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OPTIMIZATION OF RISK PROCESSES*

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STATEMENT OF THE PROBLEM

Mathematical and statistical problems of insurance, together with their solution methods, constitute the subject of so-called actuarial or insurance mathematics. One of the subdivisions of insurance mathematics is risk theory, which deals with stochastic processes (risk processes) of the form [1-5]

$$u(t) = u + \alpha(t) - \beta(t), \tag{1}$$

where $\alpha(t)$, $\beta(t)$, $t \geq 0$, are stochastic processes with monotone nondecreasing paths, $\alpha(0) = \beta(0) = 0$, and $u > 0$. Here u is the initial capital of the insurance company; the process $\alpha(t)$ describes the total inflows and $\beta(t)$ the total outflows during the time $[0, t]$. Assume that the risk process depends on parameters, i.e.,

$$u(t, x) = u + \alpha(t, x) - \beta(t, x), \tag{2}$$

where $x \in X$, $X = \left\{x \in \mathbf{R}^n : -\infty < a_i \leq x_i \leq b_i < \infty \ \forall i = \overline{1, n}\right\}$, $\mathbf{R}^n$ is the n-dimensional Euclidean space.

It is relevant to solve the problem of capital maximization at the end of a given time interval $[0, T]$. In other words, we solve an optimization problem of the form

$$\max_{x \in X} \mathbf{M}\, u(T, x), \tag{3}$$

where $\mathbf{M}$ is the expectation over the measure $\mathbf{P}$ of the probability space on which the risk process is defined.

As an additional constraint on the solution of problem (3), we can consider the inequality

$$1 - \mathbf{P}(u(t, x) > 0 \ \ \forall t \in [0, T]) \leq \varepsilon, \tag{4}$$

where ε is a small parameter (for instance, $\varepsilon = 10^{-3}$ [1]). Here $1 - \mathbf{P}(\cdot)$ is the probability of ruin.

Let x^* be one of the solutions of problem (3), (4). Then the process $u(t, x^*)$ is optimal in the sense that it maximizes the capital (in the mean) and guarantees a certain financial stability by making the probability of ruin of the insurance company sufficiently small. Note that the solution of problem (3) without constraint (4) does not guarantee this stability. It is sufficient to consider the example of an insurance company with a capital of $1 million that insures risks for the amount of $150 million. It successfully increases its capital until the first insurance event (the first claim).

A problem of the form (3), (4) for a certain class of risk processes was first stated in [2, Ch. 13] alongside other problems incorporating the constraint (4).

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BRIEF ANALYSIS OF METHODS OF SOLUTION OF PROBLEM (3), (4)

Lagrange Function and Existence Conditions of the Extremum. We rewrite problem (3), (4) in the following equivalent form:

$$\min \{f(x) \mid g(x) \leq 0, \ x \in X\}, \tag{5}$$

where $f(x) = -\mathbf{M}\mu(T, x)$, $g(x) = \varepsilon^{-1}(1 - \mathbf{P}(u(t, x) > 0 \ \forall\, t \in [0, T])) - 1$.

The Lagrange function $L(x, y) = f(x) + yg(x)$, $x \in X$, $y \geq 0$ plays a special role in deriving the existence conditions of a solution of problem (5) and describing the algorithm to find the solution. Specifically, suppose that there exist $x^* \in X$ and $y^* \geq 0$ such that $L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*)$. Then x^* is a solution of problem (5) (see [6]). The pair (x^*, y^*) is called a saddle point of the Lagrange function. Thus, the solution of problem (5) reduces to finding the first component of a saddle point of the Lagrange function.

We assume that $f(x)$, $g(x)$ are convex functions of $x \in X$ and the Slater regularity condition is satisfied for problem (5). In this case, a saddle point exists [6].

Algorithms to Find Saddle Points. The choice of an algorithm to solve problem (5) depends primarily on whether we know how to construct an exact (or an approximate) analytical expression for the functions $f(x)$, $g(x)$ or how to estimate the values of these functions by the Monte-Carlo method. For the first case, there are fairly many saddle-point algorithms that use the Lagrange function and its various modifications [7, 8].

For the second case, we do not have such a variety of algorithms. The solution is usually reduced to application of a stochastic analogue of the Arrow–Hurwitz method [9]. The nonlinear constraint of problem (5) contains a bound on the probability of a rare event – ruin in a given time interval. Solution of such problems (extremal problems with rare events) by stochastic programming methods has been examined in [10-14].

Multiplier Method. The multiplier method relies on the notion of modified Lagrange function [7, 8]. There are various classes of such functions. One of the popular modifications for problem (5) is defined as follows:

$$L_c(x, y) = \begin{cases} f(x) + yg(x) + 0.5\,cg^2(x), & \text{if} \quad cg(x) \geq -y, \\ f(x) - 0.5\,y^2/c, & \text{if} \quad cg(x) < -y. \end{cases}$$

For $c = 0$, this function reduces to an ordinary Lagrange function. The functions $L(x, y)$ and $L_c(x, y)$ have identical saddle-point sets [7, 8], and we can thus use $L_c(x, y)$ instead of $L(x, y)$ to construct more efficient computational algorithm of solving problem (5).

An iterative step of the multiplier method is defined as follows. For fixed values c_k and y_k we solve the problem

$$\min_{x \in X} L_{c_k}(x, y_k). \tag{6}$$

Let x^{k^*} be a solution of problem (6). Then $y_{k+1} = \max\{0, y_k + c_k g(x^{k^*})\}$, and $c_{k+1} \geq c_k$ (here $c_1 > 0$ and $\lim_{k\to\infty} c_k = \infty$).

In each step of the multiplier method, we solve the minimization problem (6) with simple constraints. This can be done by various efficient algorithms [7, Ch. 1], but their application requires evaluation of the gradient of the function $L_c(x, y)$ with respect to x. At the same time, the evaluation of the function $g(x)$ involves solving fairly complex equations, and the error involved in this procedure is not conducive to efficient application of finite differences for determination of the gradient. Therefore, despite their low rate of convergence, coordinatewise or random descent methods are quite acceptable for solving problem (6).

Random Search of Solution of Problem (6). An iterative step of the minimizing sequence is defined as

$$z^{m+1} = \pi_X\left(z^m - s_m \xi^m\right), \quad m \geq 1.$$

Here $\pi_X(\cdot)$ is the projector on the set X, $z^1 = x^{(k-1)^*}$, $s_m = O(m^{-1})$, the random vector ξ^m is defined below.

If the gradient of the function $f(x)$ is easily evaluated, we may set

$$\xi^m = \begin{cases} \nabla f(x^m) + 3\Delta^{-1}(m)\,\mathbf{b}(m)[y_k g(x^m + \Delta(m)\mathbf{b}(m)) + \\ + 0.5\, c_k g^2(x^m + \Delta(m)\mathbf{b}(m))], & \text{if} \quad c_k g(x^m) \geq -y_k, \\ \nabla f(x^m), & \text{if} \quad c_k g(x^m) < -y_k. \end{cases}$$

Otherwise $\xi^m = 3\Delta^{-1}(m)\mathbf{b}(m)L_{c_k}(x^m + \Delta(m)\mathbf{b}(m), y^k)$.

Here $\Delta(m) = \Delta/m^{0.2}$, $\delta > 0$; $\mathbf{b}(m)$ is the m-th realization of a random vector with independent $(-1, 1)$-uniform coordinates. In both cases, if the modified Lagrange function is twice continuously differentiable with respect to x, we have the equality (see Appendix) $\mathbf{M}\ \xi^m = \nabla_x, L_{c_k}(x^m, y^k\} + O(\Delta(m))$, $O(\Delta(m))$ is a vector with norm of order $\Delta(m)$.

Under the above assumptions, this random search algorithm converges almost surely to a solution of problem (6) (see [14]).

AN EXAMPLE OF RISK PROCESS OPTIMIZATION

Optimization of a Reinsurance Contract. We assume that the insurance company writes n types of insurance contracts, and the number of claims under insurance contracts of type i in time interval $(0, t)$ is Poisson distributed with the parameter $\mu_i t$, $i = \overline{1, n}$. The amount of type i claims has the distribution function $1 - \exp(-t/\tau_i)$, $t \geq 0$.

If the expected claims under certain types of insurance contracts are very large, the insurance company purchases reinsurance protection. For instance, according to a surplus agreement, the insurance company pays an amount not exceeding one line x_i, and the remainder is paid by the reinsurer. The amount paid by the insurance company under contracts of type i in this case has the distribution function

$$P_i(t, x_i) = \begin{cases} 1 - \exp(-t/\tau_i), & \text{if} \quad t < x_i, \\ 1, & \text{if} \quad t \geq x_i. \end{cases}$$

Therefore, the stochastic process $\beta(t, x)$ in (2) is a compound Poisson process, i.e., the number of claims in the interval $[0, t]$ has the Poisson distribution $(\mu t)^k \exp(-\mu t)/k!$, $k \geq 1$, and the claim amount has the distribution function $F(t, x)$, $t \geq 0$, with mean $\tau(x)$ (and $F(+0, x) = 0$). Here $\mu = \sum_{i=1}^{n} \mu_i$, $F(t, x) = \sum_{i=1}^{n} p_i P_i(t, x_i)$, $p_i = \mu_i / \mu$.

Also note that $\tau(x) = \sum_{i=1}^{n} p_i \tau_i (1 - \exp(-x_i/\tau_i))$. Indeed, if $\mathbf{P}(\xi < t) = \Phi(t)$, then $\mathbf{M}\min(\xi, x) = \int_0^x t d\Phi(t) + x(1 - \Phi(t))$. Substituting for $\Phi(t)$ the distribution function $1 - \exp(-t/\tau)$ and integrating by parts, we obtain $\mathbf{M}\min(\xi, x) = \tau(1 - \exp(-t/\tau))$. It now remains to note that $\tau(x)$ is a linear combination of the corresponding means (with weights p_i).

The arrival process in (2) is defined as $\alpha(t, x) = c(x)t$, where $c(x) > 0$ is a constant that expresses the rate of arrival of insurance premiums. This constant is determined as follows.

Let c_i be the rate of arrival of insurance premiums for contracts of type i. According to the reinsurance agreement, part of the insurance premium is transferred to the reinsurer, i.e., the fraction retained by the insurance company is given by the function $\rho_i(x) = \int_0^\infty t\, dP_i(t, x_i) / \int_0^\infty t\, dF_i(t)$, and the fraction transferred to the reinsurer is $1 - \rho_i(x)$. The rate of arrival of net insurance premiums for the insurance company is thus given by $c(x) = \sum_{i=1}^{n} c_i \rho_i(x)$. Since $\int_0^\infty t\, dF_i(t) = \tau_i$, and $\int_0^\infty t\, dP_i(t, x_i) = \tau_i(1 - \exp(-x/\tau_i))$, we finally obtain $c(x) = \sum_{i=1}^{n} c_i(1 - \exp(-x/\tau_i))$.

Objective Function Evaluation. Let us now analyze problem (5) in application to the given class of risk processes. Let $T = \infty$. The mean $\mathbf{M}u(T, x)$ is infinite in this case, and we accordingly consider the objective function $\lim_{T\to\infty} T^{-1}\mathbf{M}u(T,$

TABLE 1

Parameters	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
μ_i	0.2	0.5	1	1.2	2	3	4
τ_i	3	2	3	2	1	0.3	0.1
c_i	0.8	1.2	3.5	2.8	2.4	1	0.5

TABLE 2

Parameters	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
x_i	3.2	2.4	3.4	2.6	1.9	0.76	0.59
x_i	3.4	2.2	3.5	2.5	1.8	0.78	0.4
x_i	3.2	2.3	3.4	2.4	1.7	0.7	0.5

x), which is the rate of growth of the capital of the insurance company in unit time. We find that in (5) $f(x) = c(x) - \mu\tau(x)$ or using the previous formulas

$$f(x) = \sum_{i=1}^{n} \left(c_i - \mu_i\tau_i\right) \left(1 - \exp\left(-x_i / \tau_i\right)\right). \tag{7}$$

Cramer–Lundberg Bound for the Probability of Ruin. Let $Q(u, x) = 1 - \mathrm{P}(u(t, x) > 0 \ \forall\ t \in [0, \infty))$ be the probability of ruin of the insurance company (u is the initial capital, see (2)). For this probability we have the Cramer–Lundberg bound [1-5] $Q(u, x) \leq \exp(-R(x)\, u)$, where the function $R(x)$ is the solution of the equation

$$\int_0^\infty \exp(R(x)\, z)(1 - F(x, z))\, dz = c(x) / \mu. \tag{8}$$

Substituting in (8) the characteristics of the above-mentioned compound Poisson process, we find that the sought function is the solution of the equation

$$\sum_{i=1}^{n} p_i \int_0^{x_i} \exp\left(R(x)\, z - z / \tau_i\right) dz = \sum_{i=1}^{n} c_i (1 - \exp(-x / \tau_i)) / \sum_{i=1}^{n} \mu_i \tag{9}$$

Taking the integrals in the left-hand side, we find

$$\sum_{i=1}^{n} p_i \left(\exp\left(\left(R(x) - 1/\tau_i\right) x_i\right) - 1\right) / \left(R(x) - 1/\tau_i\right) = \sum_{i=1}^{n} c_i (1 - \exp(-x / \tau_i)) / \sum_{i=1}^{n} \mu_i$$

Numerical Example. The input data for solving the optimization problem with $n = 7$ are presented in Table 1. The results of three executions of the computational process with initial capital $u = 40$ are given in Table 2. In all three cases, the random search method executed 2,000 iterations, with 10 iterations by the variable y. Substitution of the optimal values of the variables x_i in formula (7) gives in all three cases 1.4 for the optimal growth of capital.

APPENDIX

Let $h(x)$ be a twice continuously differentiable function, $x \in (-\infty, +\infty)$. Then by Taylor's formula $h'(x) = 0.5(\alpha\Delta)^{-1}\,(h(x + \alpha\Delta) - h(x - a\Delta)) + O(\alpha\Delta)$. Left- and right-multiplying by α^2, we obtain

$$\alpha^2 h'(x) = 0.5\, \alpha\, \Delta^{-1} (h(x + \alpha\Delta) - h(x - \alpha\Delta)) + O(\alpha^3\Delta). \tag{10}$$

Let α in (10) be a (0, 1)-uniform random variable. Noting that $\nu\alpha$ is (−1, 1)-uniform if $\mathbf{P}(\nu = 1) = \mathbf{P}(\nu = -1) = 1/2$ and denoting $\beta = \nu\alpha$, we obtain the bound (passing to expectations on both sides in (10))

$$h'(x) = 3\,\Delta^{-1}\mathbf{M}\,\beta\, h(x + \beta\,\Delta) + O(\Delta). \tag{11}$$

Now let $h(x)$ be a twice continuously differentiable function, $x \in \mathbb{R}^n$. From (11) we have

$$\partial h(x) / \partial x_i = 3\,\Delta^{-1} M\,\beta_i\, h(x_1, \ldots, x_i + \beta_i\Delta, \ldots, x_n) + O(\Delta). \tag{12}$$

It follows from (12) that to estimate the gradient we need to find the vector $\mathbf{b} = (\beta_1, \ldots, \beta_n)$ of independent (−1, 1)-uniform random variables and also to evaluate the function at n random points. The latter is not practicable for large problems.

We also have the formula

$$\nabla h(x) = 3\,\Delta^{-1}\,\mathbf{M}\,\mathbf{b}\, h(x + \mathbf{b}\,\Delta) + O_\Delta, \tag{13}$$

where O_Δ is a vector with norm of order Δ. This formula differs from (12) in that it requires only one evaluation of the function. To prove formula (13), note that we have the equality

$$3\Delta^{-1}\mathbf{M}\beta_1\, h(x + \beta\,\Delta) = 3\Delta^{-1}\mathbf{M}\, \Big[\beta_1 h\big(x_1 + \beta_1\Delta, x_2, x_3 + \beta_3\Delta, \ldots, x_n + \beta_n\Delta\big) +$$

$$+ \Delta\beta_1\beta_2\,\partial h\big(x_1 + \beta_1\Delta, x_2, x_3 + \beta_3\Delta, \ldots, x_n + \beta_n\Delta\big) / \partial x_2 + O(\Delta^2)\Big] =$$

$$= 3\,\Delta^{-1}\mathbf{M}\beta_1\, h\big(x_1 + \beta_1\Delta, x_2, x_3 + \beta_3\Delta, \ldots, x_n + \beta_n\Delta\big) + O(\Delta^2).$$

Similarly expanding in a Taylor series in the variables $x_3, \ldots, x_n$, we obtain

$$3\Delta^{-1}\mathbf{M}\beta_1\, h(x + \beta\,\Delta) = 3\Delta^{-1}\mathbf{M}\beta_1\, h\big(x_1 + \beta_1\Delta, x_2, x_3, \ldots, x_n\big) + O(\Delta).$$

By (12), the right-hand part of this equality is a bound of the derivative with respect to x_1. By symmetry, the same equalities hold for other partial derivatives, which proves equality (13).

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