Philosophia Vol. 5 No. 3 Pp. 213-217 July 1975

ON REPRESENTING 'TRUE-IN-L' IN L

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Given Tarski's familiar treatment of the semantic paradoxes, no formal language can adequately represent its own truth-concept.[1] But natural languages do, apparently, express their own truth-concepts and this fact alone has been enough to motivate some to seek alternative treatments of the paradoxes. In this paper we demonstrate that a language construed according to the "category" approach[2], modified in certain respects, can indeed express its own truth-concept.

Part I specifies the language to be studied; Part II contains the proof of the truth-representation theorem for the language. It will be seen that the possibility of truth-representation of the kind under consideration depends only on the satisfaction of rather simple conditions – conditions which clearly may be met in ways other than given here.

I

L is a usual first-order quantificational language, including a one-place predicate constant 'T' (to be interpreted as the truth-predicate for L), and with conjunction, negation, and the universal quantifer taken as primitive. There is one difference: the individual variables come in a finite number of sorts (associated with the variables is a function s from the positive integers into the k integers 1, 2, ... k, where, for positive integer i, s(i) is the *sort* of the variable x_i). The sorting of variables plays no role in the definition of wff. We identify L with the set of its wffs.

For the truth-functional connectives we use Kleene's weak three-valued truth-tables,[3] labelled by $\wedge$ and $-$:

	$A \wedge B$	B t u f	$\bar{A}$
	t	t u f	f
A	u	u u u	u
	f	f u f	t

We use the quantifier $\forall$ so that for any $X \subseteq \{t, u, f\}$, $\forall X = t$ iff $X = \{t\}$, $\forall X = u$ iff $u \in X$, $\forall X = f$ otherwise.

As ranges for the sorted variables we provide a sortally segmented domain. A function v is a *valuation* on a domain $D = U_1 \cup U_2 \cup ... \cup U_k$ iff

(1) for every i, $1 \leqslant i \leqslant k$, U_i is a non-empty set.

(2) v assigns to each individual constant an element of D.

(3) v assigns to each n-ary predicate an element of $\{t, u, f\}^{D^n}$.

A *value assignment* α is an assignment to each individual variable x_i of an element of $U_{s(i)}$. Then we may define a unique value $v\,\alpha$ for each term and wff as follows (where α_d^x is like α except that $\alpha_d^x x = d$):

$v\,\alpha\,x = \alpha$ for individual variable x
$v\,\alpha\,a = va$ for individual constant a
$v\,\alpha\,Ft_1 \ldots t_n = vF(v\,\alpha\,t_1, v\,\alpha\,t_2, \ldots, v\,\alpha\,t_n)$
$v\,\alpha \sim A = \overline{v\,\alpha\,A}$ for wff A
$v\,\alpha\,(A \,\&\, B) = v\,\alpha\,A \wedge v\,\alpha\,B$
$v\,\alpha\,(x_i)A = \forall\{v\,\alpha_d^{x_i}A : d \in U_{s(i)}\}$ [4]

We note that the usual local determination lemma holds:

Lemma O: If v and v′, α and α' coincide on the constants, predicates and free variables of A, then $v\alpha A = v'\alpha' A$.

An immediate corollary is that for sentences (closed wffs) A, $v\alpha A$ is independent of α, and we may write simply vA.

II

For each predicate F we define the truth and falsity ranges (on a valuation v) as follows:

$vF^+ = \{\langle d_1, \ldots, d_n\rangle : vF(d_1, \ldots, d_n) = t\}$
$vF^- = \{\langle d_1, \ldots, d_n\rangle : vF(d_1, \ldots, d_n) = f\}$

If v and v′ are valuations in the same domain D, we say that v′ is a

T-extension of v if they coincide except that $vT^+ \subseteq v'T^+$ and $vT^- \subseteq v'T^-$. 'T', it will be recalled, is to be our truth-predicate. We designate this relation by '<'; it is clearly a partial ordering.

Lemma 1: If $v<v'$, then for every α in D, and wff A, $v\alpha A = t \Rightarrow v'\alpha A = t$ *and* $v\alpha A = f \Rightarrow v'\alpha A = f$.

Proof: By induction on A.

Let V be a <-chain of valuations in D. By $\bar{V}$ we mean the valuation which coincides with the elements of V except at T, and such that $\bar{V}T^+ = \bigcup_{v \in V} vT^+$ and $\bar{V}T^- = \bigcup_{v \in V} vT^-$. It is clear that $\bar{V}$ is an upper bound of V; i.e., for each $v \in V$, $v < \bar{V}$.

Let v be a valuation in a domain $D = U_1 \cup U_2 \cup ... \cup U_k$ such that, for some j, $1 \leqslant j \leqslant k$, $L = U_j$. Thus the wffs of L are made to constitute one of the sortal segments of the domain of v. We say that v *partially represents truth* (by 'T') for L iff both $vT(A) = t \Rightarrow vA = t$ and $vT(A) = f \Rightarrow vA = f$ for each sentence $A \epsilon L$. Let PR be the set of all v satisfying this condition. (If both implications are bi-conditionals, the qualification "partially" may be dropped.)

We wish to show not *just* that L has truth-representing interpretations, for this holds even where the semantics of L is classical (as long as certain conditions are met; for example that there is no individual constant *a* of L such that $va = {\sim}Ta$). We wish to show that L has truth-representing interpretations even where there are no restrictions against self-reference.

Lemma 2: Let v be an <-chain of valuations in D. If $V \subseteq$ PR, then $\bar{V} \epsilon$ PR.

Proof: Suppose $\bar{V}T(A) = t$. Then $A \epsilon \bar{V}T^+$, so for some $v \epsilon V$, $A \epsilon vT^+$; hence, $vT(A) = t$, and since $v \epsilon$ PR, we have $vA = t$, so (since $v < \bar{V}$) $\bar{V}A = t$ (by lemma 1). The argument is similar if $\bar{V}T(A) = f$.

As a consequence of lemma 2 we have by Zorn's lemma:

Lemma 3: Every partially representing valuation has a maximal partially representing T-extension.

Our final lemma shows that maximal elements of PR have the right property:

Lemma 4: If v is maximal in PR, then v represents truth (by T).

Proof: Suppose $vA = t$ and v' coincides with v except that $v'T(A) = t$. Then $v < v'$ by construction: we show that $v' \in PR$. Indeed, if $v'T(B) = t$, and $B = A$, then $vB = t$ by hypothesis; if $B \neq A$, $vT(B) = v'T(B) = t$, so since $v \in PR$, $vB = t$. In either case, by lemma 1, $v'B = t$. The argument is similar if $v'T(B) = f$; hence $v' \in PR$. Since v is maximal in PR, we must have $v = v'$, so $vT(A) = t$. A symmetrical argument shows that if $vA = f$, $vT(A) = f$.

We may now state our theorem:

Theorem: Let v be any valuation in a domain $D = U_1 \cup U_2 \cup \ldots \cup U_k$ such that, for some j, $1 \leqslant j \leqslant k$, $L = U_j$. Then there is a valuation v' which coincides with v on all sentences not containing 'T', and which represents truth (by 'T'). (v may be a classical valuation: i.e., for every predicate F we may have only t and f in the range of vF.)

Proof: Let v'' be like v except that $v''T^+ = v''T^- = \Lambda$. Then v'' coincides with v on non-T sentences (by lemma 0) and the same will be true of every T-extension of v''. Furthermore, v'' is trivially in PR. But by the preceding lemmata there is a T-extension v' of v'' which represents truth by 'T' (and coincides with v'', and hence with v, except on 'T').

We close with a couple of brief applications.

(1) Let L contain an individual constant *a*, and let v_0 be a valuation of the T- and *a*-free fragment of L. Let v_1 and v_2 be like v_0 except that $v_1 a = v_2 a = Ta$, $v_1 T^+ = \{Ta\}$, $v_1 T^- = \phi$, $v_2 T^+ = \phi$, and $v_2 T^- = \{Ta\}$. Then it is easy to see that both v_1 and v_2 partially represent truth by T, and hence have truth-representing extensions. This is a reflection of the observation that 'This sentence is true.' is true if true and false if false.

(2) Let v be truth-representing and let $va = {\sim}Ta$. Then $v{\sim}Ta = t$ iff $vTa = f$ iff $va \in vT^-$ iff ${\sim}Ta \in vT^-$ iff (since v is truth-representing) $v{\sim}Ta = f$. Hence $v{\sim}Ta = u$ on any such valuation; the Liar is neither true nor false.

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NOTES

[1] Tarski (1956).

[2] See Martin (1967), (1968), (1970).

[3] Kleene (1950, p.334). The use of the weak tables is not necessary for the proof of our theorem. An argument for their use, based upon category considerations and independent of the paradoxes, is given in Martin (1974). Our argument in the present paper may be adapted to any set of truth-functions ϕ which include t and f in their field and have the following properties (enunciated in Fine (1974)):
Stability: If ϕ has the value t or f for given truth-values as arguments, it retains that value when any argument not in {t,f} is replaced by one of the latter.
Fidelity: Whenever all arguments are in {t,f,} ϕ behaves classically (e.g. in the present case we have $t \wedge t = t$, $t \wedge f = f \wedge t = f \wedge f = f$).

[4] We wish to preserve the usual relationship between universal quantification and conjunction. This, along with the adoption of Kleene's weak truth-tables, accounts for the requirement that $\forall X = u$ whenever $u \in X$. If the domain were not divided into sortal segments, each containing the values of one sort of variable, the above requirement would lead to excessively counter-intuitive consequences. For example, if the domain consisted of abstract and concrete objects, and the sortal range of the predicate F ('is yellow') were restricted to concrete objects, then even if the domain contained a yellow object, the sentence '$(\exists x)Fx$' would be without truth-value. With the segmented domain, and the variable x ranging over the segment containing the concrete objects, '$(\exists x)Fx$' would be true.

REFERENCES

[1] Fine, K.: 1974, 'Vagueness, Truth and Logic', unpublished manuscript.

[2] Kleene, S.C.: 1950, *Introduction to Metamathematics*, Van Nostrand, New York.

[3] Martin, R.L.: 1967, "Toward a Solution to the Liar Paradox", *The Philosophical Review* 76, 279-311.

[4] –: 1968, "On Grellings' Paradox", *The Philosophical Review* 77, 321-331.

[5] –: 1970, "A Category Solution to the Liar", in: R.L. Martin, ed., *The Paradox of the Liar*, Yale University Press, London and New Haven, 91-112.

[6] –: 1974, "Sortal Ranges for Complex Predicates", *The Journal of Philosophical Logic* 3,159-167.

[7] Tarski, Alfred: 1956, "The Concept of Truth in Formalized Languages", in: *Logic, Semantics, Metamathematics*, Oxford University Press.