A new class of polynomially convex sets

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A compact subset $E \subset \mathbf{C}^N$ is said to be *polynomially convex* if for every point $z \in \mathbf{C}^N \setminus E$, there is a holomorphic polynomial P with $P(z)=1$ and $\|P\|_E = \sup\{|P(w)|: w \in E\} < 1$. In general it is difficult to determine whether a set is polynomially convex, but questions of polynomial convexity come up repeatedly in function-theoretic considerations in $\mathbf{C}^N$. In the present paper we exhibit some new examples of polynomially convex sets in $\mathbf{C}^2$, which are contained in the boundary of the ball $\mathbf{B}_2 = \{z \in \mathbf{C}^2: |z_1|^2 + |z_2|^2 < 1\}$. This work is closely related to work on removable singularities for $\bar{\partial}_b$.

Given a relatively compact domain D in a Stein manifold $\mathcal{M}$, call the compact subset E of bD *removable* provided $bD \setminus E = \Gamma$ is a $\mathscr{C}^1$ submanifold of $\mathcal{M} \setminus E$ such that for every continuous function f on Γ that satisfies there the tangential Cauchy—Riemann equations in the weak sense, there is F holomorphic on D and continuous on $D \cup \Gamma$ such that $F = f$ on Γ. Such removable sets have been studied in several recent papers [8, 12—16, 20]. In particular, Jöricke [8] proved that a compact totally real 2-disc of class $\mathscr{C}^2$ in $b\mathbf{B}_2$ is removable.

The nexus between the theory of polynomial convexity and that of removable singularities is provided by the result that a *compact set $E \subset b\mathbf{B}_2$ is removable if and only if it is polynomially convex* [20, Th. II.10]. An analogous result is valid on strongly pseudoconvex domains: If D is a strongly pseudoconvex domain in a Stein manifold of dimension two, bD smooth, then $E \subset bD$ is removable if and only if it is convex with respect to the algebra $\mathcal{O}(\bar{D})$ of functions holomorphic on $\bar{D}$.

We begin with a general theorem that shows certain sets in strongly pseudoconvex boundaries to be removable and hence to enjoy the convexity property indicated above. It has as an immediate corollary Jöricke's theorem. The proof of Jöricke's theorem thereby obtained is not long, but it does depend in an essential way on some recent work [1] of Bedford and Klingenberg on the hulls of 2-spheres

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in Stein manifolds. We also show that certain discs in $b\mathbf{B}_2$ that are not totally real are, nonetheless, polynomially convex, *viz.*, those with at most a finite number of complex tangents each of which is hyperbolic. Our general theorem also yields the polynomial convexity of certain Cantor sets. The paper concludes with a proof that a hyperbolic point in a two-dimensional submanifold of $\mathbf{C}^2$ has a neighborhood basis of polynomially convex sets.

We are indebted to Professor J. Vrabec for help with the topological parts of this paper.

We shall use the notation that if E is a compact subset of the open set Ω, then $\mathcal{O}(\Omega)$-hull E denotes the set

$$\{z\in\Omega:\ |f(z)| \leqq \sup_E |f| \text{ for all } f\in\mathcal{O}(\Omega)\}.$$

If $E=\mathcal{O}(\Omega)$-hull E, we say that E is $\mathcal{O}(\Omega)$-convex.

I. Theorem. *Let D be a relatively compact, strongly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in the two-dimensional Stein manifold $\mathscr{M}$. Let $E\subset bD$ be a compact set with the following properties:*

A) *There is a Stein neighborhood Ω of E such that E is $\mathcal{O}(\Omega)$-convex.*

B) *If $p\in E$, there exists a neighborhood U of p in bD with $\Omega\supset U$, with $bU\cap E=\emptyset$, and with U homeomorphic to a Euclidean ball in $\mathbf{R}^3$.*

Then the set E is removable and so $\mathcal{O}(\bar{D})$-convex.

The condition A implies that the set E is an intersection of Stein domains, so by a result of Lupacciolu [15], if we were in dimension three or above rather than in dimension two, it by itself would suffice to guarantee the removability of the set E.

Note also that condition A implies that E is small topologically: Its *topological* dimension (as distinguished from its *metric* or *Hausdorff* dimension) is not more than two: As $E\subset bD$, $\dim E\leqq 3$; if $\dim E=3$, then [7] E contains an open subset of bD. Granted that E is $\mathcal{O}(\Omega)$-convex, this is impossible.

Proof of the theorem. We consider a continuous function f on $bD\setminus E$ that satisfies $\bar\partial_b f=0$ in the weak sense; we are to show that there is $F\in\mathcal{O}(D)$ that assumes continuously the boundary values f on $bD\setminus E$.

We show first that f continues holomorphically into D in a neighborhood of each point of the set E. Thus, let $p\in E$, and choose a neighborhood U_p of p in accordance with condition B. As U_p is diffeomorphic to a Euclidean ball in $\mathbf{R}^3$ and as bU_p is disjoint from the set E, it follows that there is a smooth two-dimensional sphere $\Sigma_p''\subset U_p$ that separates the compact set bU_p from the compact set $E\cap U_p$. (Note: We use here the nontrivial fact that a three-dimensional manifold of class $\mathscr{C}^2$ that is homeomorphic to $\mathbf{R}^3$ is ($\mathscr{C}^2$)-diffeomorphic to $\mathbf{R}^3$. See [18] and the references cited there. The analogous statement for $\mathbf{R}^4$ is known to be false.)

There is a sphere Σ'_p in D that is of class $\mathscr{C}^\infty$ and that is very near Σ''_p, so near that every CR-function g on $bD\setminus E$ extends holomorphically into a neighborhood of Σ'_p. According to Bedford and Klingenberg [1], we can perturb Σ'_p slightly to obtain a 2-sphere Σ_p that lies very near Σ'_p and that bounds a Levi-flat 3-ball Σ^+_p, which is the envelop of holomorphy of Σ_p in the sense that if h is holomorphic on a neighborhood of Σ_p (in $\mathscr{M}$), then h continues holomorphically into a neighborhood of Σ^+_p. If we have made our perturbations sufficiently small, then our function f continues holomorphically into a neighborhood of Σ^+_p.

Let Δ_p be the domain in bD that is bounded by Σ''_p and that contains $E\cap U_p$. Let $W_p\subset D$ be a domain obtained as follows: Perturb $\Delta_p\cup\Sigma''_p$ into D slightly leaving it fixed near E so as to obtain a smooth manifold with boundary $\tilde\Delta_p$. If we have made our perturbations of Σ''_p into Σ'_p into Σ_p sufficiently small, we can obtain $\tilde\Delta_p$ that is again strongly pseudoconvex, that is so close to bD that each CR-function h on $bD\setminus E$ extends holomorphically into a neighborhood of $\tilde\Delta_p$, so that $b\tilde\Delta_p=\Sigma_p$ and so that $b\tilde\Delta_p\cap\Sigma^+_p=\Sigma_p$. The subdomain W_p of D bounded by $\tilde\Delta_p\cup\Sigma^+_p$ is pseudoconvex. Consequently, there is a strongly pseudoconvex domain W'_p with these properties: 1) $W'_p\subset W_p$, 2) near $E\cap U_p$, bW_p and bW'_p coincide, and 3) outside a neighborhood of $E\cap U_p$, bW'_p is contained in the domain where f is known to be holomorphic.

Since $W'_p\subset\Omega$ and E is $\mathscr{O}(\Omega)$-convex, the set $E\cap U_p$ is convex with respect to $\mathscr{O}(\overline{W}'_p)$. It follows that f continues holomorphically through W'_p: In case $M=\mathbf{C}^2$, we may invoke a result of Lupacciolu [14]; in the more general case, we refer to the paper [12] of Laurent—Thiebaut.

As a consequence of the construction of the neighborhoods W_p and W'_p and the compactness of the set E, we see that there are finitely many pairwise disjoint relatively open subsets $V_1, \ldots, V_r$, of $\bar D$ with $E\cap bV_j=\emptyset$ and such that f continues into $V_j\cap D$ as a holomorphic function. There is, in addition, a neighborhood V of $bD\setminus E$ in $\bar D\setminus E$ such that the function f continues holomorphically into $V\cap D$. It follows that for some neighborhood V_0 of bD in $\bar D$, f continues holomorphically into $V\cap D$. As D is pseudoconvex, f continues through the whole of D — by a version of Hartogs' theorem. (For this, we can cite [4, Th. VII.D.4].)

This completes the proof of the theorem.

Having established the general result I, we turn to some special cases.

II. Corollary. *Let D be a strongly pseudoconvex domain in a two-dimensional Stein manifold $\mathscr{M}$. Every two-dimensional compact totally real disc Δ of class $\mathscr{C}^1$ in bD is removable and so $\mathscr{O}(\bar D)$-convex.*

In connection with this corollary it is worth noting that not all totally real discs in $\mathbf{C}^2$ are polynomially convex. An example was given in Wermer [21]; another example is given in [5]: Define $g(\zeta)=(1-\zeta\bar\zeta)\bar\zeta e^{i\zeta\bar\zeta}$. Then $\partial g/\partial\bar\zeta$ is zero-free on $\mathbf{C}$, and $g(\zeta)=0$ if $|\zeta|=1$. Thus, the polynomially convex hull of the totally real disc $E=\{(\zeta, g(\zeta)): |\zeta|\leqq 1\}$ contains the disc $\{(\zeta, 0): |\zeta|\leqq 1\}$: E is not polynomially convex.

Proof. According to [6], there is a neighborhood Ω of Δ in M on which there is a nonnegative strongly plurisubharmonic function ϱ of class $\mathscr{C}^2$ with $\Delta=\{\varrho=0\}$. If for small $\varepsilon>0$, $\Omega_\varepsilon=\{z\in\Omega: \varrho(z)<\varepsilon\}$, then for small ε, the component of Ω_ε that contains Δ is a Stein neighborhood of Δ with respect to which Δ is convex.

It remains only to see that Δ has a neighborhood $U\subset\Omega\cap bD$ that is diffeomorphic to a ball in $\mathbf{R}^3$. This is essentially clear. We may suppose given a $\mathscr{C}^1$ diffeomorphism ψ from $\Delta_2=\{(x_1, x_2)\in\mathbf{R}^2: x_1^2+x_2^2<2\}$ onto a closed submanifold of an open subset of bD such that ψ takes the closed disc $\bar\Delta_1\subset\Delta_2$ of radius one diffeomorphically onto Δ. We can then find a $\mathscr{C}^1$-vector field ξ defined on a neighborhood of Δ in bD that is tangent to bD and that is transverse to Δ. If then Ψ: $\Delta_2\times\mathbf{R}$ satisfies $\Psi(x_1, x_2, 0)=\psi(x_1, x_2)$ and $\Psi_*(\frac{\partial}{\partial x_3})=\xi$, then Ψ takes some neighborhood of $\bar\Delta_1\times\{0\}\subset\Delta_2\times\mathbf{R}$ diffeomorphically onto an open neighborhood of Δ in bD. As $\bar\Delta_1\times\{0\}$ has arbitrarily small neighborhoods diffeomorphic to a ball, we are done.

If Σ is a two-dimensional submanifold of class $\mathscr{C}^2$ in a two-dimensional complex manifold, $\mathscr{M}$, and if Σ is totally real in a deleted neighborhood of the point $p_0\in\Sigma$ but the tangent space $T_{p_0}\Sigma$ is complex, then it is possible to choose local holomorphic coordinates (z, w) in $\mathscr{M}$ near p_0 so that p_0 is the origin and so that near 0, Σ is given by an equation

$$w = z\bar z+\lambda(z^2+\bar z^2)+o(z^2).$$

The number λ is uniquely determined, and p_0 is said to be an *elliptic point if* $\lambda\in[0, \frac{1}{2})$, a *parabolic point* if $\lambda=\frac{1}{2}$, and a *hyperbolic point* if $\lambda\in(\frac{1}{2}, \infty)$. This classification was introduced by Bishop [3].

Near an elliptic point, the local hull of holomorphy of Σ contains a real hypersurface with boundary [11]. We shall see below that near a hyperbolic point, Σ is polynomially convex.

III. Corollary. *Let Δ be a compact two-dimensional disc of class $\mathscr{C}^2$ in bD, D a strongly pseudoconvex domain with $\mathscr{C}^2$ boundary in a two-dimensional Stein manifold $\mathscr{M}$. If Δ is totally real except at a finite number of points each of which is a hyperbolic point, then Δ is removable and so $\mathscr{O}(\bar D)$-convex.*

The crux of the corollary, given what has gone before, is to see that Δ is the zero locus of a continuous nonnegative plurisubharmonic function.

IV. Lemma. *With Δ as in Corollary* III, *there is a neighborhood Ω of Δ in $\mathscr{M}$ on which is defined a nonnegative plurisubharmonic function ϱ such that $\Delta=\{z\in\Omega: \varrho(z)=0\}$.*

Proof. Fix a smooth Hermitian metric on $\mathcal{M}$, and denote by $\operatorname{dist}_{\mathcal{M}}(p, q)$ the distance from p to q, $p, q \in \mathcal{M}$, in this metric.

Let $p_1, \ldots, p_n$ be the points in Δ at which the tangent to Δ is complex, and let $\Delta' = \Delta \setminus \{p_1, \ldots, p_n\}$, a totally real closed submanifold with boundary of $\mathcal{M} \setminus \{p_1, \ldots, p_n\}$. The function $\varrho_1(z) = \operatorname{dist}^2_{\mathcal{M}}(z, \Delta)$ is of class $\mathscr{C}^2$ in a small neighborhood Ω_1 of Δ in $\mathcal{M}$ and is strongly plurisubharmonic on a smaller open set $\Omega_2 \subset \Omega_1$ containing Δ'. It is not plurisubharmonic near the points p_j, so we will modify it there. The modification will be local near each p_j. Thus, we may fix a p_j and work in local coordinates in which $p_j = 0$.

According to Theorem VI below the set $\Delta \cap r\bar{\mathbf{B}}_2$ is polynomially convex for each sufficiently small $r > 0$. Hence, there is a smooth plurisubharmonic function $\varphi\colon \mathbf{C}^2 \to [0, \infty)$ the zero locus of which is the set $\Delta \cap r\bar{\mathbf{B}}_2$. If $r > 0$ is sufficiently small, $\Delta \cap r\bar{\mathbf{B}}_2$ is totally real except at 0.

Choose numbers $0 < r'' < r' < r$, and let $\tau \leqq 0$ be a smooth function satisfying

(i) $\operatorname{supp} \tau \subset\subset r'\mathbf{B}_2$, and

(ii) $\tau < 0$ near $\Delta \cap r''\bar{\mathbf{B}}_2$.

If τ is sufficiently small in the $\mathscr{C}^2$-norm, then the function $\varrho_2 = \varrho_1 + \tau$ is still strongly plurisubharmonic in a neighborhood Ω_3 of $\Delta \cap (r\bar{\mathbf{B}}_2 \setminus r''\mathbf{B}_2)$. The nonnegative function $\varrho_3 = \max(\varrho_2, 0)$ is continuous and plurisubharmonic on Ω_3 and vanishes on a neighborhood of $\Delta \cap b(r''\mathbf{B}_2)$. Thus, we can extend ϱ_3 as a plurisubharmonic function on a neighborhood Ω_4 of $\Delta \cap r\bar{\mathbf{B}}_2$ by taking the extension to be zero in a neighborhood of $\Delta \cap r''\bar{\mathbf{B}}_2$.

Recall that φ is a nonnegative plurisubharmonic function on $\mathbf{C}^2$ vanishing on $\Delta \cap r\bar{\mathbf{B}}_2$. The function $\varrho_4 = \varrho_3 + \varphi$ is a nonnegative continuous plurisubharmonic function on Ω_4 that has $\Delta \cap r\bar{\mathbf{B}}_2$ as zero locus. Moreover, on $\Omega_4 \cap (r\bar{\mathbf{B}}_2 \setminus r'\mathbf{B}_2)$, ϱ_3 agrees with ϱ_1, so on this set, $\varrho_4 = \varrho_1 + \varphi$. This function is smooth and plurisubharmonic on $\Omega_4 \cap (r\bar{\mathbf{B}}_2 \setminus r'\mathbf{B}_2)$.

Finally, we patch ϱ_1 and ϱ_4 to obtain

$$\varrho = h\varrho_1 + (1-h)\varrho_4$$

where $h\colon \mathbf{C}^2 \to [0, 1]$ is a smooth function that equals 0 on $r'\bar{\mathbf{B}}_2$ and equals 1 outside $r\mathbf{B}_2$. Since both ϱ_1 and ϱ_4 are smooth and strongly plurisubharmonic along $\Delta \cap (r\mathbf{B}_2 \setminus r'\mathbf{B}_2)$ where the patching occurs, the resulting function ϱ is also strongly plurisubharmonic there. The function ϱ is the required modification of ϱ_1 near p_j.

We repeat this procedure for each p_j and obtain finally the plurisubharmonic function we seek.

This completes the proof.

There is an extension of the results obtained above for discs to certain more general sets. A subset X of an n-dimensional manifold is called *cellular* if for every open neighborhood U of X there is an open set V homeomorphic to $\mathbf{R}^n$ with $X\subset V\subset U$. The proof of Theorem I implies that *if D is a strongly pseudoconvex domain in a two-dimensional Stein manifold, bD of class $\mathscr{C}^2$, then every compact set $X\subset bD$ that is cellular and that is $\mathcal{O}(\Omega)$-convex for some neighborhood Ω of X is removable.*

The final application we make of Theorem I is to certain Cantor sets. By a *Cantor set* we understand a compact perfect subset of some Euclidean space. Alternatively, a Cantor set is a subset of a Euclidean space that is homeomorphic to the usual Cantor middle-third set constructed in the interval [0, 1]. For these matters one can consult the topology text [10].

The following question has not been settled yet: Is every Cantor set in $b\mathbf{B}_N$ removable? We shall exhibit below a class of removable Cantor sets.

A Cantor set $E\subset\mathbf{R}^n$ is called *tame* if there is a homeomorphism of $\mathbf{R}^n$ onto itself that carries E onto the usual middle-third Cantor set contained in a coordinate line of $\mathbf{R}^n$. In [2], Bing gave a condition under which a Cantor set E in $\mathbf{R}^3$ is tame. To state it, we need a definition: Call a set $X\subset\mathbf{R}^3$ 1-*LCC* (1-locally connected complement) if for each $x\in X$ and for every neighborhood U of x, there is a neighborhood V of x, $V\subset U$ such that every simple closed curve $\gamma\subset V\setminus X$ is null-homotopic in $U\setminus X$. Bing proves that a 1-LCC Cantor set in $\mathbf{R}^3$ is tame. (Not every Cantor set has this property as is shown by Antoine's necklace [17].) Alternatively, it suffices for each point $x\in X$ to have a neighborhood V that is homeomorphic to $\mathbf{R}^3$ and that has the property that $bV\cap X=\varphi$.

V. Corollary. *Let D be a strongly pseudoconvex domain in a two-dimensional Stein manifold $\mathcal{M}$, bD of class $\mathscr{C}^2$. Suppose $E\subset bD$ is a Cantor set such that*

A) *There is a Stein neighborhood Ω of E in $\mathcal{M}$ such that E is $\mathcal{O}(\Omega)$-convex.*

B) *If $p\in E$, there is a neighborhood V_p of p in bD that is homeomorphic to $\mathbf{R}^3$ and that contains a compact neighborhood E_p of p in E that is a tamely embedded Cantor set in $\mathbf{R}^3=V_p$.*

Then E is removable.

Proof. Let $p_0\in E$. The hypothesis B) yields a homeomorphism $\Phi\colon V_{p_0}\to\mathbf{R}^3$ that carries E_{p_0} onto the standard Cantor middle-third set K contained in the x_1-axis of $\mathbf{R}^3$. Let p_0 correspond to $x_0\in K$ under Φ. There are open Euclidean balls B in $\mathbf{R}^3$ that contain x_0 and that satisfy $bB\cap\Phi(E\cap V_{p_0})=\emptyset$. It follows that in $\Phi^{-1}(B)$ there are 2-spheres of class $\mathscr{C}^2$ disjoint from E that bound 3-balls containing p_0.

We turn finally to the proof of the result we have used above to the effect that hyperbolic points have polynomially convex neighborhoods.

VI. Theorem. *Let Σ be the surface in $\mathbf{C}^2$ given by the equation*

$$z_2 = z_1\bar{z}_1+\gamma(z_1^2+\bar{z}_1^2)+F(z_1)$$

with $\gamma > \frac{1}{2}$ *and where F is of class* $\mathscr{C}^2$ *and satifies* $F(z_1) = o(z_1^2)$, $z_1 \to 0$. *If* $r > 0$ *is sufficiently small then* $\Sigma \cap r\bar{\mathbf{B}}_2$ *is polynomially convex and satisfies* $P(\Sigma \cap r\bar{\mathbf{B}}_2) = \mathscr{C}(\Sigma \cap r\bar{\mathbf{B}}_2)$.

Here for a compact set X in $\mathbf{C}^2$, $\mathscr{C}(X)$ denotes the space of continuous $\mathbf{C}$-valued functions on X, and $P(X)$ is the subspace consisting of those functions that can be approximated uniformly by polynomials.

We are indebted to Sidney Webster for drawing this problem to our attention.

To prove the result, we introduce the surface Σ^0 with equation

$$z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$$

and the totally real two-dimensional planes

$$V_1 = \{(\zeta, \bar{\zeta}) \in \mathbf{C}^2 \colon \zeta \in \mathbf{C}\}$$

and

$$V_2 = \left\{\left(\zeta, -\frac{1}{\gamma}\zeta - \bar{\zeta}\right) \colon \zeta \in \mathbf{C}\right\}.$$

In addition, we define a map $\Phi \colon \mathbf{C}^2 \to \mathbf{C}^2$ by

$$\Phi(z_1, z_2) = \big(z_1, z_1 z_2 + \gamma(z_1^2 + z_2^2)\big).$$

As Dan Burns pointed out to us in a slightly different context, Φ is a proper map from $\mathbf{C}^2$ onto itself of multiplicity two, which carries V_1 and V_2 injectively onto Σ^0.

Proof. We propose to find surfaces S_1 and S_2 that osculate V_1 and V_2, respectively, at the origin, and that satisfy

$$\Phi^{-1}(\Sigma) = S_1 \cup S_2.$$

To this end, it is convenient to introduce $\varphi \colon \mathbf{C}^2 \to \mathbf{C}$ by

$$\varphi(z_1, z_2) = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + F(z_1) - z_2$$

so that $\varphi = 0$ defines Σ.

We construct S_1 as follows. The surface S_1 is to be of the form

$$S_1 = \{(\zeta, \bar{\zeta} + f(\zeta)) \colon \zeta \in \mathbf{C},\ \zeta \text{ small}\}$$

with $f(\zeta) = o(\zeta)$, $\zeta \to 0$. The condition that $\Phi(S_1) \subset \Sigma$ is that $\varphi\big(\Phi(\zeta, \bar{\zeta} + f(\zeta))\big) = 0$, which, when written out explicitly, is the quadratic in f

$$\gamma f(\zeta)^2 + (2\gamma\bar{\zeta} + \zeta) f(\zeta) - F(\zeta) = 0.$$

Solve this for f using the quadratic formula to find

$$f = \frac{1}{2\gamma}\left\{-(\zeta + 2\gamma\bar{\zeta}) \pm \sqrt{(\zeta + 2\gamma\bar{\zeta})^2 + 4\gamma F}\right\}. \tag{1}$$

We want $f(\zeta)=o(\zeta)$, $\zeta\to 0$, so we choose the plus sign in (1). With this choice of sign, $df(0)=0$, and f is of class $\mathscr{C}^1$ everywhere except possibly at zero. We show that f is of class $\mathscr{C}^1$ at 0 as follows. Write

$$\sqrt{(\zeta+2\gamma\bar\zeta)^2+4\gamma F(\zeta)} = (\zeta+2\gamma\bar\zeta)+\tilde f(\zeta).$$

The function $\tilde f$ satisfies the equation

$$\tilde f^2+2(\zeta+2\gamma\bar\zeta)\tilde f = 4\gamma F.$$

Differentiate this with respect to ζ to get

$$(2)\qquad (\zeta+2\gamma\bar\zeta+\tilde f)\tilde f_\zeta = 2\gamma F_\zeta-\tilde f.$$

We have $F(\zeta)=o(\zeta^2)$ and $\tilde f(\zeta)=o(\zeta)$, $\zeta\to 0$. Consequently, by (2) we see that $\tilde f_\zeta(\zeta)=o(1)$, $\zeta\to 0$. Similarly, $f_{\bar\zeta}(\zeta)=o(1)$, $\zeta\to 0$. It follows that f is of class $\mathscr{C}^1$, as we wished to see.

Thus, with f determined in this way, the surface S_1 is of class $\mathscr{C}^1$, and near the origin, it is a small perturbation of the totally real plane V_1. As such, it is totally real. We know that totally real surfaces are locally polynomially convex and that we have, locally at least, polynomial approximation on such surfaces. Thus, for small $\delta>0$, if

$$S_1(\delta) = \{(\zeta, \bar\zeta+f(\zeta))\colon |\zeta|<\delta\}$$

with f given by (1), then $S_1(\delta)$ is polynomially convex and satisfies $P(S_1(\delta))=\mathscr{C}(S_1(\delta))$.

We now make a similar analysis for the surface S_2. It is to be of the form

$$S_2 = \left\{\left(\zeta, -\frac{1}{\gamma}\zeta-\bar\zeta+g(\zeta)\right)\colon \zeta\in\mathbf{C}\right\}$$

with $g(\zeta)=o(\zeta)$, $\zeta\to 0$. The condition that $\Phi(S_2)\subset\Sigma$ is found to be expressed by the equation

$$\gamma g^2(\zeta)-(2\gamma\bar\zeta+\zeta)g(\zeta)-F(\zeta) = 0.$$

The quadratic formula yields

$$g = \frac{1}{2\gamma}\{(2\gamma\bar\zeta+\zeta)\pm\sqrt{(2\gamma\bar\zeta+\zeta)^2+4\gamma F}\}.$$

We take the minus sign to obtain $g(\zeta)=o(\zeta)$, $\zeta\to 0$. The function g is continuous, is of class $\mathscr{C}^1$ except possibly at the origin and satisfies $dg(0)=0$. We claim that g is of class $\mathscr{C}^1$ at 0 as well. Write

$$\sqrt{(2\gamma\bar\zeta+\zeta)^2+4\gamma F} = (2\gamma\bar\zeta+\zeta)+\tilde g.$$

Then the function $\tilde{g}$ satisfies the quadratic equation

$$\tilde{g}^2+2(2\gamma\bar{\zeta}+\zeta)\tilde{g} = 4\gamma F.$$

We differentiate this with respect to ζ and with respect to $\bar{\zeta}$ and find that $\tilde{g}_\zeta(\zeta)$ and $\tilde{g}_{\bar{\zeta}}(\zeta)$ are both $o(1)$, $\zeta\to 0$, whence $\tilde{g}$ is of class $\mathscr{C}^1$ at 0. Thus, g is of class $\mathscr{C}^1$ as we wished.

If for small $\delta>0$,

$$S_2(\delta) = \left\{\left(\zeta, -\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta)\right): |\zeta| \leqq \delta\right\},$$

then $S_2(\delta)$ is polynomially convex and satisfies $P(S_2(\delta))=\mathscr{C}(S_2(\delta))$.

Notice that if δ is small, then $S_1(\delta)\cap S_2(\delta)=\{0\}$ as follows from the assumption that $\gamma>1/2$.

Now fix $\varepsilon>0$, define $\psi\colon \mathbf{C}^2\to\mathbf{C}$ by

$$\psi(z_1, z_2) = \frac{1}{4}(z_1^2-z_2^2)+\varepsilon z_1 z_2,$$

and consider the sets $\psi(S_1(\delta))$ and $\psi(S_2(\delta))$. For small ζ, we have, taking $\zeta=\xi+i\eta$,

$$\psi(\zeta, \bar{\zeta}+f(\zeta)) = i\xi\eta+\varepsilon(\xi^2+\eta^2)+o(\zeta^2), \tag{3}$$

for $f(\zeta)=o(\zeta)$, $\zeta\to 0$. The equation (3) implies that for small δ, $\psi(S_1(\delta))$ is contained in the cone

$$\{u+iv\colon |v| \leqq Cu\}$$

in the $u+iv=\psi(z_1, z_2)$-plane for some $C>0$. We also have

$$\psi\left(\zeta, -\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta)\right) = \frac{1}{4}\left(\zeta^2-\left(-\left(\frac{1}{\gamma}\zeta+\bar{\zeta}\right)+g(\zeta)\right)^2\right)+\varepsilon\zeta\left(-\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta)\right).$$

As g is $o(\zeta)$, $\zeta\to 0$, we can write this as

$$\psi\left(\zeta, -\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta)\right) = \frac{1}{4}\left(\zeta^2\left(\frac{1}{\gamma}\zeta+\bar{\zeta}\right)^2\right)-\varepsilon\zeta\left(\frac{1}{\gamma}\zeta+\bar{\zeta}\right)+o(\zeta^2)$$

$$= \frac{1}{4}\left[-\left(\frac{1}{\gamma^2}+\frac{2}{\gamma}\right)\xi^2-\left(\frac{2}{\gamma}-\frac{1}{\gamma^2}\right)\eta^2+\frac{2i}{\gamma^2}\xi\eta\right]-\varepsilon\zeta\left(\frac{1}{\gamma}\zeta+\bar{\zeta}\right)+o(\zeta^2).$$

The coefficient of ξ^2 is plainly negative, and so is the coefficient of η^2, because $\gamma>1/2$. Thus, provided ε and δ are sufficiently small, we have that $\operatorname{Re}\psi\,(\zeta-\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta))\leqq 0$ and that

$$\left|\operatorname{Im}\psi\left(\zeta-\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta)\right)\right| \leqq -c\operatorname{Re}\psi\left(\zeta-\frac{1}{\gamma}\zeta-\bar{\zeta}+g(\zeta)\right).$$

That is, the set $\psi(S_2(\delta))$ is contained in a cone with vertex the origin and lying otherwise in the open left half of the $(u+iv)$-plane.

Notice that $\psi^{-1}(0)\cap S_1(\delta)\cap S_2(\delta)$ is the polynomially convex set consisting of the origin in $\mathbf{C}^2$.

Thus, a theorem of Kallin [9; 19, Lemma 29.21] implies that, provided δ is small, the union $S_1(\delta)\cup S_2(\delta)$ is polynomially convex.

Moreover, as $P(S_1(\delta))=\mathscr{C}(S_1(\delta))$ and $P(S_2(\delta))=\mathscr{C}(S_2(\delta))$, again provided δ is small, it follows that

$$P(S_1(\delta)\cup S_2(\delta)) = \mathscr{C}(S_1(\delta)\cup S_2(\delta)). \tag{4}$$

It not, there is a nonzero measure μ on $S_1(\delta)\cup S_2(\delta)$ such that $\mu(P)=0$ for every holomorphic polynomial P. Since ψ takes $S_1(\delta)$ and $S_2(\delta)$ into disjoint sectors, Mergelyan's theorem provides a sequence $\{P_j\}_{j=1,2,\dots}$ of polynomials such that $|P_j\circ\psi|\leqq 1$ on $S_1(\delta)\cup S_2(\delta)$ and $\{P_j\circ\psi\}_{j=1,2,\dots}$ converges pointwise to one on $S_1(\delta)$ and to zero on $S_2(\delta)\setminus\{0\}$. If P is any polynomial, we have

$$0 = \int (P_j\circ\psi)P\,d\mu$$

for all j. By the dominated convergence theorem we may conclude that μ_1, the restriction of μ to $S_1(\delta)$, satisfies $\mu_1(P)=0$ for all polynomials P. As $P(S_1(\delta))=\mathscr{C}(S_1(\delta))$ it follows that μ_1 is the zero measure. In the same way, μ_2, the restriction of μ to $S_2(\delta)$, is the zero measure, and thus $\mu=0$.

The equality (4) follows. It implies that every compact subset of $S_1(\delta)\cup S_2(\delta)$ is polynomially convex.

If now $E\subset\Sigma$ is a compact neighborhood in Σ of the origin chosen so small that $\Phi^{-1}(E)\subset S_1(\delta)\cup S_2(\delta)$, the set $\Phi^{-1}(E)$ is polynomially convex. But as $\Phi\colon \mathbf{C}^2\to\mathbf{C}^2$ is a proper holomorphic mapping, the polynomial convexity of $\Phi^{-1}(E)$ implies that of E. Let $z_0\in\mathbf{C}^2\setminus E$. As $\Phi^{-1}(E)$ is polynomially convex and disjoint from the set $\Phi^{-1}(z_0)=E_0$, which consists of one point or of two points since Φ is a two-sheeted branched covering, the set $E_0\cup\Phi^{-1}(E)$ is polynomially convex, so there is a polynomial P on $\mathbf{C}^2$ with $P=1$ on E_0 and $|P|<1/2$ on $\Phi^{-1}(E)$. The theory of analytic covers [4] yields a polynomial equation

$$P^2+(p\circ\Phi)P+q\circ\Phi = 0$$

for some choice of entire functions, actually polynomials, p and q on $\mathbf{C}^2$. We have then that q is holomorphic on $\mathbf{C}^2$, $q(z_0)=1$ and $|q|<1/4$ on E. Thus, E is polynomially convex.

Finally, we must show that $P(E)=\mathscr{C}(E)$. Granted that E is polynomially convex, this follows from a result of Wermer's [21]. As Wermer's proof is not simple — of course it covers situations much more general than ours, we offer the following short proof. Consider $h\in\mathscr{C}(E)$. The function $h\circ\Phi$ is in $\mathscr{C}(\Phi^{-1}(E))$, so there is a sequence $\{P_j\}_{j=1,2,\dots}$ of polynomials on $\mathbf{C}^2$ that converges uniformly on $\Phi^{-1}(E)$ to $h\circ\Phi$, since $\mathscr{C}(S_1(\delta)\cup S_2(\delta))=P(S_1(\delta)\cup S_2(\delta))$ and $\Phi^{-1}(E)\subset S_1(\delta)\cup S_2(\delta)$. The function $Q_j(z)=\frac{1}{2}(P_j(z')+P_j(z''))$ for all $z\in\mathbf{C}^2$ such that $\Phi^{-1}(z)$ consists of exactly two points z' and z'' is holomorphic on $\mathbf{C}^2$ — again we invoke the theory of analytic covers. The sequence $\{Q_j\}_{j=1,2,\dots}$ converges uniformly on E to h.

This completes the proof.

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