ANNALS OF BIOMEDICAL ENGINEERING 1, 481–488 (1973)

Hexadecapolar Shift Equations Applicable to Equivalent Cardiac Generators of Lower Degree[1]

DANIEL A. BRODY AND JOHN W. COX, JR.

Division of Clinical Physiology, University of Tennessee Medical Units, Memphis, Tennessee 38103

LEO G. HORAN

Veterans Administration Hospital and the Department of Medicine, Medical College of Georgia, Augusta, Georgia 30902

Received March 18, 1973

Electrocardiographic shift equations for manifest hexadecapolar behavior due to eccentric location of equivalent generators of lesser degree have been derived and numerically validated. Under these conditions hexadecapolar properties emerge as first-order positional moments of eccentric octapole coefficients, second-order moments of quadripole coefficients, and third-order moments of dipole coefficients. Assuming that hexadecapolar field information can be successfully evaluated for beating heart preparations, including the intact human, the formulations presented herein should prove useful in determining the location of an equivalent cardiac generator of lesser degree.

INTRODUCTION

The possibility of determining the location of the equivalent cardiac generator has posed a stimulating challenge to experimental electrocardiographers ever since Gabor and Nelson (1954) published a set of equations which indicated that this feat could be accomplished in the case of a dipolar source. Later, Geselowitz (1960) showed that the apparent quadripolar content of an equivalent generator depended in part upon where the origin of the reference system was located within the torso, which had been idealized into an electrically homogeneous, isotropic, realistically bounded volume conductor. In the pure case of a wholly dipolar source, true dipolar and manifest quadripolar content could be quantitatively evaluated from the distribution of potentials over a known torso surface configuration. The location of the generator could then be determined from solution of five linear simultaneous equations which contained the three components of dipole moment and five components of quadripole content. Translation of the reference system to the origin thus determined caused apparent quadripolar content to vanish.

Eventually the Gabor–Nelson and the Geselowitz formulations, although appearing to differ somewhat from each other, were demonstrated to express virtually identical relationships (Brody *et al.*, 1961). Within them appear first-order

[1] This work was supported by Grant Nos. HL-11667, HL-14032, HL-01362, and HL-09495 of the National Heart and Lung Institute, National Institutes of Health, U. S. Public Health Service, and an award from the Veterans Administration.

positional moments of dipole moment, analogous to the static moments of mechanics, from which the coordinates x, y, z of dipole location are determined. Because of their intimate relationship to generator location they have come to be called the "shift equations" of electrocardiography. Subsequently, other formulations of this genre have been developed, namely for the manifest octapolar content of eccentric dipoles and quadripoles (Brody, 1968). Recent experiences in the laboratory suggest that eventually the next order of generator approximation may prove experimentally applicable (Brody *et al.*, 1971; Horan *et al.*, 1972). Therefore, in this report we address ourselves to the problem of determining the hexadecapolar behavior of an eccentric dipole, quadripole, and octapole.

DERIVATION OF SHIFT EQUATIONS

Burger and van Milaan (1946) showed that electrocardiographic potential V could be expressed, at least to a first approximation, as

$$V = F_i M^i, \quad i = 1,2,3, \tag{1}$$

where M^i and F_i are components of heart and lead vector, respectively. We (Brody *et al.*, 1961) later generalized this result to

$$V = F_i M^i + F_{ij} M^{ij} + F_{ijk} M^{ijk} + F_{ijkl} M^{ijkl} + \cdots, \quad i,j,k,l, \cdots = 1,2,3. \tag{2}$$

In this expression the indexed F and M symbols represent, respectively, lead and heart tensors whose rank is expressed by the number of indexes related to each symbol. The first scalar product on the right of the equation gives dipole contribution to electrocardiographic potential (as in Eq. (1)), the second scalar product in the running sum expresses quadripole contribution, etc. The equation emphasizes not only the idea of successive approximations, but also the concept that lead and generator characteristics can be separated from each other.

Our earlier development of lead tensor theory was based on the concept of the electrocardiographic lead field (McFee *et al.*, 1952; Brody and Romans, 1953). By definition the lead field is that scalar potential (voltage, as distinguished from a streamline field) function u which is produced in the torso when an electrocardiographic connection is reciprocally energized with one unit of current from an external source. In this context, then, lead tensors of increasing rank are defined by

$$\begin{aligned} F_i &= \partial u/\partial x^i, \\ F_{ij} &= 1/2!\ \partial^2 u/\partial x^i \partial x^j, \\ F_{ijk} &= 1/3!\ \partial^3 u/\partial x^i \partial x^j \partial x^k, \text{ etc.} \\ i,j,k, \cdots &= 1,2,3, \end{aligned} \tag{3}$$

where the x^i are Cartesian coordinates $x^1 = z$, $x^2 = x$, $x^3 = y$. The heart tensor components were shown to be formed from the outer products of unit direction-vector components, each such element multiplied by an appropriate scalar value of dipole or multipole moment (Brody *et al.*, 1961). The exact structure of the dipole–multipole current singularities need not concern us in this communication.

TABLE 1
FIRST RANK TENSOR COMPONENTS OF FIRST DEGREE IDEAL LEAD FIELDS
(cf. Eqs. (3) and (4))

m	Series	F_1	F_2	F_3	Associated coefficient
0	cos	1			a_{10}
1	cos		1		a_{11}
1	sin			1	b_{11}

Further, in developing lead tensor theory, we found it useful to deal with restricted forms of the reciprocally generated potential functions which we termed "ideal lead fields." These may be expressed as

$$u = \frac{2}{1+\delta_m{}^0}\,\frac{(n-m)!}{(n+m)!}\,r^n \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix} P_n{}^m(\cos\theta), \tag{4}$$

where $r\cos\theta = z$, $r\sin\theta\cos\phi = x$, $r\sin\theta\sin\phi = y$; $P_n{}^m(\cos\theta)$ are associated Legendre functions of the nth degree and mth order; $\delta_m{}^0 = 1$ if $m = 0$ and is otherwise zero. Each of these functions is transformable to a homogeneous algebraic expression of the same degree. By repeated partial differentiation of these various expressions down to constant terms, tableaux can be generated which indicate the exact "interaction" between each ideal lead configuration and the corresponding generator component which it senses. Although such tables have been published previously (Brody *et al.,* 1961), they are repeated here (Tables 1, 2, and 3) because the information which they contain is centrally important to the development which follows.

In Tables 1,2, and 3, lead tensors are gathered together, line-by-line, into usable packets of information. The generator coefficient associated with each such grouping is listed in the right hand column of the table on the same line. Since the lead tensors were derived from ideally sensing connections, it can be deduced from the second line of Table 3, for example, that for some sort of octapole generator the a_{31} coefficient will amount to

$$a_{31} = (M^{211} + M^{112} + M^{121})/3 - M^{222}/4 - (M^{233} + M^{332} + M^{323})/12. \tag{5}$$

By this kind of manipulation heart tensors are likewise gathered into packets

TABLE 2
SECOND RANK TENSOR COMPONENTS OF SECOND DEGREE IDEAL LEAD FIELDS
(cf. Eqs. (3) and (4))

m	Series	F_{11}	F_{22}	F_{33}	F_{12}	F_{23}	F_{31}	Associated coefficient
0	cos	1	$-1/2$	$-1/2$				a_{20}
1	cos				$1/2$			a_{21}
1	sin						$1/2$	b_{21}
2	cos		$1/4$	$-1/4$				a_{22}
2	sin					$1/4$		b_{22}

TABLE 3

THIRD RANK TENSOR COMPONENTS OF THIRD DEGREE IDEAL LEAD FIELDS
(cf. Eqs. (3) and (4))

m	Series	F_{111}	F_{122}	F_{133}	F_{211}	F_{222}	F_{233}	F_{311}	F_{322}	F_{333}	F_{123}	Associated coefficient
0	cos	1	$-1/2$	$-1/2$								a_{30}
1	cos				$1/3$	$-1/4$	$-1/12$					a_{31}
1	sin							$1/3$	$-1/12$	$-1/4$		b_{31}
2	cos		$1/12$	$-1/12$								a_{32}
2	sin										$1/12$	b_{32}
3	cos					$1/24$	$-1/24$					a_{33}
3	sin								$1/24$	$-1/24$		b_{33}

which express their respective contributions to individual generator components. The necessary permutations of the indices are included in the above expression. With two identical indices there are three permutations, as shown. With all three indices different there would be six permutations.

The hexadecapolar behavior of eccentric, lower-degree generators may now be determined. As an illustrative example we will take the case of the a_{41} coefficient. The ideal lead field for this coefficient is to be found in the cosine series of Eq. (4), with $n = 4$, and $m = 1$. Transformed to Cartesian coordinates, this expression becomes

$$u = xz[z^2 - 3/4\ (x^2 + y^2)]. \tag{6}$$

Eccentric Dipole

The first-rank lead tensors are, by Eq. (3),

$$\begin{aligned} F_1 &= 3x[z^2 - (x^2 + y^2)/4], \\ F_2 &= z[z^2 - 3(3x^2 + y^2)/4], \\ F_3 &= -3/2\ xyz. \end{aligned} \tag{7}$$

From Eq. 1 and Table 1 it follows directly that

$$a_{41} = F_1 a_{10} + F_2 a_{11} + F_3 b_{11}, \tag{8}$$

where a_{10}, a_{11}, and b_{11} are the Z-, X-, and Y-oriented components of dipole moment.

Eccentric Quadripole

The second-rank lead tensors, by Eq. (3), are

$$\begin{aligned} F_{11} &= 3xz, \\ F_{22} &= -9/4\ xz, \\ F_{33} &= -3/4\ xz, \\ F_{12} &= 3[z^2 - (3x^2 + y^2)/4]/2 \\ F_{13} &= -3/4\ xy, \\ F_{23} &= -3/4\ yz. \end{aligned} \tag{9}$$

Note that the sum of the F_{ii} is zero, as would be expected in dealing with a conservative field. F_{21}, F_{31}, and F_{32} are not given because of lead tensor symmetry. Gathering terms together in accordance with the groupings shown in Table 2 gives

$$a_{41} = 3xz(a_{20} - a_{22}) + 3[z^2 - (3x^2 + y^2)/4]a_{21} - 3xyb_{21}/2 - 3yzb_{22}, \tag{10}$$

where the a_{2m} and b_{2m} are the coefficients of an eccentric quadripole located at x, y, z.

Eccentric Octapole

Proceeding according to $F_{ijk} = 1/3\ \partial F_{ij}/\partial x^k$

$$\begin{aligned} F_{111} &= x, \\ F_{122} &= -3x/4 \\ F_{133} &= -x/4, \end{aligned} \tag{11}$$

$$
\begin{aligned}
F_{211} &= z,\\
F_{222} &= -3z/4,\\
F_{233} &= -z/4,\\
F_{311} &= F_{322} = F_{333} = 0,\\
F_{123} &= -y/4.
\end{aligned}
$$

Third-rank tensor terms may now be gathered together according to the grouping shown in Table 3, giving

$$a_{41} = x(a_{30} - 3a_{32}) + 3za_{31} - 3yb_{32}, \tag{12}$$

where the a_{3m} and b_{3m} are generator coefficients of an eccentric octapole located at x, y, z.

The remaining a_{4m} and b_{4m} were similarly determined for current signularities of less than fourth degree. The resulting formulations are listed in the appendix. As can be seen, manifest hexadecapolar behavior emerges as first-order moments of octapolar coefficients, second-order moments of quadripolar coefficients, and third-order moments of dipolar components. In order to validate the derivations, as well as to guard against inadvertent error, all orders of dipole, quadripole, and octapole sources were successively placed (numerically) at a representative eccentric location within a Gaussian sphere of unit radius. Resulting voltages were computed for several hundred locations over the spherical reference surface, and, from these, even-degree multipole coefficients through the eighth degree were inversely determined by least-squares solution of the several hundred equations which were derived from the above information. Truncation error in series representation of the various current sources was minimized by limiting eccentricity to less than 20%. The fourth-degree (hexadecapolar) coefficients, thus obtained, did not differ materially from the values predicted by the shift equations, thus, serving to confirm their accuracy.

DISCUSSION

In electrocardiography the cardiac region is commonly treated as the site of a time-varying distribution of active electrical sources. For regions external to a Gaussian sphere of radius, R, which contains all of the active sources, field potentials may be efficiently and uniquely represented by the multipole expansion

$$v = \frac{1}{4\pi\gamma} \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^{n+1} Y_n(\theta,\phi);\ Y_n(\theta,\phi) = a_{n0}P_n^{0}(\cos\theta)$$

$$+ \sum_{m=1}^{n} (a_{nm} \cos m\phi + b_{nm} \sin m\phi)\ P_n^{m}\ (\cos\theta);\ r > R, \tag{13}$$

where r, θ, ϕ are conventional spherical coordinates, v is field potential external to the Gaussian reference sphere, γ is the specific conductivity of the medium, the $Y_n(\theta,\phi)$ are surface spherical harmonics, and the a_{nm} and b_{nm} are equivalent generator coefficients. The medium is implicitly treated as being homogeneous and isotropic, at least external to the surface $r = R$. Because net flux is zero, Eq. (13) contains no unipole term, and the series begins with $n = 1$ and its corresponding three dipole terms. There are five quadripole terms for $n = 2$, seven octapole terms for $n = 3$, etc. Beginning with the dipole terms, field potentials are attenuated, respectively, according to the inverse square, cubic, quartic, etc. laws.

Despite the compactness and generality of Eq. (13), it is still not known to what degree the series formulation would have to be carried to avoid loss of significant field information through truncation error. It seems clear from previous experimental observation (Horan *et al.*, 1963) that a single fixed-location dipole, or even a single moving dipole, cannot reliably account for the distribution of field potentials over a conductor surface. Clinical investigation of the effect of removing dipolar content from the surface potential distribution found in maps of normal humans has yielded large residual signals which have not, as yet, been characterized as to the fractional content of higher order components (Horan *et al.*, 1972). Laboratory experience with isolated rabbit and turtle hearts contained within an accurately fabricated, electrolyte-filled spherical chamber indicates that a significant increment of field information is gained by quantitative evaluation of octapolar content (Brody *et al.*, 1971). Even in this favorable situation, however, some of the preparations show appreciable amounts of greater-than-octapole residual signal. We, therefore, believe that evaluation of the next degree of current singularity, namely the hexadecapole, may prove rewarding. In anticipation of the need to process such information we have derived, and present herein, the electrocardiographic shift equations which pertain to the hexadecapole.

APPENDIX—HEXADECAPOLAR SHIFT EQUATIONS[2]

For Eccentric Dipole

$$a_{40} = 2z(2z^2 - 3x^2 - 3y^2)a_{10} - 3/2\ (4z^2 - x^2 - y^2)(xa_{11} + yb_{11}),$$
$$a_{41} = 3x(z^2 - x^2/4 - y^2/4)a_{10} + z(z^2 - 9/4\ x^2 - 3/4\ y^2)a_{11} - 3/2\ xyz\ b_{11},$$
$$b_{41} = 3y(z^2 - x^2/4 - y^2/4)a_{10} - 3/2\ xyz\ a_{11} + z(z^2 - 3/4\ x^2 - 9/4\ y^2)b_{11},$$
$$a_{42} = z/2\ (x^2 - y^2)a_{10} + x/6\ (3z^2 - x^2)a_{11} - y/6\ (3z^2 - y^2)b_{11},$$
$$b_{42} = xyz\ a_{10} + y/2\ (z^2 - x^2/2 - y^2/6)a_{11} + x/2\ (z^2 - x^2/6 - y^2/2)b_{11},$$
$$a_{43} = x/24\ (x^2 - 3y^2)a_{10} + z/8\ (x^2 - y^2)a_{11} - xyz/4\ b_{11},$$
$$b_{43} = y/24\ (3x^2 - y^2)a_{10} + xyz/4\ a_{11} + z/8\ (x^2 - y^2)b_{11},$$
$$a_{44} = x/48\ (x^2 - 3y^2)a_{11} - y/48\ (3x^2 - y^2)b_{11},$$
$$b_{44} = y/48\ (3x^2 - y^2)a_{11} + x/48\ (x^2 - 3y^2)b_{11}.$$

For Eccentric Quadripole

$$a_{40} = 3(2z^2 - x^2 - y^2)a_{20} - 12xz\ a_{21} - 12yz\ b_{21} + 3(x^2 - y^2)a_{22} + 6xy\ b_{22},$$
$$a_{41} = 3xz(a_{20} - a_{22}) + 3(z^2 - 3x^2/4 - y^2/4)a_{21} - 3xy/2\ b_{21} - 3yz\ b_{22},$$
$$b_{41} = 3yz(a_{20} + a_{22}) - 3xy/2\ a_{21} + 3(z^2 - x^2/4 - 3y^2/4)b_{21} - 3xz\ b_{22},$$

[2] In accord with modern practice, Z serves as the polar axis in these formulations. In the previously published octapolar shift equations (Brody, 1968) X was employed as the polar axis. The earlier equations can readily be made to conform to the present formulations by replacing x by z, y by x, and z by y.

$$a_{42} = (x^2 - y^2)/4\ a_{20} + z(xa_{21} - yb_{21}) + (z^2 - x^2/2 - y^2/2)a_{22},$$
$$b_{42} = xy/2\ a_{20} + z(ya_{21} + xb_{21}) + (z^2 - x^2/2 - y^2/2)b_{22},$$
$$a_{43} = (x^2 - y^2)/8\ a_{21} - xy/4\ b_{21} + z/2\ (x\ a_{22} - y\ b_{22}),$$
$$b_{43} = xy/4\ a_{21} + (x^2 - y^2)/8\ b_{21} + z/2\ (y\ a_{22} + x\ b_{22}),$$
$$a_{44} = (x^2 - y^2)/8\ a_{22} - xy/4\ b_{22},$$
$$b_{44} = xy/4\ a_{22} + (x^2 - y^2)/8\ b_{22}.$$

For Eccentric Octapole

$$a_{40} = 4z\ a_{30} - 6x\ a_{31} - 6y\ b_{31},$$
$$a_{41} = x(a_{30} - 3a_{32}) + 3z\ a_{31} - 3y\ b_{32},$$
$$b_{41} = y(a_{30} + 3a_{32}) + 3z\ b_{31} - 3x\ b_{32},$$
$$a_{42} = 2z\ a_{32} + x/2\ (a_{31} - 2a_{33}) - y/2\ (b_{31} + 2b_{33}),$$
$$b_{42} = 2z\ b_{32} + y/2\ (a_{31} + 2a_{33}) + x/2\ (b_{31} - 2b_{33}),$$
$$a_{43} = x/2\ a_{32} - y/2\ b_{32} + z\ a_{33},$$
$$b_{43} = y/2\ a_{32} + x/2\ b_{32} + z\ b_{33},$$
$$a_{44} = x/2\ a_{33} - y/2\ b_{33},$$
$$b_{44} = y/2\ a_{33} + x/2\ b_{33}.$$

REFERENCES

BRODY, D. A., AND ROMANS, W. E. A model which demonstrates the quantitative relationship between the electromotive forces of the heart and the extremity leads. *American Heart Journal* 1953, **45**, 263–276.

BRODY, D. A., BRADSHAW, J. C., AND EVANS, J. W. The elements of an electrocardiographic lead tensor theory. *Bulletin of Mathematical Biophysics* 1961, **23**, 31–42.

BRODY, D. A. The inverse determination of simple generator configurations from equivalent dipole and multipole information. IEEE Transactions on Bio-Medical Engineering 1968, **BME-15**, 106–110.

BRODY, D. A., WARR, O. S., WENNEMARK, J. R., COX, J. W., KELLER, F. W., AND TERRY, F. H. Studies of the equivalent cardiac generator behavior of isolated turtle hearts. *Circulation Research* 1971, **29**, 512–524.

BURGER, H. C., AND VAN MILAAN, J. B. Heart vector and leads, Part I. *British Heart Journal* 1946, **8**, 157–161.

GABOR, D., AND NELSON, C. V. Determination of the resultant dipole of the heart from measurements on the body surface. *Journal of Applied Physics* 1954, **25**, 413–416.

GESELOWITZ, D. B. Multipole representation for an equivalent cardiac generator. *Proceedings of the Institute of Radio Engineers* 1960, **48**, 75–79.

HORAN, L. G., FLOWERS, N. C., AND BRODY, D. A. Body surface potential distribution: Comparison of naturally and artificially produced signals as analyzed by digital computer. *Circulation Research* 1963, **13**, 373–387.

HORAN, L. G., FLOWERS, N. C., AND MILLER, C. B. A rapid assay of dipolar and extradipolar content of the human electrocardiogram. *Journal of Electrocardiology* 1972, **5**, 211–223.

MCFEE, R., STOW, R. M., AND JOHNSTON, F. D. Graphic representation of electrocardiographic leads by means of fluid mappers. *Circulation* 1952, **6**, 21–29.