Entire functions of several variables and their asymptotic growth

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§ 1. Introduction

Let $f(z)$ be an entire function of several complex variables of finite order ϱ and normal type (in what follows, we shall always let the variable z represent an n-tuple $(z_1, \ldots, z_n)$ and $\|z\| = (\sum_{j=1}^{n} z_j \bar{z}_j)^{1/2}$ is the Euclidean norm).

Following the classical case of functions of one variable [7], we introduce the functions $h(z) = \overline{\lim_{r\to\infty}} \frac{\ln|f(rz)|}{r^\varrho}$ and $h^*(z) = \overline{\lim_{z'\to z}} h(z')$, and we call $h^*(z)$ the radial indicator (of growth) function [4, 5, 6].

Both $h(z)$ and $h^*(z)$ are positively homogeneous of order ϱ; that is for $t > 0$, $h(tz) = t^\varrho h(z)$ and $h^*(tz) = t^\varrho h^*(z)$. Lelong has further shown that $h(z) = h^*(z)$ except on a set of $\mathbf{R}^{2n}$ Lebesgue measure zero, and since both are positively homogeneous of order ϱ, $h(z) = h^*(z)$ for almost all $z \in S^{2n-1}$, the unit sphere in $\mathbf{C}^n$. The function $h^*(z)$ is plurisubharmonic and is independent of the point in $\mathbf{C}^n$ chosen as origin (thus, if $f(z) \not\equiv 0$, it will always be possible to assume, without loss of generality, that $f(0) \neq 0$).

There are certain properties of the classical indicator function of one variable which have no counterpart for n variables ($n \geqq 2$). The classical indicator function is continuous [7, p. 54], but Lelong [6] has shown that this is no longer necessarily true for $n \geqq 2$. His method was to construct (for all ϱ) a non-continuous plurisubharmonic function complex homogeneous of order ϱ.

For functions of one variable, the growth of the function $f(z)$ is determined by the density and distribution of the zeros [7]. In particular, the regularity of the distribution of the zeros determines the regularity of the function $f(z)$ and the regularity of the indicator function. Our criteria for regularity of growth will be the following: Let E be a measurable set of positive numbers and $E_r = E \cap [0, r)$. If $\lim_{r\to\infty} \frac{\text{meas}\,(E_r)}{r} = 0$, E is said to be a set of zero relative measure (an E^0-set).

Definition. A function $f(z)$ of finite order ϱ and normal type will be said to be of *completely regular growth along a ray* (rz), $r \in [0, \infty)$, if $\lim_{r\to\infty} \frac{\ln|f(rz)|}{r^\varrho} = h^*(z)$ where r takes on all values in the complement of some E^0-set; $f(z)$ will be said to be of *completely regular growth for a (measurable) set* D if $f(z)$ is of completely regular growth for almost all $z \in D$ (i.e. except perhaps on a set of Lebesgue measure zero). The E^0-set will in general depend upon z.

Paragraphs 2 and 3 extend several results known to be true for the classical indicator function to the case of several variables. In § 4, it is shown that if in some region there are not too many zeros of the function, then the indicator function satisfies a Lipschitz condition in the projection on the unit sphere, and hence by homogeneity is continuous in the cone containing the region with vertex at the origin.

§ 2. Global properties of the distribution of zeros

To investigate the behavior of the function, we shall need some inequalities relating the functions $h_{f,r} = \frac{\ln|f(rz)|}{r^\varrho}$ and $h^*(z)$ on compact sets.

LEMMA 2.1. *If $f(z)$ is of order ϱ and finite type, there is a constant T_0 such that $|h^*(z)| \leqq T_0\|z\|^\varrho$.*

Proof. If B is the type of $f(z)$, then $h^*(z) \leqq B\|z\|^\varrho$. Furthermore, $h^*(z) \geqq h(z)$, so there exists a $T > 1$ such that in every complex line in which $f(z) \not\equiv 0$, $h(z) \geqq -TB\|z\|^\varrho$ (see [7, p. 21]). But then, by the upper semicontinuity of $h^*(z)$, $|h^*(z)| \leqq TB\|z\|^\varrho$. Q.E.D.

LEMMA 2.2. *Let $k_t(z)(t \in [0, \infty))$ be a family of plurisubharmonic functions uniformly bounded above on each compact set and let $k(z) = \overline{\lim_{t\to\infty}} k_t(z)$. Then for every compact set K and every continuous $l(z) \geqq k(z)$, given $\varepsilon > 0$, there exists T_ε such that $t \geqq T_\varepsilon$ implies $k_t(z) \leqq l(z) + \varepsilon$ for all $z \in K$.*

Proof. The proof can be found in Hörmander [1], p. 283.

There exists a real valued positive C^∞ function $\alpha(z)$ with compact support such that $\alpha(z)$ depends only on $\|z\|$ and $\int \alpha(z)dV = 1$. Let $\alpha_\delta(z) = \frac{1}{\delta^{2n}}\alpha(z/\delta)$ $(\delta \leqq 1)$ and $h^*_\delta(z) = \int \alpha_\delta(z' - z)h^*(z')dV'$. Then $h^*_\delta(z)$ is C^∞. It follows from the mean value property of the plurisubharmonic function $h^*(z)$ that $h^*_\delta(z) \geqq h^*(z)$, and it follows from the upper semicontinuity of $h^*(z)$ that $\lim_{\delta\to 0} h^*_\delta(z) = h^*(z)$ (pointwise convergence).

LEMMA 2.3. *For every compact set K in $\mathbf{C}^n$, given $\varepsilon > 0$, $1 \geqq \delta > 0$, there exists an $R_{\delta,\varepsilon}$ such that*

$$h_{f,r}(z) = \frac{\ln|f(rz)|}{r^\varrho} \leqq h_\delta^*(z) + \varepsilon \quad \text{for} \quad r \geqq R_{\delta,\varepsilon} \quad \text{and} \quad z \in K .$$

Proof. The proof follows immediately from Lemma 2.2.

We now establish some of the global properties of the zero set of an entire function of finite order and normal type. To do this, we shall need a measure of the zero set. We follow here the development of Lelong [3]. Let

$$d_z = \sum_{j=1}^n \frac{\partial}{\partial z_j} dz_j \quad \text{and} \quad d_{\bar z} = \sum_{j=1}^n \frac{\partial}{\partial \bar z_j} d\bar z_j .$$

We introduce the current of integration $\tau = \dfrac{i}{\pi} d_z d_{\bar z} \ln|f(z)|$. Let $\beta = \dfrac{i}{2} d_z d_{\bar z} \|z\|^2$ and $\alpha_a = \dfrac{i}{2} d_z d_{\bar z} \ln\|z - a\|^2$ $(z \neq a)$.

The powers $\beta^p/p!$ are elements of volume of complex dimension p, and the powers $\alpha_a^p/p!$ are elements of volume of complex dimension p in the complex projective space CP^{n-1} of complex lines emanating from a. Assuming $f(a) \neq 0$, we pose the measures

$$\sigma = \tau \wedge \frac{\beta^{n-1}}{(n-1)!} \quad \text{and} \quad \nu_a = \frac{1}{\pi^{n-1}} \tau \wedge \alpha_a^{n-1} .$$

The common support of σ and ν is just the zero set of $f(z)$. We introduce the functions $\sigma_a(r) = \displaystyle\int_{\|z-a\| \leq r} \sigma$ and $\nu_a(r) = \displaystyle\int_{\|z-a\| \leq r} \nu_a$. Both are positive increasing functions of r. Then

$$\nu_a(r) = \frac{(n-1)!}{\pi^{n-1}} \frac{\sigma_a(r)}{r^{2n-2}} . \tag{2.00}$$

If we write $z_j = x_j + iy_j$, then the Laplacian is

$$\Delta = \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) = 4 \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar z_j} .$$

An easy calculation shows that

$$\sigma = \tau \wedge \frac{\beta^{n-1}}{(n-1)!} = \frac{1}{2\pi} (\Delta \ln|f(z)|) \frac{\beta^n}{n!} \tag{2.01}$$

The volume of the unit ball in $\mathbf{C}^n$ is $\dfrac{\pi^n}{n!}$ and the area of the unit sphere is $\dfrac{2\pi^n}{(n-1)!}$. Let $\omega_{2n-1}^{(a)}$ be the measure of area of the unit sphere centered at a.

LEMMA 2.4. (Jensen formula) *If* $f(a) \neq 0$, *then*

$$\int_0^r \frac{\nu_a(t)dt}{t} = \frac{(n-1)!}{\pi^{n-1}} \int_0^r \frac{\sigma_a(t)dt}{t^{2n-1}} = \frac{(n-1)!}{2\pi^n} \int_{\|w\|=1} \ln|f(a+rw)|\omega_{2n-1}^{(a)} - \ln|f(a)| .$$

Proof. The classical Jensen formula is valid in every complex line. The lemma then follows from integrating over all complex lines. Q.E.D.

Since we shall be primarily concerned with the case $a = 0$, we shall write $\nu(r)$ to mean $\nu_0(r)$, $\sigma(r)$ to mean $\sigma_0(r)$, and ω_{2n-1} to mean $\omega_{2n-1}^{(0)}$.

THEOREM 2.1. *Let* $f(z)$ *be a function of finite order* ϱ *and normal type such that* $f(0) \neq 0$. *Let* $N(r) = \int_0^r \frac{\nu(t)dt}{t}$. *Then* $\overline{\lim_{r\to\infty}} \frac{N(r)}{r^\varrho} \leqq \frac{(n-1)!}{2\pi^n} \int_{\|w\|=1} h^*(w)\omega_{2n-1}$.

Proof. By Lemma 2.4 we have

$$\frac{N(r)}{r^\varrho} = \frac{(n-1)!}{2\pi^n} \int_{\|w\|=1} \frac{\ln|f(rz)|}{r^\varrho} \omega_{2n-1} - \frac{\ln|f(0)|}{r^\varrho} .$$

By Lemma 2.3, given $\varepsilon > 0$, $\delta > 0$, for r sufficiently large $\frac{\ln|f(rz)|}{r^\varrho} \leqq h_\delta^*(z) + \varepsilon$ on the unit sphere. Combining and letting $r \to \infty$ and $\varepsilon \to 0$, we get $\overline{\lim_{r\to\infty}} \frac{N(r)}{r^\varrho} \leqq \frac{(n-1)!}{2\pi^n} \int_{\|w\|=1} h_\delta^*(w)\omega_{2n-1}$.

By Lemma 2.1, we can apply the Lebesgue dominated convergence theorem, from which it follows that

$$\overline{\lim_{r\to\infty}} \frac{N(r)}{r^\varrho} \leqq \lim_{\delta\to 0} \frac{(n-1)!}{2\pi^n} \int_{\|w\|=1} h_\delta^*(w)\omega_{2n-1} = \frac{(n-1)!}{2\pi^n} \int_{\|w\|=1} h^*(w)\omega_{2n-1} .$$ Q.E.D.

COROLLARY. $\overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^\varrho} \leqq \frac{e\varrho(n-1)!}{2\pi^n} \int_{\|w\|=1} h^*(w)\omega_{2n-1}$.

Proof. $\frac{\nu(r)}{r^\varrho} \leqq \frac{\varrho}{r^\varrho} \int_r^{re^{1/\varrho}} \frac{\nu(t)dt}{t} \leqq \varrho \frac{N(e^{1/\varrho}r)}{r^\varrho} = e\frac{\varrho N(e^{1/\varrho}r)}{(e^{1/\varrho}r)^\varrho}$ from which one has

$\overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^\varrho} \leqq e\varrho \overline{\lim_{r\to\infty}} \frac{N(r)}{r^\varrho}$ Q.E.D.

Remark. It follows from the corollary that if $f(z)$ is an entire function of finite order ϱ and finite type B that

$$\overline{\lim_{r\to\infty}}\frac{\nu(r)}{r^{\varrho}}=\frac{(n-1)!}{\pi^{n-1}}\overline{\lim_{r\to\infty}}\frac{\sigma(r)}{r^{\varrho+2n-2}}\leqq e\varrho B\,.$$

LEMMA 2.5. *If* $\phi(r)$ *is a locally bounded function and*

$$\overline{\Delta}=\overline{\lim_{r\to\infty}}\frac{\phi(r)}{r^{\varrho}}\,,\ \underline{\Delta}=\underline{\lim_{r\to\infty}}\frac{\phi(r)}{r^{\varrho}}\,,\quad \textit{we have for}\quad \alpha>0\,,$$

(i) $$\overline{\lim_{r\to\infty}}\left\{\frac{1}{r^{\varrho}}\int_{\alpha}^{r}\frac{\phi(t)dt}{t}\right\}\leqq\frac{\overline{\Delta}}{\varrho}$$

(ii) $$\underline{\lim_{r\to\infty}}\left\{\frac{1}{r^{\varrho}}\int_{\alpha}^{r}\frac{\phi(t)dt}{t}\right\}\geqq\frac{\underline{\Delta}}{\varrho}\,.$$

Proof. See Levin [7, p. 34].

THEOREM 2.2. *Let* $f(z)$ *be an entire function of order* ϱ *and normal type. If* $f(z)\not\equiv 0$ *and if* $f(0)\neq 0$, *then*

$$\underline{\lim_{r\to\infty}}\frac{\nu(r)}{r^{\varrho}}\leqq\frac{\varrho(n-1)!}{2\pi^{n}}\int_{\|w\|=1}h^{*}(w)\omega_{2n-1}$$

and equality holds if $f(z)$ *is of completely regular growth in* $\mathbf{C}^n$.

Proof. By Lemma 2.5, $\underline{\lim_{r\to\infty}}\frac{\nu(r)}{r^{\varrho}}\leqq\varrho\,\underline{\lim_{r\to\infty}}\frac{N(r)}{r^{\varrho}}$. If $f(z)$ is of completely regular growth, it is of completely regular growth along almost all rays and so is of completely regular growth along almost every ray in almost every complex line passing through the origin. Let (uw) $(\|w\|=1)$ be a complex line in which $f(z)$ is of completely regular growth along almost every ray. Since it is of completely regular growth on a dense set of rays, it is of completely regular growth in the sense of one variable [7, pp. 141—142] and so by the theorem for one variable, [7, p. 173], setting $n_w(r)=\{\text{number of zeros of } f(uw) \text{ of modulus less than } r\}$, we have

$$\underline{\lim_{r\to\infty}}\frac{n_w(r)}{r^{\varrho}}=\frac{\varrho}{r^{\varrho}}\int_{0}^{2\pi}h(e^{i\theta}w)d\theta\,.$$

This holds for almost all complex lines. Integrating over all complex lines and applying Fatou's lemma (since all the integrands are positive)

$$\frac{\varrho(n-1)!}{2\pi^{n}}\int_{\|w\|=1}h^{*}(w)\omega_{2n-1}\geqq\underline{\lim_{r\to\infty}}\frac{\nu(r)}{r^{\varrho}}=\underline{\lim_{r\to\infty}}\int_{\|z\|\leqq r}\frac{1}{r^{\varrho}}\frac{\tau\wedge\alpha_0^{n-1}}{\pi^{n-1}}$$

$$\geqq\underline{\lim_{r\to\infty}}\int_{CP^{n-1}}\frac{(n-1)!}{\pi^{n-1}}\frac{\alpha_0^{n-1}}{(n-1)!}\frac{n_w(r)}{r^{\varrho}}$$

$$\geqq \frac{(n-1)!}{\pi^{n-1}} \int_{CP^{n-1}} \frac{\alpha_0^{n-1}}{(n-1)!} \varlimsup_{r\to\infty} \frac{n_w(r)}{r^\varrho}$$

$$\geqq \frac{(n-1)!}{\pi^{n-1}} \int_{CP^{n-1}} \frac{\alpha_0^{n-1}}{(n-1)!} \frac{\varrho}{2\pi} \int_0^{2\pi} h(e^{i\theta}w)d\theta$$

$$\geqq \frac{\varrho(n-1)!}{2\pi^n} \int_{\|w\|=1} h(w)\omega_{2n-1} .$$

But $h^*(z) = h(z)$ almost everywhere on the unit sphere, so equality must hold everywhere in the above equations. Q.E.D.

§ 3. Functions of completely regular growth in a region

We now introduce a notation which will be useful in what follows. Let D be any subset of $\mathbf{C}^n$. We denote by

$$D_r = \{tz\,;\ z \in D\,,\ t \in [0\,, r]\}$$
$$D_\infty = \{tz\,;\ z \in D\,,\ t \in [0\,, \infty)\}$$

We show that if $f(z)$ is of completely regular growth in a region in which the indicator function is harmonic, then the density of zeros in that region is quite small.

LEMMA 3.1. *Let $n \geqq 2$ and let (rw_0) be a ray of completely regular growth ($\|w_0\| = 1$). Let $D = \{w\,; \|w\| = 1,\ \|w - w_0\| < \eta/2\}$ and define $\sigma_D(r) = \int_{D_r} \sigma$. Then there is a constant $k(\eta)$ such that*

$$\varlimsup_{r\to\infty} \frac{\sigma_D(r)}{r^{\varrho+2n-2}} = k(\eta)\left[\frac{(n-1)!}{2\pi^n} \int_{\|a\|=1} h^*(w_0 + \eta a)\omega_{2n-1} - h^*(w_0)\right]$$

Proof. Since $f(z)$ is of completely regular growth along the ray (rw_0), for any fixed s_0, $f(sw_0) \neq 0$ for almost all $s \geqq s_0$. For such $s \geqq s_0$, we form the sphere of radius ηs centered at (sw_0). Then by Lemma 2.4

$$\frac{(n-1)!}{2\pi^n} \int_{\|a\|=1} \ln|f(sw_0 + \eta sa)|\omega_{2n-1} - \ln|f(sw_0)|$$

$$\geq \frac{(n-1)!}{\pi^{n-1}} \int_0^{\eta s} \frac{\sigma_{sw_0}(t)dt}{t^{2n-1}} \geq \frac{(n-1)!}{\pi^{n-1}} \sigma_{sw_0}(3s\eta/4) \int_{\frac{3s\eta}{4}}^{\eta s} \frac{dt}{t^{2n-1}}$$

$$\geq k_1(\eta) \frac{\sigma_{sw_0}(3s\eta/4)}{s^{2n-2}}$$

We now divide by s, integrate from s_0 to r, and interchange the order of integration:

$$\frac{(n-1)!}{2\pi^n}\int_{\|a\|=1}\omega_{2n-1}\int_{s_0}^{r}\frac{\ln|f(sw_0+\eta sa)|ds}{s}-\int_{s_0}^{r}\frac{\ln|f(sw_0)|ds}{s}$$

$$\geq k_1(\eta)\int_{s_0}^{r}\frac{\sigma_{sw_0}(3s\eta/4)ds}{s^{2n-1}}.$$

We first examine the right hand side of this inequality. Let $r=s_0(1+\eta/4)^m+c$ for some integer m such that $\dfrac{r}{s_0(1+\eta/4)^m}<(1+\eta/4)$. Let

$$D_q=D_{s_0(1+\eta/4)^q}\setminus D_{s_0(1+\eta/4)^{q-1}}\quad(q=1,\ldots,m).$$

If $s_0(1+\eta/4)^{q-1}\leq s\leq s_0(1+\eta/4)^q$, then $D_q\subset B(sw_0;\,3s\eta/4)$ (the ball of radius $3s\eta/4$ centered at sw_0), since $z'\in D_q$ means that $z'=\dfrac{sz}{\|z\|}$ satisfies $\|z'-z\|=\left\|\dfrac{sz}{\|z\|}-z\right\|=|s-\|z\||<\eta s/4$, and $\|z'-sw_0\|<\eta s/2$ so $\|z-sw_0\|<3s\eta/4$.

Hence, if $a_q=s_0(1+\eta/4)^q$, $(a_q-a_{q-1})\displaystyle\int_{D_q}\sigma\leq\int_{a_{q-1}}^{a_q}\sigma_{sw_0}(3s\eta/4)ds$.

Dividing by a_q^{2n-1} and noting that $a_{q-1}\leq s\leq a_q=a_{q-1}(1+\eta/4)$, we have

$$\frac{(1-a_{q-1}/a_q)}{a_q^{2n-2}}\int_{a_{q-1}}^{a_q}d\sigma_D(s)\leq\frac{1}{a_q^{2n-1}}\int_{a_{q-1}}^{a_q}\sigma_{sw_0}(3s\eta/4)ds\leq\int_{a_{q-1}}^{a_q}\frac{\sigma_{sw_0}(3s\eta/4)ds}{s^{2n-1}}$$

or

$$\frac{\eta/4}{(1+\eta/4)^{2n-1}}\int_{a_{q-1}}^{a_q}\frac{d\sigma_D(s)}{s^{2n-2}}\leq\int_{a_{q-1}}^{a_q}\frac{\sigma_{sw_0}(3s\eta/4)ds}{s^{2n-1}}$$

Summing over all q, we have

$$k_2(\eta)\int_{s_0}^{r/(1+\eta/4)}\frac{d\sigma_D(s)}{s^{2n-2}}\leq\frac{(n-1)!}{2\pi^n}\int_{\|a\|=1}\omega_{2n-1}\int_{s_0}^{r}\frac{\ln|f(sw_0+\eta sa)|ds}{s}-\int_{s}^{r}\frac{\ln|f(sw_0)|ds}{s}$$

Integrating the left hand side of this last inequality by parts, we have

$$\int_{s_0}^{r/(1+\eta/4)}\frac{d\sigma_D(s)}{s^{2n-2}}=\frac{\sigma_D(s)}{s^{2n-2}}\Big]_{s_0}^{r/(1+\eta/4)}+(2n-2)\int_{s_0}^{r/(1+\eta/4)}\frac{\sigma_D(s)ds}{s^{2n-1}}\geq\frac{\sigma_D(s)}{s^{2n-2}}\Big]_{s_0}^{r/(1+\eta/4)}$$

We now examine the right hand side of the inequality. For all $\varepsilon > 0$ and δ, $1 > \delta > 0$, and for s sufficiently large, $\ln|f(sw_0 + \eta sa)| \leq (h_\delta^*(w_0 + \eta a) + \varepsilon) \cdot s^\varrho$ by lemma 2.3, which implies that asymptotically

$$\int_{\|a\|=1} \omega_{2n-1} \int_{s_0}^{r} \frac{\ln|f(sw_0 + \eta sa)|ds}{s} < o(r^\varrho) + \int_{\|a\|=1} (h_\delta^*(w_0 + \eta a) + \varepsilon)\omega_{2n-1} \int_{s_0}^{r} s^{\varrho-1} ds$$

$$\leq o(r^\varrho) + \frac{r^\varrho}{\varrho} \int_{\|a\|=1} (h_\delta^*(w_0 + \eta a) + \varepsilon)\omega_{2n-1}.$$

Since (rw_0) is a ray of completely regular growth, by [7, p. 144], for r sufficiently large, $\frac{1}{r^\varrho} \int_{s_0}^{r} \frac{\ln|f(sw_0)|ds}{s} \geq \frac{1}{\varrho} h^*(w_0) - \varepsilon$. Hence, for r sufficiently large,

$$\left(\frac{\sigma_D(r/(1+\eta/4))}{r^{\varrho+2n-2}} - \frac{\sigma_D(s_0)}{r^\varrho s_0^{2n-2}}\right) \leq k_3(\eta)\left(\frac{(n-1)!}{2\pi^n} \int_{\|a\|=1} h_\delta^*(w_0 + \eta a)\omega_{2n-1} - h^*(w_0)\right) + k\varepsilon$$

for some constant k. Letting $r \to \infty$ and $\varepsilon \to 0$, we find

$$\overline{\lim_{r\to\infty}} \frac{\sigma_D(r)}{r^{\varrho+2n-2}} \leq k_4(\eta)\left(\frac{(n-1)!}{2\pi^n} \int_{\|a\|=1} h_\delta^*(w_0 + \eta a)\omega_{2n-1} - h^*(w_0)\right)$$

for all δ. Applying the Lebesgue dominated convergence theorem, since $h^*(w_0 + \eta a) = \lim_{\delta\to 0} h_\delta^*(w_0 + \eta a)$,

$$\overline{\lim_{r\to\infty}} \frac{\sigma_D(r)}{r^{\varrho+2n-2}} \leqq k_4(\eta)\left(\frac{(n-1)!}{2\pi^n} \int_{\|a\|=1} h^*(w_0 + \eta a)\omega_{2n-1} - h^*(w_0)\right) \quad \text{Q.E.D.}$$

THEOREM 3.1. *Let $f(z)$ be of finite order ϱ and normal type. Let D be an open subset of the unit sphere such that $f(z)$ is of completely regular growth in D. If in addition $h^*(z)$ is pluriharmonic in D_∞, then for any set K relatively compact in D,* $\lim_{r\to\infty} \frac{\sigma_K(r)}{r^{\varrho+2n-2}} = 0$.

Proof. For each $w \in K$, there is an η_w such that the set $D_w = \{w' ; \|w'\| = 1, \|w' - w\| < 8\eta_w\}$ is contained in D. Since K is compact, it is covered by a finite number of the sets $D'_w = \{w' ; \|w'\| = 1, \|w' - w\| < \eta_w\}$. We shall index these sets D'_{w_i}, $i = 1, \ldots, N$. Since the set of rays on which $f(z)$ is of completely regular growth is dense in D, there is a w_i'' such that $\|w_i'' - w_i'\| < \eta_{w_i}$ and $f(z)$ is of completely regular growth along the ray (rw_i''). Then $D'_{w_i} \subset D''_{w_i} = \{w' ; \|w'\| = 1, \|w' - w_i''\| < 2\eta_{w_i}\}$ and $\{w' ; \|w'\| = 1, \|w' - w_i''\| < 4\eta_{w_i}\} \subset D$.

By Lemma 3.1, we have

$$\lim_{r\to\infty} \frac{\sigma_{D''_{w_i}}(r)}{r^{\varrho+2n-2}} = 0 .$$

But
$$\frac{\sigma_K(r)}{r^{\varrho+2n-2}} \leq \sum_{i=1}^{N} \frac{\sigma_{D''_{w_i}}(r)}{r^{\varrho+2n-2}} .$$ Q.E.D.

In the next section we shall prove a converse to this theorem.

Remark 1. If $h^*(z)$ is pluriharmonic in a region D, it follows from the homogeneity of $h^*(z)$ that it is pluriharmonic in the region D_∞.

Remark 2. It is actually sufficient to assume $h^*(z)$ only harmonic in D (see Lemma 4.10).

§ 4. Regions of low density of zeros

As was remarked after Theorem 2.1, for an entire function of finite order and normal type, the quantities $v(r)/r^\varrho$ and $\sigma(r)/r^{\varrho+2n-2}$ are bounded. In this chapter, we shall investigate regions in which the density of zeros as determined by the measure σ is small — that is, regions D for which $\sigma_D(r) = o(r^{\varrho+2n-2})$. We base our work on the paper of Lelong [3] in which he works out an n-dimensional analogue of the classical Hadamard theorem. We shall always assume that $f(0) \neq 0$.

We form the canonical potential by taking $h_n(a, z) = \frac{1}{\|z-a\|^{2n-2}}$ $(n \geqq 2)$ and expanding it in a neighborhood of the origin $z = 0$ in a series of homogeneous polynomials $(a \neq 0)$

$$h_n(a, z) = \frac{1}{\|a\|^{2n-2}} + P_1(a, z) + P_2(a, z) + \ldots P_i(a, z) + \ldots .$$

The polynomials $P_i(a, z)$ are harmonic in z. We define

$$e_n(a, z, q) = -h_n(a, z) + \frac{1}{\|a\|^{2n-2}} + P_1(a, z) + \ldots + P_q(a, z) .$$

Then $e_n(a, z, q)$ as well as all its partial derivatives up to order q are zero in a neighborhood of the origin. Letting $k_n = (n-2)!/2\pi^{n-1}$, we form the integrals $I_q(z) = k_n \int \sigma(a) e_n(a, z, q)$, where q is the smallest integer such that $\int_0^\infty \frac{dv(t)}{t^{q+1}}$ converges (which is equivalent to the condition that q be the smallest integer such that $\lim_{t\to\infty} \frac{v(t)}{t^{q+1}} = 0$ and $\int_0^\infty \frac{v(t)dt}{t^{q+2}} < \infty$ hold simultaneously).

If $f(z)$ is of finite order ϱ, then we can write $f(z) = Q(z) \exp(P(z))$, where $I_q(z) = \ln |Q(z)|$, $q \leqq \varrho$, and $P(z)$ is a polynomial of degree $q' \leqq \varrho$. If $f(z)$ is of normal type, then $Q(z)$ is also of order (at most) ϱ and normal type (with respect to order ϱ).

LEMMA 4.1. *Let $h^*(z)$ be the indicator function of $f(z)$ and let $k^*(z)$ be that of $Q(z)$ (with respect to order ϱ). Then* (i) *$f(z)$ is of completely regular growth along a ray (rz) if and only if $Q(z)$ is of completely regular growth along (rz);* (ii) *$h^*(z)$ is continuous at a point z_0 if and only if $k^*(z)$ is continuous at z_0;* (iii) *$h^*(z)$ satisfies a Lipschitz condition $|h^*(w) - h^*(w')| \leqq K\|w - w'\|^s$ $(s \leqq 1)$ on a subset of the unit sphere if and only if $k^*(z)$ does;* (iv) *if $q \leqq \varrho - 1$, then $k^*(z) \equiv 0$.*

Proof. Let q' be the degree of $P(z)$. We decompose $P(z)$ into homogeneous polynomials: $P(z) = \sum_{i=0}^{q'} p_i(z)$. Let $l(z) = \overline{\lim_{r\to\infty}} \operatorname{Re} \frac{P(rz)}{r^\varrho}$. If $q' < \varrho$, then $l(z) = 0$ and the limit exists along all rays. If $q' = \varrho$, $l(z) = \operatorname{Re} p_{q'}(z)$, and again the limit exists along all rays. Hence

$$h(z) = k(z) + l(z) \quad \text{and so} \quad h^*(z) = k^*(z) + l(z)$$

from which (i) and (ii) follow.

Let $j = (j_1, \ldots, j_n)$ be a multi-index with $|j| = j_1 + \ldots + j_n$. We expand $p_{q'}$ in its Taylor series expansion around w, $\|w\| = 1$, $p_{q'} = \sum_{j=0}^{q'} a_j(w)(z - w)^j$. In particular, if w' is such that $\|w'\| = 1$, since $a_j(w)$ is bounded on the unit sphere and $\|w' - w\| \leqq 2$,

$$|\operatorname{Re} p_{q'}(w') - \operatorname{Re} p_{q'}(w)| \leqq |p_{q'}(w') - p_{q'}(w)| = |\sum_{j=1}^{q'} a_j(w)(w' - w)^j| \leqq T\|w' - w\|$$

for some appropriate constant T, from which (iii) follows.

It is shown in [3] that $q = \varrho - 1$ implies that $Q(z)$ is of minimal type with respect to ϱ, so $k(z) \equiv k^*(z) \equiv 0$ in this case. If $q < \varrho - 1$, then $Q(z)$ is of order less than ϱ, so again $k(z) \equiv k^*(z) \equiv 0$, from which (iv) follows. Q.E.D.

Thus, in considering questions of continuity, Lipschitz conditions on subsets of the unit sphere, or regions of completely regular growth for $h^*(z)$, it is enough to consider the function $Q(z)$, or alternately $I_q(z) = \ln |Q(z)|$.

For entire functions of one variable, the Cartan estimate [7, p. 19] is used to estimate the function off the zero set. We develop here an n-dimensional counterpart $(n \geqq 2)$ which will serve the same purpose.

LEMMA 4.2. *Let D be an open set contained in the ball of radius r about the origin and let $A = \int_D \sigma$. Then given W, $1 > W > 0$, there exists a set Q of measure less than Wr^{2n} such that for $z \notin Q$, $\int_D \frac{\sigma(a)}{\|z-a\|^{2n-2}} \leqq \frac{CA}{Wr^{2n-2}}$.*

Proof. We may assume that $\|z\| \leq 2r$. Let $\sigma' = \sigma|_D$, the measure σ restricted to D, and $\sigma'_z(t) = \int_{\|z-a\| \leq t} \sigma'$.

We pose $\tilde{\sigma}'(z) = \sup_{B \ni z} \frac{\sigma'(B)}{\lambda(B)}$, B a ball containing z and λ Lebesgue measure. Then

$$F(z) = \int_D \frac{\sigma(a)}{\|z-a\|^{2n-2}} = \int_0^{3r} \frac{d\sigma'_z(t)}{t^{2n-2}} = \frac{\sigma'_z(t)}{t^{2n-2}}\Big|_0^{3r} + (2n-2)\int_0^{3r} \frac{\sigma'_z(t)}{t^{2n-2}}\,dt \leqq \tilde{\sigma}(z) r^2 K_n$$

so $$\left\{z\,;\, F(z) > \frac{CA}{Wr^{2n-2}}\right\} \subset \left\{z\,;\, K_n \tilde{\sigma}_D(z) > \frac{CA}{Wr^{2n}}\right\}$$

By the Hardy maximal theorem (cf. [2], p. 67)

$$\lambda\{z\,;\, \tilde{\sigma}'(z) > s\} \leq \frac{4^{2n}}{s}\nu(1) = \frac{4^{2n}}{s}A$$

so $\lambda\left\{z\,;\, F(z) > \frac{CA}{Wr^{2n-2}}\right\} \leq \frac{4^{2n}K_n}{C} Wr^{2n}$. It remains to choose $C = 4^{2n}K_n$. Q.E.D.

We shall be dealing with functions which are harmonic in a given region, and we shall need an estimate of their growth locally. The following lemma will be useful.

LEMMA 4.3. *If $g(z)$ is a harmonic function for $\|z\| \leq R$ and $A_g(r) = \max_{\|z\|=r} g(z)$, then there is a constant K such that*

$$A_g(r) \leq [A_g(R) - g(0)]K \frac{r/R}{[1-r/R]^{2n}} + |g(0)| \quad \textit{for} \quad r < R\,.$$

Proof. We first assume $R = 1$. We begin with the Poisson integral representation

$$g(z) = \frac{1}{\Omega_{2n-1}} \int_{\|w\|=1} g(w) \frac{1-\|z\|^2}{\|w-z\|^{2n}}\,\omega_{2n-1}$$

where Ω_{2n-1} is the surface area of the unit sphere $\|w\| = 1$. Subtracting $0 = \frac{1}{\Omega_{2n-1}} \int_{\|w\|=1} g(w)\omega_{2n-1} - g(0)$ we have

$$g(z) = \frac{1}{\Omega_{2n-1}} \int_{\|w\|=1} g(w) \left[\frac{1 - \|z\|^2 - \|w - z\|^{2n}}{\|w - z\|^{2n}} \right] \omega_{2n-1} + g(0) .$$

If we set $g(z) = 1$,

$$0 = \frac{1}{\Omega_{2n-1}} \int_{\|w\|=1} \left[\frac{1 - \|z\|^2 - \|w - z\|^{2n}}{\|w - z\|^{2n}} \right] \omega_{2n-1} .$$

From which

$$-g(z) = \frac{1}{\Omega_{2n-1}} \int_{\|w\|=1} [A_g(1) - g(w)] \left[\frac{1 - \|z\|^2 - \|w - z\|^{2n}}{\|w - z\|^{2n}} \right] \omega_{2n-1} - g(0) .$$

Taking the absolute value of both sides, we have

$$|g(z)| \leq [A_g(1) - g(0)] \left| \frac{(1 + r)^{2n} - 1 + r^2}{(1 - r)^{2n}} \right| + |g(0)| .$$

If $R \neq 1$, we merely divide z by R to reduce it to the above case. Q.E.D.

LEMMA 4.4. *Given $\varepsilon > 0$, there exists $s > 0$ such that if $\varrho < q + 1$ and D is a measurable set with $a \in D$ implying $\|a\| \geq sr$ ($\|z\| = r$), then $\left| \int_D \sigma(a) e_n(a, z, q) \right| \leq \varepsilon r^\varrho$ for r sufficiently large.*

Proof. In [3] it is shown that there exist constants τ and $C_1(n, q)$ such that for $\|a\| \geq \dfrac{\|z\|}{\tau}$,

$$|e_n(a , z , q)| \leq \frac{C_1 \|z\|^{q+1}}{\|a\|^{2n-1+q}} .$$

Thus, if $s \geq 1/\tau$,

$$\left| \int_D \sigma(a) e_n(a , z , q) \right| \leq C_1 r^{q+1} \int_{\|a\| \geq sr} \frac{\sigma(a)}{\|a\|^{2n-1+q}} .$$

Integrating the right hand side by parts, we have

$$\left| \int_D \sigma(a) e_n(a, z, q) \right| \leq \frac{-C_1 r^{q+1} \sigma(sr)}{(sr)^{2n+q-1}} + C_2 r^{q+1} \int_{sr}^{\infty} \frac{\sigma(t) dt}{t^{2n+q}} \leq C_2 r^{q+1} \int_{sr}^{\infty} \frac{\sigma(t) dt}{t^{2n+q}}$$

since $\lim_{r \to \infty} \dfrac{\sigma(r)}{r^{2n-1+q}} = 0$. Since $f(z)$ is of order at most ϱ and normal type, for t sufficiently large, $\sigma(t) \leqq C_3 t^{\varrho+2n-2}$ and so

$$\left| \int_D \sigma(a) e_n(a, z, q) \right| \leqq C_2 C_3 r^{q+1} \int_{sr}^{\infty} t^{\varrho-q-2} dt \leqq C_4 r^{q+1} (sr)^{\varrho-q-1} = C_4 r^{\varrho} s^{\varrho-q-1} .$$

Finally we choose s so large that the inequality is satisfied. Q.E.D.

LEMMA 4.5. *Given* $s \geqq 0$, *there is a constant* $C_0(n, q, s)$ *such that if* $D_0 = D \cap \{a ; \|a\| \leqq sr\}$ $(\|z\| = r)$, *then*

$$\left| \int_{D_0} \sigma(a)[P_1(a, z) + \ldots + P_q(a, z)] \right| \leqq C_0 \left| \int_{D_0} \frac{\sigma(a)}{\|a\|^{2n-2+q}} \right| r^q .$$

Proof. It is shown in [3] that there are positive constants $b_{n,m}$ such that

$$|P_1(a, z) + \ldots + P_q(a, z)| \leqq \frac{\|z\|^q s^q}{\|a\|^{2n-2+q}} [1 + b_{n,1} s^{-1} + \ldots + b_{n,q} s^{-q}]$$

from which the lemma follows. Q.E.D.

LEMMA 4.6. *Let* $\|w_0\| = 1$. *Then given* $\varepsilon > 0$ *and* $\delta > 0$, *there exists an* $\eta > 0$ *such that*

$$\text{measure}_{\omega_{2n-1}} \{w' ; \|w'\| = 1 , \ \|w' - w_0\| < \delta , \ k^*(w') \geqq k^*(w_0) - \varepsilon\} > \eta$$

Proof. Since $k^*(z)$ is upper semicontinuous, there is a $\delta_0 > 0$ such that $k^*(z) \leqq k^*(w_0) + \varepsilon/4^{2n+2}$ for $\|z - w_0\| < \delta_0$. Then there is a $\delta_1 \leqq \min(\delta_0, \delta)$ such that the polydisc $\Delta(w_0; \delta_1) \subset \{z; \|z - w_0\| < \delta_0\}$.

By plurisubharmonicity,

$$k^*(w_0) \leqq \frac{1}{\pi^n \delta_1^{2n}} \int_{\Delta(w_0; \delta_1)} k^*(z) dV \leqq k^*(w_0) + \frac{\varepsilon}{4^{2n+2}}$$

and so

$$\text{measure}_{\mathbf{C}^n} \{z ; z \in \Delta(w_0; \delta_1), k^*(z) \leqq k^*(w_0) - \varepsilon/4\} \leqq \frac{\pi^n \delta_1^{2n}}{4^{2n+1}} .$$

If we now consider the polydisc $\Delta' = \Delta(w_0; \delta_1/4)$, we have

$$\text{measure}_{\mathbf{C}^n} \{z ; z \in \Delta' , k^*(z) \geqq k^*(w_0) - \varepsilon/4\} \geqq \frac{3\pi^n \delta_1^{2n}}{4^{2n}} .$$

Since $\Delta' \subset B(w_0 ; \sqrt{n}\,\delta_1/2)$ and since the diameter is the longest line segment in $B(w_0 ; \sqrt{n}\,\delta_1/2)$,

$$\text{measure}_{\omega_{2n-1}} \{w' ; \|w'\| = 1 , \ w' \in \Delta' \text{ and } tw' \in \Delta' \text{ for some } t , k^*(tw') \geqq k^*(w_0) - \varepsilon/4\}$$

$$\geqq \frac{3\pi^n \delta_1^{2n}}{4^{2n}} \frac{2}{\sqrt{n}\,\delta_1} = \frac{\pi^n \delta_1^{2n-1}}{2\sqrt{n}\,4^{2n-1}} .$$

Let B be the type of the function $Q(z)$ $(B < \infty)$. Then for $\|w\| = 1$, $|k^*(w)| \leqq T_0$ by Lemma 2.1. Thus

$$\left| k^*(z) - k^*\left(\frac{z}{\|z\|}\right)\right| = \left| k^*\left(\frac{z}{\|z\|}\right)(1 - \|z\|^\varrho)\right| \leqq T_0 |1 - \|z\|^\varrho| .$$

If we restrict δ_1 so that $|1 - \|z\|^\varrho| < \varepsilon/2T_0$ for $\|z - w_0\| < \delta_1$, then $|k^*(z) - k^*(z/\|z\|)| < \varepsilon/2$ and

$$\text{measure}_{\omega_{2n-1}} \{w' ; \|w'\| = 1 , \ k^*(w') \geqq k^*(w_0) - \varepsilon\} \geqq \frac{3\pi^n \delta_1^{2n-1}}{2\sqrt{n}\, 4^{2n-1}} \quad \text{Q.E.D.}$$

LEMMA 4.7. *Let B be a set of measure less than Wr^{2n} and let $\zeta > 0$. For $\|w\| = 1$, let $E_w = \{t ; tw \in B\}$ and $E = \{w ;$ measure $E_w \geqq \zeta r\}$. Then measure$_{\omega_{2n-1}}(E) < \dfrac{T'W}{\zeta^{2n}}$ for some constant T'.*

Proof. Let χ be the characteristic function of B and $T_0 r^{2n-1}$ the area of the sphere of radius r. Then

$$T_0 \int_{\zeta r/2}^{\infty} \chi t^{2n-1} dt \leq T_0 \int_0^{\infty} \chi t^{2n-1} dt \leq W r^{2n} .$$

For any w such that measure $(E_w) \geq \zeta r$, we have measure $\{t : t \geqq \zeta r/2 , \ tw \in B\} \geq \zeta r/2$, so if $\eta = \text{measure}_{\omega_{2n-1}}(E)$

$$T_0 \left[\frac{\zeta}{2}\right]^{2n-1} r^{2n-1} \frac{\eta \zeta r}{2} < W r^{2n} \quad \text{or} \quad \eta < \frac{2^{2n} W}{T_0 \zeta^{2n}} . \qquad \text{Q.E.D.}$$

LEMMA 4.8. *Let D be an open subset of the unit sphere S^{2n-1} and $D' = S^{2n-1} \setminus D$. Let $J(z) = k_n \int_{D'_\infty} \sigma(a) e_n(a, z, q)$. Then $J(z)$ is harmonic in D_∞.*

Proof. Since any point in D_∞ lies outside the support of the measure $\sigma(a)$ restricted to D'_∞, we can differentiate under the integral sign. But $-1/\|z - a\|^{2n-2}$ is harmonic for $z \neq a$. Q.E.D.

THEOREM 4.1. *Let $f(z)$ be an entire function of order ϱ and normal type such that ϱ is not an integer and $f(0) \neq 0$. Let D be an open subset of the unit sphere. If for every relatively compact subset K of D, $\sigma_K(r) = \int_{K_r} \sigma = o(r^{\varrho+2n-2})$, then $h^*(z)$ is continuous in D_∞ and satisfies a Lipschitz condition $|h^*(w) - h^*(w')| \leq T(K)\|w - w'\|$ in K.*

Proof. By homogeneity, it is enough to prove $h^*(z)$ continuous in D. By Lemma 4.1, it is sufficient to prove the theorem for $Q(z)$ and $k^*(z)$. If $q \leq \varrho - 1$, $k^*(z) \equiv 0$ and the theorem is immediate, so we assume $\varrho < q + 1$.

The proof of the theorem, which is quite long, will be divided into several parts.

(i) Since K is relatively compact, there is an open subset D' of D such that K is relatively compact in D' and D' is relatively compact in D. Let $D'' = S^{2n-1} \setminus D'$. We begin by defining the function

$$J(z) = k_n \int_{D''_\infty} \sigma(a) e_n(a, z, q)$$

which is well defined by virtue of Lemma 4.4. If $\|z\| = r$, it also follows from Lemma 4.4 that for r and s sufficiently large $\left| k_n \int_{D''_\infty \setminus D''_{sr}} \sigma(a) e_n(a, z, q) \right| \leqq r^\varrho$. Furthermore, by lemma 4.5,

$$\left| \int_{D''_{sr}} \sigma(a)[P_1(a, z) + \ldots + P_q(a, z)] \right| \leqq C_0(n, q, s) r^q \int_{D''_{sr}} \frac{\sigma(a)}{\|a\|^{2n-2+q}} .$$

It follows from the remark after Theorem 2.1 that $\sigma_{D''}(r) = \int_{D''_r} \sigma \leqq \sigma(r) \leqq C r^{\varrho+2n-2}$ for some constant C.

Integrating by parts, we have

$$\begin{aligned} r^q \int_{D''_{sr}} \frac{\sigma(a)}{\|a\|^{2n-2+q}} &= \left[\int_0^{sr} \frac{d\sigma_{D''}(t)}{t^{2n-2+q}} \right] r^q \\ &= \frac{\sigma_{D''}(t)}{t^{2n-2+q}} \Big]_0^{sr} r^q + (2n - 2 + q) r^q \int_0^{sr} \frac{\sigma_{D''}(t) dt}{t^{2n-1+q}} \\ &\leqq C s^{\varrho-q} r^\varrho + (2n - 2 + q) r^q C \int_0^{sr} t^{\varrho-q-1} dt \\ &\leqq C_1(n, q, s) r^\varrho \end{aligned}$$

so for r large enough, $J(z) \leqq C_2 r^\varrho$ for some constant C_2. We now define $j(z) = \overline{\lim_{r\to\infty}} J(rz)/r^\varrho$ and $j^*(z) = \overline{\lim_{z'\to z}} j(z')$, which are bounded above on the unit sphere and positively homogeneous of order ϱ.

(ii) We have the identity $\ln |Q(z)| = k_n \int \sigma(a) e_n(a, z, q)$, so $\ln |Q(rw')| = k_n \int_{D'_\infty} \sigma(a) e_n(a, rw', q) + J(rw')$ for $\|w'\| = 1$. Again applying Lemma 4.4, for r and s' sufficiently large, $\left| k_n \int_{D'_\infty \setminus D'_{s'r}} \sigma(a) e_n(a, rw', q) \right| \leqq \frac{\varepsilon}{4} r^\varrho$; and by Lemma 4.5,

$$\left| \int_{D'_{s'r}} \sigma(a)[P_1(a, rw') + \ldots + P_q(a, rw')] \right| \leqq C_0(n, q, s) r^q \int_{D'_{s'r}} \frac{\sigma(a)}{\|a\|^{2n-2+q}} .$$

Integrating by parts and making use of the hypothesis that for large enough t, $\sigma_{D'}(t) < \mu t^{\varrho+2n-2}$ for arbitrarily small μ, we have

$$\begin{aligned} r^q \int_{D'_{s'r}} \frac{\sigma(a)}{\|a\|^{2n-2+q}} &= \int_0^{s'r} \frac{d\sigma_{D'}(t)}{t^{2n-2+q}} \\ &= \frac{\sigma_{D'}(t)}{t^{2n-2+q}} \Big]_0^{s'r} r^q + (2n - 2 + q) r^q \int_0^{s'r} \frac{\sigma_{D'}(t)dt}{t^{2n-1+q}} \\ &\leqq \frac{\varepsilon r^\varrho}{4C_0} + \mu(2n - 2 + q) r^q \int_0^{s'r} t^{\varrho-q-1} dt \\ &\leqq \frac{r^\varrho \varepsilon}{2C_0} \end{aligned}$$

for r sufficiently large so that μ is sufficiently small. Thus

$$\ln |Q(rw')| \leqq k_n \int_{D'_{s'r}} \sigma(a) \left[\frac{-1}{\|a - rw'\|^{2n-2}} \right] + \frac{3\varepsilon r^\varrho}{4} + J(rw')$$

and so $k(w') \leqq j(w')$. But then $k^*(z) \leqq j^*(z)$ since both are homogeneous of order ϱ. We also observe that by Lemma 2.1, this also implies that $j(z)$ is bounded below on the unit sphere. We note that from the above estimates, we also have

$$\ln |Q(rw')| + k_n \int_{D'_{s'r}} \frac{\sigma(a)}{\|a - rw'\|^{2n-2}} + \frac{3\varepsilon r^\varrho}{4} \geqq J(rw') , \quad \|w'\| = 1 .$$

(iii) Let $\alpha = d(K, D'') > 0$ and let $w \in K$. Let w' be such that $\|w' - w\| < \alpha/4$. By Lemma 4.8, $J(z)$ is harmonic in D'_∞.

By (i), there is a constant C_2 such that $J(z) \leq C_2 r^\varrho$ for $\|z\| = r$, r sufficiently large.

Let $\phi(\xi) = J(\xi + rw)$ for $\|\xi\| \leq \alpha r/2$. Then $\phi(\xi) \leq C_2(1 + \alpha/2)^\varrho r^\varrho$ and $J(rw') = \phi(r[w' - w])$, $J(rw) = \phi(0)$. Applying Lemma 4.3 to the harmonic function $\phi(\xi)$,

$$\begin{aligned} J(rw') - J(rw) = \phi(r[w' - w]) - \phi(0) &\leq A_{\varphi-\varphi(0)}(r[w' - w]) \leq \\ &\leq A_{\varphi-\varphi(0)} \left(\frac{\alpha r}{2} \right) \frac{2K}{\alpha} \frac{\|w' - w\|}{(\frac{1}{2})^{2n}} \\ &\leq K(\alpha) r^\varrho \|w' - w\| . \end{aligned}$$

On dividing by r^ϱ and letting $r \to \infty$ through a sequence of values of which $J(rw')/r^\varrho \to j(w')$, we have that $j(w') - j(w) \leq T_0\|w' - w\|$ for some constant T_0. On reversing the roles of w and w' in the above reasoning, we get that $j(w) - j(w') \leq T_0\|w - w'\|$ so $|j(w') - j(w)| \leqq T_0\|w' - w\|$. Since $J(z)$ is harmonic in D', by Fatou's lemma, $j^*(z)$ is subharmonic there, and hence $j(z) = j^*(z)$ almost everywhere in D' [5]. But then by continuity, $j(z) = j^*(z)$ everywhere in D', so $j^*(z)$ is continuous in D'. If $\|w - w'\| \geq \alpha/4$, surely there is a constant T_1 such that $|j^*(w) - j^*(w')| \leqq T_1\|w - w'\|$ since $j^*(z)$ is bounded on the unit sphere. Thus, there is a $T(K)$ such that $|j^*(w) - j^*(w')| \leq T(K)\|w - w'\|$ for $w \in K$.

(iv) Returning to the function $\phi(\xi)$, we let $\xi = \lambda r w$, $0 \leq \lambda \leq \alpha/4$. Then $J([1 + \lambda]wr) - J(wr) \leq \dfrac{K(\alpha)C_2(1 + \alpha/2)^\varrho r^\varrho \lambda}{\alpha} = T_0 r^\varrho \lambda$. Let $\varepsilon > 0$ be given, and let r_n be a sequence such that $r_n \to \infty$ and $j^*(w) - \dfrac{\varepsilon}{4} \leqq \dfrac{J(r_n w)}{r_n^\varrho}$. Then $J(wr_n) - J\left(\dfrac{wr_n}{1 + \lambda}\right) \leqq \dfrac{T_0 r_n^\varrho \lambda}{(1 + \lambda)^\varrho}$ so $j^*(w) - \dfrac{\varepsilon}{4} - T_0\lambda \leqq \dfrac{J(wr_n/1 + \lambda)}{r_n^\varrho}$ for $0 \leqq \lambda \leqq \alpha/4$. For $\alpha' \leqq \alpha$ so small that $T_0\lambda < \dfrac{\varepsilon}{4}$ for $\lambda \leqq \alpha'/4$ and $j^*(w) - \dfrac{3\varepsilon}{4} \leqq (1 + \alpha'/4)^\varrho \left[j^*(w) - \dfrac{\varepsilon}{2}\right]$ for $\lambda < \alpha'/4$, we see that there exists a sequence $r_n' = \dfrac{r_n}{(1 + \alpha'/4)}$ such that $j^*(w) - \dfrac{3\varepsilon}{4} \leqq \dfrac{J(wr)}{r^\varrho}$ holds for all r such that $r_n' \leqq r \leqq r_n'\left(1 + \dfrac{\alpha'}{4}\right)$. Since $j^*(w)$ is bounded on the unit sphere, the choice of α' is independent of w. Since $J(wr) - J(rw') \leqq T_0 r^\varrho\|w - w'\|$ for $\|w - w'\| < \alpha'/4$ and since $|j^*(w) - j^*(w')| \leqq T(K)\|w - w'\|$, there is a $\beta > 0$ (independent of w) such that $j^*(w') - \varepsilon \leqq \dfrac{J(w'r)}{r^\varrho}$ holds on a sequence $r_n' \to \infty$ for all r such that $r_n' \leqq r \leqq \left(1 + \dfrac{\alpha'}{4}\right) r_n'$ for all w' such that $\|w' - w\| < \beta$.

(v) Let $w \in K$ and let $\varepsilon > 0$ be given. We begin with the inequality established in (ii)

$$\ln |Q(rw')| + k_n \int_{D'_{s'r}} \frac{\sigma(a)}{\|a - rw'\|^{2n-2}} + \frac{3\varepsilon r^\varrho}{4} \geqq J(w'r), \quad \|w'\| = 1.$$

Let $\delta = \min\left(\beta, \dfrac{\varepsilon}{T(K)}\right)$. By Lemma 4.6, there is an $\eta_\delta > 0$ such that there exists a set E, $E = \{w'; \|w'\| = 1, \|w' - w\| < \delta, |k^*(w') - k^*(w)| < \varepsilon\}$ with $\text{measure}_{\omega_{2n-1}}(E) > \eta_\delta$.

Since $\sigma_{D'}(r) = o(r^{\varrho+2n-2})$, by Lemma 4.2 and Lemma 4.7, there exists an R_0 such that for $r \geq R_0$, $\int\limits_{D'_{s'r}} \frac{\sigma(a)}{\|a - rw'\|^{2n-2}} \leq r^{\varrho}\varepsilon$ for $w' \in E'$ with $\text{measure}_{\omega_{2n-1}}(E') < \eta_\delta/2$ except perhaps for a set of r of measure less than $\alpha' r/8$. Since we have convergence almost everywhere there exists an R'_0 such that for $r \geq R'_0$, $k^*(w') + \varepsilon \geq \frac{\ln|Q(rw')|}{r^{\varrho}}$ except on a set of ω_{2n-1} measure less than $\eta_\delta/4$. Then, for $r_n \geq \max(R_0, R'_0)$, there exists a $w'_0 \in E$ and an r, $r_n \leqq r \leqq (1 + \alpha'/4)r_n$ such that

$$\begin{aligned} k^*(w) + 4\varepsilon \geq k^*(w'_0) + 3\varepsilon &\geq \frac{\ln|Q(rw'_0)|}{r^{\varrho}} + 2\varepsilon \\ &\geq \frac{\ln|Q(rw'_0)|}{r^{\varrho}} + \frac{1}{r^{\varrho}} k_n \int\limits_{D'_{s'r}} \frac{\sigma(a)}{\|a - rw'_0\|^{2n-2}} + \frac{3\varepsilon}{4} \\ &\geq \frac{J(rw'_0)}{r^{\varrho}} \geq j^*(w'_0) - \varepsilon \geq j^*(w) - 2\varepsilon\,, \end{aligned}$$

and since ε was arbitrary, $k^*(w) \geq j^*(w)$ in D'. Since both are homogeneous of order ϱ, $k^*(z) \geq j^*(z)$ in D'_∞, and by (i) they are equal in D'_∞, from which the desired result follows. Q.E.D.

As in the case of one variable, the case $\varrho = q$ must be treated separately. It was established in [3] that for $\varrho = q$, $f(z)$ is of normal type if and only if the integrals $\int\limits_{\|a\| \leq R} \sigma(a) P_q(a, z)$ are bounded in absolute value on all compacts independent of R.

LEMMA 4.9. *Let* $\|w\| = \|w'\| = 1$. *Then there exists a constant* T' *such that* $|P_q(a, w) - P_q(a, w')| \leqq \frac{T'\|w - w'\|}{\|a\|^{2n-2+q}}$.

Proof. The polynomial $P_q(a, w)$ is homogeneous of order q in w. Let k be a multi-index of order $2n$, and let $w = (x_1, y_1, \ldots, x_n, y_n)$ be a $2n$-tuple. Then $P_q(a, w) = \sum\limits_{|k|=q} c_k(a) w^k$, and there are constants T_k such that $|c_k(a)| \leq \frac{T_k}{\|a\|^{2n-2+q}}$; T_k is independent of a. By performing a rotation (if necessary) we may assume that $w = (1, 0, \ldots, 0, 0)$. Let a' be the rotated a, and let $k_0 = (q, 0, \ldots, 0, 0)$. Then

$$\begin{aligned} |P_q(a, w) - P_q(a, w')| &= |c_{k_0}(a') - c_{k_0}(a') w'^{k_0} - \sum_{\substack{|k|=q \\ k \neq k_0}} c_k(a') w'^k| \\ &\leq |c_{k_0}(a')|\,|1 - x'_1|\,|x'^{q-1}_1 + \ldots + 1| + \sum_{\substack{|k|=q \\ k \neq k_0}} |c_k(a')|\,|w'^k| \\ &\leq \frac{1}{\|a\|^{2n-2+q}} \left[q T_{k_0} + \sum_{\substack{|k|=q \\ k \neq k_0}} T_k\right] \|w - w'\| \end{aligned}$$

since $|1 - x_1'| \leq \|w - w'\|$ and $|x_1'| \leq 1$, and since $|y_1'| \leq \|w - w'\|$ for all i, $|x_1'| \leq \|w - w'\|$ for $i \neq 1$. Q.E.D.

THEOREM 4.2. *Let $f(z)$ be an entire function of order ϱ and normal type such that ϱ is an integer and $f(0) \neq 0$. Let D be an open subset of the unit sphere. If for every relatively compact subset K of D, $\sigma_K(r) = \int_{K_r} \sigma$ satisfies $A(K) = \int_0 \frac{d\sigma_K(t)}{t^{2n-2+q}} < \infty$, then $h^*(z)$ is continuous in D_∞ and satisfies a Lipschitz condition*

$$|h^*(w) - h^*(w')| \leq T(K)\|w - w'\| \quad \text{in} \quad K.$$

Proof. By homogeneity, its is enough to prove $h^*(z)$ continuous in D. By Lemma 4.1, it is sufficient to prove the theorem for $Q(z)$ and $k^*(z)$. If $q \leq \varrho - 1$, $k^*(z) \equiv 0$, and the theorem is immediate, so we assume $\varrho = q$. Since $Q(z)$ is also of normal type, we have $\int_{\|a\| \leq s} \sigma(a)P_q(a, z)$ bounded in absolute value on every compact set independent of s. Since $P_q(a, z)$ is homogeneous of order q, there is a constant C_q such that $\left| \int_{\|a\| \leq s} \sigma(a)P_q(a, z) \right| \leq C_q\|z\|^q$. Furthermore, there is a constant T_q such that $|P_q(a, z)| \leq \frac{T_q\|z\|^q}{\|a\|^{2n-2+q}}$ [3, p. 376].

(i) Since K is relatively compact, there is a compact subset D' of D such that K is relatively compact in D'. We set $D'' = S^{2n-1} \setminus D'$ and introduce the functions

$$J_s(z) = k_n \int_{D''_\infty} \sigma(a)e_n(a, z, q) + k_n \int_{D'_s} \sigma(a)P_q(a, z, q) = J_s^{(1)}(z) + J_s^{(2)}(z).$$

This is well defined by virtue of Lemma 4.4. Also, by Lemma 4.4, for s' sufficiently large and for $\|z\| = r$ so large that $s'r \geq s$, we have $\left| k_n \int_{D''_\infty \setminus D''_{s'r}} \sigma(a)e_n(a, z, q) \right| \leq r^q$; and by Lemma 4.5,

$$k_n \int_{D''_{s'r}} \sigma(a)e_n(a, z, q - 1) \leq \left| k_n \int_{D''_{s'r}} \sigma(a)[P_1(a, z) + \ldots + P_{q-1}(a, z)] \right|$$

$$\leq C_0(n, q - 1, s)r^{q-1} \int_{D''_{s'r}} \frac{\sigma(a)}{\|a\|^{2n-3+q}}.$$

Making use of the majoration $\sigma_{D''}(t) \leq \sigma(t) \leq Ct^{q+2n-2}$ and integrating by parts, we have

$$r^{q-1}\int_{D''_{s'r}}\frac{\sigma(a)}{\|a\|^{2n-3+q}} = r^{q-1}\int_0^{s'r}\frac{d\sigma_{D''}(t)}{t^{2n-3+q}}$$

$$= \frac{\sigma_{D''}(t)}{t^{2n-3+q}}\Big]_0^{s'r} r^{q-1} + (2n-3+q)r^{q-1}\int_0^{s'r}\frac{\sigma_{D''}(t)dt}{t^{2n-2+q}}$$

$$\leq Cs'r^q + (2n-3+q)Cs'r^q = C'r^q$$

and

$$\left|k_n\int_{D''_{s'r}\setminus D''_s}\sigma(a)P_q(a\,,z) + k_n\int_{\|a\|\leq s}\sigma(a)P_q(a\,,z)\right|$$

$$= \left|k^n\int_{\|a\|\leq rs'}\sigma(a)P_q(a,z) - k_n\int_{D'_{s'r}\setminus D'_s}\sigma(a)P_q(a,z)\right| \leq [C_q + A(D')T_q]r^q\,.$$

Thus, for r sufficiently large, there is a positive constant A_0 independent of s such that $J_s(z) \leq A_0\|z\|^q$. We introduce the functions

$$j_s(z) = \overline{\lim_{r\to\infty}}\frac{J_s(rz)}{r^q} \leq A_0\|z\|^q\,,\quad j(z) = \overline{\lim_{s\to\infty}}\, j_s(z)\,,\quad j^*(z) = \overline{\lim_{z'\to z}}\, j(z') \leq A_0\|z\|_q\,.$$

All of these are positively homogeneous of order q.

(ii) $$\ln|Q(rw')| - J_s(rw') = k_n\int_{D'_\infty\setminus D'_s}\sigma(a)e_n(a, rw', q) + k_n\int_{D'_s}\sigma(a)e_n(a, rw', q-1)$$

for $\|w'\| = 1$. Let $\varepsilon > 0$ be given. By Lemma 4.4, there is an s'' such that for r sufficiently large, $\left|k_n\int_{D'_\infty\setminus D'_{s''r}}\sigma(a)e_n(a\,, rw'\,, q)\right| < \frac{\varepsilon r^q}{3}$. We choose s_0 so large that, posing $C_0 = C_0(n, q, s'')$ of Lemma 4.5, $\int_{s_0}^{\infty}\frac{d\sigma_{D'}(t)}{t^{2n-2+q}} \leq \frac{\varepsilon}{3C_0}$. For $s \geq s_0$ and r so large that $s''r \geq s$,

$$k_n\int_{D'_{s''r}\setminus D'_s}\sigma(a)e_n(a, rw', q) \leq \left|k_n\int_{D'_{s''r}\setminus D'_s}\sigma(a)[P_1(a, rw') + \ldots + P_q(a, rw')]\right|$$

$$\leq C_0(n, q, s'')r^q\int_s^{s''r}\frac{d\sigma_{D'}(t)}{t^{2n-2+q}} \leq \frac{\varepsilon r^q}{3}$$

by Lemma 4.5. Also, by Lemma 4.5, for r sufficiently large

$$k_n \int_{D'_s} \sigma(a) e_n(a\,,\, rw'\,,\, q-1) \leq \left| k_n \int_{D'_s} \sigma(a)[P_1(a\,,\, rw') + \ldots + P_{q-1}(a\,,\, rw') \right|$$

$$\leq C_0(n\,,\, q-1\,,\, 1) r^{q-1} \int_0 \frac{d\sigma_{D'}(t)}{t^{2n-3+q}} < \frac{\varepsilon r^q}{3}\,.$$

Thus $\dfrac{\ln|Q(rw')|}{r^q} \leq \dfrac{J_s(rw')}{r^q} + \varepsilon$ for $s \geq s_0$ and r sufficiently large; hence, for $s \geq s_0$, $j_s(z) + \varepsilon \geq k(z)$ so $j^*(z) \geq k^*(z)$, and $j_s(z)$ and $j^*(z)$ are bounded below on the unit sphere for $s \geq s_0$ since $k(z)$ is.

(iii) Let $\alpha = d(K\,,\, D'')$ and assume $\|w - w'\| < \alpha/4$. By Lemma 4.9, $\left| \dfrac{J_s^{(2)}(rw)}{r^q} - \dfrac{J_s^{(2)}(rw')}{r^q} \right| \leq A(D')T''\|w - w'\|$, and since $P_q(a, z)$ is homogeneous of order q, the limit exists along all rays. By Lemma 4.8, $J_s^{(1)}(z)$ is harmonic in D'_∞ and by (i), for $s \geq s_0$ and r large enough, there is a constant A'_0 such that $\dfrac{J_s^{(1)}(rw')}{r^q} \leq A'_0$ uniformly in s. By applying the same reasoning as in (iii) of Theorem 4.1, we conclude that there is a constant T_0 independent of s and w such that $|j_s(w) - j_s(w')| \leq T_0\|w - w'\|$. It then follows that $j(z)$ and $j^*(z)$ satisfy the same inequality.

(iv) Since $\lim\limits_{r\to\infty} \dfrac{J_s^{(2)}(rw)}{r^q}$ exists along all rays, by virtue of Lemma 4.9, we can repeat the reasoning in section (iv) of Theorem 4.1; we see that for all $w \in K$, there exist sequences $r_n \to \infty$ and constants α' and β (independent of w and s) such that

$$j_s(w') - \varepsilon \leq \frac{J_s(rw')}{r^q} \leq j_s(w') + \varepsilon \quad \text{for} \quad \|w' - w\| < \beta$$

and all r such that $r_n \leq r \leq r_n(1 + \alpha'/4)$.

(v) The inequality $\displaystyle\int_0^\infty \frac{d\sigma_{D'}(t)}{t^{2n-2+q}} < \infty$ implies $\lim\limits_{r\to\infty} \dfrac{\sigma_{D'}(r)}{r^{2n-2+q}} = 0$ [3, p. 373]. Given $\varepsilon > 0$, by choosing s_0 sufficiently large, we have for $s \geq s_0$ and r sufficiently large

$$\ln|Q(rw')| + \varepsilon r^q + \int_{D'_{s^*r}} \frac{\sigma(a)}{\|a - rw'\|^{2n-2}} \geq J_s(rw')$$

for all $\|w'\| = 1$ by (ii). By repeating the reasoning in (v) of Theorem 4.1, we conclude that

$$k^*(w) + 4\varepsilon \geq j_s(w) - 2\varepsilon \quad \text{for} \quad w \in K\,. \tag{4.00}$$

Thus $k^*(w) + 6\varepsilon \geq \overline{\lim_{s\to\infty}} j_s(w)$, and since ε was arbitrary, $k^*(w) \geq j^*(w)$ since $k^*(w)$ is upper semicontinuous. Since both are homogeneous of order q, $k^*(z) = j^*(z)$ in K, from which the theorem follows. Q.E.D.

LEMMA 4.10. *If $g(z)$ is a real valued function harmonic and plurisubharmonic in an open set E, then $g(z)$ is pluriharmonic in E.*

Proof. If $g(z)$ is harmonic in E, it is C^∞, hence continuous. Let $z_0 \in E$. Then for any sphere $S^{2n-1}_{z_0}$ contered at z_0, the average over $S^{2n-1}_{z_0}$ is $g(z_0)$. If in some complex line through z_0, $g(z_0)$ were strictly less than its average over the disc centered at z_0, by continuity it would be less over a whole neighborhood of complex lines, and hence $g(z_0)$ would be strictly less than its average over $S^{2n-1}_{z_0}$, which would be a contradiction. Thus equality holds in every complex line, and $g(z)$ is pluriharmonic. Q.E.D.

THEOREM 4.3. *Let $f(z)$ be an entire function of order ϱ and normal type such that $f(0) \neq 0$. Let D be an open subset of the unit sphere such that for any relatively compact subset K of D*

(i) $\sigma_K(r) = o(r^{\varrho+2n-2})$ *for ϱ not an integer*

(ii) $\int_0^\infty \frac{d\sigma_K(t)}{t^{2n-2+\varrho}} < \infty$ *for ϱ an integer.*

Then if $f(z)$ is of completely regular growth in D, $h^*(z)$ is pluriharmonic in D_∞.

Proof. We write $f(z) = Q(z)\exp(P(z))$, and we decompose $P(z) = \sum_{j=0}^{q} p_i(z)$ into homogeneous polynomials. Let $l(z) = \lim_{r\to\infty} \frac{\operatorname{Re} P(rz)}{r^\varrho}$. Then $h^*(z) = k^*(z) + l(z)$. If $\varrho < q$, $l(z) = 0$, and if $\varrho = q$, $l(z) = \operatorname{Re} p_q(z)$; hence, it is sufficient to consider $k^*(z)$. If $q \leq \varrho - 1$, $k^*(z) \equiv 0$ and we are through, so we may assume that $\varrho < q + 1$.

Let K be a compact subset of D_∞. Given $\varepsilon > 0$, for almost all z, there exists an E^0-set E_z such that for $r \notin E_z$ and r sufficiently large

$$-2\varepsilon + k^*(z) \leq -\varepsilon + \frac{Q(rz)}{r^\varrho} \leq \frac{J(rz)}{r^\varrho} \leq j^*(z) + \varepsilon = k^*(z) + \varepsilon$$

But then by (iv) of Theorem 4.1

$$-3\varepsilon + k^*(z) \leq \frac{J(rz)}{r^\varrho} \leq k^*(z) + \varepsilon$$

for all r sufficiently large (depending on z).

By (iii) of Theorem 4.1 and the continuity of $k^*(z)$

$$-4\varepsilon + k^*(z') \leq \frac{J(rz')}{r^\varrho} \leq k^*(z') + 2\varepsilon \quad \text{for} \quad \|z' - z\| < \delta_z .$$

Since $\frac{J(rz)}{r^\varrho}$ is harmonic in D_∞, $k^*(z)$ in D_∞ is the uniform limit on compact sets of harmonic functions and hence is harmonic in D_∞. By Lemma 4.11, it is pluriharmonic.

Part (ii) is proved the same way using the equivalent parts of Theorem 4.2. Q.E.D.

Remark. It is clear that the Lelong construction [3] is valid for all subharmonic functions of finite order ϱ (the polynomial $P(x)$ will just be harmonic in this case). Since we have only made use of the properties of harmonic and subharmonic functions, the results of § 4 are also applicable to the case of subharmonic functions with the obvious modifications on conditions on Δu, the measure. One must then replace the word pluriharmonic by harmonic, of course.

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