Arch. Rational Mech. Anal. 112 (1990) 339–362. © Springer-Verlag 1990

Closed Orbits of Fixed Energy for Singular Hamiltonian Systems

ANTONIO AMBROSETTI & VITTORIO COTI ZELATI

Communicated by H. BREZIS

§ 1. Introduction

This paper deals with the existence of periodic solutions of

$$q'' + V'(q) = 0 \tag{1.1}$$

such that

$$\tfrac{1}{2}|q'|^2 + V(q) = h \tag{1.2}$$

where $q \in \mathbf{R}^N$, h is a given number, $V \in C^2(\mathbf{R}^N \setminus \{0\}, \mathbf{R})$ has a singularity at $x = 0$, and V' denotes the gradient of V.

Our main results are collected in Theorems 3.6, 4.12 and 5.1. In Theorem 3.6 we deal with potentials which, roughly, behave like $-\frac{1}{|x|^a}$ with $a > 2$ (referred to in the sequel as the "strong-force" case) and prove the existence of solutions q of (1.1), (1.2) such that $q(t) \neq 0$ for all $t \in \mathbf{R}$ (so that there are no collisions). In Theorems 4.12 and 5.1 we are concerned with the case in which $V(x) \approx -\frac{1}{|x|^b}$ with $0 < b < 2$, and prove the existence of solutions q which can possibly pass through the singularity $x = 0$ (*i.e.*, which can have collisions).

To have an idea of the kind of problem we can handle, let us state two specific results concerning potentials of the type

$$V(x) = -\frac{1}{|x|} + W(x),$$

for which (1.1) becomes the perturbed Kepler's equation

$$q'' + \frac{q}{|q|^3} + W'(q) = 0.$$

Theorem 1.1. *Suppose that* $V(x) = -\frac{1}{|x|} + W(x)$ *with* $W \in C^2(\mathbf{R}^N; \mathbf{R})$ *satisfying*

(W1) $3W'(x) \cdot x + W''(x)\, x \cdot x \geqq 0;$

(W2) $W'(x) \cdot x > -\frac{1}{|x|};$

(W3) $W'(x) \cdot x + W(x) \geqq 0;$

(W4) $\liminf_{|x| \to \infty} [W(x) + \frac{1}{2} W'(x) \cdot x] \geqq 0;$

Then for all $h < 0$, *problem* (1.1), (1.2) *has a periodic solution.*

Theorem 1.2. *Let* $h < 0$ *be given and let* $V(x) = -\frac{1}{|x|} + \varepsilon U(x)$ *with* U *smooth in all* $\mathbf{R}^N$. *Then there exists an* $\bar{\varepsilon}$ *(depending on* h *and* $\|U\|_{C^2}$*) such that for all* $|\varepsilon| < \bar{\varepsilon}$, *the problem*

$$q'' + \frac{q}{|q|^3} + \varepsilon U'(x) = 0,$$

$$\tfrac{1}{2}|q'(t)|^2 - \frac{1}{|q(t)|} + \varepsilon U(q(t)) = h$$

has a periodic solution.

Theorems 1.1 and 1.2 follow from the much more general results contained in Theorems 4.12 and 5.1, respectively.

In the last few years there has been a remarkable amount of work on the existence of periodic solutions of systems with singular potentials having a given number $T > 0$ as period, see, *e.g.*, [2], [4], [7], [8], [11], [6], [14], [18]. But much less is known on problem (1.1), (1.2) where the energy rather than the period is prescribed. As far as we know, only the papers [12], [5] deal with such a problem in the large. (For perturbation results see, for example, [15].) Paper [12] covers a rather restricted class of potentials satisfying the strong-force condition, only. Paper [5] deals with the existence of solutions of (1.1), (1.2) confined in an annulus $\mathcal{A}$ where the shape of V differs strongly from that of $-\frac{1}{|x|^b}$. For example, neither the potentials $V(x) \approx -\frac{1}{|x|} - \frac{1}{|x|^b}$, $0 < b < 2$ (covered by Theorem 4.12, or for $1 < b < 2$, by Theorem 1.1) nor any $V(x) = -\frac{1}{|x|} + \varepsilon U(x)$, $\varepsilon \geqq 0$ (see Theorem 1.2) can be handled by [5]. On the other hand, we do not confine the solutions to any annulus, and in the case of Theorems 4.12, 5.1, the solutions we find could have collisions.

When $V \in C^1(\mathbf{R}^N, \mathbf{R})$ a way to seek solutions of (1.1), (1.2) is to look for stationary points of

$$\tilde{f}(u) = \tfrac{1}{2} \int_0^1 |u'|^2 \, dt \cdot \int_0^1 [h - V(u)] \, dt$$

on $H^1(S^1, \mathbf{R}^N \setminus \{0\})$ with $\int_0^1 |u'|^2 \, dt > 0$. If u is such a stationary point, then $q(t) = u(\omega t)$ where

$$\omega^2 = \frac{\int_0^1 [h - V(u)] \, dt}{\tfrac{1}{2} \int_0^1 |u'|^2 \, dt}$$

satisfies (1.1), (1.2). See, for example, [19].

In the present paper the idea of the proof relies on a variational principle, discussed in Section 2, which amounts of finding solutions of (1.1), (1.2) as critical points at a positive level of the functional

$$f(u) = \tfrac{1}{4} \int_0^1 |u'|^2 \, dt \cdot \int_0^1 V'(u) \cdot u \, dt$$

on the set

$$M_h = \left\{ u \in \Lambda : \int_0^1 [V(u) + \tfrac{1}{2} V'(u) \cdot u] \, dt = h \right\}$$

where

$$\Lambda = \{u \in H^1(S^1; \mathbf{R}^N) \text{ such that } u(t) \neq 0 \ \forall \, t\}.$$

Our principle is related to the preceding one by the fact that $\tilde{f}_{|M_h} = f$ and $\nabla f_{|M_h} = 0$ if and only if $\nabla \tilde{f} = 0$.

A similar approach has been used in [3], and earlier, for semilinear elliptic boundary value problems in [13], [1], [16], but it is new in connection with singular Hamiltonian systems.

To clarify why our approach is appropriate for our purposes and seems more suitable for a rather direct application of the Lusternik-Schnirelman (LS, for short) theory let us briefly outline the arguments of the proof.

Assuming $V(x) \approx -\dfrac{1}{|x|^\alpha}$, $\alpha > 0$ near $x = 0$ we distinguish between $\alpha > 2$ and $\alpha < 2$.

In the strong-force case, studied in Section 3, it is natural to take $h > 0$. It turns out that for such an h: (i) M_h, which $\neq \emptyset$, is a smooth manifold; (ii) cat $M_h = +\infty$ (here cat denotes the LS category); (iii) f is bounded below on M_h; (iv) f satisfies the Palais-Smale (PS for short) condition on M_h; (v) M_h is complete. Then the LS theory applies, yielding infinitely many critical points for f on M_h with $f(u) > 0$.

The case in which V does not satisfy the strong-force condition (as is the case in Theorems 1.1 and 1.2) is discussed in Section 4 and requires some care. Taking $h < 0$, which is now the "natural" value of the energy, one still has that (i), (ii) and (iii) hold, but now M_h is no longer complete.

To overcome such a problem (and the related lack of the PS condition) we modify V by setting

$$V_\varepsilon(x) = V(x) - \varepsilon \frac{1}{|x|^2}.$$

A remarkable feature of our approach is that the manifold M_h^ε, corresponding to the potential V_ε, coincide with the manifold M_h corresponding to V, and one is led to seek critical points of

$$f_\varepsilon = \tfrac{1}{4} \int_0^1 |u'|^2 \, dt \cdot \int_0^1 V_\varepsilon'(u) \cdot u \, dt,$$

on M_h. Since V_ε satisfies the strong-force condition, the preceding arguments yield a critical point $u_\varepsilon \in M_h$, such that $f_\varepsilon(u_\varepsilon) > 0$. A limiting procedure, based on some energy estimates, allows us to show that $u_\varepsilon \to u$ as $\varepsilon \to 0$ and this gives a solution of (1.1), (1.2).

The hypotheses of Theorems 3.6 and 4.12 are global in nature. In the last section we state a result, Theorem 5.1 (which is related to Theorem 4.12), where such assumptions are made in $\{V \leq h\}$ only.

The same variational approach can be used to handle a class of Hamiltonian systems including the N-body problem. These systems will be the object of a forthcoming paper.

Notation. Throughout the paper we let

$$x \cdot y = \sum_{i=1}^N x_i y_i, \qquad \forall\, x, y \in \mathbf{R}^N,$$

$$|x| = \sqrt{x \cdot x} \qquad \forall\, x \in \mathbf{R}^n,$$

$$H = H^1(S^1, \mathbf{R}^N),$$

$$(u \mid v) = \int_0^1 u' \cdot v' + \int_0^1 u \cdot v \qquad \forall\, u, v \in H,$$

$$\|u\|^2 = (u \mid u) \qquad \forall\, u \in H,$$

$$\Omega = \mathbf{R}^N \setminus \{0\},$$

$$\Lambda = \{u \in H \text{ such that } u(t) \neq 0 \ \forall\, t\}.$$

§ 2. The Variational Principle

In this section we state the Variational Principle. In the sequel we always assume that

$$V \in C^2(\Omega; \mathbb{R}).$$

We define $f \in C^1(\Lambda, \mathbf{R})$ by

$$f(u) = \tfrac{1}{4} \int_0^1 |u'|^2 \, dt \cdot \int_0^1 V'(u) \cdot u \, dt \tag{2.1}$$

and $g \in C^1(\Lambda, \mathbf{R})$ by

$$g(u) = \int_0^1 [V(u) + \tfrac{1}{2} V'(u) \cdot u] \, dt. \tag{2.2}$$

For $h \in \mathbf{R}$ we set

$$M_h = \{u \in \Lambda : g(u) = h\}. \tag{2.3}$$

We remark that from the Sobolev embedding theorem it immediately follows that

$$\text{if } u_n \rightharpoonup \bar{u}, \quad u_n, \bar{u} \in \Lambda, \quad \text{then} \quad g(u_n) \to g(\bar{u}) \quad \text{and} \quad \nabla g(u_n) \to \nabla g(\bar{u}). \tag{2.4}$$

In the rest of this section it is understood that $M_h \neq \emptyset$. It will be shown in §§ 3, 4 that this is actually the case for suitable values of h related to the behaviour of V at $x = 0$ and at $|x| \to \infty$.

Lemma 2.1. *Let V satisfy*

(A1) $$3V'(x) \cdot x + V''(x)\, x \cdot x \neq 0 \quad \forall\, x \in \Omega.$$

Then M_h is a C^1-manifold of codimension 1 *in Λ. More precisely,*

$$(\nabla g(u) \mid u) \neq 0 \quad \forall\, u \in M_h. \tag{2.5}$$

Moreover, if V also satisfies

(A2) $$V'(x) \cdot x > 0 \quad \forall\, x \in \Omega,$$

then $f(u) \geqq 0$ on M_h and $f(u) = 0$, $u \in M_h$, if and only if u is a constant.

Proof. By direct calculation one has

$$(\nabla g(u) \mid u) = \int_0^1 [\tfrac{3}{2} V'(u) \cdot u + \tfrac{1}{2} V''(u)\, u \cdot u] \, dt,$$

and the first statement follows from (A1). The second statement easily follows from (A2). □

Remark 2.2. M_h is obviously closed with respect to Λ, but not necessarely with respect to H, as we will see for the class of potentials discussed in § 4. However, if $\overline{M}_h$ denotes the closure of M_h in H, one has that $\overline{M}_h \setminus M_h \subset \partial\Lambda$.

Lemma 2.3. *Suppose* (A1) *and* (A2) *hold. Let $u \in M_h$ be a critical point of f constrained on M_h such that $f(u) > 0$. Let*

$$\omega^2 = \frac{\int_0^1 V'(u) \cdot u \, dt}{\int_0^1 |u'|^2 \, dt}. \tag{2.6}$$

Then $q(t) = u(\omega t)$ is a (non-constant) periodic solution of (1.1), (1.2).

Proof. Let $u \in M_h$ be a critical point of f constrained on M_h. Then there exists $\lambda \in \mathbf{R}$ such that

$$\nabla f(u) = \lambda \nabla g(u). \tag{2.7}$$

Taking the scalar product of (2.7) with u, we obtain

$$(\nabla f(u) \mid u) = \lambda (\nabla g(u) \mid u).$$

From this we easily deduce that

$$\lambda = \tfrac{1}{2} \int_0^1 |u'|^2 \, dt.$$

Inserting this value into (2.7) we obtain

$$\int_0^1 u' \cdot v' \, dt \cdot \int_0^1 V'(u) \cdot u \, dt = \int_0^1 |u'|^2 \, dt \int_0^1 V'(u) \cdot v \qquad \forall\, v \in H.$$

Note that $\int_0^1 |u'|^2 \, dt > 0$ if $f(u) > 0$. Then it follows that

$$\omega^2 u'' + V'(u) = 0 \tag{2.8}$$

with ω^2 given by (2.6), and $q(t) = u(\omega t)$ satisfies (1.1). Moreover, since (2.8) is autonomous, the conservation of energy yields

$$\tfrac{1}{2} \omega^2 |u'(t)|^2 + V(u(t)) = c. \tag{2.9}$$

Integrating this equation we find that

$$\tfrac{1}{2} \omega^2 \int_0^1 |u'|^2 \, dt + \int_0^1 V(u) \, dt = c$$

and thus that $c = h$ since $u \in M_h$. This implies that q (which is non-constant since $\int_0^1 |u'|^2 \, dt > 0$) satisfies (1.2). □

§ 3. Existence Results. (Strong Forces)

In the proof of the existence of critical points of f on M_h an important role is played by the behaviour of V as $x \to 0$. Let us explain this fact with the "model" case

$$V(x) = -\frac{1}{|x|^\alpha}, \qquad \alpha > 0.$$

Note that for all $\alpha \neq 2$, (A1) and (A2) hold and the variational principle applies. Here the "natural" values of the energy are: $h > 0$ if $\alpha > 2$ and $h < 0$ if $0 < \alpha < 2$. In fact, if $q(t)$ is a radial, periodic solution of

$$q'' + \alpha \frac{q}{|q|^{\alpha+2}} = 0,$$

then the corresponding energy is

$$h = \tfrac{1}{2}\,|q'(t)| - \frac{1}{|q(t)|^{\alpha}} = \left(\frac{\alpha}{2} - 1\right)\frac{1}{|q(t)|^{\alpha}}.$$

On M_h (nonempty by Lemmas 3.3 and 4.3), the functional f takes the form

$$f(u) = \tfrac{1}{4}\int_0^1 |u'|^2\,dt \cdot \int_0^1 \frac{\alpha}{|u|^{\alpha}}.$$

If $u_n \to \bar{u} \in \overline{M}_h \setminus M_h$, then (see Remark 2.2) $\bar{u} \in \partial\Lambda$. Then it is well known (see Lemma 3.1) that

$$\int_0^1 \frac{1}{|u_n|^{\alpha}} \to \infty \qquad \text{if } \alpha > 2$$

while, when $0 < \alpha < 2$, the integral above can converge to a finite value.

This model case shows that it is worthwhile to distinguish between potentials which behave (as $|x| \to 0$) like $-\dfrac{1}{|x|^a}$, $a > 2$ (strong-forces) or like $-\dfrac{1}{|x|^b}$, $0 < b < 2$ (weak forces). The former is discussed in this section; the latter will be discussed in § 4. It is worth noticing that a virtue of the strong-force case is that the weak-force case will be handled by perturbing V with a strong-force potential.

We first treat with potentials satisfying (A1), (A2) and

(A3) $\exists\, \gamma > 2$, such that $V'(x) \cdot x \leqq -\gamma V(x) \quad \forall\, x \in \Omega$;

(A4) $\exists\, \beta > 2$ and $r > 0$ such that $V'(x) \cdot x \geqq -\beta V(x) \quad \forall\, 0 < |x| \leqq r$;

(A5) $\limsup_{|x| \to \infty} [V(x) + \tfrac{1}{2} V'(x) \cdot x] \leqq 0$.

We note that (A2) and (A3) imply

$$V(x) < 0 \qquad \forall\, x \in \Omega. \tag{3.1}$$

First of all we show that (A4) implies the so-called "strong-force condition" [10]. We recall that in [10] it is proved:

Lemma 3.1. *Suppose that V satisfies the "strong-force condition", i.e., that there are an $r > 0$ and an $\alpha \geqq 2$ such that*

$$V(x) \leqq -\frac{c}{|x|^{\alpha}} \qquad \forall\, 0 < |x| < r.$$

Then $\int_0^1 V(u_n)\,dt \to -\infty$ for any sequence u_n in Λ converging weakly and uniformly to $\bar{u} \in \partial\Lambda$.

Lemma 3.2. *If* (A4) *holds, then there exists a* $c_1 > 0$ *such that*

$$V(x) \leqq -\frac{c_1}{|x|^\beta} \quad \forall\, 0 < |x| \leqq r, \tag{3.2}$$

so that

$$\int_0^1 V(u_n)\, dt \to -\infty \quad \forall\, u_n \rightharpoonup u \in \partial\Lambda. \tag{3.3}$$

Proof. For $|y| = r$ define $\varrho_y : (0, 1] \to \mathbf{R}$ by

$$\varrho_y(\lambda) = -V(\lambda y) > 0.$$

From (A4) it follows that

$$\varrho_y'(\lambda) = -V'(\lambda y) \cdot y \leqq \beta \frac{V(\lambda y)}{\lambda} = -\beta \frac{\varrho_y(\lambda)}{\lambda};$$

hence

$$\varrho_y(\lambda) \geqq \varrho_y(1)\, \lambda^{-\beta} \geqq c_2 \lambda^{-\beta} \quad \forall\, 0 < \lambda \leqq 1$$

where

$$c_2 = \min\{-V(y) \mid |y| = r\}.$$

Then, letting $y = \frac{x}{|x|} r$, we obtain

$$V(x) = V\left(\frac{|x|}{r} y\right) = -\varrho_y\left(\frac{|x|}{r}\right) \leqq -\frac{c_1}{|x|^\beta} \quad \text{for } 0 < |x| \leqq r$$

with $c_1 = c_2 r^\beta$. The last statement follows from Lemma 3.1. □

Next we prove

Lemma 3.3. *Let* (A1)–(A5) *hold and let* $h > 0$. *Then*

(1) $M_h \neq \emptyset$ *and* M_h *is complete;*

(2) $\operatorname{cat}_{M_h}(M_h) = \infty$; *more precisely, for each* $m \geqq 0$, *there is a compact* $X \subset M_h$ *such that* $\operatorname{cat}_{M_h}(X) \geqq m$.

Proof. Let $u \in \Lambda$ be fixed. For $a > 0$,

$$g_u(a) \equiv g(au) = \int_0^1 [V(au) + \tfrac{1}{2} V'(au) \cdot au]\, dt.$$

According to (2.5), $\frac{d}{da} g_u(a) \neq 0$; hence g_u is strictly monotone. Using (A5) we immediately find that

$$\lim_{a \to \infty} g_u(a) \leqq 0.$$

Let $a \to 0^+$. Then $au(t) \to 0$ uniformly and (A4), (3.2) imply that

$$g_u(a) \geqq \left(1 - \frac{\beta}{2}\right) \int_0^1 V(au)\, dt \geqq \left(\frac{\beta}{2} - 1\right) \frac{c_1}{a^\beta} \int_0^1 \frac{1}{|u|^\beta}\, dt$$

and thus $g_u(a) \to +\infty$ as $a \to 0^+$. Then for all $h > 0$ the equation $g_u(a) = h$ has a unique solution $a(u)$ and $a(u)\, u \in M_h$.

Again from (2.5) it follows that a depends continuously on u. Then M_h is a deformation retract of Λ. From [9] it is known that $\mathrm{cat}_\Lambda(\Lambda) = \infty$ and that for all m there exists a compact $\tilde{X} \subset \Lambda$ with $\mathrm{cat}_\Lambda(\tilde{X}) \geqq m$. Thus (2) follows.

To show that M_h is complete, let us take a sequence $\{u_n\} \subset M_h$ such that $u_n \to \bar{u}$ in H. (Actually, it suffices to take $\{u_n\} \subset M_h$ such that $u_n \to \bar{u}$ weakly and uniformly in [0, 1]). We assert that $\bar{u} \in \Lambda$. Otherwise, there is an interval $I \subset [0, 1]$ and an integer $\bar{n} > 0$ such that $|u_n(t)| < r$ for all $t \in I$, and for all $n > \bar{n}$. Then, using (A4), we readily find for n large that

$$
\begin{aligned}
h &= \int_0^1 [V(u_n) + \tfrac{1}{2} V'(u_n) \cdot u_n]\, dt \\
&= \int_{[0,1]\setminus I} [V(u_n) + \tfrac{1}{2} V'(u_n) \cdot u_n]\, dt + \int_I [V(u_n) + \tfrac{1}{2} V'(u_n) \cdot u_n]\, dt \\
&\geqq \left(1 - \frac{\beta}{2}\right) \int_0^1 V(u_n)\, dt + c_1,
\end{aligned}
$$

in contradiction to Lemma 3.2. So $\bar{u} \in \Lambda$ and $g(\bar{u}) = \lim g(u_n) = 0$ (since $u_n \to \bar{u}$ uniformly). Then $\bar{u} \in M_h$, as required. □

To investigate the PS condition, we first show:

Lemma 3.4. *Let* (A1)–(A5) *hold and let* $u_n \in M_h$ *be such that*

$$f(u_n) \leqq C. \tag{3.4}$$

Then both $\|u_n'\|_{L^2}$ *and* $\|u_n\|_{L^\infty}$ *are bounded.*

Proof. Inserting the expression for f into (3.4) we find

$$\tfrac{1}{4} \int_0^1 |u_n'|^2\, dt \cdot \int_0^1 V'(u_n) \cdot u_n\, dt \leqq C. \tag{3.5}$$

Moreover, using (A3) ,we find

$$h = \int_0^1 [V(u_n) + \tfrac{1}{2} V'(u_n) \cdot u_n]\, dt \leqq \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_0^1 V'(u_n) \cdot u_n\, dt;$$

hence

$$\int_0^1 V'(u_n) \cdot u_n \geqq k, \quad \text{where} \quad k = \frac{h}{\dfrac{1}{2} - \dfrac{1}{\gamma}} > 0. \tag{3.6}$$

From (3.5) and (3.6) we deduce that

$$\|u_n'\|_{L^2} \leqq c_1. \tag{3.7}$$

To show that $\|u_n\|_{L^\infty}$ is bounded, we argue by contradiction. Let $u_n = \xi_n + w_n$, with

$$\xi_n = \int_0^1 u_n \, dt.$$

From (3.7) it follows that $\|w_n'\|_{L^2} \leqq c_1$, and up to a subsequence, $w_n \to \bar{w}$ uniformly. Consequently, if $\|u_n\|_{L^\infty} \to \infty$, then $|\xi_n| \to \infty$. Then from

$$\min |u_n(t)| \geqq |\xi_n| - \max |w_n(t)| \geqq |\xi_n| - c_2$$

it would follow that $|u_n(t)| \to \infty$ uniformly. Using (A5) we would then find that

$$\limsup_{n\to\infty} \int_0^1 [V(u_n) + \tfrac{1}{2} V'(u_n) \cdot u_n] \, dt \leqq 0, \tag{3.8}$$

contradicting $g(u_n) = h > 0$. □

Lemma 3.5. *Let* (A1)–(A5) *hold. Then f satisfies PS on M_h: for all $\{u_n\} \subset M_h$ such that*

$$f(u_n) \leqq c, \tag{3.9}$$

$$\nabla f_{|M_h}(u_n) \to 0 \tag{3.10}$$

there exists $u_{n_k} \to \bar{u} \in M_h$ such that $\nabla f_{|M_h}(\bar{u}) = 0$.

Proof. From (3.9) and Lemma 3.4 it follows that $\|u_n\| \leqq$ const. Then, up to a subsequence, $u_n \to \bar{u}$ uniformly and weakly in H, and $\bar{u} \in M_h$ (see the proof of Lemma 3.3). Thus

$$\nabla f_{|M_h}(u) = \nabla f(u_n) - \lambda_n \nabla g(u_n). \tag{3.11}$$

From (3.10) it follows that

$$\nabla f(u_n) - \lambda_n \nabla g(u_n) \to 0. \tag{3.12}$$

Multiplying (3.12) by u_n we have

$$(\nabla f(u_n) \mid u_n) - \lambda_n(\nabla g(u_n) \mid u_n) \to 0.$$

Since

$$(\nabla f(u) \mid u) - \lambda(\nabla g(u) \mid u) = \tfrac{1}{2}\left[\tfrac{1}{2}\int_0^1 |u'|^2 \, dt - \lambda\right] \cdot \int_0^1 [3V'(u) \cdot u + V''(u) \, u \cdot u,] \, dt$$

and since, by (2.4) and (A1),

$$\int_0^1 [3V'(u_n) \cdot u_n + V''(u_n) \, u_n \cdot u_n] \, dt \to \int_0^1 [3V'(\bar{u}) \cdot \bar{u} + V''(\bar{u}) \, \bar{u} \cdot \bar{u}] \, dt \neq 0$$

we have that

$$\tfrac{1}{2}\int_0^1 |u_n'|^2 \, dt - \lambda_n \to 0. \tag{3.13}$$

Finally, from (3.11) it follows that

$$\nabla f_{|M_h}(u_n) = -\left(\tfrac{1}{2}\int_0^1 V'(u_n)\cdot u_n\,dt\right)u_n'' + \left(\tfrac{1}{2}\int_0^1 |u_n'|^2\,dt - \tfrac{3}{2}\lambda_n\right)V'(u_n)$$
$$+\tfrac{1}{2}\left(\tfrac{1}{2}\int_0^1 |u_n'|^2\,dt - \lambda_n\right)V''(u_n)\,u_n.$$

Since $\int_0^1 V'(u_n)\cdot u_n\,dt \to \int_0^1 V'(\bar{u})\cdot\bar{u} > 0$, $V'(u_n)\to V'(\bar{u})$ and $V''(u_n)\,u_n \to V''(\bar{u})\,\bar{u}$, we deduce, using (3.12), that u_n'' converges (up to a subsequence) in H, and the result follows. □

We can now state the main result of this section.

Theorem 3.6. *Suppose that V satisfies* (A1)–(A5). *Then, for all $h > 0$, problem* (1.1), (1.2) *has a periodic solution $q(t)$, with $q(t) \neq 0$ for all t.*

Proof. According to the variational principle (Lemma 2.3) it suffices to find critical points u of $f_{|M_h}$ with $f(u) > 0$. These critical points will be found by using the LS theory. Let

$$\mathscr{K}_m = \{X \subset M_h \mid \; X \text{ is compact and } \quad \mathrm{cat}_{M_h}(X) \geqq m\}$$

and

$$b_m = \inf_{X\in\mathscr{K}_m} \max_X f. \tag{3.14}$$

Note that $\mathscr{K}_m \neq \emptyset$ for all m by Lemma 3.3(2). Moreover b_m is a non-decreasing sequence with $0 \leqq b_m < +\infty$ for all m.

Since PS holds for $f_{|M_h}$ (Lemma 3.5), the LS theory, extended to C^1 manifolds in [17], implies that each b_m is a critical level and, if

$$b \equiv b_m = b_{m+1} = \ldots = b_{m+k},$$

then

$$\mathrm{cat}_{M_h}(Z_b) \geqq k+1,$$

where

$$Z_b = \{u \in M_h \mid f(u) = b, \quad \nabla f_{|M_h}(u) = 0\}.$$

We assert that $b_3 > 0$. If not, the preceding remark with $b \equiv b_1 = b_2 = b_3$ yields

$$\mathrm{cat}_{M_h}(Z_0) \geqq 3.$$

But Lemma 2.1 implies that $Z_0 = \{u \equiv \text{const}\} \cap M_h$. The arguments of Lemma 3.3 show then that

$$Z_0 \cong S^{N-1},$$

which implies that $\mathrm{cat}_{M_h}(Z_0) = 2$, a contradiction. Then the level b_3 carries a critical point u of $f_{|M_h}$ such that $f(u) > 0$, and this completes the proof. □

Example 3.7. If $V(x) = -\dfrac{1}{|x|^b}$ with $b > 2$ then (A1)–(A5) hold. Note that for $b = 2$, $V_2(x) = -\dfrac{1}{|x|^2}$ satisfies

$$3V_2'(x) \cdot x + V_2''(x)\, x \cdot x = 0 \quad \forall\, x \neq 0. \tag{3.15}$$

On the other hand, all the periodic solutions of

$$q'' + 2\frac{q}{|q|^4} = 0$$

have energy $h = 0$.

Example 3.8. Let

$$V(x) = -\frac{1}{|x|^b} + W(x)$$

with $b = 2$ and $W \in C^2(\Omega, \mathbf{R})$. Then (A1) and (A5) become respectively

$$3W'(x) \cdot x + W''(x)\, x \cdot x < \frac{b(b-2)}{|x|^b} \quad \forall\, x \neq 0, \tag{3.16}$$

$$W(x) + \tfrac{1}{2} W'(x) \cdot x \leqq 0 \quad \text{as } |x| \to \infty. \tag{3.17}$$

Conditions (A2), (A3) and (A4) are verified provided that

$$W'(x) \cdot x > -\frac{b}{|x|^b} \quad \forall\, x \neq 0, \tag{3.18}$$

$$\exists\, \gamma > 2\colon W'(x) \cdot x + \gamma W(x) \leqq \frac{\gamma - b}{|x|^b}, \tag{3.19}$$

$$\exists\, \beta \in (2, \gamma],\ r > 0\colon \quad W'(x) \cdot x + \beta W(x) \geqq \frac{\beta - b}{|x|^b} \quad \forall\, 0 < |x| \leqq r. \tag{3.20}$$

For example, if $W(x) = -\dfrac{1}{|x|^c}$, $c > 2$, then (3.16)–(3.20) hold. In fact, (3.16), (3.17) and (3.18) are trivially satisfied; to satisfy (3.19) and (3.20) it suffices to take $\gamma > \max\{b, c\}$ and $2 < \beta < \min\{b, c\}$.

Moreover, let us remark that if W is smooth on all $\mathbf{R}^N$, then (3.16), (3.18)–(3.20) impose no restrictions near $x = 0$. This is clear for (3.16) and (3.18) because $b > 2$. So for (3.19) and (3.20) it suffices to take any β, γ satisfying $2 < \beta < b < \gamma$.

§ 4. Existence Results (Weak Forces)

We study here the case when V behaves like $-\dfrac{1}{|x|^a}$ with $0 < a < 2$. In particular, this will include the interesting case of perturbations of the Kepler potential $-\frac{1}{|x|}$, such as

$$V(x) = -\frac{1}{|x|} + W(x).$$

Note that in the present case the meaning of a solution of (1.1)–(1.2) must be specified, because solutions passing through the singularity $x = 0$ could arise. The following definition has been introduced in [4].

Definition 4.1. $q \in H^1(S^1; \mathbf{R}^N)$ is a *solution* of (1.1), (1.2) if

(1) the set $\{t \in S^1 \mid q(t) = 0\}$ has zero measure;

(2) in the set $\{t \in S^1 \mid q(t) \neq 0\}$ q is of class C^2 and satisfies (1.1), (1.2).

In addition to (A1) and (A2) we suppose that

(A3′) $\exists\, \alpha \in (0, 2)$ such that $V'(x) \cdot x \geqq -\alpha V(x) \quad \forall\, x \in \Omega$,

(A4′) $\exists\, \delta \in (0, 2)$ and $r > 0$ such that $V'(x) \cdot x \leqq -\delta V(x) \quad \forall\, 0 < |x| \leqq r$,

(A5′) $\liminf_{|x| \to \infty} [V(x) - + \frac{1}{2} V'(x) \cdot x] \geqq 0$.

We will follow here a procedure similar to that of § 3. First, as in Lemma 3.1 we have:

Lemma 4.2. *If* (A3′), (A4′) *hold, then there is a* $c_2 > 0$ *such that*

$$V(x) \leqq -\frac{c_2}{|x|^\alpha} \quad \forall\, 0 < |x| \leqq r.$$

Proof. It suffices to repeat the arguments of Lemma 3.1 using (A3′) and taking into account that (A4′) yields $\min\{-V(y) : |y| = r\} < 0$. □

We explicitly remark that (3.1) and (3.3) do not necessarily hold in the present situation.

The next lemma replaces Lemma 3.3.

Lemma 4.3. *Let* (A1), (A2), (A3′)–(A5′) *hold and let* $h < 0$. *Then*

(1) $M_h \neq \emptyset$;

(2) $\operatorname{cat}_{M_h}(M_h) = \infty$; *more precisely, for all* $m \geqq 0$ *there is a compact* $X \subset M_h$ *such that* $\operatorname{cat}_{M_h}(X) \geqq m$.

Proof. From (A1) it still follows that

$$g_u(a) \equiv g(au) = \int_0^1 [V(au) + \tfrac{1}{2} V'(au) \cdot au]\, dt$$

is strictly monotone, and from (A5′) it follows that

$$\liminf_{a \to \infty} g_u(a) \geqq 0.$$

Then, using (A3′) and Lemma 4.2 we deduce that, for each $u \in \Lambda$,

$$g_u(a) \leqq \left(1 - \frac{\delta}{2}\right) \int_0^1 V(au)\, dt \leqq -\left(1 - \frac{\delta}{2}\right) \frac{c_2}{a^\alpha} \int_0^1 \frac{1}{|u|^\alpha}\, dt$$

and thus $g_u(a) \to -\infty$ as $a \to 0^+$. Therefore for all $h < 0$, the equation $g_u(a) = h$ has a unique solution and $M_h \neq \emptyset$. The remainder of the proof is the same as that of Lemma 3.3. □

In the present situation, as we have already remarked, M_h is no longer complete, so the LS theory cannot be directly applied. To deal with such a situation we modify V setting

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{|x|^2}, \qquad \varepsilon > 0.$$

Remark 4.4. Let

$$g_\varepsilon(u) = \int_0^1 [V_\varepsilon(u) + \tfrac{1}{2} V'_\varepsilon(u) \cdot u] \quad \text{and} \quad M_{h,\varepsilon} = \{u \in \Lambda : g_\varepsilon(u) = h\}.$$

According to Example 3.7,

$$g_\varepsilon(u) = g(u),$$

and hence $M_{h,\varepsilon} = M_h$. Moreover,

$$V'_\varepsilon(x) \cdot x = V'(x) \cdot x + 2 \frac{\varepsilon}{|x|^2} \geqq V'(x) \cdot x > 0.$$

Therefore Lemma 2.3 applies and for all $h < 0$, the critical points of f_ε on M_h such that $f_\varepsilon(u) > 0$ give rise to periodic solutions of

$$q'' + V'(q) + 2\varepsilon \frac{q}{|q|^4} = 0 \tag{4.1}$$

with energy

$$\tfrac{1}{2} |q'(t)|^2 + V(q(t)) - \frac{\varepsilon}{|q|^2} = h. \tag{4.2}$$

In order to find critical points of f_ε on M_h we state some lemmas which are the counterparts of Lemmas 3.4, 3.5. We always suppose that (A1), (A2), (A3′)–(A5′) hold.

Lemma 4.5. *If* $u_n \in M_h$ *is such that*

$$f_\varepsilon(u_n) \leqq C, \tag{4.3}$$

then $\|u'_n\|_{L^2}$ *and* $\|u_n\|_{L^\infty}$ *are bounded.*

Proof. From (4.3) it follows that

$$C \geqq f_\varepsilon(u_n) = \tfrac{1}{4} \int_0^1 |u'_n|^2\, dt \cdot \int_0^1 \left[V'(u_n) \cdot u_n + 2 \frac{\varepsilon}{|u_n|^2}\right] dt$$

$$\geqq \tfrac{1}{4} \int_0^1 |u'_n|^2\, dt \cdot \int_0^1 V'(u_n) \cdot u_n\, dt. \tag{4.4}$$

If $u_n \in M_h$, then (A3′) implies that

$$h = \int_0^1 [V(u_n) + \tfrac{1}{2} V'(u_u) \cdot u_n]\, dt \geqq \left(\tfrac{1}{2} - \frac{1}{\alpha}\right) \int_0^1 V'(u_n) \cdot u_n \, dt;$$

hence

$$\int_0^1 V'(u_n) \cdot u_n \geqq k := \frac{h}{\tfrac{1}{2} - \dfrac{1}{\alpha}} > 0.$$

This inequality and (4.4) yield

$$\int_0^1 |u_n'|^2 \, dt \leqq c_1 := \frac{4C}{k}. \tag{4.5}$$

From (4.5) and (A5′) it then follows, as in Lemma 4.3, that $\|u_n\|_{L_\infty} \leqq c_2$. □

Lemma 4.6. *f_ε satisfies PS on M_h.*

Proof. Let $u_n \in M_h$ be a PS-sequence. By Lemma 4.5 one has

$$\|u_n\| \leqq C;$$

hence $u_n \to \bar{u}$ uniformly and weakly in H. We assert that $\bar{u} \in M_h$. Indeed, in view of (2.4), it suffices to show that $\bar{u} \in \Lambda$. We shall prove this by contradiction. First, let $\bar{u} \equiv 0$. Then $u_n \to 0$ uniformly and (A4′) and Lemma 4.2 imply, for n large, that

$$h = \int_0^1 [V(u_n) + \tfrac{1}{2} V'(u_n) \cdot u_n]\, dt \leqq \left(1 - \frac{\delta}{2}\right) \int_0^1 V(u_n)\, dt$$

$$\leqq -c_2 \left(1 - \frac{\delta}{2}\right) \int_0^1 \frac{1}{|u_n|^\alpha}\, dt.$$

Since the last term tends to $-\infty$, we have a contradiction.

Next, let $\bar{u} \in \partial\Lambda$ with $\bar{u} \not\equiv 0$ (hence $\bar{u} \not\equiv$ const.). Then

$$f_\varepsilon(u_n) = \tfrac{1}{2} \int_0^1 |u_n'|^2 \cdot \int_0^1 [h - V_\varepsilon(u_n)].$$

Since

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{|x|^2} \leqq -\frac{\varepsilon}{|x|^2}, \quad \forall\, 0 < |x| < r$$

V_ε satisfies the strong-force condition and

$$\int_0^1 [h - V_\varepsilon(u_n)] \to \infty.$$

Moreover,

$$0 < \int_0^1 |\bar{u}'|^2 \leqq \liminf \int_0^1 |u_n'|^2,$$

and we reach a contradiction, proving that $\bar{u} \in M_h$. The rest of the proof follows as in Lemma 3.5. □

We explicitly remark that the arguments of Lemma 4.6 actually show that the sublevels $\{u \in M_h \mid f_\varepsilon(u) < c\}$ are complete.

The preceding result allows us to apply the LS theory to f_ε on M_h. Repeating the arguments of Theorem 3.6 we find

Lemma 4.7. *For arbitrary* $\varepsilon > 0$ *there exists* $u_\varepsilon \in M_h$ *such that* $\nabla f_{\varepsilon|M_h}(u_\varepsilon) = 0$,

$$f_\varepsilon(u_\varepsilon) = b_m = \inf_{X \in \mathscr{K}_m} \max_X f_\varepsilon, \qquad m \geqq 3,$$

and $u_\varepsilon \neq$ const. Let

$$\omega_\varepsilon^2 = \frac{\int_0^1 V_\varepsilon'(u_\varepsilon) \cdot u_\varepsilon \, dt}{\int_0^1 |u_\varepsilon'|^2 \, dt}; \tag{4.6}$$

then $y_\varepsilon(t) = u_\varepsilon(\omega_\varepsilon t)$ *satisfies*

$$y_\varepsilon''(t) + V_\varepsilon'(y_\varepsilon(t)) = 0, \tag{4.7}$$

$$\tfrac{1}{2} \omega_\varepsilon^2 \, |u_\varepsilon'(t)|^2 + V_\varepsilon(u_\varepsilon(t)) = h. \tag{4.8}$$

Remark 4.8. It will be convenient to take u_ε in such a way that

$$f_\varepsilon(u_\varepsilon) = \inf_{X \in \mathscr{K}_m} \max_X f_\varepsilon$$

where $m \geqq 3$ is fixed independently of ε. In particular, in the following lemma we will take $m = 3$. □

In the sequel, our plan is to show that u_ε converges to some u^* which gives rise to a solution y^* of (1.1)–(1.2). For this program we require some estimates.

Lemma 4.9. *There exists a* $k > 0$ *such that* $\|u_\varepsilon\| \leqq k$ *and* $u_\varepsilon \to u^*$ *uniformly.*

Proof. As anticipated in Remark 4.8, we have that

$$b_\varepsilon := f_\varepsilon(u_\varepsilon) = \inf_{\mathrm{cat}_{M_h}(A) \geqq 3} \max_{u \in A} f_\varepsilon(u).$$

Since $V_\varepsilon(x) = V(x) - \dfrac{\varepsilon}{|x|^2} \geqq V(x) - \dfrac{1}{|x|^2} \quad \forall\, \varepsilon \leqq 1$, it follows that

$$f_\varepsilon(u) \leqq f_1(u) \qquad \forall\, 0 < \varepsilon \leqq 1, \ \forall\, u \in \Lambda.$$

Thus

$$f_\varepsilon(u_\varepsilon) = b_\varepsilon \leqq b := \inf_{\mathrm{cat}_{M_h}(A) \geqq 3} \max_{u \in A} f_1 \qquad \forall\, \varepsilon \in (0, 1], \tag{4.9}$$

and the result follows from Lemma 4.5. □

Lemma 4.10. (1) $V(u^*(t)) \not\equiv h$;

(2) $u^*(t) \not\equiv 0$.

Proof. (1) If not, $u^*(t) \neq 0$ for all t so that

$$V(u_\varepsilon(t)) \to V(u^*(t)) = h,$$

$$V'(u_\varepsilon(t)) \cdot u_\varepsilon(t) \to V'(u^*(t)) \cdot u^*(t)$$

uniformly. Therefore

$$h = g(u_\varepsilon) = \int_0^1 [V(u_\varepsilon) + \tfrac{1}{2} V'(u_\varepsilon) \cdot u_\varepsilon] \, dt$$

$$\to \int_0^1 [V(u^*) + \tfrac{1}{2} V'(u^*) \cdot u^*] \, dt = h + \tfrac{1}{2} \int_0^1 V'(u^*) \cdot u^* \, dt.$$

Hence $\int_0^1 V'(u^*) \cdot u^* = 0$, in contradiction with (A2).

(2) If not, $u^* \equiv 0$ and $u_\varepsilon \to 0$ uniformly. Then, using (A4′) and the fact that $u_\varepsilon \in M_h$, we find (for ε small enough) that

$$h = \int_0^1 [V(u_\varepsilon) + \tfrac{1}{2} V'(u_\varepsilon) \cdot u_\varepsilon] \, dt \leqq \left(1 - \frac{\delta}{2}\right) \int_0^1 V(u_\varepsilon),$$

so that

$$\int_0^1 V(u_\varepsilon) \geqq \frac{h}{1 - \frac{\delta}{2}}.$$

On the other hand, since $u_\varepsilon \to 0$ uniformly, from Lemma 4.1 we infer that

$$\int_0^1 V(u_\varepsilon) \to -\infty \qquad \text{as } (\varepsilon \to 0^+),$$

a contradiction. □

Lemma 4.11. *There are numbers δ, $\Delta > 0$ such that*

$$\delta \leqq \omega_\varepsilon \leqq \Delta.$$

Proof. From Lemma 4.10 we deduce that there exists a closed interval I such that I has positive measure and such that

$$u^*(t) \neq 0, \qquad V(u^*(t)) \neq h \qquad \forall\, t \in I.$$

Integrating (4.8) over I, we find that

$$\tfrac{1}{2} \omega_\varepsilon^2 \int_I |u_\varepsilon'|^2 \, dt + \int_I V_\varepsilon(u_\varepsilon) \, dt = h\,|I|. \tag{4.10}$$

Since

$$\int_I |u_\varepsilon'|^2\,dt \leqq \int_0^1 |u_\varepsilon'|^2\,dt \leqq C,$$

equation (4.10) implies that

$$\tfrac{1}{2}\omega_\varepsilon^2 \geqq \frac{\int_I [h - V_\varepsilon(u_\varepsilon)]\,dt}{C}.$$

But $u_\varepsilon \to u^*$ uniformly on I and $u^* \neq 0$, so that

$$\int_I [h - V_\varepsilon(u_\varepsilon)]\,dt \;\to\; \int_I [h - V(u^*)]\,dt \qquad (\varepsilon \to 0^+).$$

From (4.8) $h - V_\varepsilon(u_\varepsilon) \geqq 0$ and thus, from the definition of I, one has

$$\int_I [h - V(u^*)]\,dt > 0.$$

This shows that $\omega_\varepsilon \geqq \delta > 0$. To prove that $\omega_\varepsilon \leqq \Delta$, we start by using (4.8) and (4.9) to find that

$$\tfrac{1}{2}\omega_\varepsilon^2 \int_0^1 |u_\varepsilon'|^2\,dt = \int_0^1 [h - V_\varepsilon(u_\varepsilon)] = \frac{f_\varepsilon(u_\varepsilon)}{\tfrac{1}{2}\int_0^1 |u_\varepsilon'|^2\,dt} \leqq \frac{b}{\tfrac{1}{2}\int_0^1 |u_\varepsilon'|^2\,dt}.$$

Then

$$\tfrac{1}{4}\left(\int_0^1 |u_\varepsilon'|^2\,dt\right)^2 \leqq \frac{b}{\omega_\varepsilon^2}.$$

If $\omega_\varepsilon \to \infty$, it follows that $\int_0^1 |u_\varepsilon'|^2\,dt \to 0$ and hence that both u_ε and y_ε converge uniformly to some constant $\xi \in \mathbf{R}^N$. From Lemma 4.10 it follows that $\xi \neq 0$ and $V(\xi) \neq h$. Using (4.7) we have (since $V_\varepsilon'(y_\varepsilon)$ converges uniformly to $V'(\xi)$) that y_ε converges in C^2 to ξ. Finally, passing to the limit into (4.8), we find that $V(\xi) = h$, a contradiction. □

We are now in a position to state the main result of this section:

Theorem 4.12. *Suppose that* (A1), (A2), (A3′), (A4′), (A5′) *hold. Then for all* $h < 0$, *problem* (1.1), (1.2) *has a non-constant periodic solution.*

Proof. We shall show that u^* gives rise to a solution of (1.1), (1.2) in the sense of Definition 4.1. Although this result follows routinely from the preceding lemmas, we present the complete proof for the convenience of the readers. Let

$$J = \{t \in [0, 1] \mid u^*(t) = 0\}.$$

From (4.10), with J replacing I, and from Lemmas 4.9 and 4.11 we deduce that

$$\int_J V_\varepsilon(u_\varepsilon)\,dt = |J|\,h - \tfrac{1}{2}\omega_\varepsilon^2 \int_J |u_\varepsilon'|^2\,dt \geqq |J|\,h - \tfrac{1}{2}\delta^2 k^2. \tag{4.11}$$

But $u_\varepsilon \to 0$ uniformly on J and hence, if J has positive measure, we obtain from Lemma 4.1 that

$$\int_J V_\varepsilon(u_\varepsilon)\,dt \to -\infty,$$

in contradiction to (4.11). Thus J has zero measure.

Let $K_n \subset [0,1] \setminus J$ be an increasing sequence of compact sets with

$$\bigcup_{n \geqq 1} K_n = [0,1] \setminus J,$$

and set

$$K_n^* = \{u^*(t) \mid t \in K_n\}.$$

Each $K_n^* \subset \Omega$ is compact and has a neighbourhood $\mathcal{N}_n$ such that $\bar{\mathcal{N}}_n \subset \Omega$. Then $V_\varepsilon \to V$ in $C^1(\mathcal{N}_n, \mathbf{R})$ and therefore

$$V_\varepsilon'(u_\varepsilon(t)) \to V'(u^*(t)) \quad \text{uniformly in } K_n.$$

Since u_ε satisfies

$$\omega_\varepsilon^2 u_\varepsilon'' + V_\varepsilon'(u_\varepsilon) = 0$$

and since $\omega_\varepsilon \to \omega^* \neq 0$ (Lemma 4.11), it follows that

$$u_\varepsilon \to u^* \quad \text{in } C^2(K_n, \mathbf{R}^N),$$

$$\omega^* u^{*\prime\prime} + V'(u^*) = 0 \quad \text{on } K_n.$$

Since $\cup K_n = [0,1] \setminus J$, it follows that

$$\omega^* u^{*\prime\prime} + V'(u^*) = 0 \quad \forall\, t \in [0,1] \setminus J$$

and $y^*(t) = u^*(\omega^* t)$ satisfies

$$y^{*\prime\prime} + V'(y^*) = 0 \quad \forall\, t \in [0,1] \setminus J.$$

The energy conservation (1.2) follows directly from (4.8). □

Example 4.13. Any $V(x) = -\dfrac{1}{|x|^a}$ with $0 < a < 1$ satisfies (A1), (A2), (A3)–(A5′).

To explain the significance of Theorem 4.11 we now discuss a specific example concerning a perturbation of the Kepler potential.

Example 4.14. Let us take

$$V(x) = -\frac{1}{|x|} + W(x)$$

with $W \in C^2(\Omega, \mathbf{R})$. In this case (A1) and (A5′) become, respectively,

$$3W'(x) \cdot x + W''(x)\,x \cdot x > -\frac{1}{|x|} \quad \forall\, x \in \Omega, \tag{4.12}$$

$$\liminf_{|x| \to \infty} \left[W(x) + \tfrac{1}{2} W'(x) \cdot x\right] \geqq 0. \tag{4.13}$$

Similarly (A2) and (A3′) become, respectively,

$$W'(x) \cdot x > -\frac{1}{|x|} \quad \forall\, x \in \Omega, \tag{4.14}$$

$$\exists\, 0 < \alpha < 2 \quad \text{such that} \quad W'(x) \cdot x + \alpha W(x) \geqq \frac{\alpha - 1}{|x|} \quad \forall\, x \in \Omega. \tag{4.15}$$

Condition (A4′) gives

$$\exists\, \alpha \leqq \delta < 2 \quad \text{and} \quad r > 0 \quad \text{such that} \quad W'(x) \cdot x + \delta W(x) \leqq \frac{\delta - 1}{|x|}$$
$$\forall\, 0 < |x| \leqq r. \tag{4.16}$$

For example, any $W(x) = -\dfrac{1}{|x|^a}$ with $0 < a < 2$ satisfies the above conditions (take α, δ such that $0 < \alpha < a < \delta < 2$ and $0 < \alpha < 1 < \delta < 2$).

The case discussed in Example 4.13 allows us to deduce Theorem 1.1:

Proof of Theorem 1.1. It suffices to note that (W1), (W2) and (W4) imply (4.12), (4.14) and (4.13), respectively. To satisfy (4.15) we take $\alpha = 1$ and use (W3). Lastly, since W is smooth at $x = 0$, it follows that $W'(x) \cdot x + \delta W(x)$ is bounded in any neighbourhood of the origin for any δ, so that (4.16) holds for $\delta > 1$. □

§ 5. Other Existence Results

From the conservation of energy it follows that any solution q of (1.1), (1.2) is such that

$$V(q(t)) \leqq h \quad \forall\, t.$$

Therefore it is natural to expect that assumptions (A1)–(A4) or (A1), (A2), (A3′), (A4′) need to be verified only in

$$\{x \in \Omega \mid V(x) \leqq h\} := \Omega_h. \tag{5.1}$$

We discuss only the weak-force case; in the strong-force case, $V < 0$ and $h > 0$ imply $\Omega_h = \Omega$.

Let us denote by D_h the connected component of Ω_h such that $0 \in \overline{D}_h$ and let $\partial D_h = \{x \in D_h \mid V(x) = h\}$.

Theorem 5.1. *Let $h < 0$ be given. Suppose that $\overline{D}_h$ is compact and that $V: \Omega \to \mathbf{R}$ satisfies* (A4′) *and*

($A1_h$) $3V'(x) \cdot x + V''(x)\, x \cdot x > 0 \quad \forall\, x \in D_h$;

($A2_h$) $V'(x) \cdot x > 0 \quad \forall\, x \in D_h$;

($A3'_h$) $\exists\, 0 < \alpha' < 2$ *such that* $V'(x) \cdot x \geqq -\alpha' V(x) \quad \forall\, x \in D_h$;

($A6_h$) $V \in C^4$ *in a neighbourhood of* ∂D_h *and* $\max_{\xi \in \partial D_h} [V''(\xi)\, \xi \cdot \xi] < 0$.

Then corresponding to such a value of h, (1.1), (1.2), *has a periodic solution.*

Proof. Since $V'(\xi)\cdot\xi > 0$ on ∂D_h, it follows that D_h is star-shaped with respect to $x = 0$. Set

$$G_h = \Omega \setminus D_h .$$

For every $x \in G_h$ there exists a unique $\xi \in \partial D_h$ and $s > 1$ such that

$$x = s\xi .$$

We assert that there exist functions $A, B, S \in C^2(\partial D_h)$ such that the modified potential

$$V(x) = \begin{cases} V(x), & x \in D_h \\ \Phi(x) = -\dfrac{A(\xi)}{|s - S(\xi)|} + B(\xi), & x \in G_h \end{cases}$$

is of class C^2 in Ω. To see this, it suffices to take

$$A(\xi) = \frac{4(V'(\xi)\cdot\xi)^3}{(V''(\xi)\,\xi\cdot\xi)^2},$$

$$B(\xi) = h - \frac{2(V'(\xi)\cdot\xi)^2}{V''(\xi)\,\xi\cdot\xi},$$

$$S(\xi) = 1 + \frac{2V'(\xi)\cdot\xi}{V''(\xi)\,\xi\cdot\xi}.$$

Let us remark that $(\mathrm{A}1_h)$ — $(\mathrm{A}6_h)$ imply that

$$A(\xi) > 0, \qquad B(\xi) > h - \theta > h, \qquad S(\xi) < 1$$

where $\delta = \frac{2}{3}\alpha' h$. We now rescale $\tilde V$ by setting

$$\hat V(x) = \tilde V(x) + L$$

where $L = \max(\theta - h, 0) \geqq 0$. Since $\hat V(x) = V(x) + L$ in D_h and since $L \geqq 0$, it follows immediately that $\hat V$ satisfies (A1), (A2) and (A3′) in D_h. For $\hat\delta > \delta$, we have

$$V'(x)\cdot x \leqq -\delta V(x) = -\hat\delta V(x) + (\hat\delta - \delta)\, V(x).$$

Since $V(x) \to -\infty$ as $x \to 0$, we can take r so small that

$$V'(x)\cdot x \leqq -\hat\delta V(x) - \hat\delta L = -\hat\delta(V(x) + L),$$

from which (A4′) follows.

Next, for $x = s\xi \in G_h$, we obtain

$$\hat V'(x)\cdot x = \frac{d}{d\lambda}\tilde V(\lambda x)_{|\lambda=1} = \frac{d}{d\lambda}\Phi(\lambda x)_{|\lambda=1} = \frac{sA(\xi)}{|s - S(\xi)|}, \tag{5.1}$$

$$\hat V''(x)\,x\cdot x = -\frac{2s^2A(\xi)}{|s - S(\xi)|^3}.$$

Hence

$$3\hat{V}'(x)\cdot x+\hat{V}''(x)\,x\cdot x=\frac{sA(\xi)\,(3\,|s-S(\xi)|-2s)}{|s-S(\xi)|^3}>0\qquad\forall\, s\geqq 1$$

and (A1) holds.

Since $A(\xi)>0$, (A2) follows from (5.1). Since $\overline{D}_h$ is compact and $B(\xi)\geqq h-\theta$, it follows that

$$\liminf_{|x|\to\infty}\,[\hat{V}(x)+\tfrac{1}{2}\,\hat{V}'(x)\cdot x]=\liminf_{s\to\infty}\left[-\frac{A(\xi)}{|s-S(\xi)|}+B(\xi)+L+\tfrac{1}{2}\,\frac{sA(\xi)}{|s-S(\xi)|^2}\right]$$
$$\geqq h-\theta+L\geqq 0,$$

so that (A5′) holds.

Finally to prove that (A3′) holds in G_h we need only notice that

$$\frac{s}{|s-S(\xi)|}\geqq\min\left\{1,\ \inf_{\xi\in\partial G_h}\frac{1}{|1-S(\xi)|}\right\}\equiv\hat{\alpha}$$

where $0<\hat{\alpha}\leqq 1$; hence

$$\begin{aligned}\hat{V}'(x)\cdot x&=\frac{sA(\xi)}{|s-S(\xi)|^2}\\&\geqq\hat{\alpha}\,\frac{A(\xi)}{|s-S(\xi)|}\\&\geqq\hat{\alpha}\left[\frac{A(\xi)}{|s-S(\xi)|}-(B(\xi)+L)\right]\\&=-\hat{\alpha}\hat{V}(x).\end{aligned}$$

We are now in a position to apply Theorem 4.12, from which follows the existence of a periodic solution of

$$\begin{aligned}&q''+\hat{V}'(q)=0,\\&\tfrac{1}{2}\,|q'|^2+\hat{V}(q)=e\end{aligned}\tag{5.2}$$

for all $e<0$. Since $L<-h$, it follows that $e=h+L<0$ is an admissible value of the energy. For such a choice of e, (5.2) becomes

$$\begin{aligned}&q''+\tilde{V}'(q)=0,\\&\tfrac{1}{2}\,|q'|^2+\tilde{V}(q)=h\end{aligned}$$

and hence $\tilde{V}(q(t))\leqq h$. For $x=s\xi\in G_h$ one has that

$$\tilde{V}(x)=\Phi(s\xi)>\Phi(\xi)=h.$$

Therefore $q(t)\in D_h$, $\tilde{V}(q)=V(q)$ and q satisfies (1.1), (1.2). □

As an application of Theorem 5.1 we can prove Theorem 1.2.

Proof of Theorem 1.2. We first notice that D_h is compact if ε is small enough. Hence there exist constants m, m', m'', M, M', M'' such that

$$m \leqq U(x) \leqq M, \qquad m' \leqq U'(x) \cdot x \leqq M', \qquad m'' \leqq U''(x)\, x \cdot x \leqq M''$$

for all $x \in \bar{D}_h$. For $|\varepsilon|$ small enough we immediately find that ($A1_h$), ($A2_h$), ($A3'_h$), ($A6_h$) are satisfied. □

Remark 5.2. We notice that $\bar{\varepsilon}$ of Theorem 1.2 can be explicitly estimated in terms of h and of the C^2 norm of U. For example, if $\|U\|_{L^\infty} = \|U'\|_{L^\infty} = \|U''\|_{L^\infty} = 1$, it is not difficult to see that $\bar{\varepsilon}$ can be taken to be $\min\left\{\frac{|h|}{2}, \frac{|h|^3}{4(3|h|+2)}\right\}$.

The following example is related to Theorem 1.1 and can be obtained as a straightforward application of Theorem 5.1:

Example 5.3. Let $h < 0$ be given and let $V(x) = -\frac{1}{|x|} + W(x)$ with W smooth in Ω and such that

(1) $V(x) \to -\infty$ as $|x| \to 0$ and $W(x) \geqq 0 \quad \forall\, x$ with $|x| = \frac{1}{|h|}$;

(2) $3W'(x) \cdot x + W''(x)\, x \cdot x > -\frac{1}{|x|} \quad \forall\, |x| \leqq \frac{1}{|h|}$;

(3) $W'(x) \cdot x > -\frac{1}{|x|} \quad \forall\, |x| \leqq \frac{1}{|h|}$;

(4) $W''(x)\, x \cdot x < \frac{1}{|x|} \quad \forall\, |x| \leqq \frac{1}{|h|}$.

Then (1.1), (1.2) has at least one periodic solution.

We point out that (1) is used only to show that $\{x \mid V(x) \leqq h\} \subset \left\{x \,\middle|\, |x| \leqq \frac{1}{|h|}\right\}$.

Acknowledgment. This work was supported by Ministero Pubblica Istruzione, Gruppo nazionale "Calcolo delle Variazioni".

References

1. A. Ambrosetti, *Esistenza di infinite soluzioni per problemi non lineari in assenza di parametro*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **52** (1972), 402–409.
2. A. Ambrosetti & V. Coti Zelati, *Critical points with lack of compactness and singular dynamical systems*, Ann. Mat. Pura e Appl. (3) **149** (1987), 237–259.
3. A. Ambrosetti & G. Mancini, *Solutions of minimal period for a class of convex Hamiltonian systems*, Math. Ann. **255** (1981), 405–421.

4. A. BAHRI & P. H. RABINOWITZ, *A minimax method for a class of Hamiltonian systems with singular potentials*, J. Funct. Anal. **82** (1989), 412–428.
5. V. BENCI & F. GIANNONI, *Periodic solutions of prescribed energy for a class of Hamiltonian systems with singular potentials*, J Differential Equations **82** (1989), 60–70.
6. V. COTI ZELATI, *Dynamical systems with effective-like potentials*, Nonlinear Anal. TMA **12** (1988), 209–222.
7. M. DEGIOVANNI & F. GIANNONI, *Periodic solutions of dynamical systems with Newtonian type potentials*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), 467–494.
8. M. DEGIOVANNI, F. GIANNONI & A. MARINO, *Dynamical systems with Newtonian type potentials*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **81** (1987), 271–278.
9. E. FADELL & S. HUSSEINI, *On the category of free loop space*, preprint, University of Wisconsin — Madison.
10. W. B. GORDON, *Conservative dynamical systems involving strong forces*, Trans. Amer. Mat. Soc. **204** (1975), 113–135.
11. C. GRECO, *Periodic solutions of a class of singular Hamiltonian systems*, Nonlinear Analysis T.M.A. **12** (1988), 259–270.
12. C. GRECO, *Remarks on periodic solutions, with prescribed energy, for singular Hamiltonian systems*, Comment. Math. Univ. Carolin. **28** (1987), 661–672.
13. J. A. HEMPEL, *Multiple solutions for a class of nonlinear boundary value problem*, Indiana Univ. Math. J. **30** (1971), 983–996.
14. P. MAJER, *Ljusternik-Schnirelman theory with local Palais — Smale condition and singular dynamical systems*, preprint, Scuola Normale Superiore — Pisa.
15. J. MOSER, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math. **23** (1970), 609–636.
16. Z. NEHARI, *Characteristic values associated with a class of nonlinear second-order differential equations*, Acta Math. **105** (1961), 141–175.
17. A. SZULKIN, *Ljusternik-Schnirelmann theory on C^1 manifolds*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **5** (1988), 119–139.
18. S. TERRACINI, *An homotopical index and multiplicity of periodic solutions to dynamical system with singular potential*, preprint, CEREMADE — Paris.
19. E. W. C. VAN GROESEN, *Analytical mini-max methods for Hamiltonian break orbits of prescribed energy*, J. Math. Anal. Appl. **132** (1988), 1–12.

Scuola Normale Superiore
Pisa

and

SISSA
Strada Costiera, 11
34014 Trieste, Italy

(Received January 9, 1989)