$ be the (unique) cube in $\mathfrak{B}(\Omega)\cup\mathfrak{B}(\Omega^c)$ containing Q. By the hypotheses of Case 5, either $\tilde{Q}\in\mathfrak{B}(\Omega)$ or $\tilde{Q}\in W$. If $\tilde{Q}\in\mathfrak{B}(\Omega)$, then $\bar{P}_Q$ is the polynomial associated to f via Π on $\tilde{Q}$. If $\tilde{Q}\in W$, $\bar{P}_Q$ is the polynomial associated to f via Π on $(\tilde{Q})^*$. Define $P_Q(x)$ to be the Taylor polynomial of order m for $\bar{P}_Q$ at the center of $\tilde{Q}$ in the first case or $(\tilde{Q})^*$ in the second, evaluated at x.

Fix some $Q_0\in F_1$, and let $\bar{P}_0=\bar{P}_{Q_0}$, $P_0=P_{Q_0}$.

Lemma 3.2. *If $Q^{(k)}\in F_k$, then*

$$\|P_0-P_{Q^{(k)}}\|_{L^\infty(Q^{(k)})} \leqq Cl(S)^\alpha\cdot \inf_{x\in Q^{(k)}} M(\mathfrak{M})(x).$$

Proof. Let us write P_k for $P_{Q^{(k)}}$. Suppose $k=1$ and $Q^{(1)}\in F_1$. It is necessary to distinguish several cases. $Q_0, Q^{(1)}$ are both contained in cubes $\tilde{Q}_0, \tilde{Q}_1\in\mathfrak{B}(\Omega)\cup\mathfrak{B}(\Omega^c)$ by (3.6). If both $\tilde{Q}_0, \tilde{Q}_1\in\mathfrak{B}(\Omega)$, it follows as in the proof of Lemma 2.9 that

$$\|\bar{P}_0-\bar{P}_1\|_{L^\infty(Q^{(1)})} \leqq C\|\bar{P}_0-\bar{P}_1\|_{L^\infty(\tilde{Q}_1)} \leqq Cl(Q^{(1)})^{\alpha-n}\int_\Gamma N_\alpha f(x)\,dx,$$

where Γ is a Whitney chain of bounded length connecting $\tilde{Q}_0$ to $\tilde{Q}_1$. Hence $\tilde{\Gamma}$ lies inside a fixed dilate of $Q^{(1)}$, so that $l(Q_1)^{-n}\int_\Gamma N_\alpha f(x)dx\leqq CM(\mathfrak{M})(y)$, for any $y\in Q^{(1)}$.

Finally, $\|P_0-P_1\|_{L^\infty(Q^{(1)})}\leqq C\,\|\bar{P}_0-\bar{P}_1\|_{L^\infty(Q^{(1)})}$, since the Taylor expansion is taken at a point lying in a fixed dilate of $Q^{(1)}$; this inequality is scale-invariant.

The second case occurs when $\tilde{Q}_0 \in \mathfrak{B}(\Omega)$ and $\tilde{Q}_1 \in W$; the hypotheses of Case 5 ensure that if either $\tilde{Q}_i \in \mathfrak{B}(\Omega^c)$, then $\tilde{Q}_i \in W$. In this case

$$\|\bar{P}_0 - \bar{P}_1\|_{L^\infty(Q^{(1)})} \leqq C \|\bar{P}_0 - \bar{P}_1\|_{L^\infty(\tilde{Q}_1^*)},$$

and again the proof of Lemma 2.9 applies. The third and fourth cases, when $\tilde{Q}_1 \in \mathfrak{B}(\Omega)$ and $\tilde{Q}_0 \in W$ or both $\tilde{Q}_0, \tilde{Q}_1 \in W$, are handled in the same way. Thus we have

$$\|P_0 - P_1\|_{L^\infty(Q^{(1)})} \leqq Cl(S)^\alpha \cdot \inf_{x \in Q^{(1)}} M(\mathfrak{M})(x).$$

Consider the general case $k > 1$. Given $Q^{(k)} \in F_k$, there is a unique cube $Q_j^{(k-1)}$ containing it. By definition of F_{k-1}, $Q_j^{(k-1)} \notin F_{k-1}$; however, there exists $Q^{(k-1)} \in F_{k-1}$ such that $Q^{(k-1)}$ and $Q_j^{(k-1)}$ were obtained by subdividing the same cube $Q_i^{(k-2)}$. Again $Q_i^{(k-2)} \notin F_{k-2}$, but proceeding as before we select $Q^{(k-2)} \in F_{k-2}$, and proceeding inductively we obtain $\{Q^{(1)}, \ldots, Q^{(k)}\}$, where each $Q^{(i)} \in F_i$. Furthermore there is a constant r such that $Q^{(i)} \subset r \cdot Q^{(i-1)}$ for each i. Since $l(Q^{(i-1)}) = 2^R l(Q^{(i)})$, for a certain larger value of r we have $r \cdot Q^{(i)} \subset r \cdot Q^{(i-1)}$, and hence in particular, $Q^{(k)} \subset r \cdot Q^{(i)}$ for $1 \leqq i < k$.

By the triangle inequality

$$\begin{aligned} \|P_0 - P_k\|_{L^\infty(Q^{(k)})} &\leqq \textstyle\sum_{j=0}^{k-1} \|P_j - P_{j+1}\|_{L^\infty(Q^{(k)})} \\ &\leqq \textstyle\sum_{j=0}^{k-1} \|P_j - P_{j+1}\|_{L^\infty(r \cdot Q^{(j)})} \leqq C \sum_{j=0}^{k-1} \|P_j - P_{j+1}\|_{L^\infty(Q^{(j)})}. \end{aligned}$$

The argument given above for the case $k = 1$ provides a bound for each term:

$$\|P_j - P_{j+1}\|_{L^\infty(Q^{(j)})} \leqq Cl(Q^{(j)})^{\alpha - n} \int_{A \cdot Q^{(j)} \cap \Omega} N_\alpha f(x)\, dx,$$

for some constants C and A independent of j and S. Summing over j yields (with a larger value of A)

$$\|P_0 - P_k\|_{L^\infty(Q^{(k)})} \leqq Cl(S)^\alpha \textstyle\sum_{j=0}^{k-1} 2^{-R\alpha j} [2^{R(k-j)} l(Q^{(k)})]^{-n} \cdot \int N_\alpha f(x)\, dx,$$

where the integral in the j-th term is taken over $\Omega \cap A 2^{R(k-j)} \cdot Q^{(k)}$. This is no larger than

$$Cl(S)^\alpha \textstyle\sum_{j=0}^{k-1} 2^{-R\alpha j} \cdot \inf_{x \in Q^{(k)}} M(\mathfrak{M})(x) \leqq Cl(S)^\alpha \inf_{x \in Q^{(k)}} M(\mathfrak{M})(x).$$

This completes the proof of Lemma 3.2.

We can now use Lemma 3.2 to conclude the proof of Case 5 of the theorem. Since $|S \setminus \bigcup_k \bigcup_{F_k} Q_j^{(k)}| = 0$,

$$(3.7) \quad \int_S |Af(y) - P_0(y)|\, dy \leqq \textstyle\sum_{k,j} \int_{Q_j^{(k)}} (|Af(y) - P_{j,k}(y)| + |P_{j,k}(y) - P_0)|)\, dy,$$

where $P_{j,k}$ is the polynomial of degree $\leqq m$ associated to f on $Q_j^{(k)}$ as defined above. By Lemma 3.2,

$$(3.8)\quad \sum_{k,j}\int_{Q_j^{(k)}}|P_{j,k}(y)-P_0(y)|\,dy \leqq C\sum_{k,j} l(Q_j^{(k)})^n\cdot l(S)^\alpha \inf_{x\in Q_j^{(k)}} M(\mathfrak{M})(x)$$

$$\leqq Cl(S)^\alpha\int_S M(\mathfrak{M})(x)\,dx.$$

To estimate $\int_{Q_j^{(k)}}|\Lambda f(y)-P_{j,k}(y)|\,dy$ for a fixed cube $Q_j^{(k)}$, we proceed as in Case 4. Let $Q=Q_j^{(k)}\in\mathfrak{B}(\Omega)\cup\mathfrak{B}(\Omega^c)$; suppose first that $Q\in\mathfrak{B}(\Omega)$. Temporarily we write $\bar{P}$ and P for $\bar{P}_{j,k}$ and $P_{j,k}$. Choose a polynomial q of degree $\leqq m$ so that

$$\int_Q |f-q|\,dx \leqq 2l(Q)^{n+\alpha}\inf_{x\in Q} N_\alpha f(x).$$

Since the operator Π is a projection, $\bar{P}-q$ is the polynomial associated to $f-q$ on Q via Π. Then by Proposition 2.5,

$$\|(\bar{P}-q)\|_{L^\infty(Q)} \leqq Cl(Q)^{-n}\|\bar{P}-q\|_{L^1(Q)} \leqq Cl(Q)^{-n}\int_Q |f(y)-q(y)|\,dy.$$

$P-q$ is the Taylor expansion of $\bar{P}-q$ to order m at a point lying in a fixed dilate of Q, so this implies

$$\|P-\bar{P}\|_{L^\infty(Q)} = \|(P-q)-(\bar{P}-q)\|_{L^\infty(Q)} \leqq Cl(Q)^{-n}\int_Q |f(y)-q(y)|\,dy$$

$$\leqq Cl(Q)^\alpha \inf_{x\in Q} N_\alpha f(x).$$

Returning to the notation of (3.7), we have

$$(3.9)\quad \int_{Q_j^{(k)}}|P_{j,k}-\bar{P}_{j,k}|\,dx \leqq Cl(Q_j^{(k)})^\alpha\int_{Q_j^{(k)}} N_\alpha f(x)\,dx = Cl(Q_j^{(k)})^\alpha\int_{Q_j^{(k)}}\mathfrak{M}(x)\,dx,$$

in the case $\tilde{Q}_j^{(k)}\in\mathfrak{B}(\Omega)$. If on the other hand $\tilde{Q}_j^{(k)}\in W$, then passing to the reflected cube $(Q_j^{(k)})^*\in\mathfrak{B}(\Omega)$ and applying the same argument yields the same estimate (3.9).

Finally, we have

$$(3.10)\quad \int_{Q_j^{(k)}}|\Lambda f(y)-\bar{P}_{j,k}(y)|\,dy \leqq Cl(Q_j^{(k)})^\alpha\int_{Q_j^{(k)}} M(\mathfrak{M})(x)\,dx.$$

This is proved exactly as in Case 4, using the fact that $\tilde{Q}_j^{(k)}\subset Q_j^{(k)}\in\mathfrak{B}(\Omega)\subset W$, where $l(Q_j^{(k)})\sim l(\tilde{Q}_j^{(k)})$. Combining (3.8), (3.9) and (3.10) demonstrates that the right-hand side of (3.7) is dominated by

$$Cl(S^\alpha)\int_S M(\mathfrak{M})(x)dx+\sum_{k,j} Cl(Q_j^{(k)})^\alpha\int_{Q_j^{(k)}} M(\mathfrak{M})(x)\,dx \leqq Cl(S)^\alpha\int_S M(\mathfrak{M})(x)\,dx.$$

Thus for any cube S in Case 5,

$$\inf_P l(S)^{-n-\alpha}\int_S |\Lambda f(y)-P(x)|\,dx \leqq C\inf_{x\in S} M\big(M(\mathfrak{M})\big)(x).$$

Case 6. S satisfies all hypotheses of Case 5, except that S is not dyadic. Any S is contained in a cube which is a union of 2^n dyadic cubes all of equal sidelengths comparable to $l(S)$. The proof of Case 5 applies equally well to such a union of dyadic cubes. Hence the proof of Theorem 3.2 is complete.

Parallel results hold for the function spaces defined for any open connected Ω by

$$(3.11) \qquad \mathcal{E}_\alpha^p(\Omega) = \{f \in L^1_{\mathrm{loc}}(\Omega)\colon\ N_\alpha f \in L^p(\Omega)\}.$$

(Again only cubes contained in Ω are used to define $N_\alpha f$.) $\mathcal{E}_\alpha^p(\Omega)$ is a Banach space of functions modulo polynomials of degree m, with norm $\|f\|_{\mathcal{E}_\alpha^p(\Omega)} = \|N_\alpha f\|_{L^p(\Omega)}$. In strict analogy with Theorem 1.1 there is

Theorem 3.3. *If Ω is an (ε, ∞) domain, then Ω is an extension domain for $\mathcal{E}_\alpha^p$.*

If Ω is an unbounded (ε, ∞) domain, then $\mathfrak{B}(\Omega)$ contains arbitrarily large cubes. Then the extension operator Λ is defined as in Theorem 1.1, except that we now let W be all of $\mathfrak{B}(\Omega^c)$. The proof of Theorem 3.1 shows that Λ is bounded from $\mathcal{E}_\alpha^p(\Omega)$ to $\mathcal{E}_\alpha^p(\mathbf{R}^n)$. If on the other hand Ω is bounded, let $Q_0 \in \mathfrak{B}(\Omega)$ be of maximal size. Define $W \subset \mathfrak{B}(\Omega^c)$ as in Theorem 1.1. Let $\{\varphi_j\}_{j>0}$ be the partition of unity subordinate to $\{Q_j \in W(\Omega^c)\}$ employed above, let $\varphi_0 = 1 - \sum_j \varphi_j$ on Ω^c, and let P_0 be the polynomial associated to f on Q_0 via Π. Π is constructed as in Proposition 2.5, with $N = m+1$, so that P_0 has degree $\leqq m$. Define, for $f \in \mathcal{E}_\alpha^p(\Omega)$,

$$\Lambda f(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ \sum_{j>0} \varphi_j(x) P_j(x) + \varphi_0(x) P_0(x) & \text{if } x \notin \Omega. \end{cases}$$

Observe that $\Lambda f(x) \equiv P_0(x)$ for x outside a bounded neighborhood of $\bar{\Omega}$. Then repeating the arguments of Theorem 3.1 proves that $\|\Lambda f\|_{\mathcal{E}_\alpha^p(\mathbf{R}^n)} \leqq C \|f\|_{\mathcal{E}_\alpha^p(\Omega)}$.

4. Necessity of the (ε, δ) condition

The sharpness of the (ε, δ) hypothesis is evinced by the existence of a partial converse to Theorem 1.1 in two dimensions, in the "conformally invariant" case $0 < \alpha \leqq 1$ and $p \cdot \alpha = n$. In this case, the norm in $\mathcal{E}_\alpha^p(\mathbf{R}^n)$ is invariant under dilation as well as under translation and rotation, and $\mathcal{E}_\alpha^p(\mathbf{R}^n)$ is preserved by the inversion $x \to \dfrac{x}{|x|^2}$. One of the principal objectives of this section is to establish

Theorem 4.1. *Suppose that $\Omega \subset \mathbf{R}^2$ is finitely connected, and $p \cdot \alpha = 2$. If there exists a bounded linear extension operator $\Lambda\colon \mathcal{E}_\alpha^p(\Omega) \to \mathcal{E}_\alpha^p(\mathbf{R}^n)$, then Ω is an (ε, ∞) domain for some $\varepsilon > 0$.*

A useful tool will be

Lemma 4.2. *Suppose* $f\in C^\infty(\mathbf{R}^2)$. *If* $p\cdot\alpha>1$, $0<\alpha\leqq 1$, $x, y\in\mathbf{R}^2$ *and* γ *is the line segment joining* x *to* y, *then*

$$|f(x)-f(y)|^p \leqq C|x-y|^{p\cdot\alpha-1}\int_\gamma N_\alpha f(t)^p\,dt.$$

Proof. Fix any $\varphi\in C_0^\infty(\mathbf{R}^2)$ with $\int\varphi=1$; and let $\varphi_s(x)=s^{-2}\varphi(s^{-1}x)$. $\frac{d}{ds}(\varphi_s(x))=s^{-1}\psi_s(x)$, where $\psi\in C_0^\infty$. Define $F(x,s)=(f*\varphi_s)(x)$. Suppose that $x=(0,0)$ and $y=(\lambda,0)$.

$$f(x)-f(y) = \big(f(x)-F(x,\lambda)\big)+\big(F(x,\lambda)-F(y,\lambda)\big)+\big(F(y,\lambda)-f(y)\big).$$

Then

$$F(x,\lambda)-f(x) = \int_0^\lambda \frac{d}{ds}F(x,s)\,ds = \int_0^\lambda s^{-1}(\psi_s*f)(0,0)\,ds.$$

Since $\psi\in C_0^\infty$ and $\int\psi(x)dx=0$, $|\psi_s*f(0,0)|\leqq Cs^\alpha\cdot N_\alpha f(s,0)$. Therefore,

$$|F(x,\lambda)-f(x)| \leqq C\int_0^\lambda s^{-1+\alpha}N_\alpha f(s,0)\,ds \leqq C\lambda^{-1+\alpha+1/p'}\cdot\Big(\int_0^\lambda N_\alpha f(s,0)^p\,ds\Big)^{1/p}$$

$$= C\lambda^{\alpha-\frac{1}{p}}\cdot\Big(\int_\gamma N_\alpha f(t)^p\,dt\Big)^{1/p}.$$

Similarly

$$F(y,\lambda)-F(x,\lambda) = \int_0^\lambda \frac{d}{ds}F(s,0,\lambda)\,ds$$

$$= \int_0^\lambda s^{-1}(\psi_\lambda*f)(s,0)\,ds$$

so

$$|F(y,\lambda)-F(x,\lambda)| \leqq C\lambda^\alpha\int_0^\lambda N_\alpha f(s,0)\,ds \leqq C\lambda^{\alpha-\frac{1}{p}}\Big(\int_\gamma N_\alpha f(t)^p\,dt\Big)^{1/p}.$$

The term $|F(y,\lambda)-f(y)|$ is dominated by the same expression, so the proof is complete.

The lemma fails if $p\cdot\alpha\leqq 1$ (and $p>1$), by the Sobolev embedding theorem in $\mathbf{R}^1$. If $1<\alpha\leqq 2$ and $p\cdot(\alpha-1)>1$, it can be generalized by replacing $f(x)-f(y)$ by $P_x(y)-f(y)$, where $P_x(y)$ is the Taylor polynomial of order 1 for f at x, evaluated at y. Similar generalizations hold for $\alpha>2$, with stronger restrictions on p and α.

To prove Theorem 4.1 we first establish a weaker property of extension domains. $B(x,r)$ will denote the open ball in $\mathbf{R}^2$. $S(x,r)$ is its boundary.

Lemma 4.3. *Suppose that* $p\cdot\alpha\geqq 2$, $1<p<\infty$ *and that* $\Omega\subset\mathbf{R}^2$ *is a (connected) extension domain for* $\mathscr{E}_\alpha^p$. *Then there is* $M<\infty$ *such that any* x_0, x_1 Ω *lie in the same component of* $B(x_0, M\cdot d(x_0,x_1))\cap\Omega$. *(*$\Omega$ *need not be assumed to be finitely connected.)*

A still weaker property is that Ω is uniformly locally connected: for any $\varepsilon>0$ there exists $\delta>0$ such that for any $x_0, x_1\in\Omega$ with $d(x_0, x_1)<\delta$, there exists a path γ Ω joining x_0 to x_1 with diameter less than ε. If Ω is bounded, finitely connected and uniformly locally connected, then $\partial\Omega$ is the disjoint union of finitely many Jordan curves and points ([13], p. 171). For our purposes the discrete points may be disregarded.

Proof. Let $d=d(x_0, x_1)$. Suppose that $M\gg 1$ is given, and x_0, x_1 fail to lie in the same component of $\Omega\cap B(x_0, Md)$. Fix $\varphi\in C_0^\infty$ such that $\varphi\equiv 1$ on $\{|x|\leqq 1/2\}$ and $\varphi\equiv 0$ on $\{|x|\geqq 3/4\}$. Let $g(x)$ be $\equiv 0$ except on the component of $\Omega\cap B(x_0, Md)$ which contains x_1, and $g(x)\equiv\varphi(M^{-1}d^{-1}\cdot(x-x_1))$ on that component.

Then $g\in\mathscr{E}_\alpha^p(\Omega)$ and $\|g\|_{\mathscr{E}_\alpha^p(\Omega)}\leqq C$, where C depends on φ but not on M, d, x_0 or x_1. For only cubes $Q\subset\Omega$ on which g is not identically zero contribute to $\tilde{N}_\alpha(g)$. Such cubes intersect the component of $\Omega\cap B(x_0, Md)$ which contains x_0; but since $Q\subset\Omega$, $Q\cap(\Omega\cap B(x_0, M\cdot d))$ is connected. Thus on any cube $Q\subset\Omega$, either $g\equiv 0$ or $g(x)\equiv\varphi(M^{-1}\cdot d^{-1}\cdot(x-x_1))$. So $\tilde{N}_\alpha g(x)\leqq\tilde{N}_\alpha(\varphi(M^{-1}d^{-1}(x-x_1)))$ pointwise in Ω, and therefore

$$\|g\|_{\mathscr{E}_\alpha^p(\Omega)}\leqq\left\|\varphi(M^{-1}d^{-1}(x-x_1))\right\|_{\mathscr{E}_\alpha^p(\mathbf{R}^2)}=C(Md)^{\frac{2}{p}-\alpha}.$$

Suppose that G were an extension of g to a function in $\mathscr{E}_\alpha^p(\mathbf{R}^2)$. If $d<r<\frac{1}{2}Md$, the components of $\Omega\cap B(x_0, Md)$ containing x_0 and x_1 respectively each meet $S(x_0, r)$. Fix $\psi\in C_0^\infty$ with $\int\psi=1$ and let $G_\varepsilon(x)=(G*\psi_\varepsilon)(x)$. For sufficiently small ε, G_ε assumes both the values 0 and 1 on $S(x_0, r)$ for each $d<r\leqq\frac{1}{2}Md$. By Lemma 4.2 (applied to an arc on $S(x_0, r)$), $\int_{S(x_0,r)}(N_\alpha G_\varepsilon)^p(y)dy\geqq C\cdot r^{1-\alpha p}$. Passing to the limit as $\varepsilon\to 0$ yields

$$\|G\|_{\mathscr{E}_\alpha^p}^p\geqq\int_d^{\frac{1}{2}\cdot Md}\int_{S(x_0,r)}(N_\alpha G)^p(y)\,dy\,dr\geqq C\int_d^{\frac{1}{2}\cdot Md}r^{1-\alpha p}\,dr$$

$$\geqq C\cdot\begin{cases}\log M & \text{if } p\cdot\alpha=2\\ d^{2-\alpha p}\cdot(1-\frac{1}{2}M)^{2-\alpha p} & \text{if } p\cdot\alpha>2.\end{cases}$$

Comparing the estimates for $\|G\|_{\mathscr{E}_\alpha^p}$ and $\|g\|_{\mathscr{E}_\alpha^p(\Omega)}$ as $M\to\infty$ concludes the proof.

It is easy to construct examples of extension domains for $\mathfrak{N}_\alpha^p$ or $\mathscr{E}_\alpha^p$ when $p\cdot\alpha<2$, for which the conclusion of Lemma 4.3 need not hold. However, the remainder of the proof of Theorem 4.1 is valid for all $p\cdot\alpha\leqq 2$ (and $p\cdot\alpha>1$). Thus if $1<p\cdot\alpha\leqq 2$ and Ω is a finitely connected extension domain for $\mathscr{E}_\alpha^p$ or $\mathfrak{N}_\alpha^p$ which satisfies the relatively weak conclusion of Lemma 4.3, then Ω is in fact an (ε, δ) domain.

Lemma 4.4. *Suppose that Ω is a simply connected extension domain for $\mathscr{E}_\alpha^p$, $1<p\cdot\alpha\leqq 2$, and Ω satisfies the conclusion of Lemma 4.3. Then $\partial\Omega$ consists of a single*

(possibility unbounded) Jordan curve which satisfies Ahlfors' three-point condition (1.5).

Proof. Since Ω is simply connected and uniformly locally connected, either Ω is the region enclosed by a bounded Jordan curve, or $\partial\Omega$ is a union of unbounded Jordan curves. Although we will simply assume for ease of exposition that $\partial\Omega$ is a single Jordan curve Γ, the same argument shows that this must indeed be the case.

Fix a constant M_0 such that any $x_0, x_1 \in \Omega$ lie in the same component of $\Omega \cap B(x_0, M_0 \cdot d(x_0, x_1))$. Suppose that $M \gg M_0$ and there exist $x_0, x_1 \in \Gamma$ and points y_0, y_1, one on each arc of Γ with endpoints x_0 and x_1, such that $d(x_i, y_j) > M \cdot d(x_0, x_1) = M \cdot d$ for each pair (i, j). Let C_i be the component of $\Omega \cap \overline{B(x_0, 10 \cdot M_0 \cdot d)}$ whose boundary contains y_i. Let $A_i = \bar{C}_i \cap S(x_0, 10 \cdot M_0 \cdot d)$. Finally fix a continuous arc $\gamma\colon [0, 1] \to \bar{\Omega}$ with $\gamma(i) = x_i$, $\gamma(t) \in \Omega$ if $0 < t < 1$, and $\gamma \subset B(x_0, 2 \cdot M_0 \cdot d)$.

Observe that $d(A_0, A_1) \geqq C \cdot d$. For given points $z_i \in A_i$, choose nearby points $z_i' \in C_i$. Any path in Ω joining z_0' to z_1' must cross γ. Since $\gamma \subset B(x_0, 2 \cdot M_0 \cdot d)$ and $d(z_i', x_0) \sim d(z_i, x_0) = 10 \cdot M_0 \cdot d$, any ball centered at z_0' containing such a path must have radius at least $6 \cdot M_0 d$. By the conclusion of Lemma 4.3, $d(z_0', z_1') \geqq M_0^{-1} \cdot 6 \cdot M_0 d = 6 \cdot d$, so that $d(A_0, A_1) \geqq 6d$.

The ensuing argument will rely on the next lemma, whose proof is left to the reader.

Lemma 4.5. *Suppose that $A_0, A_1 \subset S(0, r)$ are closed sets, and that $d(A_0, A_1) \geqq C_0 \cdot r$ $(C_0 > 0)$. Then there exists $\varphi \in C_0^\infty(\mathbf{R}^2)$, supported in $B(0, 10 \cdot r)$ such that $\|\varphi(r^{-1}x)\|_{C^\infty} \leqq C_1$, $\varphi \equiv i$ on A_i, and all derivatives of φ are identically 0 on A_i. C_1 depends on C_0 but not on r or the A_i.*

Next in the above situation, let φ be the function given by Lemma 4.5 with $r = 10 \cdot M_0 d$. Define

$$g(x) = \begin{cases} \varphi(x) & \text{if} \quad x \in \Omega \setminus (C_0 \cup C_1) \\ i & \text{if} \quad x \in C_i. \end{cases}$$

For any $10 \cdot M_0 d < r < Md$, g attains both the values 0 and 1 on $S(x_0, r)$. Hence as in the proof of Lemma 4.3, for any extension G of g to $\mathbf{R}^2$ we have

$$\|G\|_{\mathscr{E}_\alpha^p}^p \geqq C \int_{10 \cdot M_0 d}^{Md} t^{1-\alpha p}\, dt \geqq C \cdot \begin{cases} \log M & \text{if} \quad p \cdot \alpha = 2 \\ d^{2-\alpha p}\left(M^{2-\alpha p} - (10 M_0)^{2-p\alpha}\right) & \text{if} \quad p \cdot \alpha < 2. \end{cases}$$

On the other hand $\|g\|_{\mathscr{E}_\alpha^p(\Omega)}^p \leqq C \cdot (M_0 d)^{2-p\alpha}$. For, by dilating we may assume that $10 M_0 d = 1$. Then since $\|\varphi\|_{\mathscr{E}_\alpha^p(\mathbf{R}^2)} \leqq C \|\varphi\|_{C^\infty(\mathbf{R}^2)} \leqq C$, only cubes $Q \subset \Omega$ which meet A_0 or A_1 can lead to difficulties in the estimation of $\tilde{N}_\alpha(g)$. But the contribution to $\tilde{N}_\alpha(g)$ made by such cubes is controlled by $\|\varphi\|_{C^\infty}$, since all derivatives of

φ vanish identically on the A_i. Letting $M\to\infty$ and comparing norms of g and G again concludes the proof.

Finally, combining Lemmas 4.3 and 4.4 and applying Theorem A, it follows that any simply connected extension domain is an (ε, δ) domain if $p\cdot\alpha=2$, and $0<\alpha\leqq 1$. The same arguments can be applied for finitely connected domains; see [10].

Theorem 1.2, in which $\mathscr{E}_\alpha^p$ is replaced by $\mathfrak{N}_\alpha^p$ and (ε, ∞) by (ε, δ), is also proved in the same fashion. As long as $d(x_0, x_1)$ is sufficiently small, the L^p norms of the functions arising in the proof can be made negligible relative to their $\mathscr{E}_\alpha^p$ norms (by multiplying by an additional cutoff function); this is how the (ε, δ) condition with $\delta<\infty$ arises.

We close by commenting without proofs on some further results. Lemma 4.3 extends to $\mathbf{R}^n$ for all n, when $p\cdot\alpha\geqq n$, with exactly the same proof. Furthermore, if $\Omega\subset\mathbf{R}^n$ is any extension domain for $\mathfrak{N}_\alpha^p$ where $0<\alpha\notin Z$ and $p\cdot\alpha>n$, then there exists $\varepsilon>0$ such that for any $x\in\bar{\Omega}$ and any sufficiently small $r>0$, $B(x, r)\cap\Omega$ contains an open ball of radius $\varepsilon\cdot r$. This is false for $\alpha\in Z^+$. Theorem 1.2 is also valid for $L_2^1(\mathbf{R}^2)$ (which is defined via ordinary weak derivatives, not via the maximal operator N_2).

It is not difficult to construct domains $\Omega\subset\mathbf{R}^2$ which are extension domains for L_1^p either for all $p<2$ or for all $p>2$, but not for $p=2$, and which illustrate that the two halves of the proof of Theorem 4.1 do indeed break down when $p\cdot\alpha<2$ or $p\cdot\alpha>2$, respectively. Let

$$\Omega_\varepsilon = \{(x, y)\in\mathbf{R}^2 : y<0, \quad \text{or} \quad |x|<\varepsilon \quad \text{and} \quad 0\leqq y<1,$$
$$\text{or} \quad \varepsilon\leqq|x|\leqq 1+\varepsilon \quad \text{and} \quad |x|-\varepsilon<y<1\}.$$

If $p>2$, then Ω_ε is an extension domain for L_1^p, with the norm of an extension operator uniformly bounded independent of ε as $\varepsilon\to 0$ (the extension operator is constructed just as in Section 3). However, if $p\leqq 2$ the norm of any extension operator for L_1^p is $\geqq c(\varepsilon)$ where $c(\varepsilon)\to\infty$ as $\varepsilon\to 0$. On the other hand, $(\bar{\Omega}_\varepsilon)^c$ is an extension domain for L_1^p with bound independent of ε as $\varepsilon\to 0$ when $p<2$, but not when $p\geqq 2$. It is possible to build out of the Ω_ε a bounded, simply connected domain $\Omega\subset\mathbf{R}^2$ such that Ω is an extension domain for L_1^p if and only if $p>2$ and $(\bar{\Omega})^c$ is an extension domain for L_1^p if and only if $p<2$. Such a domain has previously been constructed by Maz'ya [12].

This example also suggests that it should be possible to construct a domain $\Omega\subset\mathbf{R}^2$ which is an extension domain for L_1^p for all $p>p_0$, for any given $p_0>2$, or for all $p<p_0$, for any given $p_0<2$. However, the details of this construction have not yet been carried out.

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