MODAL LOGIC AND THE THEORY OF MODAL AGGREGATION

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Introduction

The recognition of the importance of the theory of modal reduction as a limb of modal logic predates Kripke's classic paper by almost three decades.[1] But before Kripke, reduction theory never constituted the body and soul of the beast. So seductive, however, was the Kripkean technique that the modal logical enterprise to which it was ideally suited became paramount. This is the cumulative description of the class of all extensions of the ground logic which is now usually called K. From the point of view of Kripke semantics it seems natural to identify 'extension of K' with 'logic having stronger reduction principles than K'. It is obvious that the class of all normal logics forms a lattice under inclusion. What is not so obvious, but is coming slowly now to light, is that the Kripkean programme can furnish at best an inadequate account of this lattice. Still the fixation upon reductive principles has left its nomenclative scars. Thus, for example, the usual rule-of-thumb strength characterization of a logic by the strength of its reduction principles, and not, for example, by its inverse ability to distinguish among modalities. So we say that S_5 is very strong, rather than 'almost trivial'.

The standard nomenclature embodies a considerable bias in favour of modal reduction theory. We have, therefore, attempted to remove this bias by adopting the following more neutral classification scheme. A logic, L, is called *normal* iff [RN] $\vdash_L \alpha \Rightarrow \vdash_L \Box\alpha$ is admissible for L. Further, we say that L is *regular* iff L admits [RR] $\vdash_L \alpha \to \beta \Rightarrow \vdash_L \Box\alpha \to \Box\beta$. Finally, we say that L is *aggregative* (or has complete aggregation) iff $\vdash_L (\Box p \wedge \Box q) \to \Box(p \wedge q)$. This terminology diverges from both Kripke's and Segerberg's.[2]

Modal reduction theory, conceived in a general way, concerns it-

$$M\alpha \to M'\alpha$$

where α is a sentence and M and M′ are strings of □'s and ◇'s. (More strictly conceived, the '→' must be replaced by '↔'.) At the same level of generality, aggregation theory concerns itself with principles of the form:

$$\Box\alpha_1 \wedge \Box\alpha_2 \wedge \ldots \Box\alpha_n \to (\leftrightarrow) \Box(\delta(\alpha_1 \ldots \alpha_n))$$

where '$\delta(\alpha_1 \ldots \alpha_n)$' abbreviates some Boolean compound in $\wedge$ and $\vee$.

Obviously reduction theory and aggregation theory will interact in certain straightforward ways. If, for example, we are to concern ourselves with the aggregation of modal functions more complex than mere necessity, we shall require help from reduction theory. But we shall also show below that the interaction has more subtle and interesting features and that these contribute in no small way to the interest and importance of the study of modal aggregation.

If we think in a general way about aggregation principles we can see that, just as in the case of reduction principles, they may be partially ordered by strength. Corresponding to the complete reduction of S_5 there is what might be called the complete aggregation of K.

$$\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n \to \Box(\alpha_1 \wedge \ldots \wedge \alpha_n)$$

Less formally, if some (finite) number of sentences are severally necessary, then they are also jointly necessary. Corresponding to the complete irreductivity of K we find the complete lack of aggregativity of the logic having [RR] and [RN], but no aggregation principles at all. We call this logic N. Thus with respect to aggregativity, K is to N as, with respect to reductivity, S_5 is to K. It is to the aggregative logics between N and K that we want to draw attention. The main problem in doing so is that we are totally conditioned, as students of the literature of modal logic, to beginning at the top. For even in K we are committed to:

$$[K] : \Box p \wedge \Box q \to \Box(p \wedge q)$$

from which the principle of complete aggregation is easily seen to follow. Thus K is a weak logic only if we are blind to every feature of modal logic other than reduction theory. For from the point of view of aggregation theory, K is as strong a logic as can be constructed.

Now no one would suggest that philosophical logic should be restricted to enterprises which can be carried out using the S_5 $\Box$. For there are numerous interpretations of the $\Box$ which are philosophically significant and for which at least one of the S_5 laws does not hold. Thus for example,

$$[B]: p \rightarrow \Box\Diamond p$$

is absurd in the context of temporal logic since it collapses the later-earlier distinction. Similarly,

$$[4]: \Box p \rightarrow \Box\Box p$$

leads to difficulties for the epistemic reading of '$\Box$' while

$$[T]: \Box p \rightarrow p$$

in deontic logic expresses a Leibnizian optimism (or pessimism) which borders upon insanity.

In much the same way ought the principle of complete aggregation to offend our sensibilities as philosophers once we depart from the analyticity or logical necessity reading of '$\Box$' Suppose, for example, that we wished to capture some features of general moral reasoning (as opposed to utilitarian or intuitionist reasoning). One would likely take as one basic principle the following law of moral consistency:

$$[Con]: \neg\Box\bot$$

in words, no logical falsehood ought to be the case. Were one to adopt as well the principle of complete aggregation, one would be committed to the view that if both α and β ought to be the case, then α and β are consistent. There clearly are, however, moral theories which allow (or even require) conflicts of obligation. So to put the matter brutally, there can be no deontic logic which takes as a primitive law the principle of complete aggregation. The best that we can do is to formalize certain particular ethical theories (namely, those which do not allow conflicts of obligation).

This view is not entirely beyond the bounds of controversy. E.J. Lemmon argued in some detail for the existence in ordinary moral life of pairs of inconsistent sentences both of which ought to be the case.[3] On the basis of these situations Lemmon argues that the principle

$$[D]: \Box p \to \neg\Box\neg p$$

(which is equivalent to [Con] in the presence of [K]) must be rejected. The argument runs: conflicts of oughts are genuine features of moral life and when such a conflict arises, some contradiction ought to be the case, since [K] serves as a basic principle. Followers of the Lemmon line clearly take aggregation theory as given in advance and think that what remains for debate is the sort of reduction theory appropriate for deontic logic. If, however, we consider ourselves as considering the various possible trade-offs among alternative sorts of principles, then the appeal of [Con] (or perhaps [D]) might well override that of [K]. If we maintain the primacy of moral consistency, then Lemmon's arguments turn out to undermine [K].

Arguments of this sort may be deployed in contexts other than that of deontic logic. In fact, the deontic realm is only one of many in which our basic intuitions seem to come from distinct if not incompatible sources. In physical logic, for example, we find ourselves confronted by a situation in which we must deal with the dictates of several distinct theories, no one of which can command our allegiance to the exclusion of all others.

Clearly we need a larger view than that provided by the Kripkean framework. But which of the multitude of possible generalizations will provide the philosophically most correct or even most plausible account? There is one which recommends itself above all others. It inherits the respectability of Kripkean semantics on two grounds. First, it is an elementary semantical theory, *i.e.* it is first-order. Secondly, its truth conditions derive from structural considerations in the same way that Kripke's do.

This can be made exact in the following way: The usual structure is an object called a frame *iFi* which is a pair (U, R) with U a non-empty set and $R \subseteq U^2$ a binary relation. Intuitively we may think of U as a set of possible cases, contexts or worlds and R as the relation of 'relative possibility' or 'accessibility' to be spelled out differently according as our interest lies in the temporal, epistemic, or whatever sphere.

The relationship between frame theory and truth conditions, (*i.e.*, model theory) may be described in a number of ways. Propositionally speaking, a truth condition corresponds to the specification of a function:

$$\Box: 2^U \to 2^U$$

i.e., a propositional operator.

Can this operator be derived in some natural way or must we rely upon some vague bundle of intuitions concerning necessity which will determine '□' indirectly? The answer is rather Schilleresque. Law and inclination unite in this matter and we find a very powerful and general derivation; one which lies at the heart of the Tarski and Jónsson representation theory.[4] Furthermore, the operator which we derive in this way matches the usual relational truth-condition. The way to derive the 'right' operator can be economically described by the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\;r\;} & 2^U \\ \{\cdot\}\ \downarrow & & \wr \\ 2^U & \xrightarrow{\;\Box\;} & 2^U \end{array}$$

Here r is the function associated with R, *i.e.*, $\forall u \epsilon U$, $r(u) = \{v | v \in U \;\&\; uRv\}$, $\{\cdot\}$ is the degenerate form of the Yoneda functor, *i.e.*, $\forall u \epsilon U$, $\{\cdot\}u = \{u\}$. Finally, '□' is the left Kan extension[5] of $\{\cdot\}$ along r. We quickly calculate that

$$\forall X, Y \in 2^U,$$
$$\Box X \cap \Box Y = \Box(X \cap Y)$$

and indeed that:

$$\Box X = \{u \in U | \forall v, v \epsilon r(u) \Rightarrow v \epsilon X\}.$$

This amounts to the truth condition which has received the mantle of intuitive acceptability.

The derivation extends in a straightforward way to the generalized version of a first order frame in which the frame relation is not restricted to being binary. Those familiar with the Tarski-Jonsson representation theory will realize that the general theory is constructed in such a way that the representation of n-ary operators in terms of n + 1-ary relations is accomplished. This part of the theory has received little attention since all of the principal applications, *e.g.*, to closure algebras and cylindric algebras, involve unary operators.

Following the more general route, let us take a frame F^n to be a pair (U, R) with $R \subseteq U^{n+1}$ for n fixed, called the rank of the frame. If we cleave to the same approach which served us so well for ordinary frames, then the generalized frame determines an n-ary propositional operator '⊙'. Furthermore such a generalized operator satisfies the following:

$$\forall X_1, \ldots X_j, X_{j+2}, \ldots, X_n, Y, Z \in 2^U,$$
$$\odot(X_1, \ldots, X_j, Y, X_{j+2}, \ldots, X_n) \cap \odot(X_1, \ldots, X_j, Z, X_{j+2}, \ldots, X_n)$$
$$= \odot(X_1, \ldots, X_j, Y \cap Z, X_{j+2}, \ldots, X_n)$$

which is the appropriate generalized algebraic version of K together with RR and has the truth condition:

$$\odot(X_1, \ldots, X_n) = \{u \in U | \forall v_1, \ldots, v_n \in U, \langle v_1, \ldots, v_n \rangle \in r(u) \Rightarrow v_1 \in X_1 \text{ or } \ldots \text{ or } v_n \in X_n\}.$$

Translating this into the usual sort of truth condition in terms of the concept of truth at a world u, in a model we have:

$$\models^m_u \odot (\alpha_1, \ldots, \alpha_n) \Leftrightarrow \forall x_1, \ldots, x_n, uRx_1, \ldots, x_n \Rightarrow \models^m_{x_1} \alpha_1 \text{ or } \ldots \text{ or } \models^m_{x_n} \alpha_n.$$

Although we have not had much experience with generalized modal operators they are not entirely unknown. So far as binary operators are concerned, *i.e.*, those appropriate to frames of rank 2, the dual of $\odot$ which has the general truth condition

$$\models^m_u \neg\odot (\neg\alpha_1, \ldots, \neg\alpha_n) = \exists x_1, \ldots, x_n \in U: \langle x_1, \ldots, x_n \rangle \in r(u) \;\&\; \models^m_{x_1} \alpha_1 \;\&\; \ldots \;\&\; \models^m_{x_n} \alpha_n.$$

has been studied by theorists in relevance logic.[6]

The axiomatization of generalized modal logic

The relevance logicians do not study the basic theory of the operator '$\neg \odot \neg$' (which they and we call '$\circ$'). That is, they do not axiomatize the logic which is determined by the class of all frames of rank 2. Instead they focus their attention on a much smaller class of frames: those for which a complex of semantic postulates hold. These are required in order that entailment defined: $\neg (\alpha \circ \neg \beta)$ behave in the right way. There are interesting insights in this work, not the least of which is the suggested interpretation of '$\circ$' as a generalized consistency operator. Nevertheless, the relevant semantics for '$\circ$' represents only one highly specialized interpretation and we focus instead upon the more general theory.

One might axiomatize the logic determined by the class of frames of rank n by translating the algebraic work of Tarski and Jonsson, although the construction of a Henkin-style completeness proof for the resulting logic is by no means a trivial task. Such a completeness proof appears to have been accomplished first for the rank 2 logic by

R.I. Goldblatt in [Goldblatt 1971]. The proof which appears in [Jennings, Schotch, Johnston a] is essentially a generalization of Goldblatt's. The logic $K\odot_n$ appropriate to the class of frames of rank n, may be obtained by adding to any adequate formulation of the classical propositional calculus all instances of:

$$[K\odot_n]\ \odot(\alpha_1, \ldots, \alpha_j, \beta, \alpha_{j+2}, \ldots, \alpha_n) \wedge \odot(\alpha_1, \ldots, \alpha_j, \gamma, \alpha_{j+2}, \ldots, \alpha_n) \rightarrow \odot(\alpha_1, \ldots, \alpha_j, \beta \wedge \gamma, \alpha_{j+2}, \ldots, \alpha_n)$$

and closing under the rules:

$$[RR\odot_n]\ \vdash_{K\odot_n} \alpha \rightarrow \beta \Rightarrow \vdash_{K\odot_n} \odot(\gamma_1, \ldots, \gamma_j, \alpha, \gamma_{j+2}, \ldots, \alpha_n)\ \odot(\gamma_1, \ldots, \gamma_j, \beta, \gamma_{j+2}, \ldots, \gamma_n)$$

$$[RN\odot_n]\ \vdash_{K\odot_n} \alpha \Rightarrow \vdash_{K\odot_n} \odot(\gamma_1, \ldots \gamma_j, \alpha, \gamma_{j+2}, \ldots, \gamma_n)$$

Goldblatt suggests a temporal interpretation for the rank 2 '$\odot$' to the effect that '$\odot(\alpha,\beta)$ is to be read 'It will be the case that α and then (after that) it will be the case that β'. This together with the relevancers' suggestion about consistency exhaust the informal interpretations so far available for generalized modal operator. Both of the above extend easily to frames of rank n but clearly much remains to be done in the way of providing informal content before the $K\odot_n$ logics are to have any strong role in philosophical logic. Doubtless no satisfactory philosophical interpretation can be found for the very spartan logic determined by the whole class of frames of rank n; what interpretation has ever been discovered for the '$\Box$' of the logic *K*? For significant applications to say deontic logic, one customarily requires a little more than just the bare bones.

One extension that we shall almost certainly require even if we have complete confidence in the informal motivation of $\odot$, is some way of adding a unary modal operator, say '$\Box$' to our language. This forms a central portion of the programme of relevance logic and their approach to the problem is instructive. In effect, those interested in matters of relevance simply lift the standard relational semantics. That is, they add an extra binary relation to their frames and introduce the usual truth-condition with respect to this new relation. It would give the shade of C.I. Lewis cause to shudder.

If we think of n-ary frames as representing a generalization of the notion of consistency, then a very natural way of introducing the unary operators becomes available. Following the great tradition we can introduce possibility as self-consistency. In the case of frames of rank 2 this amounts to:

$$\Diamond\alpha =_{df.} \bigcirc(\alpha, \alpha)$$
$$\Box\alpha =_{df.} \odot(\alpha, \alpha)$$

From these and the original truth conditions we obtain:

$$\models^{m}_{u} \Diamond\ \alpha \Leftrightarrow \exists x,y\text{: } uRxy\ \&\ \models^{m}_{x} \alpha\ \&\ \models^{m}_{y} \alpha$$

$$\models^{m}_{u} \Box\alpha \Leftrightarrow \forall x,y\text{: } uRxy \Rightarrow \models^{m}_{x} \alpha \text{ or } \models^{m}_{y} \alpha$$

The generalization to the case of rank n frames is obvious. An immediate consequence of this generalization is that we are no longer committed to the principle of complete aggregation. One can esaily construct a model in which K fails. Now in case we feel the first stirrings of panic at this we should notice that the new approach gives rise to a logic which is both normal and regular. Thus generalized modal semantics as we conceive it is conservative in all but aggregativity.

The principles of partial aggregation

What principles are left to us once [K] has been abandoned? This is by no means an easy question to answer. The obvious procedure is to discover an axiomatization of the logic determined by the class of n-ary frames and then see what aggregation principles are entailed. A little reflection will convince us that some scheme of aggregation will be forthcoming no matter how high up in the [$K\odot_n$] hierarchy we locate our notion of necessity. Such a conviction turns out to be well founded even though the description of the new theory turns out to be far more complex than anything to which we are likely to have become accustomed. This additional mass of complexity is what lurks behind the seeming simplicity of complete aggregation. It seems that unless we opt for no aggregativity at all, opting for anything less than [K] enmeshes us in al almost impenetrable skein of lesser aggregation principles. Part of the reason for this is given below.

The most direct way to investigate the generalization of the concept of necessity is to attempt an axiomatization of the logic got by the generalization of the truth-conditions, ignoring the fact that '□' is, at every level, defined in terms of ⊙. This procedure has the advantage of suppressing just that portion of the language which we feel presents difficulties for informal rendering.

In one respect, however, it is essential to bear in mind the connextion between '□' and '⊙'. We already know that the language of

$\odot$ is appropriate to the class of n-ary frames; we know that we can express enough first order relational properties by means of some $\odot$ axiom or other. The same is not true of '$\square$'. To take a very simple example, there is no sentence in the language of '$\square$' which matches the property:

$$\forall u, x, y : uRxy \equiv uRyx.$$

Clearly the axiom we need for this is

$$\odot(\alpha, \beta) \rightarrow \odot(\beta, \alpha)$$

Since we have thus got hold of what is in effect the wrong language there is no guarantee that the $\square$-logic is even axiomatizable.

To some extent our fears can be allayed; it is possible to axiomatize the $\square$-logic which is derived from $K\odot_n$ (for any n) but as yet no finite axiomatization has been found. Indeed there is some reason to think that none is available. We begin by giving names to the members of the hierarchy of logics. We call the logic K which has full aggregation K_1, the $\square$-logic obtained from $K\odot_2$, K_2, that from $K\odot_3$, K_3 *etc.* With this in mind, the logic K_n will consist of PC with the two modal rules [RR] and [RN]. Together with these are all instances of every member of i distinct sequences of axioms. Some of these sequences may be described, if somewhat laboriously, as follows:

(A) The n-multiple sequence

For every integer r such that n divides r without remainder there is a member of the (A) sequence [Kn] r, having the overall form:

$$\square p_1 \wedge \ldots \wedge \square p_r \rightarrow \square\gamma$$

Here γ is a disjunction having three sorts of disjuncts. First is $\frac{r}{n}$ disjuncts each of which is an n-fold conjunction of $p_i (1 \leqslant i \leqslant r)$ such that no two disjuncts in this group have any variable in common. Second is a group of $\frac{r}{n}\binom{n}{2}$ disjuncts, which have the form of binary conjuncts and represent all possible pairs taken from the initial disjuncts. The last is a group of $2(\frac{r}{n} - 1)$ disjuncts each of which is an $\frac{r}{n}$-fold conjunction. These last disjuncts list all possible selections of $\frac{r}{n}$ of p_i's which agree on at least one variable (say p_1)

(B) The finite sequence

For every integer r such that $[\frac{r}{n}]$ = n-1 (where $[\frac{a}{b}]$ indicates Euclidean division, *i.e.*, the number of times b divides a ignoring any remainder) and $\frac{r}{n}$ has remainder 1, there is a member of the (B) sequence. Each axiom has the form:

$$[K_n]_r^1 \quad \Box p_1 \wedge \ldots \wedge \Box p_r \rightarrow \Box\gamma$$

For this sequence, γ has the form of a $\binom{r}{n}$-fold disjunction of n-fold conjunctions of the p_i ($1 \leqslant i \leqslant r$). These conjunctions exhaust all the possible ways of selecting n p_i's.

(C) The remainder sequence

For every integer r such that $[\frac{r}{n}] \leqslant n$ and $\frac{r}{n}$ has remainder t there is a member of the (C) sequence of the form:

$$[K_n]_r^t \; \Box p_1 \wedge \ldots \wedge \Box p_r \rightarrow \Box\gamma$$

Here γ is an $([\frac{r}{n}] \overset{r}{+} t)$-fold disjunction of $([\frac{r}{n}] + t)$-fold conjunctions of the p_i. These conjunctions exhaust all possible ways of selecting $[\frac{r}{n}] + t$ of the p_i.

Unfortunately, there is no proof that any member of any one of the three sequences is derivable from any other member of that sequence or from any member of any other sequence. That this is surprising is due to our previous commitment to K_1. The K_1 sequences all collapse into a single sequence in which every member is derivable from every other. This may be taken as an indication of the great aggregative strength of the axiom [K]. Other indications are given below.

One of our objectives has been achieved. We now know what sorts of aggregation theories lie between K and N. Moreover, the principles of incomplete aggregation though infinite in number for any logic K_n, exhibit overall similarities and regularities which are relatively easy to grasp.

Extensions of the K_n logics and other matters

Within the new framework we are able to draw a great many distinctions which were not available to us previously. Considering just those sorts of principles which are likely to occur in deontic logic, we see first that we lose the indiscernability of

$$[D] \; \Box p \rightarrow \neg\Box\neg p$$

and

$$[Con] \; \neg\Box\bot$$

The reason for this is that it is only in the presence of [K] that [D] amounts to a genuine consistency principle. In the general setting [D] makes a much stronger demand. What is required is a restriction to frames of rank n which satisfy:

$$\forall u, \exists x: uR\underbrace{x, ..., x}_{n \text{ times}}$$

In contrast to this, it is clear that in order to obtain the validity of Con we require frames which satisfy the obvious generalization of *seriality viz*:

$$\forall u, \exists x_1, ..., x_n: uRx_1, ..., x_n$$

Frequently writers on Deontic Logic take [D] as an axiom and then derive [Con] with the aid of [K]. This has always seemed unnatural to us since [Con] is more obviously a consistency principle (more often what ethicists are after) while it is not clear what sort of principle [D] is supposed to be. Sometimes it is said that [D] is the formal counterpart of the dictum 'ought implies can', but this claim is false. The force of this slogan is far better represented by a formula such as

$$\overset{a}{O}p \rightarrow \overset{a}{\langle P \rangle}p$$

where O represents ought and $\langle P \rangle$ represents physical possibility in some crude sense and where both these modalities are relativized to the agent a. It might be argued that the required formula is a consequence of some version of [D], but this depends upon some such problematic assumption as that

$$\neg O^a \neg p \rightarrow \overset{a}{\langle P \rangle}p$$

represents a correct principle relating deontic and physical modalities.

From a purely formal point of view, we might wonder what frame condition will give us [D] in the generalized setting. It is possible and quite easy to show that K_nCon is determined by the class of all serial n-ary frames (see [Jennings, Schotch, Johnston b]) which uses up the usual frame condition employed by binary relational semantics. It might seem at first that the correct frame restriction would be the requirement that every world be related to at least one diagonal tuple. However, this condition is too strong for completeness. In fact, it is possible, though not so easy, to show, using an adaptation of Goldblatt's ultraproduct construction in [Goldblatt 1975], that [D] does not correspond to any first order condition on n-ary frames. [D] is not first-order definable in generalized modal logic (see [Jennings, Schotch, Johnston c]). Of course, for [D] enthusiasts, the utility of the generalized approach will be made thereby to seem doubtful. But the burden is on such philosophers to show that [D] actually conveys any information that is not conveyed at least as well by [Con]. Certainly, in the deontic context, the absence of [D] does not seem at all troublesome.

The connexion between aggregation theory and reduction theory is not yet fully understood. It is known that there are first-order conditions on n-ary frames which determine the logics K_nT, K_nB K_n4 and K_n5 (see [Jennings, Schotch, Johnston b]).

It is clear that as an area of research the theory of generalized modal logic is as yet underdeveloped. At best we have skimmed off a few of the easier results, but enough to show that the project is attractive from both a technical and a philosophical perspective. It may turn out that many of the things that one can do in the generalized setting can be done equally well by some complication of existing methods. In spite of this there seems to be no other approach which will serve as well in the study of aggregation theory. Certainly neighbourhood semantics represents no simplification. For the conditions upon neighbourhood structures required for the K_n axioms merely explain infinite sequences of aggregation principles by appeal to infinite sequences of closure conditions upon neighbourhood families. The problem of finding any suggestive neighbourhood semantics for the K_n logics remains open.

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NOTES

1 See e.g. [Becker 1930], [Becker 1952] as well as the papers of H.B. Smith, Churchman and [Rosenbloom 1950] pp. 61-63.

2 See [Segerberg 1971]

3 See [Lennon 1965]

4 See [Tarski, Jónsson 1951]

5 See [MacLane 1971] Chap X.

6 See [Rontley, Meyer 1973].

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