Convexity of means and growth of certain subharmonic functions in an n-dimensional cone

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1. Preliminaries

This paper extends some results by Norstad [9] on subharmonic functions in the complex plane, cut along a half-ray, to an n-dimensional cone.

Cartesian coordinates of a point x of R^n, $n \geqq 3$, are denoted $(x_1, \ldots, x_n)$. We introduce spherical coordinates for x by

$$|x| = r, \quad x_1 = r\cos\theta_1, \quad x_i = r\cos\theta_i \prod_{j=1}^{i-1} \sin\theta_j \quad \text{for} \quad i = 2, \ldots, n-1$$

and

$$x_n = r \prod_{j=1}^{n-1} \sin\theta_j.$$

Here $0 \leqq \theta_i \leqq \pi$ for $i = 1, \ldots, n-2$ and $0 \leqq \theta_{n-1} \leqq 2\pi$. When integrating, we shall also use the parameter ω, defined by $x = r\omega$. Then $d\omega = \sqrt{g}\, d\theta_1 \ldots d\theta_{n-1}$ with $\sqrt{g} = \prod_{j=1}^{n-1} (\sin\theta_j)^{n-j-1}$.

Let $\Omega = \Omega(\psi_0)$ be the cone $\{x;\, 0 \leqq \theta_1 < \psi_0\}$, where ψ_0 is given, $0 < \psi_0 < \pi$. If v is a function, defined in Ω, we shall let $v(r, 0)$ denote the value of v at the point $x = (r, 0, \ldots, 0)$. Also, if v is independent of $\theta_2, \ldots, \theta_{n-1}$, we shall write $v(r, \theta_1)$ for the value of v at any point whose first two spherical coordinates are r, θ_1.

In spherical coordinates the Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\delta, \tag{1.1}$$

where the Beltrami operator δ is given by

$$\delta = \frac{1}{\sqrt{g}} \sum_{j=1}^{n-1} \frac{\partial}{\partial \theta_j}\left(\frac{\sqrt{g}}{g_j}\frac{\partial}{\partial \theta_j}\right).$$

Here $g_1=1$ and $g_j=\prod_{i=1}^{j-1}(\sin\theta_i)^2$ for $j=2,\ldots,n-1$, so $g=\prod_{j=1}^{n-1}g_j$. If the function F only depends on θ_1,

$$\delta F = F''(\theta_1)+(n-2)\cot\theta_1 F'(\theta_1). \tag{1.2}$$

For two C^2 functions u and v we also let

$$(\nabla u, \nabla v) = \sum_{j=1}^{n-1}\frac{1}{g_j}\frac{\partial u}{\partial\theta_j}\frac{\partial v}{\partial\theta_j}.$$

Let u be subharmonic in Ω. We are going to study the means $L_\alpha(r), \alpha\geqq 1$, and $J(r)$, defined by

$$L_\alpha(r,u) = \left(\int_S\left(\frac{u(r\omega)}{f_\lambda(\theta_1)}\right)^\alpha f_\lambda(\theta_1)g_\lambda(\theta_1)\,d\omega\right)^{1/\alpha},$$

where S is the part of the unit sphere $|\omega|=1$ where $0\leqq\theta_1<\psi_0$, and

$$J(r,u) = \sup_S\frac{u(r\omega)}{f_\lambda(\theta_1)}.$$

Here f_λ and g_λ are certain eigenfunctions of the Beltrami operator. Some of their properties are listed in the next section. When $1<\alpha<\infty$, u is required to be non-negative.

We shall also examine the relation between $M(r)=\sup_S u(r\omega)$, $J(r)$ and $L(r)=L_1(r)$.

2. The functions $\mathbf{f}_\lambda$ and $\mathbf{g}_\lambda$

We first consider the case $n\geqq 3$. If k is a given number, $k>0$, we denote by $F_k=F_k(\theta)$ the unique solution of the problem

$$\delta F+k(k+n-2)F = 0 \quad\text{for}\quad 0\leqq\theta<\pi, \tag{2.1}$$

$F_k(0)=1$ and $F_k'(0)=0$. It is known that F_k depends continuously on k and has a first zero $\psi(k)$ in $(0,\pi)$. As a function of $k\in(0,\infty)$ $\psi(k)$ is strictly decreasing with range $(0,\pi)$. Let $k(\psi)$ denote its inverse. Now fix $k=k(\psi_0)$. Then

$$v(x) = v(r,\theta_1) = r^kF_k(\theta_1) \tag{2.2}$$

is harmonic in Ω and exhibits the Phragmén—Lindelöf growth for subharmonic functions in Ω, vanishing at $\partial\Omega$. When $\psi_0=\pi/2$ so that Ω is a half-space, $k=1$ for all n.

With a given λ, $0<\lambda<1$, let $f_\lambda(\theta)=F_{k\lambda}(\theta)F_{k\lambda}(\psi_0)^{-1}$. ($F_{k\lambda}(\psi_0)>0$ since $\psi_0=\psi(k)<\psi(k\lambda)$.) Hence $f_\lambda(\psi_0)=1$ and f_λ solves

$$\delta F+k\lambda(k\lambda+n-2)F = 0. \tag{2.3}$$

It follows from the minimum principle that f_λ is strictly decreasing for $0\leqq\theta\leqq\psi(k\lambda)$. Let $w(r,\theta_1)=r^{k\lambda}f_\lambda(\theta_1)$. Then w is harmonic in Ω, $w(x)=|x|^{k\lambda}$ at $\partial\Omega$ and on $|x|=1$,

$$1\leqq w(x)\leqq f_\lambda(0)=C(\lambda)^{-1}, \tag{2.4}$$

by which $C(\lambda)$ is defined.

Since the indicial equation at $\theta=0$ of (2.1) is $\mu(\mu+n-3)=0$, (2.3) also has solutions g_λ, unbounded at $\theta=0$ and such that $(\sin\theta)^{n-2}g_\lambda(\theta)\to 0$ as $\theta\to 0$. We may choose g_λ such that $g_\lambda(\theta)\to+\infty$ when $\theta\to 0$ and $g_\lambda(\psi_0)=0$. An application of Sturm's comparison theorem shows that g_λ has no zeros in $(0,\psi_0)$. The minimum principle then gives that g_λ is strictily decreasing for $0\leqq\theta\leqq\psi_0$, so $g'_\lambda(\theta)\leqq 0$ for these values of θ. Actually, $g'_\lambda(\psi_0)\neq 0$, since otherwise g_λ would be identically zero. Thus we may prescribe $g'_\lambda(\psi_0)=-1$. These conditions determine g_λ uniquely.

We shall also need

$$f'_\lambda(\theta)g_\lambda(\theta)-f_\lambda(\theta)g'_\lambda(\theta)=(\sin\theta)^{2-n}(\sin\psi_0)^{n-2}. \tag{2.5}$$

To see this, let h be the left member of (2.5). Then, by (1.2), $h'=f''_\lambda g_\lambda-f_\lambda g''_\lambda=-(n-2)\cot\theta\ h$, which gives $h(\theta)=C(\sin\theta)^{2-n}$. Since $h(\psi_0)=1$, we get (2.5).

Above we assumed $n\geqq 3$. When $n=2$ and $k=1$, $\cos\lambda\theta$ and $\sin\lambda(\pi/2-\theta)$ are two linearly independent solutions of (2.3).

When n is even, it is possible to obtain explicit expressions for f_λ. For example, for $n=4$, we have $\psi_0=\pi/(k+1)$,

$$f_\lambda(\theta)=\frac{\sin\dfrac{\pi}{k+1}}{\sin\pi\dfrac{k\lambda+1}{k+1}}\,\frac{\sin(k\lambda+1)\theta}{\sin\theta}.$$

Also,

$$g_\lambda(\theta)=\frac{\sin\dfrac{\pi}{k+1}}{k\lambda+1}\,\frac{\sin(k\lambda+1)(\psi_0-\theta)}{\sin\theta}.$$

Especially

$$C(\lambda)=\frac{\sin\pi\dfrac{k\lambda+1}{k+1}}{(k\lambda+1)\sin\dfrac{\pi}{k+1}}.$$

A recurrence formula, from which f_λ can be evaluated by means of residues, is given in Hayman [7, p. 160].

3. Statement of results

Let u be subharmonic in Ω and λ a given number, $0<\lambda<1$. Throughout the paper we assume that u satisfies the boundary condition

$$u(y) \leqq C(\lambda)u(|y|, 0) \quad \text{when} \quad y \in \partial\Omega \setminus \{0\}. \tag{3.1}$$

Here $u(y)$ is defined when $y \in \partial\Omega$ as $\lim u(x)$ when $x \to y$, $x \in \Omega$. $C(\lambda)$ is given by (2.4). We shall prove

Theorem 1. *Let $u \not\equiv -\infty$ be subharmonic in Ω and satisfy* (3.1). *Then the mean $L_\alpha(r)$, $\alpha \geqq 1$, is a convex function with respect to the family $Ar^{k\lambda}+Br^{-k\lambda+2-n}$, $r>0$. If $\alpha>1$, u is supposed to be non-negative.*

Theorem 2. *If u is subharmonic in Ω and satisfies* (3.1) *then $J(r)$ is convex with respect to the family $Ar^{k\lambda}+Br^{-k\lambda+2-n}$, $r>0$.*

Theorem 1 corresponds to theorems I and IV of Norstad [9] and Theorem 2 is a generalization of Theorem III of [9]. Transferred to the right half-plane the two-dimensional results are that

$$\left(\int_{-\pi/2}^{\pi/2} \left(\frac{u(re^{i\theta})}{\cos \lambda\theta}\right)^\alpha \cos \lambda\theta \sin \lambda \left(\frac{\pi}{2} - |\theta|\right) d\theta\right)^{1/\alpha}$$

and

$$\sup_{|\theta|<\pi/2} \frac{u(re^{i\theta})}{\cos \lambda\theta}$$

are convex with respect to $Ar^{\lambda}+Br^{-\lambda}$. Here $C(\lambda) = \cos\dfrac{\lambda\pi}{2}$. Continuity on the axis of symmetry and on the boundary is implicit in [9].

The limiting case $\lambda=1$, which corresponds to boundary values $u(y) \leqq 0$, was treated, for a half-space of R^n, by Dinghas [4]. His result is that

$$r^{n-1}\left(\int_{\substack{|\omega|=1\\|\theta|<\pi/2}} \left(\frac{u(r\omega)}{\cos \theta}\right)^\alpha \cos^2 \theta \, d\omega\right)^{1/\alpha},$$

is a convex function of r^n, which is the conclusion of Theorem 1 in case $k=\lambda=1$.

When $\alpha=1$ and $u=\log^+ |f(z)|$ with f analytic in the right half-plane and such that $|f(z)| \leqq 1$ on the imaginary axis, the result is a classical theorem by Ahlfors [1].

From the convexity we get

Corollary. *Under the assumptions of Theorem 1 and 2 and if $u(0)<\infty$, $r^{-k\lambda}L_\alpha(r)$ and $r^{-k\lambda}J(r)$ are non-decreasing, so the limits*

$$\lim_{r\to\infty} r^{-k\lambda} L_\alpha(r) \quad \text{and} \quad \lim_{r\to\infty} r^{-k\lambda} J(r)$$

exist, possibly $=\infty$.

Let $w(r, \theta_1)=r^{k\lambda}f_\lambda(\theta_1)$ and put $L(1, w)=d(\lambda)$, that is

$$d(\lambda) = 2\pi \int_0^{\psi_0} f_\lambda(\theta) g_\lambda(\theta) (\sin\theta)^{n-2} d\theta \prod_{j=2}^{n-2} \int_0^{\pi} (\sin\theta)^{n-j-1} d\theta$$

with obvious interpretation if $n=3$. We then clearly have

$$L(r) \leqq d(\lambda) J(r). \tag{3.2}$$

From our assumptions it follows that $r^{-k\lambda}M(r)$ has a positive limit as $r\to\infty$ if u is non-negative somewhere. A proof is given in Dahlberg [3] or Essén—Lewis [6]. If $u(x_0)\geqq 0$, we conclude from the Corollary that u is non-negative at some point at $|x|=r$ for all $r\geqq |x_0|$. We then have

$$J(r) \leqq M(r) \leqq C(\lambda)^{-1} J(r). \tag{3.3}$$

From (3.2) and (3.3) we get some trivial relations between the three limits. A precise result is

Theorem 3. *If u satisfies the conditions of Theorem 1, if $u(0)<\infty$ and if $u(r, 0)=0(r^{k\lambda})$ when $r\to 0$, then* $\lim_{r\to\infty} r^{-k\lambda}J(r)=\infty$ *or*

(a) $$\lim_{r\to\infty} r^{-k\lambda}L(r) = d(\lambda) \lim_{r\to\infty} r^{-k\lambda}J(r).$$

If further u is non-negative somewhere, then

(b) $$\lim_{r\to\infty} r^{-k\lambda}M(r) = C(\lambda)^{-1} \lim_{r\to\infty} r^{-k\lambda}J(r),$$

while, if $u\leqq 0$ throughout Ω,

(c) $$\lim_{r\to\infty} r^{-k\lambda}M(r) = \lim_{r\to\infty} r^{-k\lambda}J(r).$$

Our boundary condition (3.1) implies

$$u(y) \leqq C(\lambda) M^+(|y|), \tag{3.4}$$

where $M^+(r)=\max\big(M(r), 0\big)$. Among the consequences of (3.4) is the generalized Ahlfors—Heins theorem in R^n, proved by Essén—Lewis [6]. Related problems are studied in Dahlberg [3] and Wanby [10]. We also refer to Hellsten, Kjellberg and Norstad [8] and Drasin and Shea [5].

4. Some results on the Green's function

Let $\Omega_R=\Omega\cap\{|x|<R\}$ and denote by $G(x, y)$ and $G_R(x, y)$ the Green's functions for Ω and Ω_R respectively. Also let $\dfrac{\partial}{\partial N}$ denote the inner normal derivative with respect to $y\in\partial\Omega$ or $\partial\Omega_R$. In the following we will need some estimates by Azarin

[2] of $\dfrac{\partial G_R}{\partial N}$ and $\dfrac{\partial G}{\partial N}$. With F_k and v as in (2.2) we have

$$\frac{\partial G_R}{\partial N}(x, R\omega) \approx \left(\frac{|x|}{R}\right)^k R^{1-n} F_k(\theta_1(x)) F_k(\theta_1(\omega)), \tag{4.1}$$

$$\frac{\partial G}{\partial N}(x, y) \approx F_k(\theta_1(x)) \frac{\partial v}{\partial N}(y/|y|) \left(\frac{|x|}{|y|}\right)^k |y|^{1-n}, \quad \text{if} \quad 0 < |x| < \frac{4}{5}|y|, \tag{4.2}$$

$$\frac{\partial G}{\partial N}(x, y) \approx F_k(\theta_1(x)) \frac{\partial v}{\partial N}(y/|y|) \left(\frac{|y|}{|x|}\right)^k |x|^{2-n} |y|^{-1}, \quad \text{if} \quad 0 < |y| < \frac{4}{5}|x|, \tag{4.3}$$

$$\frac{\partial G}{\partial N}(x, y) \approx F_k(\theta_1(x)) \frac{\partial v}{\partial N}(y/|y|) |x-y|^{-n}, \quad \text{if} \quad \frac{4}{5} \leqq \frac{|x|}{|y|} \leqq \frac{5}{4}. \tag{4.4}$$

Here $f \approx g$ means that there are positive constants C_1 and C_2, only depending on Ω, such that $C_1 \leqq f/g \leqq C_2$.

Let $d\sigma(y)$ denote Lebesgue measure on $\partial\Omega$. Following [6, pp. 117—118] we let μ be the measure on $\partial\Omega$, defined by

$$t^{n-2}\, dt\, d\mu(y) = d\sigma(y), \quad |y| = t,$$

and

$$B(t, \theta_1) = \int_{\partial\Omega \cap \{|y|=t\}} \frac{\partial G}{\partial N}\left(\frac{x}{|x|}, y\right) d\mu(y).$$

We then get, with $|x| = r$,

$$\int_{\partial\Omega \cap \{|y|=t\}} \frac{\partial G}{\partial N}(x, y)\, d\mu(y) = r^{1-n} B(t/r, \theta_1).$$

We also note that

$$t^n B(t, \theta_1) = B(1/t, \theta_1). \tag{4.5}$$

In the following we shall also use the following notation:

$$D_R = \Omega \cap \{|x| = R\}, \quad K_R = \partial\Omega \cap \{|x| < R\}$$

and

$$\Omega_{r_1, r_2} = \Omega \cap \{r_1 \leqq |x| \leqq r_2\}, \quad K_{r_1, r_2} = \partial\Omega_{r_1, r_2} \cap \partial\Omega.$$

5. Proof of Theorem 1

We shall first prove

Lemma. *Let u be a subharmonic C^2 function in $\bar{\Omega}$ and suppose that u satisfies* (3.1). *Then*

$$\int_S \left(\frac{u(r\omega)}{f_\lambda(\theta_1)}\right)^{\alpha-1} (\delta u(r\omega) + k\lambda(k\lambda + n - 2) u(r\omega)) g_\lambda(\theta_1)\, d\omega \leqq 0. \tag{5.1}$$

Here S denotes the part of the unit sphere $|\omega|=1$, where $0\leqq\theta_1<\psi_0$. When $\alpha>1$, u is supposed to be positive.

Proof. We first assume $\alpha>1$. Denote the integrand of (5.1) by D_α and put $u(x)=q(x)f_\lambda(\theta_1)$. We get

$$D_\alpha = q^{\alpha-1}(f_\lambda\delta q+q\delta f_\lambda+2(\nabla q, \nabla f_\lambda)+k\lambda(k\lambda+n-2)qf_\lambda)g_\lambda.$$

Since f_λ satisfies (2.3), we have

$$\int_S D_\alpha\, d\omega = \int_S q^{\alpha-1}(f_\lambda\delta q+2(\nabla q, \nabla f_\lambda))g_\lambda\, d\omega$$

$$= \int q^{\alpha-1}\left(f_\lambda\frac{1}{\sqrt{g}}\sum_{j=1}^{n-1}\frac{\partial}{\partial\theta_j}\left(\frac{\sqrt{g}}{g_j}\frac{\partial q}{\partial\theta_j}\right)+2\sum_{j=1}^{n-1}\frac{1}{g_j}\frac{\partial q}{\partial\theta_j}\frac{\partial f_\lambda}{\partial\theta_j}\right)g_\lambda\sqrt{g}\, d\theta_1\dots d\theta_{n-1}$$

$$= \int\sum_{j=1}^{n-1}\left[\frac{\sqrt{g}}{g_j}\frac{\partial q}{\partial\theta_j}q^{\alpha-1}f_\lambda g_\lambda\right]_{\theta_j=0}^{\theta_j=a_j} d\theta_1\dots d\theta_{j-1}d\theta_{j+1}\dots d\theta_{n-1}$$

$$-\int\sum_{j=1}^{n-1}\frac{\sqrt{g}}{g_j}\frac{\partial q}{\partial\theta_j}\left(\frac{\partial}{\partial\theta_j}(q^{\alpha-1}f_\lambda g_\lambda)-2q^{\alpha-1}\frac{\partial f_\lambda}{\partial\theta_j}g_\lambda\right)d\theta_1\dots d\theta_{n-1}.$$

Here $a_1=\psi_0$, $a_j=\pi$ when $2\leqq j\leqq n-2$ and $a_{n-1}=2\pi$. In the first sum all terms are zero. For $j=1$ we use that $g_\lambda(\psi_0)=0$ and that $g_\lambda(\theta_1)(\sin\theta_1)^{n-2}\to 0$ as $\theta_1\to 0$. When $2\leqq j\leqq n-2$ we note that $\dfrac{\sqrt{g}}{g_j}=0$ for $\theta_j=0$ or π. Finally $\dfrac{\sqrt{g}}{g_{n-1}}$ is independent of θ_{n-1} and $\dfrac{\partial g}{\partial\theta_{n-1}}q^{\alpha-1}$ has the same values for $\theta_{n-1}=0$ and 2π for fixed $\theta_1, \dots, \theta_{n-2}$ with $0<\theta_1<\psi_0$, $0<\theta_j<\pi$ when $2\leqq j\leqq n-2$. Thus we get

$$\int_S D_\alpha\, d\omega = (1-\alpha)\int_S q^{\alpha-2}f_\lambda g_\lambda(\nabla q, \nabla q)\, d\omega$$

$$+\int\sqrt{g}\, q^{\alpha-1}\frac{\partial q}{\partial\theta_1}(f_\lambda'(\theta_1)g_\lambda(\theta_1)-f_\lambda(\theta_1)g_\lambda'(\theta_1))\, d\theta_1\dots d\theta_{n-1}$$

$$\leqq (\sin\psi_0)^{n-2}\int q^{\alpha-1}\frac{\partial q}{\partial\theta_1}\prod_{j=2}^{n-1}(\sin\theta_j)^{n-1-j}\, d\theta_1\dots d\theta_{n-1}$$

$$= \alpha^{-1}(\sin\psi_0)^{n-2}\int[q^\alpha(r, \theta_1, \dots, \theta_{n-1})]_{\theta_1=0}^{\theta_1=\psi_0}\prod_{j=2}^{n-1}(\sin\theta_j)^{n-1-j}\, d\theta_2\dots d\theta_{n-1}.$$

Here we used (2.5).

Now, if y is the point with polar coordinates $(r, \psi_0, \theta_2, \dots, \theta_{n-1})$,

$$[q^\alpha(r, \theta_1, \dots, \theta_{n-1}]_{\theta_1=0}^{\theta_1=\psi_0} = (u(y))^\alpha-(u(|y|, 0)C(\lambda))^\alpha,$$

which is non-positive because of (3.1). Thus the lemma is proved if $\alpha>1$. Small changes are needed in case $\alpha=1$. We omit the details.

Remark. It is clear from the proof that the lemma is true under the somewhat weaker boundary condition

$$\left(a^{-1}\int u^{\alpha}(r, \Psi_0, \theta_2, \dots, \theta_{n-1})\,\prod_{j=2}^{n-1}(\sin\theta_j)^{n-1-j}\,d\theta_2\dots d\theta_{n-1}\right)^{1/\alpha} \leqq C(\lambda)u(r, 0).$$

Here the domain of integration is given by $0<\theta_j<\pi$ for $2\leqq j\leqq n-2$ and $0<\theta_{n-1}<2\pi$ and

$$a = \prod_{j=2}^{n-1}\int(\sin\theta_j)^{n-1-j}\,d\theta_j = \begin{cases} \dfrac{(2\pi)^{(n-1)/2}}{(n-3)!!} & \text{if } n \text{ is odd} \\ \dfrac{2(2\pi)^{(n-2)/2}}{(n-3)!!} & \text{if } n \text{ is even.} \end{cases}$$

This corresponds to Norstad's boundary condition

$$\left[\frac{1}{2}\big(u^{\alpha}(ir)+u^{\alpha}(-ir)\big)\right]^{1/\alpha} \leqq \cos\frac{\pi\lambda}{2}\,u(r).$$

Now suppose $\alpha>1$ and $L_\alpha(r, u)=Ar^{k\lambda}+Br^{-k\lambda+2-n}$ for $r=r_1$ and $r_2, r_1<r_2$. A and B are constants. The assertion of Theorem 1 is that $L_\alpha(r, u)\leqq Ar^{k\lambda}+Br^{-k\lambda+2-n}$ for $r_1<r<r_2$. We shall first approximate u by C^2 subharmonic functions u_m which also satisfy the boundary condition (3.1). If the restriction of u to the positive x_1-axis is not continuous, we first replace u by its least harmonic majorant in a small cylinder around the x_1-axis: $r_1-\eta\leqq x_1\leqq r_2+\eta$, $\sum_2^n x_j^2\leqq\eta^2$. The new function is then subharmonic in Ω, satisfies (3.1) and is continuous on the x_1-axis for $r_1\leqq x_1\leqq r_2$. Now, if $\varepsilon>0$ is given, there are points $x^{(1)}, \dots, x^{(N)}$ on $\partial\Omega$ and a $\delta>0$ such that each point in the set K_{r_1,r_2} belongs to some ball $|x-x^{(k)}|<\delta$ and such that

(5.2) $$u(x) < u(x^{(k)})+\varepsilon \quad\text{if}\quad |x-x^{(k)}| < 2\delta.$$

This follows from the semicontinuity at the boundary and a compactness argument. Since u is continuous on the positive x_1-axis for $r_1\leqq x_1\leqq r_2$, we may take δ so small that

(5.3) $$|u(t, 0)-u(s, 0)| < \varepsilon \quad\text{if}\quad |t-s| < 2\delta,\quad r_1 \leqq t \leqq s \leqq r_2.$$

Let $u_\varepsilon(x)=u(x_1+\delta, x_2, \dots, x_n)-\dfrac{2\varepsilon}{1-C(\lambda)}$. Then u_ε is subharmonic in an open domain D which contains Ω_{r_1,r_2}. If $x\in K_{r_1,r_2}$ we get according to (5.2), (3.1) and (5.3),

$$u_\varepsilon(x) \leqq C(\lambda)u_\varepsilon(|x|, 0)-\varepsilon(1-C(\lambda)).$$

Let $v_\varepsilon(x)=u_\varepsilon(x)+\dfrac{3\varepsilon}{1-C(\lambda)}\left(\dfrac{r}{r_1}\right)^{k\lambda}f_\lambda(\theta_1)$ to make $v_\varepsilon>0$. Note that $r^{k\lambda}f_\lambda(\theta_1)$ satisfies (3.1) with equality. Now choose a sequence u_m of subharmonic functions which decrease to v_ε in D.

We have to show that the functions u_m satisfy (3.1). For x in K_{r_1,r_2} we first observe that $u_m(x)<v_\varepsilon(x)+\varepsilon(1-C(\lambda))<C(\lambda)v_\varepsilon(|x|,0)$, if $m\geqq$ some $m_1=m_1(\varepsilon,x)$. Since u_m and v_ε are continuous on the positive x_1-axis for $r_1\leqq x_1\leqq r_2$, we get

$$u_{m_1}(x+y)<C(\lambda)v_\varepsilon(|x+y|,0) \quad \text{if} \quad |y|< \text{ some } \eta(\varepsilon,x).$$

Now, by another compactness argument, we see that there are finitely many $x^{(k)}\in\partial\Omega$ such that each $x\in K_{r_1,r_2}$ may be written as $x=x^{(k)}+y$ and such that

$$u_{m_k}(x^{(k)}+y)<C(\lambda)v_\varepsilon(|x^{(k)}+y|,0) \quad \text{for some } k.$$

With $M=\max m_k$, by using that the sequence u_m decreases, we obtain

$$u_m(x)<C(\lambda)v_\varepsilon(|x|,0)\leqq C(\lambda)u_m(|x|,0)$$

for all $x\in K_{r_1,r_2}$, if $m\geqq M$.

It is easy to see that, for $r=r_1$ or r_2, $L_\alpha(r,u_m)\leqq L_\alpha(r,u)+0(\varepsilon)$, so $L_\alpha(r,u_m)\leqq$ $\leqq A_\varepsilon r^{k\lambda}+B_\varepsilon r^{-k\lambda+2-n}$, where $A_\varepsilon\to A$ and $B_\varepsilon\to B$ as $\varepsilon\to 0$. If the theorem is proved for the C^2 function u_m, the rest is standard, letting in order $m\to\infty$, $\varepsilon\to 0$ and $\eta\to 0$.

If $\alpha=1$, to make u finite, we start by replacing u by $\max(u,-N)$, which tends to u when $N\to\infty$. This does not affect the boundary condition.

When u is C^2 we have

$$\Delta u=u''_{rr}+\frac{n-1}{r}u'_r+\frac{1}{r^2}\delta u\geqq 0. \tag{5.4}$$

Let $G(r)=L_\alpha^\alpha(r)=\int_S q^\alpha f_\lambda g_\lambda\,d\omega$. Then

$$G'(r)=\int_S \alpha q^{\alpha-1}u'_r g_\lambda\,d\omega \tag{5.5}$$

and, if $\alpha>1$,

$$G''(r)=\int_S \alpha(\alpha-1)q^{\alpha-2}(u'_r)^2\frac{g_\lambda}{f_\lambda}\,d\omega+\int_S \alpha q^{\alpha-1}u''_{rr}g_\lambda\,d\omega.$$

From (5.5) it follows by use of the Cauchy—Schwartz inequality that

$$(G'(r))^2\leqq\alpha^2 G(r)\int_S q^{\alpha-2}(u'_r)^2\frac{g_\lambda}{f_\lambda}\,d\omega.$$

Hence, by (5.4)

$$G''(r)\geqq\frac{\alpha-1}{\alpha}\frac{(G'(r))^2}{G(r)}-\frac{n-1}{r}G'(r)-\frac{\alpha}{r^2}\int_S q^{\alpha-1}g_\lambda\delta u\,d\omega,$$

so from the lemma we get

$$G''(r)-\frac{\alpha-1}{\alpha}\frac{(G'(r))^2}{G(r)}+\frac{n-1}{r}G'(r)-\frac{\alpha}{r^2}k\lambda(k\lambda+n-2)G(r)\geqq 0$$

or

$$L''_\alpha(r)+\frac{n-1}{r}L'_\alpha(r)-\frac{k\lambda(k\lambda+n-2)}{r^2}L_\alpha(r)\geqq 0. \tag{5.6}$$

If $\alpha=1$, we have $G'(r)=\int_S u'_r g_\lambda d\omega$ so we arrive at (5.6) by another differentiation and the lemma.

Equality in (5.6) occurs if and only if $L_\alpha(r)=C_1 r^{k\lambda}+C_2 r^{-k\lambda+2-n}$. The result therefore follows from the (one-dimensional) maximum principle.

Remark 1. An equivalent formulation of the conclusion is that $r^{n-2+k\lambda}L_\alpha(r)$ is a convex function of $r^{n-2+2k\lambda}$.

Remark 2. The above mentioned theorem by Dinghas follows from ours by letting $\lambda\to 1$. In fact, first replace f_λ by $C(\lambda)f_\lambda$ so that $f_\lambda(0)=1$ and $f_\lambda(\pi/2)=C(\lambda)$. As $\lambda\to 1$, $f_\lambda(\theta)\to\cos\theta$ and also $g_\lambda(\theta)\to\cos\theta$ in C^1 on compact parts of $(0,\pi)$. Further $\int_{0\leqq\theta_1\leqq\eta} g_\lambda d\omega\to 0$ when $\eta\to 0$. This is seen by observing that $g_\lambda(\theta)$ is a decreasing function of λ.

6. Proof of Theorem 2 and the Corollary

Let $h(r)=Ar^{k\lambda}+Br^{-k\lambda+2-n}$ and assume $J(r)=h(r)$ for $r=r_1$ and r_2, $r_1<r_2$. Solving for A and B we get

$$h(r)=D^{-1}\big(J(r_1)(r^{k\lambda}r_2^{-k\lambda+2-n}-r^{-k\lambda+2-n}r_2^{k\lambda})+J(r_2)(r^{-k\lambda+2-n}r_1^{k\lambda}-r^{k\lambda}r_1^{-k\lambda+2-n})\big),$$

where $D=r_1^{k\lambda}r_2^{-k\lambda+2-n}-r_2^{k\lambda}r_1^{-k\lambda+2-n}$.

Let $H(x)=H(r,\theta_1)=h(r)f_\lambda(\theta_1)$. Since $r^{k\lambda}f_\lambda$ and $r^{-k\lambda+2-n}f_\lambda$ are harmonic in Ω, H is. We shall see that H majorizes u in Ω_{r_1,r_2}. In order to apply the maximum principle, we note that $v=u-H\leqq 0$ when $|x|=r_1$ or r_2. Then either $v\leqq 0$ throughout Ω_{r_1,r_2} or v has a positive maximum at $x_0\in\partial\Omega$. But since H satisfies (3.1) with equality, $v(x_0)\leqq C(\lambda)v(|x_0|,0)$, so the maximum cannot be positive. Thus $\dfrac{u(x)}{f_\lambda(\theta_1)}\leqq h(r)$ when $|x|=r$, $r_1\leqq r\leqq r_2$. Consequently $J(r)\leqq h(r)$ for these values of r, and we are through.

Proof of the Corollary

Since $\overline{\lim}_{x\to 0}u(x)=u(0)<\infty$, $J(r)$ is bounded above when r is small. Also, $h(r)$ is a positive linear combination of $J(r_1)$ and $J(r_2)$ for $r_1<r<r_2$. Thus we may let $r_1\to 0$ in the inequality $J(r)\leqq h(r)$. We obtain $J(r)\leqq r^{k\lambda}J(r_2)r^{-k\lambda}$ which is the assertion. The proof for L_α is the same.

Remark. Theorem 2 is actually true with (3.1) replaced by $u(y)<\infty$ and

(6.1) $$u(y) \leqq C(\lambda)M^{+}(|y|, u) \quad \text{when} \quad y\in\Omega$$

provided that $u(x)\geqq 0$ somewhere on D_r for $r\geqq r_1$, so that $J(r)\geqq 0$. To see this, we note that $M^{+}(H, r)=H(r, 0)$ so v satisfies (6.1), and the conclusion is reached as above.

In general, (6.1) is not sufficient for Theorem 2 to be valid. Let $\lambda'\in(\lambda, 1)$. A trivial example is then $u=-r^{k\lambda'}f_{\lambda'}(\theta_1)$, which is harmonic in Ω, satisfies (6.1) and has $r^{-k\lambda}J(r)=Cr^{k(\lambda'-\lambda)}$ where C is a negative constant. (Actually $C=-1$.)

7. Proof of (a) of Theorem 3

Assume that $A=\lim_{r\to\infty} r^{-k\lambda}J(r)<\infty$. We use the notation of Section 4. The function

(7.1)
$$H_R(x)=\int_S \frac{\partial G_R}{\partial N}(x, R\omega)u(R\omega)R^{n-1}\,d\omega+\int_{K_R}\frac{\partial G_R}{\partial N}(x, y)C(\lambda)u(|y|, 0)d\sigma(y),$$

is harmonic in Ω_R with boundary values $u(R\omega)$ at D_R and $C(\lambda)u(|y|, 0)$ at K_R. H_R obviously majorizes u. Thus, if $y\in\partial\Omega$,

$$H_R(y)=C(\lambda)u(|y|, 0)\leqq C(\lambda)H_R(|y|, 0),$$

so H_R satisfies (3.1) in Ω_R. Since $u\leqq J(r)f_\lambda$,

(7.2) $$H_R(x)\leqq A|x|^{k\lambda}f_\lambda(\theta_1).$$

Especially $H_R(0)\leqq 0$. Consequently

(7.3) $$r^{-k\lambda}L(r, H_R)\leqq R^{-k\lambda}L(R, H_R) \quad \text{for} \quad r<R.$$

Now, an application of the maximum principle in Ω_R shows that $H_{R'}(x)\geqq H_R(x)$ if $R'>R$. So, by (7.2) and the Harnack principle, $H_R(x)$ increases to a harmonic function $H(x)\leqq A|x|^{k\lambda}f_\lambda(\theta_1)$ in Ω, as $R\to\infty$. Taking the limit in (7.1), we want to show that

(7.4) $$\int_S \frac{\partial G_R}{\partial N}(x, R\omega)u(R\omega)R^{n-1}\,d\omega\to 0, \quad \text{when} \quad R\to\infty \quad \text{and } x \text{ is fixed.}$$

We have $u(x)\leqq AC(\lambda)^{-1}|x|^{k\lambda}$. If we also knew that, for some B, $u(x)\geqq B|x|^{k\lambda}$ when x is large, (7.4) would follow from (4.1). Otherwise we may argue as follows. By (4.1) it is enough to prove (7.4) for $u_1=u+C|x|^{k\lambda}f_\lambda(\theta_1)$, where C is chosen so that u_1 is positive somewhere. Then $\lim_{r\to\infty} r^{-k\lambda}M(r, u_1)$ exists and is finite. If

$$v(x) = C(\lambda)\int_{\partial\Omega}\frac{\partial G}{\partial N}(x, y)M^+(|y|, u_1)\,d\sigma(y),$$

it follows that $v(x)=0(|x|^{k\lambda})$ when x tends to ∞, so it suffices to show (7.4) for $p=v-u_1$. The function p is superharmonic and non-negative in Ω. Following [6, pp. 120—121] we note that for r large there exists x_r, with $|x_r|=r$,

$$r^{-k\lambda}p(x_r)\to 0 \quad \text{as} \quad r\to\infty$$

and

(7.5) $$\theta_1(x_r)\leqq \text{constant} < \psi_0.$$

From the maximum principle and (4.1) we deduce

$$p(x_r)\geqq \int_S \frac{\partial G_R}{\partial N}(x_r, R\omega)p(R\omega)R^{n-1}d\omega \geqq C_1\left(\frac{|x_r|}{R}\right)^k F_k(\theta_1(x_r))\int_S F_k(\theta_1(\omega))p(R\omega)\,d\omega.$$

Denote the latter integral by $I(R)$. Taking $r=R/2$, we obtain from (7.5)

$$R^{-k\lambda}I(R)\leqq C_2r^{-k\lambda}p(x_r),$$

which tends to 0 as $R\to\infty$. Thus, by (4.1),

$$0\leqq \int_S \frac{\partial G_R}{\partial N}(x, R\omega)\,p(R\omega)R^{n-1}d\omega \leqq C_3|x|^k\big(I(R)R^{-k\lambda}\big)R^{-k(1-\lambda)}\to 0,$$

when $R\to\infty$, so (7.4) is verified.

Since $\dfrac{\partial G_R}{\partial N}\uparrow\dfrac{\partial G}{\partial N}$ as $R\to\infty$, we note, with $u^+=\max(u, 0)$, that

$$\int_{K_R}\frac{\partial G_R}{\partial N}(x, y)C(\lambda)u^+(|y|, 0)\,d\sigma(y)\uparrow\int_{\partial\Omega}\frac{\partial G}{\partial N}(x, y)C(\lambda)u^+(|y|, 0)\,d\sigma(y),$$

which is finite, due to (4.2) (and (4.3)). Since $H(x)$ is finite, also

$$\lim_{R\to\infty}\int_{K_R}\frac{\partial G_R}{\partial N}(x, y)C(\lambda)u^-(|y|, 0)\,d\sigma(y) = \int_{\partial\Omega}\frac{\partial G}{\partial N}(x, y)C(\lambda)u^-(|y|, 0)\,d\sigma(y)$$

is finite. Here $u=u^+-u^-$. Thus

$$H(x) = \int_{\partial\Omega}\frac{\partial G}{\partial N}(x, y)C(\lambda)u(|y|, 0)\,d\sigma(y).$$

It is easily seen that H satisfies (3.1) and $H(0)\leqq 0$, so $r^{-k\lambda}L(r, H)$ and $r^{-k\lambda}J(r, H)$ have finite limits as r tends to ∞. Since $u(x)\leqq H(x)\leqq A|x|^{k\lambda}f_\lambda(\theta_1)$, $r^{-k\lambda}J(r, u)\leqq r^{-k\lambda}J(r, H)\leqq A$. Hence $\lim_{r\to\infty}r^{-k\lambda}J(r, H)=A$. By (7.3) we get

$$r^{-k\lambda}L(r, H_R)\leqq R^{-k\lambda}L(R, H_R)\leqq R^{-k\lambda}L(R, H).$$

From the definition of H_R it is seen that $L(R, u)=L(R, H_R)$. Hence, letting $R\to\infty$,

$$r^{-k\lambda}L(r, H) \leqq \lim_{R\to\infty} R^{-k\lambda}L(R, u) \leqq \lim_{R\to\infty} R^{-k\lambda}L(R, H).$$

Thus

$$\lim_{r\to\infty} r^{-k\lambda}L(r, H) = \lim_{R\to\infty} r^{-k\lambda}L(R, u) = a$$

(So it suffices to prove the theorem for H.)

Now repeat the procedure with $H^{(1)}=H$ instead of u, etc. We get an increasing sequence of harmonic functions $H^{(n)}(x)$ in Ω with

$$H^{(n)}(x) = \int_{\partial\Omega}\frac{\partial G}{\partial N}(x, y)C(\lambda)H^{(n-1)}(|y|, 0)\,d\sigma(y)$$

and $H^{(n)}(x)\leqq A|x|^{k\lambda}f_\lambda(\theta_1)$. Hence $H^{(n)}$ has a finite harmonic limit

$$h(x) = \int_{\partial\Omega}\frac{\partial G}{\partial N}(x, y)C(\lambda)h(|y|, 0)\,d\sigma(y), \tag{7.6}$$

when $n\to\infty$. We also observe that h satisfies (3.1) with equality.

Below we shall prove

$$h(x) = A|x|^{k\lambda}f_\lambda(\theta_1). \tag{7.7}$$

Supposing this done, we have $d(\lambda)^{-1}r^{-k\lambda}L(r, h)=A$. If ε is given >0 and r_0 fixed,

$$d(\lambda)^{-1}r_0^{-k\lambda}L(r_0, H^{(n)}) > A-\varepsilon \quad \text{for some} \quad n = n(\varepsilon, r_0).$$

Since $r^{-k\lambda}L(r, H^{(n)})$ increases to a when $r\to\infty$, we obtain $ad(\lambda)^{-1}> A-\varepsilon$, and so we are through.

8. Proof of (7.7)

To prove that h is a multiple of $r^{k\lambda}f_\lambda(\theta_1)$, it is by (7.6) enough to show that $h(r, 0)=Cr^{k\lambda}$. With B as in Section 4 we have

$$h(r, 0) = \int_0^\infty C(\lambda)h(t, 0)B(t/r, 0)t^{n-2}r^{1-n}\,dt.$$

From the construction of h we know $h(r, 0)\leqq Ar^{k\lambda}$. Using part of the proof of the generalized Ahlfors—Heins theorem in R^n ([6, pp. 119—123]), we see that $u(r, 0)\geqq Cr^{k\lambda}$ when x is large. Here (4.2) — (4.4) are needed. Since we have assumed that $u(r, 0)\geqq C'r^{k\lambda}$ when r is near 0, we have $|h(r, 0)|\leqq C''r^{k\lambda}$ for all $r>0$. Let $f(t)=h(t, 0)t^{-k\lambda}$. Then f is C^∞ and bounded on R^+ and

$$f(r) = \int_0^\infty C(\lambda)f(t)(t/r)^{k\lambda}t^{n-2}r^{1-n}B(t/r, 0)\,dt.$$

Put $r=e^{-x}$, $t=e^{-s}$ and $f(e^{-x})=\varphi(x)$. Hence

$$\varphi(x)=\int_{-\infty}^{\infty}\varphi(s)C(\lambda)e^{(x-s)(n-1+k\lambda)}B(e^{x-s},0)\,ds. \tag{8.1}$$

With $K(s)=C(\lambda)e^{s(n-1+k\lambda)}B(e^s,0)$ we then have $\varphi=\varphi*K$. Here $\frac{d^m\hat{K}(\xi)}{d\xi^m}$ exists for every m, since $\int_{-\infty}^{\infty}|s|^m K(s)\,ds$ is finite, which is readily checked. Thus $(1-\hat{K})\hat{\varphi}=0$. Since $\varphi\not\equiv 1$ solves (8.1), $\hat{K}(0)=1$. Further we observe that $\hat{K}(\xi)\not\equiv 1$ if $\xi\neq 0$, so $\hat{\varphi}$ has its support at the origin. Now

$$\hat{K}'(0)=\int_{-\infty}^{\infty}(-is)K(s)\,ds=-iC(\lambda)\int_{-\infty}^{\infty}se^{s(n-1+k\lambda)}B(e^s,0)\,ds,$$

which, by a change of variables and (4.5), equals

$$-iC(\lambda)\int_1^{\infty}(t^{k\lambda+n-2}-t^{-k\lambda})B(t,0)\ln t\,dt.$$

This is obviously $\neq 0$, so we conclude that $\hat{\varphi}(\xi)=C\delta(\xi)$. Hence φ is constant, which means that $h(r,\theta_1)=Cr^{k\lambda}f_\lambda(\theta_1)$. From the construction of h we have $C\leqq A$. But $u\leqq h$ so $J(r,u)\leqq Cr^{k\lambda}$. Thus $C=A$ and the proof is finished.

9. Proof of (b) and (c) of Theorem 3

To prove (b) we first observe that $u(x)\leqq\min\big(M(r),J(r)f_\lambda(\theta_1)\big)$, so

$$L(r,u)\leqq L\big(r,\min\,(M(r),J(r)f_\lambda(\theta_1))\big).$$

Let $m(r)=r^{-k\lambda}M(r)$, $j(r)=r^{-k\lambda}J(r)$ and $e(r)=j(r)d(\lambda)-r^{-k\lambda}L(r)$ so that $e(r)\to 0$ as $r\to\infty$. We have $0\leqq j(r)\leqq m(r)\leqq C(\lambda)^{-1}j(r)$. Thus there is a $\psi_1=\psi_1(r)$, $0\leqq\psi_1\leqq\psi_0$, such that $m(r)=j(r)f_\lambda(\psi_1)$. Hence

$$j(r)\,d(\lambda)-e(r)$$

$$\leqq j(r)a\Big(\int_0^{\psi_1}f_\lambda(\psi_1)g_\lambda(\theta_1)(\sin\theta_1)^{n-2}\,d\theta_1+\int_{\psi_1}^{\psi_0}f_\lambda(\theta_1)g_\lambda(\theta_1)(\sin\theta_1)^{n-2}\,d\theta_1\Big)$$

where

$$a=\begin{cases}\dfrac{(2\pi)^{(n-1)/2}}{(n-3)!!} & \text{if } n \text{ is odd}\\[2ex] \dfrac{2(2\pi)^{(n-2)/2}}{(n-3)!!} & \text{if } n \text{ is even.}\end{cases}$$

It follows that

$$e(r)a^{-1}\geqq j(r)\int_0^{\psi_1}\big(f_\lambda(\theta_1)-f_\lambda(\psi_1)\big)g_\lambda(\theta_1)(\sin\theta_1)^{n-2}\,d\theta_1.$$

The assertion of the theorem is that $\psi_1(r)\to 0$ as $r\to\infty$. If not so, there would exist an $\eta>0$ and a sequence $r_i\to\infty$ as $i\to\infty$, such that $\psi_1(r_i)\geqq\eta$. It would follow that

$$e(r_i)a^{-1}\geqq j(r_i)\int_0^{\eta}\big(f_\lambda(\theta_1)-f_\lambda(\eta)\big)g_\lambda(\theta_1)(\sin\theta_1)^{n-2}\,d\theta_1.$$

Hence $\lim_{r\to\infty} j(r)=A\leqq 0$, which is a contradiction unless $A=0$ in which case there is nothing to prove.

In case $u\leqq 0$, we have $0\leqq -j(r)\leqq -m(r)\leqq -C(\lambda)^{-1}j(r)$. With ψ_1 as above, the aim is to show that $\psi_1\to\psi_0$ as $r\to\infty$. Proceeding by contradiction as before, we get

$$-e(r) \leqq j(r)a\int_{\psi_0-\eta}^{\psi_0}(f_\lambda(\psi_0-\eta)-f_\lambda(\theta_1))g_\lambda(\theta_1)(\sin\theta_1)^{n-2}\,d\theta_1$$

on some sequence $r=r_i$, where $r_i\to\infty$ as $i\to\infty$, and some $\eta>0$. This gives $A\geqq 0$ and the proof is finished.

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