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# *On Positive Solutions of Emden Equations in Cone-like Domains*

CATHERINE BANDLE & MATTS ESSÉN

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## 1. Introduction

Let $x = (x_1, \dots, x_n)$ be a typical point in $\mathbf{R}^N$ and denote by $(r, \theta)$, $\theta \in \mathbf{S}^{N-1} := \{x: |x| = 1\}$, $r = |x|$, its polar coordinates. Let $\Omega$ be a domain on the unit sphere $\mathbf{S}^{N-1}$ with a boundary $\partial\Omega$ of class $C^2$. We define a *cone* in $\mathbf{R}^N$ to be a set of the type

$$\mathscr{C} = \{x: r \in \mathbf{R}^+, \theta \in \Omega\}.$$

In this paper we study the positive solutions of the *generalized Emden equation*

$$\Delta u + r^\sigma u^p = 0 \quad \text{in } \mathscr{C}, \quad \sigma \in \mathbf{R}, p > 1 \tag{1.1}$$

satisfying Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\mathscr{C} - \{0\}. \tag{1.2}$$

Because of its relevance to physics and its rich mathematical structure the Emden equation has attracted the interest of many mathematicians, starting with Fowler (*cf.*, *e.g.*, the survey of WONG [Wo].)

The structure of the radially symmetric solutions of (1.1), (1.2) is now completely understood [GS, BM]. General positive solutions in $\mathbf{R}^N$ were studied by GIDAS & SPRUCK in a substantial paper [GS]. They were able to determine the precise behaviour near isolated singularities and deduce some nonexistence results.

We say that a solution of our problem is *regular* if it belongs to $C^2(\mathscr{C}) \cap C^0(\mathscr{C} \cup \partial\mathscr{C})$ and satisfies (1.1) and (1.2). We also consider *singular solutions*, denoted by $u_s$. They are of class $C^2(\mathscr{C}) \cap C^0(\mathscr{C} \cup \partial\mathscr{C} - \{0\})$ and discontinuous at 0. It is known that for certain ranges of $p$ such solutions exist [BL]. In Section 2 we extend a nonexistence theorem of GIDAS & SPRUCK [GS]. For a sector, *i.e.*, a cone in $\mathbf{R}^2$, it is expressed as follows.

**Theorem 1.1.** *Let* $S(\nu) = \{(r, \theta) : r > 0, 0 < \theta < \nu\pi, \nu \in (0, 2]\}$ *be a sector. For* $1 < p \leqq \max\{1 + (2 + \sigma)\nu, 1 - (2 + \sigma)\nu\} =: p^*$, *problem* (1.1), (1.2) *possesses neither regular nor singular solutions besides* $u \equiv 0$.

The bound is sharp since for $p > p^*$ solutions of the form $u = r^{-(2+\sigma)/(p-1)}\alpha(\theta)$ are known to exist [BL]. For $\sigma = 0$ this result is already found in [BL]. We give a simpler proof, which also includes the non-existence of singular solutions.

It should be noticed that if we perform a *Kelvin transformation*

$$y = \frac{x}{|x|^2}, \quad v(y) = |x|^{N-2} u(x), \tag{1.3}$$

then (1.1), (1.2) becomes

$$\Delta v(y) + |y|^{-N-2-\sigma+p(N-2)} v^p = 0 \quad \text{in } \mathscr{C} \tag{1.1'}$$

$$v = 0 \quad \text{on } \partial\mathscr{C}. \tag{1.2'}$$

From this observation it follows that every statement concerning the behaviour of a solution near zero leads to a statement on its behaviour near infinity. A solution of (1.1), (1.2) will be called *regular at infinity*, if the transformed function $v(y)$ is regular at zero, or equivalently if

$$\lim_{|x| \to \infty} u(x)\, |x|^{N-2} = 0. \tag{1.4}$$

In Section 3 we determine the precise asymptotic behaviour of the solutions which are regular at zero or at infinity. For a sector we get

**Theorem 1.2.** *If u is regular and* $\sigma \geqq -2$, *then there exists a positive constant* $u_0$ *such that* $\lim_{r\to 0} r^{-\nu} u(re^{i\theta}) = u_0 \sin(\theta/\nu)$ *uniformly in* $\theta$. *If u is regular at infinity and* $\sigma \leqq -2$, *then there exists a positive constant* $u_\infty$ *such that* $\lim_{r\to\infty} r^{\nu} u(re^{i\theta}) = u_\infty \sin(u/\theta\nu)$ *uniformly in* $\theta$.

In Section 4 we derive a non-existence theorem based on an identity of PUCCI & SERRIN [PS] and on the results of Section 3. We also study the question whether solutions can be ordered.

The following notation will be used throughout this paper. If two positive functions $f$ and $g$ satisfy in $X$ the inequalities $c_1 \leqq \frac{f(x)}{g(x)} \leqq c_2$ for some positive constants $c_1$ and $c_2$, we shall write $f \approx g$ in $X$. The letter $C$ stands for generic constants depending only on $\Omega$ and $N$.

## 2. Nonexistence result and asymptotic estimates

**2.1.** In polar coordinates the Laplace operator has the form

$$\Delta = \frac{1}{r^{N-1}}\left(r^{N-1}\frac{\partial}{\partial r}\right)_r + \frac{1}{r^2}\Delta_\theta,$$

where $\Delta_\theta$ is the Beltrami operator on the sphere $S^{N-1}$. Let $\psi > 0$ be the first eigenfunction of

$$\Delta_\theta \psi + \omega\psi = 0 \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

It is supposed to be normalized such that

$$\int_\Omega \psi \, d\theta = 1, \tag{2.2}$$

$d\theta$ denoting the area element on $\mathbf{S}^{N-1}$.

If we multiply (1.1) by $\psi$, integrate the resulting expression over $\Omega$ and use Jensen's inequality we get for the "mean value"

$$\tilde{u}(r) := \int_\Omega u(r, \theta)\, \psi(\theta)\, d\theta$$

the differential inequality

$$\Delta_r \tilde{u} - \frac{\omega}{r^2} \tilde{u} + r^\sigma \tilde{u}^p \leqq 0, \tag{2.3}$$

where $\Delta_r$ is the radial part of $\Delta$ and $\omega$ is the lowest eigenvalue of (2.1). Observe that

$$\Delta_r - \frac{\omega}{r^2} = r^{-(\beta-\lambda)} \frac{\partial}{\partial r}\left[r^\beta \frac{\partial}{\partial r}(r^{-\lambda})\right] =: L \tag{2.4}$$

where

$$\lambda = \gamma_\pm = -\frac{N-2}{2} \pm \sqrt{\omega + \left(\frac{N-2}{2}\right)^2},$$
$$\beta = N - 1 + 2\lambda.$$

We note that if $x$ is a given variable and $y$ is defined by (1.3), we have

$$u(x)\,|x|^{-\gamma_+} = v(y)\,|y|^{-\gamma_-}, \qquad u(x)\,|x|^{-\gamma_-} = v(y)\,|y|^{-\gamma_+}.$$

**2.2.** Next we derive an elementary lemma for positive radially symmetric solutions of

$$Lv \leqq 0 \quad \text{for } r > 0. \tag{2.5}$$

**Lemma 2.1.** (i) *If* $\beta > 1$, *then the function* $w(r) := r^{-\lambda} v(r)$ *is decreasing.*
(ii) *If* $\beta \leqq 1$, *then* $w(r)$ *is an increasing function.*

**Proof.** From (2.5) we have

$$r^\beta w'(r) \leqq \varrho^\beta w'(\varrho) \quad \text{if } r \geqq \varrho. \tag{2.6}$$

Hence if $w'(\varrho) \leqq 0$ for some $\varrho$, then $w'(r) \leqq 0$ for all $r \geqq \varrho$. Suppose that the statement (i) is false. Then $w'(r) > 0$ on an interval $(0, r_1]$, which together with the positivity of $w$ implies that

$$\lim_{r \to 0} w(r) = w_0 < \infty. \tag{2.7}$$

From (2.6) we get for $r < R < r_1$,

$$w(R) - w(r) \geqq \frac{r_1^\beta w'(r_1)}{\beta - 1} [r^{-\beta+1} - R^{-\beta+1}].$$

For $r \to 0$ the left-hand side is bounded because of (2.7), while the right-hand side tends to infinity. This is a contradiction and (i) is thus established.
(ii) Suppose that $w \neq 0$ does not increase everywhere. Then by (2.6) there exists a positive number $\varrho$ such that $w'(\varrho) < 0$ and we have

$$w'(r) \leqq r^{-\beta} (\varrho^\beta w'(\varrho)) =: -ar^{-\beta}, \quad r \geqq \varrho$$

where $a > 0$, and

$$w(r) \leqq w(\varrho) - a \int_\varrho^r t^{-\beta} \, dt \to -\infty, \quad (r \to \infty).$$

Since $w$ is non-negative, this is impossible and we have proved the lemma.

Let us consider the functions $w_+(r) := r^{-\gamma_+}\tilde{u}(r)$ and $w_-(r) := r^{-\gamma_-}\tilde{u}(r)$, $\tilde{u}(r)$ being defined in Section 2.1. We note that

$$\beta_+ := N - 1 + 2\gamma_+ > 1, \quad \beta_- := N - 1 + 2\gamma_- < 1. \tag{2.8}$$

From the previous arguments we obtain

**Corollary 2.2.** (i) *$w_+(r)$ is non-increasing and* $\lim_{R\to\infty} w_+(R) =: w_\infty^+ < \infty$.
(ii) *$w_-(r)$ is non-decreasing and* $\lim_{r\to 0} w_-(r) =: w_0^- < \infty$.

**2.3.** We shall prove the following non-existence result.

**Theorem 2.3.** *If* $p \leqq \max\left\{1 - \frac{\sigma+2}{\gamma_+}, 1 - \frac{\sigma+2}{\gamma_-}\right\} =: p^*$, *no regular or singular positive solutions of* (1.1), (1.2) *exist in* $\mathscr{C}$.

**Proof.** Let us assume that our problem has a nontrivial solution $u$. If $w_+ = r^{-\lambda}\tilde{u}(r)$, where $\lambda = \gamma_+$, it follows from (2.3) and (2.4) with $\beta = \beta_+ = N - 1 + 2\gamma_+$ that

$$R^\beta w_+'(R) - r^\beta w_+'(r) + \int_r^R s^{\beta-\lambda+\sigma+p\lambda} w_+(s)^p \, ds \leqq 0, \quad 0 < r < R. \tag{2.9}$$

By Corollary (2.2)(i), $w_+$ is non-increasing, hence

$$R^\beta w_+'(R) + w_+(R)^p \int_r^R s^{\beta-\lambda+\sigma+p\lambda} \, ds \leqq 0, \quad r < R.$$

Letting $r \to 0$, we see that $\beta - \lambda + \sigma + p\lambda > -1$ and

$$w_+'(R) \, w_+(R)^{-p} + CR^{(p-1)\lambda+\sigma+1} \leqq 0. \tag{2.10}$$

Integrating (2.10) from $r$ to $R$, we obtain

$$(2.11)\qquad (p-1)^{-1}\left(w_+(r)^{1-p}-w_+(R)^{1-p}\right)+C\int_r^R s^{(p-1)\gamma_++\sigma+1}\,ds \leqq 0.$$

Since the left-hand member is bounded as $r \to 0$, it follows that

$$(2.12)\qquad (p-1)\gamma_+ + \sigma + 2 > 0.$$

If $w_\infty^+ > 0$, the left-hand member in (2.11) is unbounded as $R \to \infty$, which contradicts the inequality. We conclude that $w_\infty^+ = 0$.

The next step is to do the same calculations for $\lambda = \gamma_-$ and $\beta = \beta_-$. By Corollary (2.2)(ii), $w_-$ is non-decreasing, and we obtain

$$-w_-'(r)\,r^\beta + w_-(r)^p \int_r^R s^{\beta-\lambda+\sigma+p\lambda}\,ds \leqq 0.$$

Letting $R \to \infty$, we see that $\beta - \lambda + \sigma + p\lambda < -1$, and

$$(2.10\text{a})\qquad -w_-'(r)\,w_-(r)^{-p} + Cr^{(p-1)\gamma_-+\sigma+1} \leqq 0.$$

Integrating (2.10a) from $r$ to $R$, we obtain

$$(2.11\text{a})\qquad (p-1)^{-1}\left(w_-(R)^{1-p}-w_-(r)^{1-p}\right)+C\int_r^R s^{(p-1)\gamma_-+\sigma+1}\,ds \leqq 0.$$

Since the left-hand member in (2.11a) is bounded as $R \to \infty$, it follows that

$$(2.12\text{a})\qquad (p-1)\gamma_- + \sigma + 2 < 0.$$

If $w_0^- > 0$, the left-hand member in (2.11a) is unbounded as $r \to 0$, which contradicts the inequality. We conclude that $w_0^- = 0$.

We have proved Theorem 2.3 and

**Corollary 2.4.** *If there exists a nontrivial regular or singular solution of* (1.1) *and* (1.2) *in* $\mathscr{C}$, *then* (2.12) *and* (2.12a) *hold and*

(i) $\lim_{R\to\infty} w_+(R) = w_\infty^+ = 0$,

(ii) $\lim_{R\to 0} w_-(R) = w_0^- = 0$.

The bounds in Theorem 2.3 are sharp because if

$$p^* < p < \begin{cases} \infty, & N = 2, 3, \\ (N+1)/(N-3), & N > 3, \end{cases}$$

there exist solutions of the form (*cf.* [BL])

$$(2.13)\qquad u(r,\theta) = r^{-(\sigma+2)/(p-1)}\,\alpha(\theta)$$

where $\alpha(\theta)$ satisfies the boundary value problem

(2.14)

$$\Delta_\theta \alpha - \frac{\sigma+2}{p-1}\left(N-2-\frac{\sigma+2}{p-1}\right)\alpha + \alpha^p = 0 \quad \text{in } \Omega, \quad \alpha = 0 \quad \text{on } \partial\Omega.$$

Another consequence of (2.11) is stated in

**Corollary 2.5.** *If u is any regular or singular solution of* (1.1), (1.2), *then there exist positive constants* $r_0$, $_1$ *and* $c_2$ *such that*

(i) *if* $0 < w_0^+ := \lim_{r\to 0} w_+(r) < \infty$, *then*

$$\tilde{u}(r) \approx w_0^+ r^{\gamma_+}, \qquad r \to 0, \tag{2.15}$$

$$\tilde{u}(r) \leqq c_1 r^{-(\sigma+2)/(p-1)}, \qquad r \geqq r_0 > 0. \tag{2.16}$$

(ii) *if* $w_0^+ = \infty$, then

$$\tilde{u}(r) \leqq c_2 r^{-(\sigma+2)/(p-1)}, \qquad 0 < r < \infty.$$

A result complementary to Corollary 2.5 is given in

**Corollary 2.6.** *Let u be a regular or singular solution of* (1.1), (1.2). *Then there exist positive constants* $c_1$ *and* $c_2$ *such that*

(i) $\tilde{u}(r) \geqq c_1 r^{\gamma_+}$ *for all* $r \leqq r_0$,
(ii) $\tilde{u}(r) \geqq c_2 r^{\gamma_-}$ *for all* $r \geqq R$.

**Proof.** From (2.4) and (2.5), we have

$$w'_-(1) \leqq r^\beta w'_-(r), \qquad 0 < r \leqq 1$$

(we note that $\beta = \beta_- < 1$), and thus by Corollary 2.4(ii)

$$(1-\beta)^{-1}\, r^{1-\beta} w'_-(1) \leqq w_-(r) = r^{-\gamma_-}\tilde{u}(r).$$

Since $-\beta + 1 + \gamma_- = -N + 2 - \gamma_- = \gamma_+$, the first statement is proved. The second follows if we apply (i) to the transformed problem (1.1'), (1.2').

## 3. Local behaviour

**3.1.** For the arguments of this Section, we shall need Green's function $K(x, y)$ for the Laplace operator in the cone $\mathscr{C}$, satisfying, for fixed $y \in \mathscr{C}$

$$\Delta_x K(x, y) = -\delta(x - y), \tag{3.1}$$

where $x \in \mathscr{C}$, $K(\cdot, y) = 0$ on $\partial\mathscr{C}$ and $K(x, y) \to 0$ as $|x| \to \infty$, $x \in \mathscr{C}$.

Such a function exists (*cf.* [B]; further references can be found in [ELe]). Moreover, for $|x| > |y|$, we have the classical representation formula of BOULIGAND [B]:

$$K(x, y) = |\Omega| \sum_{n=1}^{\infty} \frac{|x|^{\beta_n}\,|y|^{\alpha_n}}{\sqrt{(N-2)^2 + 4\omega_n}}\, \Psi_n(\theta_x)\, \Psi_n(\theta_y) \tag{3.2}$$

where $|\Omega|$ is the Lebesgue measure of $\Omega$, $\omega_n$ is the $n^{\text{th}}$ eigenvalue of (2.1) and $\Psi_n(\theta)$ is the corresponding eigenfunction, normalized with respect to the $L^2$-

norm (*cf.* [B, LF]). Furthermore, $\beta_n < 0 < \alpha_n$ are roots of $t(t+N-2)-\omega_n = 0$. We note that $\alpha_1 = \gamma_+$ and $\beta_1 = \gamma_-$. If $|x| < |y|$, the roles of $x$ and $y$ in (3.2) are interchanged.

We shall need several estimates.

**Lemma 3.1.** *Let $\varepsilon > 0$ be given. Let $\Psi_1$ be denoted by $\Psi$. If $x, y \in \mathscr{C}$, then*

$$K(x,y) \approx \Psi(\theta_x)\,\Psi(\theta_y)\,|x|^{\gamma_+}\,|y|^{\gamma_-}, \qquad 0 < |x| < (1-\varepsilon)\,|y|, \tag{3.3}$$

$$K(x,y) \approx \Psi(\theta_x)\,\Psi(\theta_y)\,|x|^{\gamma_-}\,|y|^{\gamma_+}, \qquad 0 < |y|\,(1+\varepsilon) < |x|. \tag{3.4}$$

*If $|x| = r$, then*

$$0 < \frac{\partial}{\partial r} K(x,y) \leqq C\Psi(\theta_x)\,\Psi(\theta_y)\,|x|^{\gamma_+-1}\,|y|^{\gamma_-}, \qquad 0 < |x| < (1-\varepsilon)\,|y|, \tag{3.5}$$

$$0 < -\frac{\partial}{\partial r} K(x,y) \leqq C\Psi(\theta_x)\,\Psi(\theta_y)\,|x|^{\gamma_--1}\,|y|^{\gamma_+}, \qquad 0 < |y|\,(1+\varepsilon) < |x|. \tag{3.6}$$

The constants of comparison depend also on $\varepsilon$.

**Proof.** For the first two estimates, we refer to [A, Lemma 1]. Estimates (3.5) and (3.6) are consequences of (3.2) and estimates (i)–(iii) in [ELe, Section 4].

Let $\mathscr{D}(R_0, R) = \mathscr{C} \cap \{R_0 < |x| < R\}$. Green's function $K_1(x, y)$ in the domain $\mathscr{D}(0, R)$ can be written as $K(x, y) - K(x_R^*, y)$, where $x_R^*$ is the reflection of $x$ in the sphere $\{|x| = R\}$. From Lemma 3.1, we see that

$$0 < \frac{\partial}{\partial n} K_1(x,y) \leqq C\Psi(\theta_x)\,\Psi(\theta_y)\,|x|^{\gamma_--1}\,|y|^{\gamma_+} \tag{3.7}$$

for $0 < |y|\,(1+\varepsilon) < |x| = R$, $x, y \in \mathscr{C}$, where $n$ is the inner normal in the domain $\mathscr{D}(0, R)$.

Similarly, if $K_2(x, y)$ is Green's function in the domain $\mathscr{D}(R_0, \infty)$, then

$$0 < \frac{\partial}{\partial n} K_2(x,y) \leqq C\Psi(\theta_x)\,\Psi(\theta_y)\,|x|^{\gamma_+-1}\,|y|^{\gamma_-} \tag{3.8}$$

for $|x| = R_0$, $|x| < (1-\varepsilon)\,|y|$, $x, y \in \mathscr{C}$.

**Lemma 3.2.**

$$\int_\Omega K(x,y)\,d\theta_x \leqq \begin{cases} C\,|x|^{\gamma_+}\,|y|^{\gamma_-}, & \textit{if } |x| \leqq |y|, \\ C\,|x|^{\gamma_-}\,|y|^{\gamma_+}, & \textit{if } |x| \geqq |y|. \end{cases}$$

**Proof.** Since we have Lemma 3.1, it suffices to estimate $K(x, y)$ in $\left\{\frac{|y|}{2} \leqq |x| \leqq 2\,|y|\right\} \cap \mathscr{C}$. We begin with the case $N \geqq 3$. By the maximum principle, we have

$$K(x,y) \leqq \frac{|x-y|^{2-N}}{A_N} := s_N(x-y)$$

where $A_N$ is the area of $S^{N-1}$. Since

$$\int_{S^{N-1}} s_N(x-y)\,d\theta_x = \min(|x|^{2-N}, |y|)^{2-N},$$

it is easy to deduce the lemma in this case. When $N=2$, we consider the function $H(y) = \int_{\Omega} K(x,y)\,d\theta_x$, which is harmonic in $\mathscr{C} \cap \{|y| < |x|\}$ and $\mathscr{C} \cap \{|y| > |x|\}$ and is continuous in $\overline{\mathscr{C}}$. Since we can control $H$ on the sets $\mathscr{C} \cap \left\{|y| = \frac{|x|}{2}\right\}$, $\mathscr{C} \cap \{|y| = |x|\}$ and $\mathscr{C} \cap \{|y| = 2|x|\}$, the lemma is true also in this case.

Let $K_0(x,y)$ be Green's function for the Laplace operator in $\mathscr{D}(R_0, R)$, where $y \in \mathscr{D}(R_0, R)$. Then the function

$$v(y) = u(y) - \int_{|x|=R} \frac{\partial K_0}{\partial n}(x,y)\,u(x)\,d\sigma(x) - \int_{|x|=R_0} \frac{\partial K_0}{\partial n}(x,y)\,u(x)\,d\sigma(x)$$

vanishes on $\partial\mathscr{D}(R_0, R)$. Here $n$ represents the inner normal in $\mathscr{D}(R_0, R)$ and $d\sigma(x)$ is the area measure. The Riesz representation formula gives

$$v(y) = \int_{\mathscr{D}(R_0,R)} K_0(x,y)\,(-\Delta v(x))\,dx = \int_{\mathscr{D}(R_0,R)} K_0(x,y)\,(-\Delta u(x))\,dx. \tag{3.9}$$

The maximum principle implies that

$$K_0(x,y) \leqq K_1(x,y), \qquad x, y \in \mathscr{D}(R_0, R)$$

and thus that

$$\frac{\partial K_0}{\partial n}(x,y) \leqq \frac{\partial K_1}{\partial n}(x,y), \qquad |x| = R,\ y \in \mathscr{D}(R_0, R). \tag{3.10}$$

In the same way, it follows that

$$\frac{\partial K_0}{\partial n}(x,y) \leqq \frac{\partial K_2}{\partial n}(x,y), \qquad |x| = R_0,\ y \in \mathscr{D}(R_0, R). \tag{3.11}$$

Furthermore, we have

$$K_0(x,y) \leqq K(x,y), \qquad x, y \in \mathscr{D}(R_0, R). \tag{3.12}$$

Let us define $u(x) = 0$, $x \notin \mathscr{C}$. Assuming that $2R_0 \leqq |y| \leqq R/2$, we use (3.9), (3.10)–(3.12), (3.7) and (3.8) to deduce that

$$u(y) \leqq \int_{\mathscr{D}(R_0,R)} K(x,y)\,|x|^\sigma u(x)^p\,dx$$
$$+ C\left\{R_0^{-\gamma_-} \int_{|x|=R_0} \Psi(\theta_x)\,u(x)\,d\theta_x\,|y|^{\gamma_-} + R^{-\gamma_+} \int_{|x|=R} \Psi(\theta_x)\,u(x)\,d\theta_x\,|y|^{\gamma_+}\right\}.$$

We define $M(r) = \sup u(y)$ for $|y| = r$. Combining the last inequality with Lemma 3.2, we obtain

$$(3.13)\qquad M(r) \leqq C\left\{r^{\gamma_-} \int_{R_0}^{r} M(s)^p s^{\sigma+1-\gamma_-}\, ds + r^{\gamma_+} \int_{r}^{R} M(s)^p s^{\sigma+1-\gamma_+}\, ds + \left(\frac{r}{R_0}\right)^{\gamma_-} \tilde{u}(R_0) + \left(\frac{r}{R}\right)^{\gamma_+} \tilde{u}(R)\right\}, \quad 2R_0 \leqq r \leqq R/2.$$

Let $\chi_\pm = (\sigma+2)/(p-1) + \gamma_\pm$. If $M(r) = \eta(r)\, r^{-(\sigma+2)/(p-1)}$ and $\tilde{u}(R) = v(R)\, R^{-(\sigma+2)/(p-1)}$, it follows from (3.13) that

$$(3.14)\qquad \eta(r) \leqq C\left\{r^{\chi_-} \int_{R_0}^{r} \eta(s)^p s^{-1-\chi_-}\, ds + r^{\chi_+} \int_{r}^{R} \eta(s)^p s^{-1-\chi_+}\, ds + v(R_0)\, R_0^{-\chi_-} r^{\chi_-} + v(R)\, R^{-\chi_+} r^{\chi_+}\right\}, \quad 2R_0 \leqq r \leqq R/2.$$

Note that if a solutions to (1.1), (1.2) exists, then Theorem 2.3 implies that

$$(3.15)\qquad \chi_- < 0 < \chi_+.$$

We are now in a position to prove the main results of this section.

**Theorem 3.3.** *Let there exist a nontrivial positive solution $u$ of* (1.1), (1.2).

(i) *If* $M(r)\, r^{(\sigma+2)/(p-1)} \to 0$ *as* $r \to 0$, *then there are positive constants $c_1$ and $c_2$ such that*

$$c_1 r^{\gamma_+} \leqq M(r) \leqq c_2 r^{\gamma_+} \quad \text{for all } r \leqq r_0.$$

(ii) *If* $M(r)\, r^{(\sigma+2)/(p-1)} \to 0$ *as* $r \to \infty$, *then there exist positive constants $c_1$ and $c_2$ such that*

$$c_1 r^{\gamma_-} \leqq M(r) \leqq c_2 r^{\gamma_-} \quad \text{for all } r \geqq r_0.$$

**Proof.** By assumption, we have $\eta(r) \to 0$ as $r \to 0$ and consequently, $v(R_0) \to 0$ as $R_0 \to 0$. Letting $R_0 \to 0$ in (3.14) and using (3.15), we obtain

$$(3.16)\qquad \eta(r) \leqq C\left\{r^{\chi_-} \int_{0}^{r} \eta(s)^p s^{-1-\chi_-}\, ds + r^{\chi_+} \int_{r}^{R} \eta(s)^p s^{-1-\chi_+}\, ds + v(R)\, R^{-\chi_+} r^{\chi_+}\right\}, \quad r \leqq R/2.$$

We wish to eliminate the first term in the right-hand member of (3.16). To do so, we introduce

$$\Psi(r) = \begin{cases} \sup\limits_{0 \leqq t \leqq r} \eta(t), & r \leqq R, \\ \Psi(R), & r \geqq R. \end{cases}$$

Then the sum of the two terms containing integrals in (3.16) is dominated by

$$r^{\chi_-}\int_0^r \Psi(s)^p s^{-1-\chi_-}\,ds + r^{\chi_+}\int_r^\infty \Psi(s)^p s^{-1-\chi_+}\,ds$$
$$= \int_0^1 \Psi(rs)^p s^{-1-\chi_-}\,ds + \int_1^\infty \Psi(rs)^p s^{-1-\chi_+}\,ds,$$

which is an increasing function of $r$. From (3.16), we then deduce that

$$\Psi(r) \leqq C\left\{r^{\chi_-}\int_0^r \Psi(s)^p s^{-1-\chi_-}\,ds + r^{\chi_+}\int_r^\infty \Psi(s)^p s^{-1-\chi_+}\,ds + c_1 r^{\chi_+}\right\}$$

where $c_1 = v(R)\,R^{-\chi_+}$.

Since $\Psi(r)\to 0$ as $r\to 0$, there exists $R_1$ such that

$$C\,|\chi_-|^{-1}\,\Psi(R_1)^{p-1} < \tfrac{1}{2},$$
$$Cr^{\chi_-}\int_0^r \Psi(s)^p s^{-1-\chi_-}\,ds \leqq \tfrac{1}{2}\Psi(r), \qquad 0<r<R_1.$$

Thus, if $R = 2R_1$, we see that

$$\Psi(r) \leqq 2Cr^{\chi_+}\left\{\int_r^\infty \Psi(s)^p s^{-1-\chi_+}\,ds + c_1\right\}. \tag{3.17}$$

Let $H(r) = \int_r^\infty \Psi(s)^p s^{-1-\chi_+}\,ds + c_1$. Since $H'(r) = -\Psi(r)^p r^{-1-\chi_+}$, it follows from (3.17) that

$$-H'(r)\,H(r)^{-p} \leqq Cr^{\chi_+(p-1)-1}.$$

Integrating this inequality from 0 to $r$, we obtain

$$H(r)^{1-p} - H(0)^{1-p} \leqq \frac{Cr^{\chi_+(p-1)}}{\chi_+}.$$

If $H(0)=\infty$, it follows that

$$0 < c \leqq H(r)\,r^{\chi_+}, \qquad r\leqq R_1.$$

On the other hand,

$$r^{\chi_+}H(r) = r^{\chi_+}\left\{\int_r^\infty \Psi(s)^p s^{-1-\chi_+}\,ds + c_1\right\} \to 0, \qquad r\to 0.$$

The contradiction shows that $H(0)$ is finite. We now use (3.17) to conclude that

$$\eta(r) \leqq r^{\chi_+}\,2CH(0), \qquad r\leqq R_1.$$

This implies that

$$M(r) = r^{-(\sigma+2)/(p-1)}\,\eta(r) \leqq c_2 r^{\gamma_+}.$$

for all sufficiently small $r$. The first estimate is now a consequence of this result and Corollary 2.6.

In order to prove the second statement, we apply assertion (i) to the transformed problem (1.1′), (1.2′) (we note that $|y|^{-\gamma_+}v(y) = |x|^{-\gamma_-}u(x)$ (*cf.* Section 2.1)).

*Remarks.*

(1) Part (i) or (ii), respectively, of Theorem 3.3 depends only on the behaviour of $u$ in $\mathscr{C} \cap \{|x| \leqq R\}$ or $\mathscr{C} \cap \{|x| \geqq R\}$ and is not affected by the behaviour of $u$ in the complementary part of the cone $\mathscr{C}$.

(2) A related result can be obtained from [KKO] for solutions belonging to certain Sobolev spaces. The estimates there do not apply under our assumptions.

**Corollary 3.4.** *Let $u$ be a positive solution of* (1.1), (1.2).

(i) *If $u$ is regular and if in addition $\sigma \geqq -2$, we have $M(r) \approx r^{\gamma_+}$ as $r \to 0$.*

(ii) *If $u$ is regular at infinity and if in addition $p(N-2) \geqq \sigma + N$, we have $M(r) \approx r^{\gamma_-}$ as $r \to \infty$.*

**Proof.** If $u$ is a regular solution, then by definition $u$ is continuous in $\mathscr{C} \cap \{|x| < R_0\}$ and hence $M(r) \to 0$ as $r \to 0$. Thus the first assertion follows from Theorem 3.3 (i).

If $u$ is regular at infinity, then the transformed function $v$ is regular at the origin (*cf.* (1.3)) and satisfies (1.1′), (1.2′). Assertion (i) applied to $v$ yields

$$\max_{|y|=r} v(y) \approx |y|^{\gamma_+} \quad \text{as } y \to 0.$$

Since $|y|^{-\gamma_+} v(y) = |x|^{-\gamma_-} u(x)$, we obtain (ii).

*Remarks.*

(1) A solution of the form $u(r, \theta) = r^{-(\sigma+2)/(p-1)} \alpha(\theta)$, considered in (2.13), is regular if $\sigma < -2$. It is regular at infinity if $p(N-2) > \sigma + N$.

(2) The most regular solutions at zero (infinity) which problem (1.1), (1.2) can admit behave like $u \approx r^{\gamma_+}$ as $r \to 0$ ($u \approx r^{\gamma_-}$ as $r \to \infty$).

**Corollary 3.5.** *Let $u > 0$ satisfy* (1.1), (1.2).

(i) *If $u = O(r^{\gamma_+})$ as $r \to 0$, then $|\nabla u| = O(r^{\gamma_+ - 1})$ as $r \to 0$.*

(ii) *If $u = O(r^{\gamma_-})$ as $r \to \infty$, then $|\nabla u| = O(r^{\gamma_- - 1})$ as $r \to \infty$.*

**Proof.** We follow an argument found in [GS, To]. Let $u$ be a regular solution and let

$$v(x) := \varepsilon^{\nu} u(\varepsilon x), \qquad \nu = -(2 + \sigma + \gamma_+ p),$$

which satisfy the equation

$$\Delta v + \varepsilon^{\nu+\sigma+2} |x|^{\sigma} u(\varepsilon x)^p = 0 \quad \text{in } \mathscr{C}, \qquad v = 0 \quad \text{on } \partial\mathscr{C}. \tag{3.18}$$

By our assumption on $u$, we have

$$\varepsilon^{\nu+\sigma+2} u(\varepsilon x)^p \leqq c \tag{3.19}$$

for all $\varepsilon \in (0, 1)$ and $|x| \leqq R_0$.

Let us consider (3.18) in the domain $\Pi = \mathscr{C} \cap \{r_0/2 < |x| < 2r_0\}$, where $2r_0 < R_0$. If $|x| = r_0/2$ or $|x| = 2r_0$, we know that $v(x) \leqq C\varepsilon^{\nu+\gamma_+} r_0^{\gamma_+}$.

Let $h$ be harmonic in $\Pi$ and such that $h = v$ on $\partial\Pi$. In a standard way (*cf.* [Wi]) we prove that

$$|\nabla h(x)| \leqq C\varepsilon^{\nu+\gamma_+}r_0^{\gamma_+-1}, \qquad |x| = r_0.$$

The function $w := v - h$ is a solution of (3.18) and vanishes on $\partial\Pi$. Since the known term $\varepsilon^{\nu+\sigma+2}u(\varepsilon x)^p$ is uniformly bounded in $\Pi$ for all $\varepsilon \in (0, 1)$, regularity theory tells us that $|\nabla w|$ is uniformly bounded on $|x| = r_0$. Hence

$$(\varepsilon r_0)^{\nu+1} |\nabla u(\varepsilon x)| \leqq r_0^{\nu+1} |\nabla v(x)| \leqq C(\varepsilon^{\nu+\gamma_+}r^{\nu+\gamma_+} + r_0^{\nu+1})$$

which proves assertion (i).

The second assertion is obtained by applying (i) to the transformed problem (1.1'), (1.2').

*Remark.* The above method applies also to higher derivatives.

Finally, we discuss the asymptotic behaviour of the very regular solutions.

**Corollary 3.6.** *Let $u$ be a positive solution of* (1.1), (1.2) *and let $\psi$ be as in* (2.1).

(i) *If $u = O(r^{\gamma_+})$ as $r \to 0$, then $\lim_{r\to 0} r^{-\gamma_+}u(r, \theta) = \|\psi\|_2^{-2} \psi(\theta)\, w_0^+$; the convergence is uniform in $\theta$.*

(ii) *If $u = O(r^{\gamma_-})$ as $r \to \infty$, then $\lim_{r\to\infty} r^{-\gamma_-}u(r, \theta) = \|\psi\|_2^{-2} \psi(\theta)\, w_\infty^-$ uniformly in $\theta$.*

*Remark.* We recall that (*cf.* Section 2)

$$w_0^+ = \lim_{r\to 0} w_+(r) = \lim_{r\to 0} r^{-\gamma_+}\tilde{u}(r),$$

$$w_\infty^- = \lim_{r\to\infty} w_-(r) = \lim_{r\to\infty} r^{-\gamma_-}\tilde{u}(r).$$

**Proof.** Let $K_R(x, y)$ be Green's function in $\mathcal{D}(0, R) = \mathcal{C} \cap \{0 < |x| < R\}$. Consider the function

$$v(x) := \int_{\mathcal{D}(0,R)} K_R(x, y)\, |y|^\sigma u(y)^p\, dy,$$

which is a solution of

$$\Delta v + r^\sigma u^p = 0$$

in $\mathcal{D}(0, R)$ vanishing on $\partial\mathcal{D}(0, R)$. By our assumption, $M(r) \leqq cr^{\gamma_+}$, $0 < r \leqq 1$. Since $K_R(x, y) \leqq K(x, y)$, we can deduce the following estimate from Lemma 3.2.

$$v(x) \leqq c_1 r^{\gamma_+}(r^\mu + R^\mu), \quad 0 < r < R \leqq 1, \quad \mu = \sigma + 2 + \gamma_+(p-1) > 0. \tag{3.20}$$

The function $u - v$ is harmonic in $\mathcal{D}(0, R)$, vanishes on $\mathcal{C}$ and coincides with $u(x)$ on $\{|x| = R\} \cap \mathcal{C}$. We conclude that

$$u(x) - v(x) = \int_\Omega \frac{\partial}{\partial n_y} K_R(x, R\theta_y)\, u(R, \theta)\, R^{N-1}\, d\theta_y, \qquad x \in \mathcal{D}(0, R).$$

From Bouligand's formula (3.2), we deduce that

$$\lim_{r\to 0} r^{-\gamma_+} \frac{\partial}{\partial n_y} K_R(x, R\theta_y) = c_2\psi(\theta_x)\,\psi(\theta_y)\,R^{\gamma_- -1}$$

uniformly in $\theta_x$ and $\theta_y$. Consequently, if $\theta = \theta_x$, then

$$\limsup_{r\to 0} |r^{-\gamma_+}u(r,\theta) - c_2\psi(\theta)\,R^{N-2+\gamma_-}\tilde{u}(R)| \leqq c_1 R^{\mu}$$

uniformly in $\theta$. We note that $\gamma_+ + \gamma_- = 2 - N$.

Choosing $R$ arbitrarily small, we obtain

$$\lim_{r\to 0} r^{-\gamma_+}u(r,\theta) = c_2\psi(\theta)\,w_0^+ .$$

To determine $c_2$, we multiply this equation by $\psi(\theta)$ and integrate the resulting expression over $\Omega$. This proves the first part of Corollary 3.6.

Again applying a Kelvin transformation, we deduce the second part of Corollary 3.6.

*Remark.* Some of these results are also true for solutions of the equation

$$\Delta u + r^{\sigma}\,|u|^{p-1}\,u = 0, \qquad \sigma \in \mathbb{R},\ p > 1 \tag{1.1''}$$

with Dirichlet boundary conditions on $\partial\mathscr{C} \setminus \{0\}$. If $u$ is a nontrivial solution of this problem, we introduce

$$M_+(r) = \sup u_+(y) \quad \textit{for } |y| = r,$$
$$M(r) = \sup |u(y)| \quad \textit{for } |y| = r,$$

where $u_+ = \max(u, 0)$ and we define $u$ to be zero outside $\mathscr{C}$. The proof of Theorem 3.3 will also give us the following result.

**Theorem 3.3'.** *Assume that there exists a nontrivial solution $u$ of* (1.1'') *with Dirichlet boundary conditions on* $\partial\mathscr{C} \setminus \{0\}$.

(i) *If* $M_+(r)\,r^{(\sigma+2)/(p-1)} \to 0$ *as* $r \to 0$, *then there is a positive constant $c$ such that*

$$M_+(r) \leqq c r^{\gamma_+} \quad \textit{for all } r \leqq r_0 .$$

(ii) *If* $M(r)\,r^{(\sigma+2)/(p-1)} \to 0$ *as* $r \to 0$, *then there is a positive constant $c$ such that*

$$M(r) \leqq c r^{\gamma_+} \quad \textit{for all } r \leqq r_0 .$$

*There are analogous results at infinity.*

Clearly, Corollary 3.5 also holds for solutions of (1.1'').

Let us assume that $u$ is a solution of (1.1'') with Dirichlet boundary conditions on $\partial\mathscr{C} \setminus \{0\}$ and that $M(r)$ is defined as above. If $M(r) = O(r^{\gamma_+})$ as $r \to 0$, the same argument as in the proof of Corollary 3.6 shows that

$$\limsup_{r\to 0} |r^{-\gamma_+}u(r,\theta) - c_2\psi(\theta)\,R^{-\gamma_+}\tilde{u}(R)| \leqq c_1 R^{\mu}$$

where $\tilde{u}(R)$ is defined as in Section 2. (This time, however, $u$ is not necessarily non-negative). Hence

$$|(R^{-\gamma_+}\tilde{u}(R) - R_1^{-\gamma_+}\tilde{u}(R))\, c_2| \leqq c_1(R^\mu + R_1^\mu).$$

Letting $R$ and $R_1 \to 0$, we conclude that $\lim_{R\to 0} R^{-\gamma_+}\tilde{u}(R) = w_0^+$ exists and that uniformly in $\theta$,

$$\lim_{r\to 0} r^{-\gamma_+}u(r,\theta) = \|\psi\|_2^{-2}\,\psi(\theta)\, w_0^+ .$$

There are analogous results at infinity (to be compared to Corollaries 2.2 and 3.6).

## 4. Consequences

**4.1.** We start with a non-existence result based on a Pohožaev-type identity introduced by ESTEBAN & LIONS [EL] and developed by PUCCI & SERRIN [PS] for domains which are starshaped. (A domain $\Pi$ is starshaped if there exists a point $x_0 \in \Pi$ such that the line-segment $\overline{x_0 x}$ is contained in $\Pi$ when $x \in \Pi$.)

From Theorem 2.3, it is known that a necessary condition for the existence of a positive solution of (1.1), (1.2) is that

$$-\gamma_+ < \frac{\sigma+2}{p-1} < -\gamma_- . \tag{4.1}$$

**Theorem 4.1.** *Assume that* (4.1) *holds and that*

$$\sigma \neq \tfrac{1}{2}N(p-1) - (p+1). \tag{4.2}$$

*Then the problem* (1.1), (1.2) *has no positive solutions such that*

$$M(r)\, r^{(\sigma+2)/(p-1)} \to 0, \qquad r \to 0,\ r \to \infty. \tag{4.3}$$

**Proof.** Suppose such a solution $u$ exists. We wish to apply the Pucci-Serrin identity in the domain $\mathscr{D}' = \mathscr{D}(R_0, R)$ (*cf.* Section 3). However, $\mathscr{D}'$ is not necessarily starshaped and $u$ does not vanish on $\partial\mathscr{D}' \cap \mathscr{C}$. Therefore, we have to modify the arguments in [PS]. Choosing $h(x) = x$ and $a$ to be constant, Proposition 1 in [PS] gives in our case that

$$\frac{\partial}{\partial x_i}\left\{x_i \mathscr{F}(x,u,\nabla u) - x_j \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} - au\frac{\partial u}{\partial x_i}\right\} \tag{4.4}$$
$$= N\mathscr{F}(x,u,\nabla u) - |x|^\sigma u^{1+p}(\sigma(1+p)^{-1} - a) - (1+a)\,|\nabla u|^2,$$

where repeated indices $i$ and $j$ are understood to be summed from 1 to $N$ and

$$\mathscr{F}(x,u,q) = \tfrac{1}{2}\,|q|^2 - |x|^\sigma u^{p+1}(p+1)^{-1}.$$

We apply the divergence theorem to (4.4). If $n$ is the outer normal, it says that

$$\int_{\mathscr{D}'} \operatorname{div} v\, dx = \int_{\partial\mathscr{D}'} v\cdot n\, ds.$$

On $\partial\mathcal{D}' \cap \partial\mathcal{C}$, we know that $u$ vanishes and we can use arguments from [PS]. Since $x \cdot n = 0$ on $\partial\mathcal{C}$, there is no contribution from this part of the boundary in the final formula. $\partial\mathcal{D}' \cap \mathcal{C}$ is the union of $\mathcal{C} \cap \{|x| = R_0\}$ and $\mathcal{C} \cap \{|x| = R$. Since we have (4.3), we can use the estimates from Theorem 3.3 and Corollary 3.5, which say that

$$M(r) = O(r^{\gamma_+}), \quad r \to 0; \quad |\nabla u| = O(r^{\gamma_+ - 1}), \quad r \to 0;$$

$$M(R) = O(R^{\gamma_-}), \quad R \to \infty; \quad |\nabla u| = O(R^{\gamma_- - 1}), \quad R \to \infty.$$

A computation shows that when $R_0 \to 0$ and $R \to \infty$, the contributions from $\partial\mathcal{D}' \cap \mathcal{C}$ tend to 0. Here (4.1) is crucial: in fact we have

$$\sigma + N + (p-1)\gamma_+ + 2\gamma_+ > N - 2 + 2\gamma_+ > 0,$$

$$\sigma + N + (p-1)\gamma_- + 2\gamma_- < N - 2 + 2\gamma_- < 0,$$

which is what is needed for these conclusions to hold.

Assuming that the integral is defined, we obtain

$$\int_{\mathcal{C}} (|\nabla u|^2 (\tfrac{1}{2}(N-2) - a) - |x|^\sigma u^{p+1}((N+\sigma)(p+1)^{-1} - a))\, dx = 0.$$

For $a = \frac{1}{2}(N-2)$, it is clear that the integral is defined. If (4.2) holds, then the coefficient in the second term is non-vanishing, and thus $u$ must be identically zero. We have proved Theorem 4.1.

*Remark.* The case $\sigma = \frac{1}{2}N(p-1) - (p+1)$ or equivalently $p = (N+2+2\sigma)/(N-2)$ is special. If $\mathcal{C} = R^N$, ground-state solutions of the form

$$u(r) = \{\lambda \sqrt{(N+\sigma)(N-2)}\,/(\lambda^2 + r^{2+\sigma})\}^{(N-2)/(\sigma+2)}$$

exist, which for $\sigma > -2$ obviously satisfy (4.3).

**4.2.** We now investigate the asymptotic behaviour of solutions which are not regular at zero or at infinity and which satisfy

$$(4.5) \qquad r^{(\sigma+2)/(p-1)} u(x) \leqq c \quad \text{for } r \leqq r_0 \quad (r \geqq R).$$

**Lemma 4.2.** *Assume that $u$ is a positive solution of* (1.1), (1.2), *that*

$$N - 2 - 2\delta < 0, \quad \textit{where } \delta = (\sigma+2)/(p-1)$$

*and that* (4.5) *holds for $r \to 0$. Then for any sequence $\{r_n\}$ tending to* 0, *there exists a subsequence $\{r_{n_j}\}$ such that*

$$\lim_{j\to\infty} r_{n_j}^{(\sigma+2)/(p-1)}\, u(x) = \alpha(\theta) \quad \textit{in } \mathcal{D}'(\Omega),$$

*where $\alpha$ is a nonnegative solution of* (2.14). *For sequences tending to infinity, the analogous conclusion holds if*

$$N - 2 - 2\delta > 0.$$

*Remarks.*
(1) $N-2-2\delta \neq 0$ is equivalent to (4.2).
(2) If $\limsup_{n\to\infty} r_n^\delta M(r_n) > 0$, the subsequence can be chosen in such a way that $\alpha$ will be a positive solution of (2.14).

**Proof.** We give the details for sequences tending to zero. Let $u = r^{-(\sigma+2)/(p-1)}v$. Then $v$ satisfies

$$v_{rr} + r^{-1}(N-1-2\delta)\,v_r + r^{-2}\,\Delta_\theta v - r^{-2}\,\delta(N-2-\delta)\,v + r^{-2}\,v^p = 0. \tag{4.6}$$

With the change of variable $t = \gamma \log r$, we obtain

$$\gamma^2 v_{tt} + \gamma v_t(N-2-2\delta) + \Delta_\theta v - \delta(N-2-\delta)\,v + v^p = 0. \tag{4.7}$$

We choose $\gamma = -(N-2-2\delta) \neq 0$ and obtain

$$\gamma^2(v_{tt} - v_t) + \Delta_\theta v - \delta(N-2-\delta)\,v + v^p = 0. \tag{4.8}$$

Multiplying (4.8) by $v_t$ and integrating the resulting expression over $\Omega$, we get

$$\frac{dE(t)}{dt} = \gamma^2 \int_\Omega v_t^2\,d\theta, \tag{4.9}$$

where

$$E(t) := \tfrac{1}{2}\gamma^2 \int_\Omega v_t^2\,d\theta - \tfrac{1}{2}\int_\Omega |\nabla_\theta v|^2\,d\theta - \tfrac{1}{2}\,\delta(N-2-\delta)\int_\Omega v^2\,d\theta + (p+1)^{-1}\int_\Omega v^{p+1}d\theta.$$

Thus, for $t < 0$,

$$E(0) - E(t) = \gamma^2 \int_t^0 \int_\Omega v_t^2\,d\theta\,d\tau =: \varphi(t). \tag{4.10}$$

Since $E(t)$ is increasing, $\varphi(t)$ is decreasing. Furthermore, we have

$$\varphi'(t) - 2\varphi(t) = -E_1(t), \tag{4.11}$$

where

$$E_1(t) = \int_\Omega |\nabla_\theta v|^2\,d\theta + \delta(N-2-\delta)\int_\Omega v^2\,d\theta - 2(p+1)^{-1}\int_\Omega v^{p+1}\,d\theta + 2E(0).$$

The same argument as in the proof of Corollaray 3.5 gives

**Lemma 4.3.** *If $v$ is uniformly bounded as $r \to 0(\infty)$, then all its derivatives with respect to $t$ and $\theta$ are uniformly bounded for associated values of $t$ (the relation depends on the sign of $\gamma$).*

Hence $E_1$ is uniformly bounded on $(-\infty, 0)$ and we have

$$\varphi(t) = \int_t^0 e^{2(t-\tau)}E_1(\tau)\,d\tau, \qquad t < 0. \tag{4.12}$$

Thus $\varphi$ is uniformly bounded and we have proved

**Lemma 4.4.** *If* (4.5) *holds, then*

$$(4.13) \qquad \int_{-\infty}^{0} \int_{\Omega} v_t^2 \, d\theta \, d\tau < \infty .$$

From (4.11), we see that

$$\varphi'(t) = 2\varphi(t) - E_1(t) = 2\varphi(-\infty) + \varepsilon(t) - E_1(t)$$

where $\varepsilon(t) \to 0$, $t \to -\infty$, and $E_1 \in \text{Lip}\,(-\infty, 0)$ (*cf.* Lemma 4.3).

If $\limsup_{t \to -\infty} |\varphi'(t)| > 0$, (4.13) must be false. We conclude that

$$(4.14) \qquad \int_{\Omega} v_t^2 \, d\theta \to 0, \qquad t \to -\infty .$$

Integrating (4.8) with respect to $t$, we obtain

$$e^t v_t(0, \theta) - v_t(t, \theta) + \int_t^0 e^{t-\tau} V(\tau, \theta) \, d\tau = 0, \qquad t < 0$$

where

$$V(t, \theta) = \gamma^{-2}(\Delta_\theta v - \delta(N - 2 - \delta)\, v + v^p).$$

For $\eta \in C_0^\infty(\Omega)$ and for $f(t) = \int_{\Omega} V(t, \theta)\, \eta(\theta) \, d\theta$, we get

$$\int_{\Omega} v_t(t, \theta)\, \eta(\theta) \, d\theta = e^t \int_{\Omega} v_t(0, \theta)\, \eta(\theta) \, d\theta + \int_t^0 e^{t-\tau} f(\tau) \, d\tau .$$

From (4.14), we see that $v(t, \cdot) \to 0$ in $\mathscr{D}'(\Omega)$ as $t \to -\infty$. Hence

$$(4.15) \qquad \int_t^0 e^{t-\tau} f(\tau) \, d\tau \to 0, \qquad t \to -\infty .$$

According to Lemma 4.3, $f$ and $f'$ are uniformly bounded on $(-\infty, 0)$. If

$$g(t) = \begin{cases} e^t, & t < 0, \\ 0, & t > 0, \end{cases} \qquad F(t) = \begin{cases} f(t), & t < 0, \\ 0, & t > 0, \end{cases}$$

then (4.15) says that $F * g(t) \to 0$ as $t \to -\infty$. According to Pitt's form of Wiener's theorem (*cf.* Theorem 10a, p. 211 in [W]), it follows that $F(t) \to 0$ as $t \to -\infty$, *i.e.*,

$$(4.16) \qquad \int_{\Omega} (\Delta_\theta v - \delta(N - 2 - \delta)\, v + v^p)\, (\theta)\, \eta(\theta) \, d\theta \ \to \ 0, \qquad t \to -\infty .$$

We can now go back to the variable $r$. If $\{r_n\}$ is a sequence tending to zero, and if $v_n(\theta) = v(r_n, \theta)$, it follows from Ascoli's theorem that $\{v_n\}$ has a subsequence $\{v_{n_j}\}$ which is uniformly convergent on compact subsets of $\Omega$ as $j \to \infty$ and $r_{n_j} \to 0$. If $v_n \to \alpha$, then the function $\alpha$ is a weak solution of (2.14) and thus a classical solution. If in particular $\limsup_{n \to \infty} M(r_n, v) > 0$, we can choose the subsequence in such a way that $\alpha$ is not the trivial solution. (In this step, we also use Lemma 4.3).

*Remark.* By choosing another subsequence, we can assume that both $\{v_n\}$ and $\{\Delta_\theta v_n\}$ are uniformly convergent on compact subsets of $\Omega$. Letting $t \to -\infty$ in (4.10), we see that

$$\tfrac{1}{2}\int_\Omega |\nabla_\theta \alpha|^2\, d\theta + \tfrac{1}{2}\,\delta(N-2-\delta)\int_\Omega \alpha^2\, d\theta - (p+1)^{-1}\int_\Omega \alpha^{p+1}\, d\theta$$
$$= \gamma^2 \int_{-\infty}^{0}\int_\Omega v_t^2\, d\theta\, dt - E(0),$$

and that the integral in the left-hand member is independent of the choice of the subsequence $\{r_{n_j}\}$.

*Remark.* It remains to treat the case $\gamma = N - 2 - 2\delta = 0$. With the change of variables $t = \log r$, (4.6) becomes

$$v_{tt} + \Delta_\theta v - (p+1)^{-1} v^p = 0. \tag{4.8'}$$

The analogue of (4.9) is

(4.9')
$$\frac{d}{dt}\left(\tfrac{1}{2}\int_\Omega v_t^2\, d\theta - \left(\tfrac{1}{2}\int_\Omega |\nabla_\theta v|^2\, d\theta + \tfrac{1}{2}\int_\Omega \delta^2 v^2\, d\theta - (p+1)^{-1}\int_\Omega v^{p+1}\, d\theta\right)\right) = 0.$$

It is easy to see that in a bounded domain, solutions of (1.1), (1.2) cannot be ordered. The question is open whether this statement is true in cones. It has been proved in [BL] that for $\sigma = 0$ and $p < (N+2)/(N-2)$, no regular solution $u$ lies under any singular solution $u_s$ of the type $r^{-2/(p-1)}\,\alpha(\theta)$. Let us add some supplementary remarks.

**Lemma 4.5.** *Let* $u_s = r^{-(\sigma+2)/(p-1)}\,\alpha_1(\theta)$ *and* $v_s = r^{-(\sigma+2)/(p-1)}\,\alpha_2(\theta)$ *be two singular solutions of* (1.1), (1.2). *If* $u_s \leqq v_s$ *in* $\mathscr{C}$, *then* $u_s \equiv v_s$.

**Proof.** The positive difference $d(\theta) = \alpha_2(\theta) - \alpha_1(\theta)$ satisfies the differential inequality

$$\Delta_\theta d - \frac{\sigma+2}{p-1}\left(N - 2 - \frac{\sigma+2}{p-1}\right) d + p\alpha_1^{p-1}\, d \leqq 0. \tag{4.17}$$

From this, it follows that

$$0 = \int_\Omega (\alpha_1\, \Delta_\theta d - d\Delta_\theta\, \alpha_1)\, d\theta \leqq (1-p)\int_\Omega d\alpha_1^p\, d\theta \leqq 0. \tag{4.18}$$

Equality holds only for $d \equiv 0$, which proves the assertion.

This proof together with Lemma 4.2 implies

**Lemma 4.6.** *Let $u$ and $v$ be two solutions which are not regular at infinity but which satisfy* (4.5) *for all* $r \geqq R$. *If*

$$\limsup_{r\to\infty} \frac{u(x)}{v(x]} \leqq 1,$$

*then*

$$\limsup_{r\to\infty} \frac{u(x)}{v(x)} = 1$$

*in the sense of Lemma 4.2.*

The next result extends the result in [BL, Theorem 6.1] on the non-existence of stationary solutions under singular solutions.

**Theorem 4.7.** *Let $U$ be a positive solution of* (1.1), (1.2) *satisfying*

$$r^\delta U(x) \leqq c, \qquad x \in \mathscr{C}$$

*where* $\delta = (\sigma + 2)/(p - 1)$.

(i) *Assume that* $N - 2 - 2\delta < 0$. *Then there is no nontrivial regular solution* $u$ *with* $u \leqq U$.

(ii) *Assume that* $N - 2 - 2\delta > 0$. *Then there is no nontrivial solution $u$ regular at infinity with* $u \leqq U$.

**Proof.** We give here a proof completely different from that in [BL, Theorem 6.1].
(i) Suppose that a regular solution $u \leqq U$ exists. Then $V = U - u$ satisfies:

$$\Delta V + p r^\sigma u^{p-1} V \leqq 0, \qquad x \in \mathscr{C}.$$

This together with Green's identity yields

$$\int_{\partial\mathscr{D}'} \left(u \frac{\partial V}{\partial n} - V \frac{\partial u}{\partial n}\right) ds = \int_{\mathscr{D}'} (u\,\Delta V - V\,\Delta u)\, dx \leqq (1-p) \int_{\mathscr{D}'} r^\sigma u^p V\, dx.$$

If we choose $\mathscr{D}' = \mathscr{D}(R_0, R)$ (*cf.* Section 3.1), then, since $u$ and $V$ vanish on $\partial\mathscr{D}' \cap \partial\mathscr{C}$,

$$\int_{\Gamma_0 \cup \Gamma} \left(u \frac{\partial V}{\partial n} - V \frac{\partial u}{\partial n}\right) ds \leqq (1-p) \int_{\mathscr{D}'} r^\sigma u^p\, \delta\, dx \tag{4.19}$$

where $\Gamma_0 = \partial\mathscr{D}' \cap \{|x| = R_0\}$ and $\Gamma = \partial\mathscr{D}' \cap \{|x| = R\}$. From Corollary 3.5, we know that $r^{-\gamma_+} u(x)$ and $r^{1-\gamma_+} |\nabla u|$ are bounded as $x \to 0$.

We assert that it follows from our assumptions on $U$ and $u$ that $r^{1+\delta} |\nabla V|$, $r^\delta u(x)$ and $r^{1+\delta} |\nabla u|$ are uniformly bounded for $0 < r < \infty$. To see this, we use the arguments in the proof of Corollary 3.5 with $\nu = -\sigma - 2 + p\,\delta$. Consequently, using (4.1), we see that

$$\left|\int_{\Gamma_0} \left(u \frac{\partial V}{\partial n} - V \frac{\partial u}{\partial n}\right) ds\right| \leqq c R^{N-2+\gamma_+-\delta} = c R^{-\gamma_- - \delta} \to 0, \qquad R_0 \to 0, \tag{4.20}$$

$$\left|\int_{\Gamma} \left(u \frac{\partial V}{\partial n} - V \frac{\partial u}{\partial n}\right) ds\right| \leqq c R^{N-2-2\delta} \to 0, \qquad R \to \infty. \tag{4.21}$$

Inserting (4.20) and (4.21) into (4.19), we get a contradiction.

The proof of (ii) is similar and we omit the details.

**Note added in proof.** The case $\sigma = \frac{1}{2} N(p-1) - (p+1)$ which was left open in Theorem 4.1, has recently been studied by H. EGNELL "Positive solutions of semilinear equations in cones" (to appear). Existence and nonexistence of solutions depend on $\sigma$ and the shape of $\Omega$.

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Mathematisches Institut
Universität Basel
Switzerland

and

Department of Mathematics
Uppsala University
Sweden

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