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# THE EXTENSION THEORY OF HERMITIAN OPERATORS AND THE MOMENT PROBLEM

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## INTRODUCTION

This paper is dedicated to further development of the theory of generalized resolvents, preresolvent, and resolvent matrices of a Hermitian operator $A$ on a separable Hilbert space $\mathfrak{h}$. The role of the formula of generalized resolvents in the extension theory was expounded (on the condition that the defect numbers are equal to unity) by Naimark [55] and Krein [32], who showed that it contains as a corollary the well-known Nevanlinna formula of the moment problem. Krein [32] introduced the notion of the resolvent matrix. These notions and the results obtained at first for the case of a Hermitian operator with the defect numbers $(1, 1)$ were generalized and developed afterward in the papers of these authors and their followers in the case of more general classes of operators and spaces. Simultaneously, the range of problems solved within the theory was extended.

In content, the given paper complements the papers of Krein et. al. [32–43, 58, 81–84] and continues our previous investigations on the extension theory of a Hermitian operator $A$ whose domain $\mathfrak{D}(A)$ is dense in $\mathfrak{h}$. We consider extensions of a nondensely defined operator $A$, and this enables us to extend the range of applicability of the operator approach to classical interpolation and boundary problems. We use systematically an abstract version of the Green identity formalized in the concept of a boundary-value space (in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ see [12, 13, 18, 6, 79].

The paper, with the exception of Sec. 7, is devoted to self-adjoint extensions of the operator $A$. Note, however, that even the investigation of $\mathfrak{L}$-resolvent matrices of a Hermitian operator $A$ with a dense domain $\mathfrak{D}(A)$ leads to the consideration of some nondensely defined Hermitian operator as well as non-self-adjoint extensions of the latter and their characteristic functions.

In Sec. 1, which is preparatory, we present the necessary facts concerning linear relations, some classes of $R$-functions, and some propositions on the extension theory of a nondensely defined Hermitian operator in a Hilbert space $\mathfrak{h}$. Here we recall the notion of a boundary-value space (BVS) $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ of a nondensely defined Hermitian operator $A$, the Weyl function $M(\lambda)$, and the forbidden manifold $\mathcal{F}_\Pi$ corresponding to the BVS $\Pi$, and also give some statements of the extension theory from [53, 18, 79]. We mention only some elementary ones, which we need to describe the main results of the paper:

(1) the mapping $\Gamma = \{\Gamma_2, \Gamma_1\} : A^* \to \mathcal{H} \oplus \mathcal{H}$ defines the bijective correspondence between the set of proper extensions $\tilde{A}$ $(A \subset \tilde{A} \subset A^*)$ of the operator $A$ and the set of linear relations $\theta$ in $\mathcal{H}$:

$$\tilde{A} = \tilde{A}_\theta \leftrightarrow \theta = \Gamma\tilde{A} = \{\{\Gamma_2\hat{f}, \Gamma_1\hat{f}\} : \hat{f} = \{f, f'\} \in \tilde{A} \subset A^*\}; \tag{0.1}$$

if $\theta = B$ is an operator, then relation (0.1) takes its usual form

$$\tilde{A}_\theta = \tilde{A}_B = \ker(\Gamma_1 - B\Gamma_2); \tag{0.2}$$

(2) the forbidden manifold $\mathcal{F}_\Pi = \Gamma\{0, \mathfrak{N}\}$, where $\mathfrak{N} = \mathfrak{D}(A)^\perp$ corresponds in formula (0.1) to the Hermitian extension of $A$

$$\tilde{A}_{\mathcal{F}_\Pi} = A \dotplus \hat{\mathfrak{N}} \qquad (\hat{\mathfrak{N}} = \{0, \mathfrak{N}\});$$

(3) an extension $\tilde{A}_\theta$ is an operator if and only if $\mathcal{F}_\Pi \cap \theta = \{0\}$. Even the two last statements demonstrate the utility of the forbidden manifold $\mathcal{F}_\Pi$, which plays an essential part in the extension theory.

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In Sec. 2 we investigate $\mathfrak{N}$-regular linear relations, that is, a closed linear relation $T$ such that the linear manifold $T+\hat{\mathfrak{N}}$ is closed. In the case of proper extensions $T=\tilde{A}(\supset A)$ of the operator $A$ and $\mathfrak{N}=\mathfrak{D}(A)^{\perp}$ this definition is equivalent to the definition of a regular extension of $A$ [31, 64], whose role is well known [60]. We find new criteria for an extension $\tilde{A}$ to be $\mathfrak{N}$-regular, and here we mention the following one: the linear manifold $P_{\mathfrak{N}}\mathfrak{D}(\tilde{A}^*)$ is closed in $\mathfrak{N}$. The $\mathfrak{N}$-regularity criteria of an extension $\tilde{A}$ in terms of the forbidden manifold $\mathcal{F}_{\Pi}$ and the Weyl function $M(\lambda)$ are obtained.

In Sec. 3 the generalized resolvents of a nondensely defined Hermitian operator $A$ ($\overline{\mathfrak{D}(A)}=\mathfrak{h}_0\subset\mathfrak{h}$) are studied. The operator-valued function $\mathbf{R}_\lambda=P(\tilde{A}-\lambda)^{-1}\restriction\mathfrak{h}$ is a generalized resolvent if $\tilde{A}=\tilde{A}^*$ is a self-adjoint extension of the operator $A$, acting in a Hilbert space $\tilde{\mathfrak{h}}\supset\mathfrak{h}$, $P$ is the orthogonal projection of $\tilde{\mathfrak{h}}$ onto $\mathfrak{h}$. Here the following analog of the Krein formula for resolvents is obtained:

$$\mathbf{R}_\lambda=(A_2-\lambda)^{-1}-\gamma(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\bar{\lambda}), \tag{0.3}$$

and all the parameters from (0.3) are expressed in terms of BVS:

$$\hat{\gamma}(\lambda):=\{\gamma(\lambda),\lambda\gamma(\lambda)\}=(\Gamma_2\restriction\hat{\mathfrak{N}}_\lambda)^{-1},\quad M(\lambda):=\Gamma_1\hat{\gamma}(\lambda),\qquad \tau(\lambda)=-\Gamma(\mathbf{R}_\lambda^{-1}+\lambda), \tag{0.4}$$

where $A_2=\ker\Gamma_2$, $M(\lambda)$ is a Weyl function, $\mathfrak{N}_\lambda$ is a defect subspace of the operator $A$. As follows from (0.4) and (0.1), $\mathbf{R}_\lambda=(\tilde{A}_{-\tau(\lambda)}-\lambda)^{-1}$, that is, for all $h\in\mathfrak{h}$, $\mathbf{R}_\lambda h$ is a solution of some "boundary-value problem" for $A^*$ with the spectral parameter $\tau(\lambda)$ in the boundary condition, which coincides with (0.1) in the case $\tau(\lambda)\equiv-\theta=\text{const}$.

Formula (0.3), as well as the Krein formula, establishes a bijective correspondence between the set of generalized resolvents $\{\mathbf{R}_\lambda\}$ and the class $\tilde{R}_{\mathcal{H}}$ of Nevanlinna functions, completed by the families of Nevanlinna linear relations. However, in contrast to the case of a densely defined Hermitian operator $A$, a self-adjoint relation $\tilde{A}$ generating the generalized resolvent $\mathbf{R}_\lambda$ may be either an operator or a linear relation. We find additional hypotheses about $\tau(\lambda)$ ($M$-admissibility conditions) for the corresponding generalized resolvent to be generated by an operator. They take the form

$$\exists\lim_{y\uparrow\infty} iy((\tau(iy)+M(iy))^{-1}h,h)=\lim_{y\uparrow\infty} iy(M^{-1}(iy)h,h)\quad\forall h\in V_\Pi(0); \tag{0.5}$$

$$s-\lim_{y\uparrow\infty} y^{-1}(\tau(iy)+M(iy))^{-1}=0. \tag{0.6}$$

If $A_2$ is an operator ($\Longleftrightarrow\mathcal{F}_\Pi(0)=\{0\}$), the $M$-admissibility conditions may be reduced to the single condition (0.6), which was obtained earlier in [51, 53]. Conversely, in the case $A_2=A\dot{+}\hat{\mathfrak{N}}$ ($\Longrightarrow\mathcal{F}_\Pi(0)=\mathcal{H}$) conditions (0.5), (0.6) are equivalent to the following one:

$$s-R-\lim_{y\uparrow\infty} y^{-1}\tau(iy)=0. \tag{0.7}$$

The last condition coincides in the scalar case $\dim\mathfrak{N}=\dim\mathcal{H}=1$ with the well-known Nevanlinna condition of the moment problem.

The formulas for generalized resolvents of a bounded Hermitian operator and a Hermitian contraction proved differently in [70] and [40] are obtained here as corollaries of relations (0.3)–(0.7).

We also thoroughly study the $M$-admissibility conditions. In particular, the criterion for an operator-valued function $\tau(\lambda)$ with values in $[\mathcal{H}]$ to be $M$-admissible is given in terms of the limit operator $\tau(i\infty)$.

In Sec. 4 the resolvent formula (0.3) is used to describe (in terms of abstract boundary conditions) the extensions of the operator $A$ with a gap $(\alpha,\beta)$, which bring to the gap a finite number of discrete levels, as well as the extensions preserving the gap. These results are analogous to those obtained in [20, 21, 79] for the case $\overline{\mathfrak{D}(A)}=\mathfrak{h}$. In particular, the extensions with a finite negative spectrum of a nonnegative operator $A\geq 0$ are described. It should be emphasized that just the existence of two forms (0.1), (0.3) of the description of self-adjoint extensions $\tilde{A}_\theta=\tilde{A}_\theta^*$ makes it possible to apply formula (0.3) not only to the classical problems (of the type of the moment problem), but also to boundary-value problems (see [20, 26, 50–52, 79]). We also describe the generalized resolvents of the operator $A$ with the gap $(\alpha, \beta)$, which are generated by self-adjoint extensions $\tilde{A}$ in $\tilde{\mathfrak{h}}(\supset \mathfrak{h})$ with the property stated above.

In Sec. 5 it is shown that each $R$-function $Q(\lambda)$ satisfying the unique condition $0 \in \rho(\operatorname{Im} Q(i))$ is a Weyl function of a Hermitian operator $A$ whose domain is, generally speaking, nondense in $\mathfrak{h}$. We propose three distinct proofs of this fact, connected with the three known functional models [71, 72, 75, 81] of a Hermitian operator $A$, which are constructed with the help of the function $Q(\lambda)$. Within each of these models we find a BVS for the linear relation $A^*$ such that the corresponding Weyl function coincides with $Q(\lambda)$. The inner description of $Q_\mu$- and $Q_M$-functions of Hermitian contractions [40] as well as of nonnegative operators [41] and also the description of spectral complements (in the sense of [69]) of bounded Hermitian operators are given here as corollaries of Theorem 5.1.

The presence of the condition $0 \in \rho(\operatorname{Im} Q(i))$ in Theorem 5.1 impelled us to consider in Sec. 6 generalized BVS's for nonclosed linear relations. Such a consideration allows us to omit the requirement $0 \in \rho(\operatorname{Im} Q(i))$ in the assumptions of the preceding theorem on the realization of the $R$-function. The utility of generalized BVS's becomes clear below in studying the inverse problems for $Q_\mu$- and $Q_M$-functions, for characteristic operator-valued functions (Sec. 7), preresolvent and resolvent matrices (Secs. 8, 9).

In Sec. 7, in view of the needs of Sec. 8, we slightly deviate from the main "self-adjoint" direction of the article. Here we introduce the class $\mathcal{A}s$ of almost solvable linear relations, which contains linear relations with two regular points $\lambda_1$, $\lambda_2$ such that $\operatorname{Im} \lambda_1 \cdot \operatorname{Im} \lambda_2 < 0$, and linear relations with a real regular point, in particular, bounded operators. We define the characteristic functions (CF) of a linear relation $T \in \mathcal{A}s$ and show (Theorem 7.3) that they exhaust the class of $J$-contractive (holomorphic on $\mathbb{C}_+$) operator-valued functions $W(\lambda)$ acting in a finite-dimensional space and a wide class of such functions in an infinite-dimensional one. In the latter case an additional hypothesis on $W(\lambda)$ is formulated in terms of rigged Hilbert spaces.

Each linear relation $T$ may be considered as a proper extension of its Hermitian part $A$. The linear relation $T$ of the class $\mathcal{A}s(A) \subset \mathcal{A}s$ is characterized by the existence of a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ for $A^*$ such that $T$ is related to a bounded linear operator $B \in [\mathcal{H}]$ via Eq. (0.2). Although the class $\mathcal{A}s(A)$ does not exhaust the class $\operatorname{Ex}_A$ of all proper extensions of $A$, it turns out to be very natural. Thus, in Theorem 7.1 we obtain an explicit formula for the calculation of the CF $W_T(\lambda)$ of the linear relation $T = \tilde{A}_B (\in \mathcal{A}s(A))$ in terms of the Weyl function $M(\lambda)$ and the boundary operator $B (\in [\mathcal{H}])$:

$$W_T(\lambda) = I + 2iK^*(B^* - M(\lambda))^{-1}KJ \quad (\operatorname{Im} B = KJK^*). \tag{0.8}$$

In the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ this formula was found by the authors in [18, 26] and, for a bounded operator $T$ it coincides with the definition of a characteristic function due to Livšic [48, 8, 9] ($M(\lambda) = \lambda I$ if $A = \{0\}$).

Precisely formula (0.8) enables us to prove for the operators (and linear relations) of the class $\mathcal{A}s$ the theorem on unitary equivalence, the multiplication theorem, and to solve the inverse problem of the theory of CF in complete analogy with the corresponding results of [48, 8, 9]. This formula is implied by the formula for resolvents (0.3) and then, in Sec. 8, is applied essentially to the proof of Theorem 8.3.

In Sec. 8, preresolvent and resolvent matrices of a Hermitian operator $A$ are investigated. These objects were introduced by Krein [32] in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, $n_\pm(A) = 1$, and naturally arise in the operator approach to classical problems of analysis of the type of the moment problem. Indeed, the set of solutions of a number of classical problems coincides with the set of $\mathfrak{L}$-resolvents of a Hermitian operator $A$, that is, with the operator-valued functions of the form $P_\mathfrak{L}\mathbf{R}_\lambda \restriction \mathfrak{L}$, where $\mathbf{R}_\lambda$ is a generalized resolvent of the operator $A$, $\mathfrak{L}$ is a subspace of $\mathfrak{h}$ ($\dim \mathfrak{L} = n_\pm(A)$), called a module one. The description of all $\mathfrak{L}$-resolvents of the operator $A$ is given by the equality

$$P_\mathfrak{L}\mathbf{R}_\lambda \restriction \mathfrak{L} = [w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda)][w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda)]^{-1}, \tag{0.9}$$

which easily follows from the formula of resolvents (0.3). The matrix-valued function

$$W_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{pmatrix}$$

is called a $\Pi\mathfrak{L}$-resolvent matrix of the operator $A$ corresponding to the BVS $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$. $W_{\Pi\mathfrak{L}}(\lambda)$ is related via a simple equality to the other main object of the theory, the $\Pi\mathfrak{L}$-preresolvent matrix of the operator $A$:

$$\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} M(\lambda) & \gamma^*(\bar\lambda)\restriction\mathfrak{L} \\ P_{\mathfrak{L}}\gamma(\lambda) & P_{\mathfrak{L}}(A_2-\lambda)^{-1}\restriction\mathfrak{L} \end{pmatrix} \tag{0.10}$$

It is useful to consider a nondensely defined Hermitian operator in studying the matrix-functions $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ and $W_{\Pi\mathfrak{L}}(\lambda)$, even in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$:

$$A_0 = A\restriction\mathfrak{L}^\perp. \tag{0.11}$$

If a linear relation $A^*$ is $\mathfrak{L}$-regular, then the operator $A_0$ has a number of "good" properties: $\mathfrak{D}(A_0) = \mathfrak{L}^\perp$, $A_0^* = A^* + \hat{\mathfrak{L}}$, the linear relation $A^*\restriction\mathfrak{L}^\perp = (A+\hat{\mathfrak{L}})^*$ is almost solvable, and, finally,

$$P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}. \tag{0.12}$$

Condition (0.12) is equivalent (see Sec. 2) to the $\mathfrak{L}$-regularity condition for $A^*$. If this condition is fulfilled (this is true, in particular, if $n_+(A) = n_-(A) < \infty$), then a $\Pi\mathfrak{L}$-preresolvent matrix turns out to be a Weyl function of the Hermitian operator $A_0$ corresponding to a some specific BVS for the linear relation $A_0^*$, and $W_{\Pi\mathfrak{L}}(\lambda)$ proves to be a CF of the linear relation $A^*\restriction\mathfrak{L}^\perp = (A+\hat{\mathfrak{L}})^* \in \mathcal{A}s(A_0)$.

One can easily deduce the last statement from formula (0.8) on account of the fact that $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ is a Weyl function of the operator $A_0$. We derive from the same formula (0.8) the following one

$$W_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} -\Gamma_2\hat{Q}^*(\lambda) & \Gamma_2\hat{\mathcal{P}}^*(\lambda) \\ -\Gamma_1\hat{Q}^*(\lambda) & \Gamma_1\hat{\mathcal{P}}^*(\lambda) \end{pmatrix}^* \tag{0.13}$$

for the calculation of $W_{\Pi\mathfrak{L}}(\lambda)$ in terms of the operator-valued functions $\mathcal{P}(\lambda)$ and $Q(\lambda)$, which are abstract analogs of polynomials of the first and second kind [here $\mathcal{P}(\lambda)$ is a skew projection onto $\mathfrak{L}$ in the decomposition $\mathfrak{h} = (A-\lambda)\mathfrak{D}(A)\dot{+}\,\mathfrak{L}$, $Q(\lambda) = P_{\mathfrak{L}}(A-\lambda)^{-1}(I-\mathcal{P}(\lambda))$]. In the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ formula (0.13) was proved by the authors in [20, 21, 79]. Note also that in the proof of (0.13) we used the expression for the Weyl function [see (8.16)] implied by the formula

$$A^* = A\dot{+}\,\hat{\mathcal{P}}^*(\lambda)\mathfrak{L}\dot{+}\,\hat{Q}^*(\lambda)\mathfrak{L}, \tag{0.14}$$

which we can consider as an analog of the first Neumann formula.

Section 9 is dedicated to inverse problems for preresolvent and resolvent matrices of an operator $A$. The necessary and sufficient conditions for a holomorphic operator-valued function $F(\lambda)$ to be a preresolvent or resolvent matrix of a Hermitian operator are found.

Finally, in Sec. 10 we apply the results from Secs. 3 and 4 to the truncated Hamburger, Stieltjes, and Hausdorff moment problems. The truncated Hamburger moment problem consists in the following [4, 6, 38]: given a sequence $\{s_k\}_0^{2n}$ of real numbers, find the necessary and sufficient conditions on $\{s_k\}_0^{2n}$ which ensure the existence of a nonnegative measure $d\sigma(t)$ such that the following representation holds

$$s_k = \int_{\mathbb{R}} t^k d\sigma(t) \quad (0 \le k \le 2n-1), \qquad s_{2n} = \int_{\mathbb{R}} t^{2n} d\sigma(t) + m, \tag{0.15}$$

with some $m \ge 0$, and describe the set $\tilde{V}(\mathbf{s};\mathbf{R})$ of all solutions $\sigma(t)$.

The number $m = s_{2n} - \int_{\mathbb{R}} t^{2n} d\sigma(t)$ is called the mass at infinity [38]. In the framework of the operator approach the description of the set $\tilde{V}(\mathbf{s},\mathbf{R})$ is reduced to the description of the set of $\mathfrak{L}$-spectral ($\mathfrak{L} = \{\mathbb{I}\}$) functions $\sigma(t) = (E_A(t)\mathbb{I}\mathbb{I}, 1)$ of a nondensely defined operator $A$ and is given by formula (0.9), which coincides in this case with the Nevanlinna formula. The appearance of the number $m$ in the truncated moment problem is implied by the existence of the linear relations $\tilde{A} = \tilde{A}^*$, which are extensions of the nondensely defined Hermitian operator $A$ and $m = 0$ if and only if $\sigma(t)$ is generated by an operator $\tilde{A}$. In other words, the function $\tau(\lambda)$ from equality (0.9) is $M$-admissible if and only if the corresponding measure $d\sigma(t)$ satisfies the equality $s_{2n} = \int_{\mathbb{R}} t^{2n} d\sigma(t)$.

Here we obtain the criterion for a solution of the problem (0.13), which has no mass on some intervals $(\alpha_j, \beta_j)$, $1 \le j \le m$, and we give a description of these solutions.

The results of the paper were partially announced in [24, 27, 28].

## 1. PRELIMINARIES

**1.** Let $\mathcal{H}$ be a separable Hilbert space. A linear relation $T$ in $\mathcal{H}$ is a linear manifold $T$ in $\mathcal{H} \oplus \mathcal{H}$. We denote by $\widetilde{C}(\mathcal{H})$ the set of closed linear relations in $\mathcal{H}$ (i.e., closed subspaces $T \subset \mathcal{H} \oplus \mathcal{H}$). For linear relations $T, S \in \widetilde{C}(\mathcal{H})$ we put (see [77–80])

$$\mathfrak{D}(T) = \{f \in \mathcal{H} : \exists f' \in \mathcal{H}, \{f, f'\} \in T\};$$

$$\mathfrak{R}(T) = \{f' \in \mathcal{H} : \exists f \in \mathcal{H}, \{f, f'\} \in T\}; \qquad \ker T = \{f \in \mathcal{H} : \{f, 0\} \in T\};$$

$$T(0) = \{f' \in \mathcal{H} : \{0, f'\} \in T\}; \qquad \widehat{T}(0) = \{\{0, f'\} : f' \in T(0)\};$$

$$\alpha T = \{\{f, \alpha f'\} : \{f, f'\} \in T\}; \qquad T^{-1} = \{\{f', f\} : \{f, f'\} \in T\};$$

$$T + S = \{\{f, f' + g'\} : \{f, f'\} \in T, \{f, g'\} \in S\};$$

$$ST = \{\{f, g\} : \exists f' \in \mathcal{H}\, \{f, f'\} \in T, \{f', g\} \in S\};$$

$$T^* = \{\{g, g'\} : (f', g) = (f, g') \qquad \forall \{f, f'\} \in T\}.$$

For a linear relation $T \in \widetilde{C}(\mathcal{H})$ we define the resolvent set $\rho(T)$ by

$$\rho(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) = \{0\},\ \mathfrak{R}(T - \lambda) = \mathcal{H}\},$$

the spectrum $\sigma(T) = \mathbb{C} \setminus \rho(T)$, and we give the classification of the spectrum in the following way:

$\sigma_c(T) = \{\lambda \in \sigma(T) : \ker(T - \lambda) = \{0\}, \overline{\mathfrak{R}(T - \lambda)} = \mathcal{H}\}$ — the continuous spectrum;

$\sigma_p(T) = \{\lambda \in \sigma(T) : \ker(T - \lambda) \neq \{0\}\}$ — the point spectrum;

$\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T))$ — the residual spectrum of $T$.

The set $\omega(T) = \{(f', f) : \{f, f'\} \in T\}$ is the numerical range of the relation $T$. A linear relation $T \in \widetilde{C}(\mathcal{H})$ is said to be Hermitian if $\omega(T) \subset \mathbb{R}$ (i.e., $T \subset T^*$); nonnegative if $\omega(T) \subset \mathbb{R}_+$; dissipative if $\omega(T) \subset \overline{\mathbb{C}}_+$ ($\mathbb{C}_+ := \{\lambda : \text{Im}\,\lambda > 0\}$). A Hermitian (dissipative) relation $T$ is said to be self-adjoint (maximal dissipative) if there does not exist an extension of $T$ in the same class, or equivalently if $\rho(T) \neq \varnothing$.

We denote the set of closed (bounded) linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ by $C(\mathcal{H}_1, \mathcal{H}_2)([\mathcal{H}_1, \mathcal{H}_2])$; in the case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ we put $C(\mathcal{H}) := C(\mathcal{H}, \mathcal{H})$, $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$. We regard $C(\mathcal{H})$ as a subset of $\widetilde{C}(\mathcal{H})$ identifying an operator $T \in C(\mathcal{H})$ with its graph gr $T = \{\{f, Tf\} : f \in \mathfrak{D}(T)\} \in \widetilde{C}(\mathcal{H})$. Let $P_{\mathcal{H}'}$ be an orthogonal projection onto subspace $\mathcal{H}' \subset \mathcal{H}$.

**Proposition 1.1** ([57]). *Let $T \in \widetilde{C}(\mathcal{H})$ and $\omega(T) \subset \{\lambda\colon \text{Im}\,(\lambda e^{i\varphi_0}) \ge 0\}$ ($\varphi_0 \in [0, 2\pi]$). Then $T(0) \perp \mathfrak{D}(T)$ and the following decomposition holds: $T = T' \oplus \widehat{T}(0)$, where $T' \in C(\mathcal{H}')$, $\mathcal{H}' = \mathcal{H} \ominus T(0)$.*

In particular, if $T$ is a self-adjoint (dissipative) linear relation, then its operator part $T$ possesses the same property and $\sigma(T') = \sigma(T)$. The spectral function of a self-adjoint linear relation $T$ is defined by the equality $E_T(\lambda) := E_{T'}(\lambda) P_{\mathcal{H}'}$.

**2. Definition 1.2** ([29]). *A family of linear relations $\tau(\lambda) \in \widetilde{C}(\mathcal{H})$ is said to be holomorphic on $\lambda \in \mathbb{C}$ if there exist a space $\mathcal{H}_1$, and operator-valued functions $\Phi(\lambda)$, $\Psi(\lambda)$ with values in $[\mathcal{H}_1, \mathcal{H}]$ holomorphic on $\lambda$ such that $\tau(\lambda)$ admits the representation*

$$\tau(\lambda) = \{\{\Phi(\lambda)h, \Psi(\lambda)h\} : h \in \mathcal{H}_1\}. \tag{1.1}$$

The kernel $k(\lambda, \mu)$, which is defined on $G \times G$ and takes values in $[\mathcal{H}]$, is said to have $\varkappa$ negative squares on $G$ if $k(\lambda, \mu) = k^*(\mu, \lambda)$ and for each $n \in \mathbb{Z}_+$ and all choices of $\lambda_j \in G$ and $h_j \in \mathcal{H}$, $j = 1, 2, \ldots, n$, the Hermitian $n \times n$-matrix $((k(\lambda_i, \lambda_j)h_i, h_j)_{i,j=1}$ has at most $\varkappa$ (and for at least one collection of $n$, $\lambda_j$, $h_j$ $(1 \le j \le n)$ exactly $\varkappa$) negative eigenvalues.

**Definition 1.3** ([36, 81]). *Denote by* $\widetilde{N}_\varkappa(\mathcal{H})$ $(\varkappa \in \mathbb{Z}_+)$ *the class of families of linear relations* $\tau(\lambda) \in \widetilde{\mathcal{C}}(\mathcal{H})$ *of form* (1.1) *holomorphic on* $\mathbb{C} \setminus \mathbb{R}$ *such that* $\tau(\lambda) = \tau^*(\bar{\lambda})$, $\rho(\tau(\lambda_0)) \neq \varnothing$ *for at least one* $\lambda_0 \in \mathbb{C}_+$ *and the kernel* $k_\tau(\lambda, \mu) = \frac{\Phi^*(\mu)\Psi(\lambda) - \Psi^*(\mu)\Phi(\lambda)}{\lambda - \bar{\mu}}$ *has* $\varkappa$ *negative squares on* $\mathbb{C} \setminus \mathbb{R}$.

Let $\widetilde{R}_\mathcal{H} := \widetilde{N}_0(\mathcal{H})$ be the class of families of linear relations $\tau(\lambda)$ of form (1.1) for which the kernel $k_\tau(\lambda, \mu)$ is nonnegative on $\mathbb{C} \setminus \mathbb{R}$. We shall write $\tau(\lambda) \in R_\mathcal{H}$ if $\tau(\lambda) \in \widetilde{R}_\mathcal{H}$ and for all $\lambda \in \mathbb{C}_+$ it takes values in the set of maximal dissipative operators. The next proposition is well known.

**Proposition 1.2.** *Let* $\tau(\lambda) \in \widetilde{R}_\mathcal{H}$ *and* $\lambda_0 \in \mathbb{C}_+$. *Then:*

(1) $\alpha = \bar{\alpha} \in \rho(\tau(\lambda_0)) \Longrightarrow \alpha \in \rho(\tau(\lambda)) \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$;

(2) $\alpha = \bar{\alpha} \in \sigma_p(\tau(\lambda_0)) \Longrightarrow \alpha \in \sigma_p(\tau(\lambda)) \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$;

(3) $\rho(\tau(\lambda_0)) \neq \varnothing \Longrightarrow \rho(\tau(\lambda)) \neq \varnothing \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$;

(4) $\tau(\lambda_0) \in [\mathcal{H}] \Longrightarrow \tau(\lambda) \in [\mathcal{H}] \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$;

(5) $[\tau(\lambda_0)](0) \neq \{0\} \Longrightarrow \tau(\lambda) = T(\lambda) \oplus [\widehat{\tau(\lambda_0)}](0), \quad T(\lambda) \in R_{\mathcal{H} \ominus [\tau(\lambda_0)](0)}$.

**Theorem** (R. Nevanlinna [4–6]). *An operator-valued function* $Q(\lambda)$ *with values in* $[\mathcal{H}]$ *belongs to the class* $R_\mathcal{H}$ *if and only if it admits the following representation:*

$$Q(\lambda) = C_Q + B_Q\lambda + \int_\mathbb{R} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2}\right) d\Sigma(t), \tag{1.2}$$

with some self-adjoint operators $C_Q$, $B_Q \in [\mathcal{H}]$, $B_Q \geq 0$, and a nondecreasing function $\Sigma(t) = \Sigma(t - 0)$ with values in $[\mathcal{H}]$ such that $\int_\mathbb{R} (1 + t^2)^{-1}\, d\Sigma(t) \in [\mathcal{H}]$.

**Definition 1.4** ([38]). *Let* $-\infty \leq \alpha < \beta \leq \infty$ *and let* $E$ *be one of the following sets: either* $[\alpha, \beta]$ *or* $\mathbb{R} \setminus (\alpha, \beta)$. *We denote by* $S^\pm_\mathcal{H}(E)$ $(\widetilde{S}^\pm_\mathcal{H}(E))$ *the class of families of linear relations* $\tau(\lambda) \in R_\mathcal{H}$ $(\tau(\lambda) \in \widetilde{R}_\mathcal{H})$ *if* $\tau(\lambda)$ *admits a holomorphic continuation to* $\mathbb{R} \setminus E$ *such that* $\pm\tau(x) \geq 0$ $\forall x \in \mathbb{R} \setminus E$.

**Definition 1.5** ([19, 22]). *Let* $E = \mathbb{R} \setminus (\alpha, \beta)$; $\varkappa \in \mathbb{Z}_+$ *and let* $\omega(\lambda) = \frac{\lambda - \beta}{\lambda - \alpha}$ *for* $-\infty < \alpha < \beta < \infty$, $\omega(\lambda) = \lambda - \beta$ *for* $\alpha = -\infty$, $\omega(\lambda) = (\alpha - \lambda)^{-1}$ *for* $\beta = +\infty$. *A family of linear relations* $\tau(\lambda) \in R_\mathcal{H}$ $(\widetilde{R}_\mathcal{H}$ *is said to belong to the class* $S^{\pm\varkappa}_\mathcal{H}(E)(\widetilde{S}^{\pm\varkappa}_\mathcal{H}(E))$ *if* $\omega(\lambda)^{\pm 1}\tau(\lambda) \in \widetilde{N}_\varkappa(\mathcal{H})$.

As was shown by Krein (see [38]) the classes $S^{\pm 0}_\mathcal{H}(E)$ coincide with the classes $S^\pm_\mathcal{H}(E)$. The authors in [22, 25, 79] characterized functions $\tau(\lambda) \in S^{\pm\varkappa}_\mathcal{H}(E)$ in terms of their zeros and "poles," the number of which in $E$ does not exceed $\varkappa$.

**3.** Let $A$ be a Hermitian operator acting in a Hilbert space $\mathfrak{h}$, generally speaking, with nondense domain $\mathfrak{D}(A)$ in $\mathfrak{h}$, $\mathfrak{h}_0 = \overline{\mathfrak{D}(A)}$. We denote by $A^*$ the adjoint linear relation, $\mathfrak{N} = \mathfrak{h} \ominus \mathfrak{h}_0$, $\widehat{\mathfrak{N}} = \{0, \mathfrak{N}\} = A^*(0)$; $\mathfrak{N}_\lambda = \ker(A^* - \lambda)$ are the defect subspaces of $A$ $(\lambda \in \mathbb{C} \setminus \mathbb{R})$; $n_\pm(A) = \dim \mathfrak{N}_{\pm i}$ are the defect numbers of $A$; $\hat{\rho}(A)$ is the set of the points of regular type.

We state some relevant definitions and propositions from [53, 79].

**Definition 1.6.** *A triple* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *in which* $\mathcal{H}$ *is a separable Hilbert space and* $\Gamma_j \in [A^*, \mathcal{H}]$ $(j = 1, 2)$, *is a boundary-value space (BVS) for a linear relation* $A^*$ *if:*

(1)

$$(f', g) - (f, g') = (\Gamma_1\hat{f}, \Gamma_2\hat{g})_\mathcal{H} - (\Gamma_2\hat{f}, \Gamma_1\hat{g})_\mathcal{H} \qquad \forall \hat{f} = \{f, f'\}, \quad \hat{g} = \{g, g'\} \in A^*, \tag{1.3}$$

(2) *the mapping* $\Gamma : \hat{f} \to \{\Gamma_2\hat{f}, \Gamma_1\hat{f}\}$ *from* $A^*$ *to* $\mathcal{H} \oplus \mathcal{H}$ *is surjective.*

For a BVS for $A^*$ to exist it is necessary and sufficient that the defect numbers of the operator $A$ coincide $(n_+(A) = n_-(A))$.

**Definition 1.7.** *A closed extension* $\tilde{A}$ *of* $A$ *is said to be proper if* $A \subset \tilde{A} \subset A^*$. *We denote the set of proper extensions of* $A$ *by* $\mathrm{Ex}_A$. *Two proper extensions* $\tilde{A}'$, $\tilde{A}'' \in \mathrm{Ex}_A$ *are called disjoint if* $\tilde{A}' \cap \tilde{A}'' = A$ *and transversal if, additionally,* $\tilde{A}' + \tilde{A}'' = A^*$.

Naturally associated with each BVS are two transversal extensions $A_j = A_j^* = \ker \Gamma_j \in \mathrm{Ex}_A$ $(j = 1, 2)$. The inverse assertion also holds.

**Proposition 1.3** ([18, 53]). *Suppose that two extensions* $A_j = A_j^* \in \mathrm{Ex}_A$ $(j = 1, 2)$ *are transversal. Then there exists a BVS* $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *for* $A^*$ *such that* $A_j = \ker \Gamma_j$ $(j = 1, 2)$.

**Definition 1.8** ([51]). *The manifold* $\mathcal{F}_\Pi = \Gamma\widehat{\mathfrak{N}} = \{\{\Gamma_2\hat{n}, \Gamma_1\hat{n}\} : \hat{n} = \{0, n\} \in \widehat{\mathfrak{N}}\}$ *is a forbidden relation corresponding to BVS* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$. *A linear relation* $\theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ *will be called admissible if* $\theta \cap \mathcal{F}_\Pi = \{0\}$.

Clearly, $\mathcal{F}_\Pi$ is a Hermitian linear relation in $\mathcal{H}$.

**Proposition 1.4** ([26, 53]). *The mapping $\Gamma : \hat{f} \to \{\Gamma_2\hat{f}, \Gamma_1\hat{f}\}$ from $A^*$ to $\mathcal{H}\oplus\mathcal{H}$ is surjective and induces a topological isomorphism between $A^*/A$ and $\mathcal{H}\oplus\mathcal{H}$ such that:*

(1) *there exists a one-to-one correspondence between the set of extensions $\tilde{A} \in \operatorname{Ex}_A$ and the set of relations $\theta \in \widetilde{C}(\mathcal{H})$*

$$\tilde{A} = \tilde{A}_\theta \leftrightarrow \theta = \Gamma\tilde{A} = \{\{\Gamma_2\hat{f}, \Gamma_1\hat{f}\} : \hat{f} \in \tilde{A}\}, \tag{1.4}$$

*in this connection $A_1 \leftrightarrow \Gamma A_1 = \{\mathcal{H}, 0\}$, $A_2 \leftrightarrow \Gamma A_2 = \{0, \mathcal{H}\}$;*

(2) *$\tilde{A}_\theta$ and $A_2$ are disjoint iff $\theta \cap \{0, \mathcal{H}\} = \{0\} (\Longleftrightarrow \theta \in C(\mathcal{H}))$;*

(3) *$\tilde{A}_{\theta_1}$ and $\tilde{A}_{\theta_2}$ are transversal iff $\theta_1 \dot{+} \theta_2 = \mathcal{H}\oplus\mathcal{H} (\theta_1, \theta_2 \in \widetilde{C}(\mathcal{H}))$, in particular, $\tilde{A}_\theta$ and $A_2$ are transversal iff $\theta \in [\mathcal{H}]$;*

(4) *if $\exists\lambda \in \rho(\tilde{A}_{\theta_1}) \cap \rho(\tilde{A}_{\theta_2})$, then the transversality of $\tilde{A}_{\theta_1}$ and $\tilde{A}_{\theta_2}$ is equivalent to the condition $0 \in \rho((\tilde{A}_{\theta_1} - \lambda)^{-1} - (\tilde{A}_{\theta_2} - \lambda)^{-1})$;*

(5) *if there exists $\zeta \in \rho(\theta_1) \cap \rho(\theta_2)$, then the transversality of $\tilde{A}_{\theta_1}$ and $\tilde{A}_{\theta_2}$ is equivalent to the condition $0 \in \rho((\theta_1 - \zeta)^{-1} - (\theta_2 - \zeta)^{-1})$;*

(6) *if $\theta_1 \in \widetilde{C}(\mathcal{H})$, $\theta_2 \in [\mathcal{H}]$, then the following equivalences hold:*

$$\theta_1 \dot{+} \theta_2 = \mathcal{H}\oplus\mathcal{H} \Longleftrightarrow 0 \in \rho(\theta_1 - \theta_2),$$

$$\overline{\theta_1 \dot{+} \theta_2} = \mathcal{H}\oplus\mathcal{H} \Longleftrightarrow 0 \in \rho(\theta_1 - \theta_2) \cup \sigma_c(\theta_1 - \theta_2);$$

(7) *an extension $\tilde{A}_\theta$ is Hermitian (self-adjoint, dissipative) iff the linear relation $\theta \in \widetilde{C}(\mathcal{H})$ possesses the same property;*

(8) *$\tilde{A}_\theta \in C(\mathfrak{h})$ (i.e., $\tilde{A}_\theta$ is an operator) $\Longleftrightarrow \theta \cap \mathcal{F}_\Pi = \{0\}$;*

(9) *the extension $\tilde{A}_{\mathcal{F}_\Pi}$ is Hermitian and takes the form*

$$\tilde{A}_{\mathcal{F}_\Pi} = A \dot{+} \hat{\mathfrak{N}} = \{\{f, Af + n\} : f \in \mathfrak{D}(A), n \in \mathfrak{N}\}. \tag{1.5}$$

4. Let $\hat{\mathfrak{N}}_\lambda = \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda\}$ $(\lambda \in \hat{\rho}(A))$, $\tilde{A}_\lambda := A \dot{+} \hat{\mathfrak{N}}_\lambda$.

**Lemma 1.1** ([53]). *Let $\lambda \in \hat{\rho}(A)$, $\tilde{A} \in \operatorname{Ex}_A$. Then the following equivalences hold:*

(1) *$\lambda \in \rho(\tilde{A}) \Longleftrightarrow \tilde{A}$ and $\tilde{A}_\lambda$ are transversal $\Longleftrightarrow A^* = \tilde{A} \dot{+} \hat{\mathfrak{N}}_\lambda$;*

(2) *$\lambda \notin \sigma_p(\tilde{A}) \Longleftrightarrow \tilde{A}$ and $\tilde{A}_\lambda$ are disjoint.*

The first assertion of Lemma 1.1 implies

**Proposition 1.5** ([53, 79]). *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for the relation $A^*$, $\pi_1$ is the orthogonal projection onto the first component in $\mathcal{H}\oplus\mathcal{H}$. The equalities*

$$\hat{\gamma}(\lambda) = (\Gamma_2|\hat{\mathfrak{N}}_\lambda)^{-1}, \qquad \gamma(\lambda) = \pi_1\hat{\gamma}(\lambda), \qquad (\lambda \in \rho(A_2)) \tag{1.6}$$

*define the operator-valued functions $\hat{\gamma}(\lambda)$, $\gamma(\lambda)$ holomorphic on $\rho(A_2)$ with values in $[\mathcal{H}, \hat{\mathfrak{N}}_\lambda]$ and $[\mathcal{H}, \mathfrak{N}_\lambda]$ respectively. Furthermore, the following relations hold:*

$$\gamma(\lambda) = \gamma(\mu) + (\lambda - \mu)(A_2 - \lambda)^{-1}\gamma(\mu), \quad (\lambda, \mu \in \rho(A_2)); \tag{1.7}$$

$$\gamma^*(\bar{\lambda}) = \Gamma_1\{(A_2 - \lambda)^{-1}, I + \lambda(A_2 - \lambda)^{-1}\} \quad (\lambda \in \rho(A_2)). \tag{1.8}$$

**Corollary 1.1.** $\mathcal{F}_\Pi(0) = \Gamma\{0, A_2(0)\} = \gamma^*(\lambda)A_2(0) \quad \forall\lambda \in \rho(A_2)$.

**Definition 1.9.** *The operator-valued function $M(\lambda)$ defined for $\lambda \in \rho(A_2)$ by the equality*

$$M(\lambda)\Gamma_2\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda, \qquad (\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda, \lambda \in \rho(A_2)) \tag{1.9}$$

*will be called a Weyl function of the operator $A$, corresponding to the BVS $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$.*

Since, by Proposition 1.5, $M(\lambda) = \Gamma_1\hat{\gamma}(\lambda)$ is holomorphic on $\rho(A_2)$, takes values in $[\mathcal{H}]$, and satisfies the equality

$$M(\lambda) - M(\mu) = (\lambda - \mu)\gamma^*(\bar{\mu})\gamma(\lambda), \qquad (\lambda, \mu \in \rho(A_2)), \tag{1.10}$$

it follows from (1.10) that $M(\lambda) \in R_{\mathcal{H}}$ and is a $Q$-function of the operator $A$, corresponding to the extension $A_2$ [18, 53, 79].

In accordance with Lemma 1.1 and Proposition 1.4 we obtain

**Proposition 1.6** ([18, 30, 53]). *Suppose that $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, $M(\lambda)$ is the corresponding Weyl function, $\theta \in \tilde{C}(\mathcal{H})$, and $\lambda \in \rho(A_2)$. Then the following equivalences hold:*

(1) $\lambda \in \rho(\tilde{A}_\theta) \Longleftrightarrow 0 \in \rho(\theta - M(\lambda))$;

(2) $\lambda \in \sigma_i(\tilde{A}_\theta) \Longleftrightarrow 0 \in \sigma_i(\theta - M(\lambda)) \quad (i = p, c, r)$.

Let $X = (X_{jk})_{j,k=1}^2 \in [\mathcal{H} \oplus \mathcal{H}]$, $J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \in [\mathcal{H} \oplus \mathcal{H}]$. We define the transformation $X[\theta]$ in $\tilde{C}(\mathcal{H})$ by the relation

$$X[\theta] := \{\{X_{21}f' + X_{22}f, X_{11}f' + X_{12}f\} : \{f, f'\} \in \theta\}. \tag{1.11}$$

The connection between various BVS's is established by

**Proposition 1.7** ([30, 26]). *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_1, \tilde{\Gamma}_2\}$ are BVS's for $A^*$, $M(\lambda)$ and $\tilde{M}(\lambda)$ are the corresponding Weyl functions, $\tilde{A} \in \mathrm{Ex}_A$, $\theta = \Gamma\tilde{A}$, $\tilde{\theta} = \tilde{\Gamma}\tilde{A}$. Then:*

(1) *there exists a $J$-unitary operator $X \in [\mathcal{H} \oplus \mathcal{H}]$ such that*

$$\begin{pmatrix} \tilde{\Gamma}_1 \\ \tilde{\Gamma}_2 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}, \qquad \tilde{\theta} = X[\theta], \qquad \tilde{M}(\lambda) = X[M(\lambda)]; \tag{1.12}$$

(2) *the transformation $X[\theta]$ can be defined as a linear-fractional transformation in the following cases:*

(a) *if $\theta \in C(\mathcal{H})$, then $X[\theta] = (X_{11}\theta + X_{12})(X_{21}\theta + X_{22})^{-1}$;*

(b) *if $0 \in \rho(X_{21})$, then $X[\theta] = (X_{21}^*)^{-1}X_{11}^* + (X_{21}^*)^{-1}(X_{21}\theta + X_{22})^{-1}$;*

(3) *if $\tilde{\theta} \in [\mathcal{H}]$, then $0 \in \rho(X_{21}\theta + X_{22})$, in particular, $0 \in \rho(X_{21}M(\lambda) + X_{22})$ and*

$$\tilde{M}(\lambda) = (X_{11}M(\lambda) + X_{12})(X_{21}M(\lambda) + X_{22})^{-1} \qquad \forall\lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.13}$$

**Proof.** (1) It follows from identity (1.3) and Proposition 1.4 that there exists a $J$-unitary operator $X$ such that

$$\tilde{\Gamma}_1 = X_{11}\Gamma_1 + X_{12}\Gamma_2, \qquad \tilde{\Gamma}_2 = X_{21}\Gamma_1 + X_{22}\Gamma_2. \tag{1.14}$$

The equality $\tilde{\theta} = X[\theta]$ easily follows from (1.4) and Proposition 1.4.

(2a) Let $\theta \in C(\mathcal{H})$ and $\{h, h'\} \in (X_{11}\theta + X_{12})(X_{21}\theta + X_{22})^{-1}$. Then there exists $f \in \mathcal{H}$ such that $h = (X_{21}\theta + X_{22})f$, $h' = (X_{11}\theta + X_{12})f$. In view of (1.11) this yields $\{h, h'\} \in X[\theta]$. The inverse inclusion is obvious.

(2b) Rewrite the $J$-unitary conditions of $X$ in the form

$$X_{11}^*X_{21} = X_{21}^*X_{11}, \qquad X_{12}^*X_{22} = X_{22}^*X_{12}, \qquad X_{11}^*X_{22} - X_{21}^*X_{12} = I; \tag{1.15}$$

$$X_{11}X_{12}^* = X_{12}X_{11}^*, \qquad X_{21}X_{22}^* = X_{22}X_{21}^*, \qquad X_{11}X_{22}^* - X_{12}X_{21}^* = I. \tag{1.16}$$

Let $\{h, h'\} \in X[\theta]$, that is, the following equality holds:

$$\{h, h'\} = \{X_{21}f' + X_{22}f,\ X_{11}f' + X_{12}f\} \qquad (\{f, f'\} \in \theta). \tag{1.17}$$

Then we obtain from (1.15) $X_{11}^*h - X_{21}^*h' = f$, i.e.,

$$\{h, X_{21}^*h'\} \in X_{11}^* - (X_{21}\theta + X_{22})^{-1},$$

$$\{h, h'\} \in (X_{21}^*)^{-1}X_{11}^* - (X_{21}^*)^{-1}(X_{21}\theta + X_{22})^{-1}.$$

Analogously, one can check the inverse inclusion

$$(X_{21}^*)^{-1}X_{11}^* - (X_{21}^*)^{-1}(X_{21}\theta + X_{22})^{-1} \subset X[\theta].$$

(3) Let $\tilde{\theta} = X[\theta] \in [\mathcal{H}]$. Assume that $\{f, f'\} \in \theta$ and $h = X_{21}f' + X_{22}f = 0$. In view of (1.17) and the condition $\tilde{\theta} \in [\mathcal{H}]$, this implies that $h' = X_{11}f' + X_{21}f = 0$, $f = X_{11}^*h - X_{21}^*h' = 0$ and hence $\ker(X_{21}\theta + X_{22}) = \{0\}$. It follows from (1.17) and the relation $\mathfrak{D}(\tilde{\theta}) = \mathcal{H}$ that $\mathfrak{R}(X_{21}\theta + X_{22}) = \mathcal{H}$, i.e., $0 \in \rho(X_{21}\theta + X_{22})$.

In particular, for $\tilde{A} = \tilde{A}_\lambda$ we have $\theta = M(\lambda)$, $0 \in \rho(X_{21}M(\lambda) + X_{22})$ $\forall\lambda \in \mathbb{C}\setminus\mathbb{R}$, and the equality $\tilde{M}(\lambda) = X[M(\lambda)]$ takes the form (1.13).

**Remark 1.1.** (1) If $\theta(0) \neq \{0\}$, then assertion 2a stops being true, for example, if $\theta = \{0, \mathcal{H}\}$, then $(I-\theta)(I+\theta)^{-1} = \mathcal{H} \oplus \mathcal{H}$.

(2) The forbidden relations $\mathcal{F}_\Pi$ and $\mathcal{F}_{\tilde{\Pi}}$ corresponding to BVS's $\Pi$ and $\tilde{\Pi}$ are also connected by the relation $\tilde{\mathcal{F}}_\Pi = X[\mathcal{F}_\Pi]$.

**5.** We characterize the forbidden relation $\mathcal{F}_\Pi$ in terms of the asymptotic behavior of the Weyl function. Letting $\mathfrak{N}''(A_2) = \mathfrak{N} \ominus A_2(0)$, we introduce the relation

$$\mathcal{F}_\Pi'' := \Gamma\hat{\mathfrak{N}}''(A_2) = \{\Gamma_2, \Gamma_1\}\hat{\mathfrak{N}}''(A_2), \qquad (\hat{\mathfrak{N}}''(A_2) = \{0, \mathfrak{N}''(A_2)\}). \tag{1.18}$$

As follows from the formulas

$$\mathcal{F}_\Pi = \Gamma\hat{\mathfrak{N}} = \Gamma\{0, A_2(0)\} \dotplus \Gamma\hat{\mathfrak{N}}''(A_2) = \{0, \mathcal{F}_\Pi(0)\} \dotplus \mathcal{F}_\Pi'' = \hat{\mathcal{F}}_\Pi(0) \dotplus \mathcal{F}_\Pi'',$$

$\mathcal{F}_\Pi''$ defines an operator in $\mathcal{H}$. In the general case $\mathcal{F}_\Pi''$ does not always coincide with the operator part $\mathcal{F}_\Pi'$ of the relation $\mathcal{F}_\Pi$, but $\mathcal{F}_\Pi'' = \mathcal{F}_\Pi' = \mathcal{F}_\Pi$ if $\mathcal{F}_\Pi(0) = \{0\}$.

**Theorem 1.1** ([53]). *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, $\mathcal{H}_j = \Gamma_j\hat{\mathfrak{N}}$ $(j = 1, 2)$, $M(\lambda)$ and $\mathcal{F}_\Pi$ are the corresponding Weyl function and forbidden relation. Then*

(1) $h \in \mathcal{H}_2 = \mathfrak{D}(\mathcal{F}_\Pi) \iff \lim_{y\uparrow\infty} y(\mathrm{Im}\, M(iy)h, h) < \infty \iff \lim_{y\uparrow\infty} y\mathrm{Im}\,(M(-iy)h, h) < \infty$;

(2) *for each $h \in \mathcal{H}_2$ there exist strong limits*

$$M(\infty)h := s - \lim_{y\uparrow\infty} M(iy)h = s - \lim_{y\uparrow\infty} M(-iy)h = \mathcal{F}_\Pi''h; \tag{1.19}$$

*in this case $M(\infty) = \mathcal{F}_\Pi$ if $\mathcal{F}_\Pi(0) = \{0\}$;*

(3) *for all $h \in \mathcal{H}$ the following equalities hold:*

$$B_M h := s - \lim_{y\uparrow\infty} \frac{M(iy)}{iy} h = \gamma^*(\lambda) P_{A_2(0)} \gamma(\lambda) h \qquad \forall\lambda \in \mathbb{C}\setminus\mathbb{R}; \tag{1.20}$$

*hence $B_M h = 0$ $\forall h \in \mathcal{H} \ominus \mathcal{F}_\Pi(0)$ and $B_M h \neq 0$ $\forall h \in \overline{\mathcal{F}_\Pi(0)} \setminus \{0\}$.*

Thus, $B_M \neq 0$ if and only if $A_2(0) \neq \{0\}$; in other words the term $B_M\lambda$ is lacking in the integral representation (1.2) of the Weyl function $M(\lambda)$ if and only if $A_2$ is an operator.

**Corollary 1.2.** *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, $\Pi_1 = \{\mathcal{H}, \Gamma_2, -\Gamma_1\}$ are BVS's for $A^*$, $M(\lambda)$ and $M_1(\lambda) = -M(\lambda)^{-1}$ are the corresponding Weyl functions, $A_1(0) = A_2(0) = \{0\} \iff B_{M_1} = B_M = 0$. Then*

$$M_1(\infty) = -M(\infty)^{-1}. \tag{1.21}$$

Note that relation (1.21) does not hold if the representation (1.2) of at least one of two functions $M(\lambda)$ or $M_1(\lambda)$ contains a linear term $B\lambda$. For example, let $\mathcal{H} = \mathbb{C}^2$, $M(\lambda) = \begin{pmatrix} \lambda & a \\ a & -\lambda^{-1} \end{pmatrix}$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we have

$$M(\infty) = \{\{\zeta e_2, a\zeta e_1\} : \zeta \in \mathbb{C}\}; \qquad M_1(\infty) = -\frac{q}{1+a^2} M(\infty)^{-1} = \left\{\left\{\zeta e_1, \frac{-a\zeta}{1+a^2} e_2\right\} : \zeta \in \mathbb{C}\right\}.$$

Therefore $M_1(\infty) \neq -M(\infty)^{-1}$.

**6.** Subspaces $\mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \cap \mathfrak{h}_0$ $(\lambda \in \mathbb{C} \setminus \mathbb{R})$ are called semidefect subspaces, and the numbers $n'_\pm(A) = \dim \mathfrak{N}'_{\pm i}$ are semidefect numbers of the operator $A$. Clearly $\mathfrak{N}'_\lambda$ are defect subspaces of the operator $P_{\mathfrak{h}_0} A \in \mathcal{C}(\mathfrak{h}_0)$.

**Proposition 1.8** ([28, 53]). $n_\pm(\mathcal{F}_\Pi) = n_\pm(A_{\mathcal{F}_\Pi}) = n'_\pm(A)$.

**Proof.** Choose a BVS for $A^*$ such that $M(i) = i$. Then $\forall f \in \mathfrak{N}'_i$ $(\hat{f} = \{f, if\})$, $\forall \hat{l} = \{0, l\} \in \hat{\mathfrak{N}}$ we obtain from (1.3)

$$0 = (if, 0) - (f, l) = (\Gamma_1 \hat{f}, \Gamma_2 \hat{l})_{\mathcal{H}} - (\Gamma_2 \hat{f}, \Gamma_1 \hat{l})_{\mathcal{H}} = -(\Gamma_2 \hat{f}, \Gamma_1 \hat{l} + i\Gamma_2 \hat{l})_{\mathcal{H}}, \tag{1.22}$$

that is, $\Gamma_2 \hat{f} \in \mathfrak{N}_i(\mathcal{F}_\Pi)$. Conversely, for all $h \in \mathfrak{N}_i(\mathcal{F}_\Pi)$ there exists $\hat{f} \in \hat{\mathfrak{N}}_i$ such that $\Gamma_2 \hat{f} = h$ (since $\Gamma_2 \hat{\mathfrak{N}}_i = \mathcal{H}$). It follows from (1.22) that $\hat{f} \in \mathfrak{h}_0$ and hence $\hat{f} \in \hat{\mathfrak{N}}'_i$. Thus, the operator $\Gamma_2$ establishes an isomorphism between $\mathfrak{N}'_i$ and $\mathfrak{N}_i(\mathcal{F}_\Pi)$; therefore $n'_\pm(A) = n_\pm(\mathcal{F}_\Pi)$. □

## 2. REGULAR OPERATORS AND REGULAR EXTENSIONS

**1.** Let $M$, $N$ be closed subspaces in $\mathfrak{h}$. As usual, the symbols $M + N$ and $M \dot{+} N$ denote the sum and the direct sum of subspaces $M$, $N$. Since the sum of linear relations is denoted by the same symbol (see Sec. 1), in order to avoid ambiguity we shall write additionally $M, N \in \mathfrak{h}^2$ for the sum of subspaces $M$, $N$ and, respectively, $M, N \in \tilde{\mathcal{C}}(\mathfrak{h})$ for the sum $M + N$ of linear relations.

**Lemma 2.1.** *Suppose that $X$, $Y$ are Banach spaces, $P$ is a linear continuous surjective mapping from $X$ to $Y$ $(PX = Y)$, $N = \ker P$. Then the range $PM$ of a closed subspace $M$ is also closed if and only if $M + N$ is a closed subspace of $X$.*

**Proof.** The *necessity* of this assertion follows from the equality

$$P^{-1}(PM) = M + N \qquad (N = \ker P)$$

and the continuity of $P$.

*Sufficiency.* If $M + N$ is a closed subspace of a Banach space $X$, the factor space $(M + N)/N$ is also a Banach space. Since $PX = Y$, we have from the Banach theorem that $P$ is an open mapping and the mapping $\tilde{P} := \tilde{X}X/N \to Y(P(x + N) = Px, x \in X)$ is a topological isomorphism. Hence $PM = P(M + N) = \tilde{P}((M + N)/N) \cong (M + N)/N$ and, therefore, $PM$ is a closed space since $(M + N)/N$ is a closed space.

We shall need the known result of T. Kato. As follows from the proof given below, this result is a consequence of the Banach theorem.

**Proposition 2.1** ([29 p. 279]). *Suppose that $M$ and $N$ are closed subspaces of a Banach space $X$, $M^\perp$ and $N^\perp$ are their annihilators in $X^*$. Then the linear manifold $M + N$ is closed in $X$ if and only if the linear manifold $M^\perp + N^\perp$ is closed in $X^*$, and in this case the following equality holds*:

$$M^\perp + N^\perp = (M \cap N)^\perp. \tag{2.1}$$

**Proof.** Let $\pi_1 : X \to X/N$ and $\pi_2 : X^* \to X^*/M^\perp$ be factor-mappings. Let $P_1 = \pi_1 \restriction M$, $P_2 = \pi_2 \restriction N^\perp$. It follows from the equality $(P_1 f, g) = (f + N, g) = (f, g) = (f, g + M^\perp) = (f, P_2 g)$ $\forall f \in M$, $g \in N^\perp$ and the relations $(X/N)^* = N^\perp$, $M^* = X^*/M^\perp$, that $P_2 = P_1^*$. By virtue of the Banach theorem on an operator with a closed range, the range of $P_1$ is closed if and only if the range of $P_2$ is closed. Further, in accordance with Lemma 2.1 we obtain that the linear manifold $M + N$ $(M^\perp + N^\perp)$ is closed if and only if the range $\pi_1(M) = P_1 M$ $(\pi_2(N^\perp) = P_2 N^\perp)$ is closed.

**2.** Let $M$ and $N$ be subspaces of Hilbert space $\mathfrak{h}$. We define (after [29, p. 276]) the minimal opening $\gamma(M, N)$ of subspaces $M$ and $N$ by the formula

$$\gamma(M, N) := \inf_{u \in M, u \notin N} \frac{\|u - P_N u\|}{\|u - P_{M \cap N} u\|} = \gamma(N, M) (\leq 1). \tag{2.2}$$

In the case $M \cap N = \{0\}$ the opening $\gamma(M,N)$ coincides with the sine of the minimal angle between $M$ and $N: \gamma(M,N) = \sin(M,N)$.

**Proposition 2.2.** *Let $M$ and $N$ be subspaces of a Hilbert space. The following assertions are equivalent:*

(1) $\gamma(M,N) > 0$;

(2) *linear manifold $M + N$ is closed in $\mathfrak{h}$;*

(3) *linear manifold $M^\perp + N^\perp$ is closed in $\mathfrak{h}$;*

(4) *linear manifold $P_{N^\perp} M$ is closed in $N^\perp$;*

(5) *linear manifold $P_N M^\perp$ is closed in $N$.*

**Proof.** The equivalence (1) $\Longleftrightarrow$ (2) was proved in [29, Theorem IV.4.2] and the equivalence (2) $\Longleftrightarrow$ (3) is contained in Proposition 2.1. Equivalences (2) $\Longleftrightarrow$ (4), (3) $\Longleftrightarrow$ (5) follow from Lemma 2.1, used for operators $P_{N^\perp}, P_N \in [\mathfrak{h}]$. □

**3.** Let $A$ be a closed linear operator in $\mathfrak{h}$ identified with its graph gr $A$, $\overline{\mathfrak{D}(A)} = \mathfrak{h}_0 \subset \mathfrak{h}$, $\mathfrak{M}_\lambda = (A-\lambda)\mathfrak{D}(A)$, $\mathfrak{N}_{\bar\lambda} = \mathfrak{M}_\lambda^\perp$, $\mathfrak{N} = \mathfrak{h} \ominus \mathfrak{h}_0$, $\hat{\mathfrak{N}}_\lambda = \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda\}$, $A^*$ be the adjoint linear relation, and let $\hat{\mathfrak{N}} = \{0, \mathfrak{N}\}$ be its multivalued part. The following direct decomposition, which is an analog of the Neumann formula ([74, 77, 80]; see also [31, 64]) holds:

$$A^* = A \dot{+} \hat{\mathfrak{N}}_\lambda \dot{+} \hat{\mathfrak{N}}_{\bar\lambda} \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}). \tag{2.3}$$

At the same time the decomposition

$$\mathfrak{D}(A^*) = \mathfrak{D}(A) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar\lambda} \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}) \tag{2.4}$$

is not a direct sum. The ambiguity in (2.4) is described by the following proposition.

**Proposition 2.3** ([31]). *Vectors $f_\lambda \in \mathfrak{N}_\lambda$ and $-f_{\bar\lambda} \in \mathfrak{N}_{\bar\lambda}$ are congruent modulo $\mathfrak{D}(A)$ (that is, $\exists f_A \in \mathfrak{D}(A) : f_A + f_\lambda + f_{\bar\lambda} = 0$) if and only if there exists a unique vector $n \in \mathfrak{N}$ such that*

$$f_\lambda = P_{\mathfrak{N}_\lambda} n, \qquad f_{\bar\lambda} = -P_{\mathfrak{N}_{\bar\lambda}} n \qquad (n \in \mathfrak{N}).$$

*In this case $\|f_\lambda\| = \|f_{\bar\lambda}\|$ and $n = (\lambda - \bar\lambda)(Af_A + \lambda f_\lambda + \bar\lambda f_{\bar\lambda})$.*

It follows from Proposition 2.3 that an operator defined by the equality

$$V_\xi P_{\mathfrak{N}_{-i}} n = P_{\mathfrak{N}_i} n \qquad (\forall n \in \mathfrak{N}) \tag{2.5}$$

is an isometric operator acting from $\mathfrak{N}_{\bar\lambda}'' = P_{\mathfrak{N}_{\bar\lambda}} \mathfrak{N}$ onto $\mathfrak{N}_\lambda'' = P_{\mathfrak{N}_\lambda} \mathfrak{N}$, named a forbidden operator. The notion of a forbidden operator was introduced in [54], its role in the extension theory of operator $A$ was clarified in [31]. As shown in [31], manifolds $\mathfrak{N}_\lambda''$ are closed (or nonclosed) only simultaneously for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In the former case the operator $A$ is called regular; in the latter case it is called singular [64].

Let $A_{op}^*$ be the operator adjoint to the operator $A \in \mathcal{C}(\mathfrak{h}_0, \mathfrak{h})$.

**Lemma 2.2.** *Let $\mathfrak{N}_\lambda' = \mathfrak{N}_\lambda \ominus \mathfrak{N}_\lambda'' = \mathfrak{N}_\lambda \cap \mathfrak{h}_0$ be a semidefect subspace of $A$. Then the following direct decomposition (analogous to the Neumann formula) holds:*

$$\mathfrak{D}(A^*) = \mathfrak{D}(A_{op}^*) = \overline{\mathfrak{D}(A)}^{(+)} \dot{+} \mathfrak{N}_i' \dot{+} \mathfrak{N}_{-i}' \dot{+} \overline{\mathfrak{N}_i''}, \tag{2.6}$$

*where $\overline{\mathfrak{D}(A)}^{(+)}$ is the closure of $\mathfrak{D}(A)$ in the graph-norm of the operator $A_{op}^*$.*

**Proof.** Making use of formula (2.4), we show that $\overline{\mathfrak{N}_{-i}''} \subset \overline{\mathfrak{N}_i''} \dot{+} \overline{\mathfrak{D}(A)}^{(+)}$. By virtue of Proposition 2.3 lineals $\mathfrak{N}_{\pm i}$ are linearly dependent modulo $\mathfrak{D}(A)$ and

$$\mathfrak{N}_{-i}'' = \mathfrak{N}_{-i} \cap (\mathfrak{D}(A) \dot{+} \mathfrak{N}_i'') \subset \mathfrak{D}(A) \dot{+} \mathfrak{N}_i''. \tag{2.7}$$

If $f_{-i} \in \overline{\mathfrak{N}}''_{-i}$, then according to (2.7) there exists $f_{-i}^{(n)} \in \mathfrak{N}''_{-i}$ $(n \in \mathfrak{N})$ such that $f_{-i}^{(n)} = f_A^{(n)} + f_i^{(n)} \to f_{-i}$ $(f_A^{(n)} \in \mathfrak{D}(A),\ f_i^{(n)} \in \mathfrak{N}_i)$ for $n \to \infty$. It follows from (2.5) that $f_{-i}^{(n)} = V_i f_{-i}^{(n)}$ and, consequently, there exist the limits

$$f_i := \lim_{n\to\infty} f_i^{(n)} = \lim_{n\to\infty} V_i f_{-i}^{(n)} \in \overline{\mathfrak{N}}''_i, \qquad f_A := \lim_{n\to\infty} f_A^{(n)} \in \overline{\mathfrak{D}(A)}^{(+)}.$$

Thus $f_{-i} = f_A + f_i$ and the inclusion $\overline{\mathfrak{N}}''_{-i} \subset \overline{\mathfrak{D}(A)}^{(+)} + \overline{\mathfrak{N}}''_i$ is proved. In view of formula (2.4) this implies that

$$\mathfrak{D}(A^*) \subset (\overline{\mathfrak{D}(A)}^{(+)} \dotplus \mathfrak{N}'_i \dotplus \mathfrak{N}'_{-i}) + \overline{\mathfrak{N}}''_i.$$

The inverse inclusion is evident since $\overline{\mathfrak{N}}''_i \subset \mathfrak{N}_i \subset \mathfrak{D}(A^*)$. □

**Corollary 2.1.** *Let $A$ be a regular Hermitian operator. Then*

$$\mathfrak{D}(A^*) = \mathfrak{D}(A) \dotplus \mathfrak{N}'_i \dotplus \mathfrak{N}'_{-i} \dotplus \mathfrak{N}''_i. \tag{2.8}$$

**Corollary 2.2.** *The following relations hold:*

$$P_{\mathfrak{N}}\mathfrak{N}_\lambda = P_{\mathfrak{N}}\overline{\mathfrak{N}}''_\lambda = P_{\mathfrak{N}}\mathfrak{D}(A^*) = P_{\mathfrak{N}}\mathfrak{D}(A^*_{op}) \qquad (\forall \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-). \tag{2.9}$$

**Proposition 2.4.** *Let $A$ be a Hermitian operator in $\mathfrak{h}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following assertions are equivalent:* (1) $\gamma(\mathfrak{M}_{\bar\lambda}, \mathfrak{N}) > 0$; (2) $\mathfrak{M}_{\bar\lambda} + \mathfrak{N}$ *is closed;* (3) $\gamma(\mathfrak{N}_\lambda, \mathfrak{h}_0) > 0$; (4) $\mathfrak{N}_\lambda + \mathfrak{h}_0 = \mathfrak{h}$; (5) $\mathfrak{N}''_\lambda \dotplus \mathfrak{h}_0 = \mathfrak{h}$; (6) $\mathfrak{N}''_\lambda$ *is closed;* (7) $P_{\mathfrak{N}}\mathfrak{N}_\lambda = P_{\mathfrak{N}}\mathfrak{N}''_\lambda = P_{\mathfrak{N}}\mathfrak{D}(A^*)$ *is closed.*

The proof follows from Proposition 2.2 and Corollary 2.2.

**Remark 2.1.** One can easily deduce from item (7) of Proposition 2.4 and relations (2.9) that the assertions (1)–(7) of Proposition 2.4 are fulfilled or are not fulfilled simultaneously for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, linear manifolds $\mathfrak{N}''_\lambda$ are closed or are not closed simultaneously for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

4. **Definition 2.1.** *Let $\mathfrak{N}$ be a subspace in $\mathfrak{h}$. A linear relation $T \in \tilde{\mathcal{C}}(\mathfrak{h})$ will be called $\mathfrak{N}$-regular if a linear manifold $T + \hat{\mathfrak{N}}$ is closed in $\mathfrak{h}^2 = \mathfrak{h} \oplus \mathfrak{h}$ (recall that $\hat{\mathfrak{N}} = \{0, \mathfrak{N}\} \subset \mathfrak{h}^2$).*

**Proposition 2.5.** *Suppose that $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, $\theta \in \tilde{\mathcal{C}}(\mathcal{H})$, $\tilde{A} = \tilde{A}_\theta$ is a proper extension of an operator $A$, $\mathcal{F}_\Pi$ is a forbidden relation, $\mathfrak{N} = \mathfrak{h}_0^\perp$. Then the following assertions are equivalent:*

(1) *$\tilde{A}$ is $\mathfrak{N}$-regular extension of $A$;*
(2) *linear relation $P_{\mathfrak{h}_0}\tilde{A}$ is closed;*
(3) *linear manifold $P_{\mathfrak{N}}\mathfrak{D}(\tilde{A}^*)$ is closed in $\mathfrak{N}$;*
(4) *linear manifold $\theta + \mathcal{F}_\Pi$ is closed in $\mathcal{H} \oplus \mathcal{H}$.*

**Proof.** (1) $\Longleftrightarrow$ (2). Since $\hat{\mathfrak{N}}^\perp = \mathfrak{h} \oplus \mathfrak{h}_0$ it follows from Proposition 2.2 that $\tilde{A}$ is an $\mathfrak{N}$-regular extension of $A$ if and only if the linear manifold

$$P_{\mathfrak{h}\oplus\mathfrak{h}_0}\tilde{A} = \{\{f, P_{\mathfrak{h}_0}f'\} : \{f, f'\} \in \tilde{A}\} \tag{2.10}$$

is closed. It remains to note that the manifold $P_{\mathfrak{h}\oplus\mathfrak{h}_0}\tilde{A}$ coincides with the product $P_{\mathfrak{h}_0}\tilde{A}$ of the linear relations $P_{\mathfrak{h}_0}$ and $\tilde{A}$.

(1) $\Longleftrightarrow$ (3). Let $J$ be an isometry in $\mathfrak{h} \oplus \mathfrak{h}$ defined by the equality $J\{h_1, h_2\} = \{h_2, -h_1\}$. According to Proposition 2.2 the linear manifolds $\tilde{A} + \hat{\mathfrak{N}}$, $\tilde{A}^\perp + \hat{\mathfrak{N}}^\perp$, $\tilde{A}^* + J\hat{\mathfrak{N}}^\perp = J(\tilde{A}^\perp + \hat{\mathfrak{N}}^\perp)$, and $P_{J\hat{\mathfrak{N}}}A^*$ are closed only simultaneously. It remains to note that $P_{J\hat{\mathfrak{N}}}A^* = P_{\mathfrak{N}}\mathfrak{D}(A^*)$.

(1) $\Longleftrightarrow$ (4). Since the mapping $\Gamma$ induces a topological isomorphism between $A^*/A$ and $\mathcal{H} \oplus \mathcal{H}$, the equivalence (1) $\Longleftrightarrow$ (4) is a consequence of the equality $\Gamma(\tilde{A}_\theta + \hat{\mathfrak{N}}) = \theta + \mathcal{F}_\Pi$. □

**Corollary 2.3.** *Under the assumptions of Proposition* 2.5 *the following assertions are equivalent:* (1) *$A$ is an $\mathfrak{N}$-regular operator;* (2) *$P_{\mathfrak{h}_0}A$ is a closed operator;* (3) $P_{\mathfrak{h}}\mathfrak{D}(A^*) = \mathfrak{N}$; (4) *$\mathcal{F}_\Pi$ is a closed linear relation in $\mathcal{H}$;* (5) *$\mathfrak{N}''_\lambda = P_{\mathfrak{N}_\lambda}\mathfrak{N}$ is closed in $\mathfrak{N}_\lambda$ $(\lambda \in \hat{\rho}(A))$.*

**Remark 2.2.** As follows from the equivalences (1) $\Longleftrightarrow$ (5) of Corollary 2.3 and (1) $\Longleftrightarrow$ (2) of Proposition 2.5, the notion of $\mathfrak{N}$-regularity coincides with the notion of regularity both for a Hermitian operator $A$ [31] and for its extension $\tilde{A} \in \mathcal{C}(\mathfrak{h})$ [64] if $\mathfrak{N} = \mathfrak{h}_0^{\perp}$. Other regularity criteria for extensions $\tilde{A} \in \mathcal{C}(\mathfrak{h})$ are given in [60].

**Corollary 2.4.** *Under the assumptions of Proposition* 2.5 *the following holds:*

(1) $A_i = \ker \Gamma_i$ *is an $\mathfrak{N}$-regular extension iff* $\mathcal{H}_i = \Gamma_i \hat{\mathfrak{N}}$ *is closed in* $\mathcal{H}$ $(i = 1, 2)$;

(2) *if $\tilde{A}_\theta$ is an $\mathfrak{N}$-regular operator, then* $\mathfrak{R}(\mathcal{F}_\Pi - \theta)$ *is closed in* $\mathcal{H}$;

(3) *if $A_2$ is an $\mathfrak{N}$-regular operator and* $B \in [\mathcal{H}]$, *then $\mathfrak{N}$-regularity of an extension* $\tilde{A}_B = \ker(\Gamma_1 - B\Gamma_2)$ *is equivalent to the condition*

$$\exists \varepsilon > 0 : \|(\mathcal{F}_\Pi - B)h\| \geq \varepsilon \|h\| \qquad \forall h \in \mathcal{H}_2 = \Gamma_2 \hat{\mathfrak{N}} \tag{2.11}$$

**Proof.** (1) Consider the case $i = 1$. Since $\theta := \Gamma A_1 = \{\mathcal{H}, 0\}$, the first assertion follows from item (4) of Proposition 2.5 and the equality $\theta + \mathcal{F}_\Pi = \mathcal{H} \oplus \mathfrak{R}(\mathcal{F}_\Pi) = \mathcal{H} \oplus \mathcal{H}_1$.

(2) Since $\theta + \mathcal{F}_\Pi$ is a closed linear relation, its multivalued part $(\theta + \mathcal{F}_\Pi)(0)$ is closed in $\mathcal{H}$. Now the assertion follows from the relation $\mathfrak{R}(\mathcal{F}_\Pi - \theta) = (\mathcal{F}_\Pi + \theta)(0)$.

(3) Condition (2.11) is equivalent to the fact that $\mathfrak{R}(\mathcal{F}_\Pi - B)$ is closed in $\mathcal{H}$. Now it remains to use the relation

$$\text{gr } \mathcal{F}_\Pi \dot{+} \text{gr } B = \text{gr } B \dot{+} \{0, \mathfrak{R}(\mathcal{F}_\Pi - B)\} \tag{2.12}$$

and to note that the minimal opening between the subspaces in (2.12) is positive. □

**Corollary 2.5.** *Suppose that* $\tilde{A} \in \text{Ex}_A$ *is a Hermitian extension of $A$ such that* $\operatorname{codim} \overline{\mathfrak{D}(\tilde{A})} < \infty$ *and* $\tilde{A}^*$ *is $\mathfrak{N}$-regular. Then all proper extensions* $A' \in \text{Ex}_{\tilde{A}}$ *of* $\tilde{A}$ *are also $\mathfrak{N}$-regular.*

**Proof.** By virtue of Proposition 2.5 $P_{\mathfrak{N}}\mathfrak{D}(\tilde{A})$ is closed in $\mathfrak{N}$. It follows from the relation $\operatorname{codim} \overline{\mathfrak{D}(\tilde{A})} = n < \infty$ that $\dim(\mathfrak{N} \ominus P_{\mathfrak{N}}\mathfrak{D}(\tilde{A})) \leq n$. For all $A' \in \text{Ex}_{\tilde{A}}$ the inclusions $\tilde{A} \subset (A')^* \subset \tilde{A}^*$ imply that the linear manifold $P_{\mathfrak{N}}\mathfrak{D}((A')^*)$ is closed and, therefore, $A'$ is an $\mathfrak{N}$-regular extension of $A$. □

**5. Proposition 2.6.** *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for* $A^*$, $\mathcal{F}_\Pi$ *is a forbidden relation,* $M(\lambda)$ *is the corresponding Weyl function with integral representation* (1.2). *Then*

$$\mathfrak{R}(B_M^{1/2}) = \mathcal{F}_\Pi(0). \tag{2.13}$$

**Proof.** Owing to Theorem 1.1 we have $B_M = \gamma^*(i)P_{A_2(0)}\gamma(i)$. This implies that $\|B_M^{1/2}h\|^2 = \|P_{A_2(0)}\gamma(i)h\|^2$ $(\forall h \in \mathcal{H})$ and the operator $U$, defined by the equality $UB_M^{1/2}h = P_{A_2(0)}\gamma(i)h$ $(h \in \mathcal{H})$, is an isometry from $\overline{\mathfrak{R}(B_M^{1/2})}$ onto $A_2(0)$ since $A_2(0) \cap \mathfrak{M}_{-i} \subset \mathfrak{N} \cap \mathfrak{M}_{-i} = \{0\}$ ([53]). Hence $B_M^{1/2}U^* = \gamma(i)^* P_{A_2(0)}$ and $U^*$ is an isometrical operator from $A_2(0)$ onto $\overline{\mathfrak{R}(B_M^{1/2})}$. In accordance with Corollary 1.1

$$\mathcal{F}_\Pi(0) = \gamma^*(i)A_2(0) = \mathfrak{R}(B_M^{1/2}U^*) = \mathfrak{R}(B_M^{1/2}). \ \square$$

**Corollary 2.6.** *Under the assumptions of Proposition* 2.6 *the following equality holds*:

$$\mathcal{F}_\Pi = \hat{\mathfrak{R}}\,(B_M^{1/2}) \dot{+} \text{gr } M(i\infty) = \{0, \mathfrak{R}\,(B_M^{1/2})\} \dot{+} \text{gr } M(i\infty). \tag{2.14}$$

The proof follows from (2.13) and relations $\mathcal{F}_\Pi = \hat{\mathcal{F}}_\Pi(0) \dot{+} \mathcal{F}_\Pi''$, $\mathcal{F}_\Pi'' = \text{gr } M(i\infty)$ (see Theorem 1.1).

**Proposition 2.7.** *Under the assumptions of Proposition* 2.6 *the following assertions are equivalent:* (1) $\mathfrak{R}\,(B_M)$ *is closed in* $\mathcal{H}$; (2) $\mathcal{F}_\Pi(0)$ *is closed in* $\mathcal{H}$; (3) $\gamma(\hat{A}_2(0), A_1) > 0$; (4) $\gamma(\overline{\mathfrak{D}(A_2)}, \mathfrak{N}_\lambda) > 0$; (5) $\gamma(A_2(0), \mathfrak{M}_{\bar{\lambda}}) > 0 \ \ \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$.

**Proof.** The equivalences (1) $\Longleftrightarrow$ (2) and (4) $\Longleftrightarrow$ (5) are consequences of relation (2.13) and Proposition 2.2. The equivalence (2) $\Longleftrightarrow$ (3) follows from Lemma 2.1 and the equality $\mathcal{F}_\Pi(0) = \Gamma_1 \hat{A}_2(0)$. Now it remains to show that (5) $\Longleftrightarrow$ (2). If $\gamma(A_2(0), \mathfrak{M}_{-i}) > 0$, then according to Proposition 2.2 $P_{A_2(0)}\mathfrak{N}_i$ is closed. Since $A_2(0) \cap \mathfrak{M}_{-i} = \{0\}$ we have $\overline{P_{A_2(0)}\mathfrak{N}_i} = A_2(0)$ and hence $P_{A_2(0)}\mathfrak{N}_i = A_2(0)$. Making use of Lemma 2.1 and relations $\ker \gamma^*(i) = \mathfrak{M}_{-i}$, $\gamma(A_2(0), \mathfrak{M}_{-i}) > 0$ we conclude that $\mathcal{F}_\Pi(0) = \gamma^*(i)A_2(0) = \gamma^*(i)P_{A_2(0)}\mathfrak{N}_i$ is closed in $\mathcal{H}$.

Conversely, if $\gamma(A_2(0), \mathfrak{M}_{-i}) = 0$, then according to Lemma 2.1 and Proposition 2.2 we have that the linear manifolds $A_2(0) \dot{+} \mathfrak{M}_{-i}$ and $\mathcal{F}_\Pi(0) = \gamma^*(0)A_2(0)$ are nonclosed. □

**Remark 2.3.** One can easily deduce Proposition 2.7 from Proposition 2.2 and the next assertion: if $M$, $N$ are subspaces in $\mathfrak{h}$, then the following equivalence holds: $M + N$ is closed $\Longleftrightarrow$ the range of the operator $P_N P_{M^\perp} \upharpoonright N$ is closed. In turn this assertion is a consequence of the equivalence

$$\|P_N P_M\| < 1 \Longleftrightarrow \|P_N P_M | N\| < 1$$

held if $M \cap N = \{0\}$.

Now we shall characterize the regularity of an operator $A$ in terms of the Weyl function.

**Proposition 2.8.** *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for* $A^*$, $M(\lambda)$ *is the corresponding Weyl function,* $\mathfrak{N} = \mathfrak{D}(A)^\perp = \mathfrak{h}_0^\perp$. *Then the operator* $A$ *is* $\mathfrak{N}$*-regular if and only if the linear manifolds* $\mathfrak{R}\,(B_M)$, $\mathrm{gr}\ M(\infty)$ *are closed and form an acute angle (here* $B_M$ *is a coefficient of* $\lambda$ *in the integral representation* (1.2) *of* $M(\lambda)$*).*

**Proof.** If $A$ is a regular operator, then $\mathcal{F}_\Pi = \Gamma\hat{\mathfrak{N}}$ and $\mathcal{F}_\Pi(0)$ are closed. Since $\mathcal{F}''_\Pi = \Gamma\{0, \mathfrak{N} \ominus A_2(0)\}$ is the range of the subspace $\{0, \mathfrak{N} \ominus A_2(0)\} \subset A^*$ and $\Gamma \in [A^*, \mathcal{H} \oplus \mathcal{H}]$ is a bounded operator, it follows from the equality

$$\mathcal{F}_\Pi = \Gamma\hat{\mathfrak{N}} = \widehat{\mathcal{F}_\Pi(0)} \dot{+} \Gamma\{0, \mathfrak{N} \ominus A_2(0)\} = \widehat{\mathcal{F}_\Pi(0)} \dot{+} \mathcal{F}''_\Pi \tag{2.15}$$

and the known theorem from [45, p. 11] that $\mathcal{F}''_\Pi$ is closed. In accordance with (2.13) and Proposition 2.2 we obtain $\gamma(\widehat{\mathcal{F}_\Pi(0)}, \mathcal{F}''_\Pi) > 0$. To complete the proof of the direct assertion it remains to note that by virtue of Theorem 1.1 and Proposition 2.6 $\mathcal{F}''_\Pi = \mathrm{gr}\ M(\infty)$ and $\mathcal{F}_\Pi(0) = \mathfrak{R}\,(B_M^{1/2})$.

The inverse assertion follows immediately from (2.15) and Corollary 2.3. □

**Remark 2.4.** (1) Suppose that $M(\infty)$ is a bounded operator. This is true, for example, if $\mathcal{H}_2$ is closed. In this case $\widehat{\mathcal{F}_\Pi(0)}$ and $\mathcal{F}''_\Pi$ form an acute angle. If, additionally, an extension $A_2$ is $\mathfrak{N}$-regular, then the operator $A$ is $\mathfrak{N}$-regular if and only if $\mathcal{F}_\Pi(0)$ is closed. In particular, $A$ is $\mathfrak{N}$-regular if $A_2(0) = \{0\}$, or more generally, if $\dim A_2(0) < \infty$. In the case $\dim \mathfrak{N} = \infty$ one can easily construct examples of $\mathfrak{N}$-regular extensions $A_2 \in \tilde{\mathcal{C}}(\mathfrak{h}) \ominus \mathcal{C}(\mathfrak{h})$ of operator the $A$, that is, not $\mathfrak{N}$-regular. To this end we must choose the Weyl function $M(\lambda)$ such that $\mathcal{H}_2 = \mathcal{H}_2(M)$ is a closed subspace and $\mathfrak{R}\,(B)$ is not.

(2) An operator $A_2$ is a nonregular extension of a regular operator $A$ if and only if an operator $M(\infty)$ is closed but unbounded. In the general case $(A_2(0) \neq \{0\})A$ is a nonregular extension of a regular operator $A$ if and only if gr $M(\infty)$ and $\mathfrak{R}\,(B)$ are closed, $\gamma(\widehat{\mathfrak{R}\,(B)}, \mathrm{gr}\ M(\infty)) > 0$ but the operator $M(\infty)$ is unbounded.

In the next example we produce a Weyl function $M(\lambda)$ such that $\gamma(\widehat{\mathcal{F}_\Pi(0)}, \mathcal{F}''_\Pi) = 0$ and $M(\infty)$ is an unbounded operator.

**Example 2.1.** Let $L_0 = L_0^* \in \mathcal{C}(\mathcal{H}_0)$, $L = \begin{pmatrix} 0 & L_0 \\ L_0 & 0 \end{pmatrix} \in \mathcal{C}(\mathcal{H})$, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$. We put

$$M(\lambda) = B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)(1+t^2)\, dE(t),$$

where $E(t)$ is the spectral function of the operator $L$, and $B = P_1$ is the orthogonal projection onto $\mathcal{H}^1 := \{0\} \oplus \mathcal{H}_0$. It is easy to see that $\mathcal{H}_2 = \mathfrak{D}(L_0) \oplus \{0\}$ and

$$M(\lambda)h = -Lh + \int_{\mathbb{R}} \frac{1}{t-\lambda}(1+t^2)\, dE(t)h \qquad \forall h \in \mathcal{H}_2.$$

Therefore, $M(\infty) = -L\upharpoonright\mathcal{H}_2 = \begin{pmatrix} 0 \\ L_0 \end{pmatrix} : \mathcal{H}_2 \to \mathcal{H}^1$ and the linear manifold gr $M(\infty)$ is closed. Since $\mathfrak{R}\,(B) = \mathfrak{R}\,(P_1) = \mathcal{H}^1$ we have

$$\mathcal{F}_\Pi = \{\{0, h_1\} + \{h_2, M(\infty)h_2\} : h_1 \in \mathcal{H}^1, h_2 \in \mathcal{H}_2\} = \mathcal{H}_2 \oplus \mathcal{H}^1$$

and $\mathcal{F}_\Pi$ is nonclosed. It follows from §5 that there exists an operator $A$ and a BVS $\{\mathcal{H},\Gamma_1,\Gamma_2\}$ such that $M(\lambda)$ coincides with the corresponding Weyl function.

**Proposition 2.9.** *Suppose that $J$, $X \in [\mathcal{H}\oplus\mathcal{H}]$ are the same as in Proposition* 1.7 *$M(\lambda) \in R_{\mathcal{H}}$, $0 \in \rho(\mathrm{Im}\ M(i))$ and $\tilde{M}(\lambda) = X[M(\lambda)]$ (see* (1.12)*). Then the linear manifold $\mathfrak{R}\,\widehat{(B_M)}\dot{+}$ gr $M(\infty)$ and $\mathfrak{R}\,\widehat{(B_{\tilde{M}})}\dot{+}$ gr $\tilde{M}(\infty)$ are either closed or not simultaneously, that is, the property of linear manifold* (2.14) *to be closed is invariant under the transformation group* (2.12).

The proof follows from Theorem 5.1 and Propositions 2.8 and 1.7.

**Corollary 2.7.** *If under the assumptions of Proposition* 2.9 *$B_M = B_{\tilde{M}} = 0$, then* gr $M(\infty)$ *and* gr $\tilde{M}(\infty)$ *are either closed or not simultaneously.*

**Remark 2.5.** If $M(\lambda)$ contains a linear term $B_M\lambda$, then the property of linear manifolds gr $M(\infty)$ and $\mathfrak{R}\ (B_M)$ to be closed is not invariant under the transformation group (1.12). Thus, in example 2.1, gr $M(\infty)$ is closed while gr $\tilde{M}(\infty)$ is not closed if

$$B_{\tilde{M}} := \lim_{\lambda=iy\to\infty} \frac{\tilde{M}(\lambda)}{\lambda} = 0.$$

We give one more $\mathfrak{N}$-regularity criterion for extensions $\tilde{A} \in \mathrm{Ex}_A$.

**Proposition 2.10.** *Let $A$ be a Hermitian operator in $\mathfrak{h}$, $\mathfrak{h}_0 = \overline{\mathfrak{D}(A)}$, $\mathfrak{N} = \mathfrak{h}\ominus\mathfrak{h}_0$, $\tilde{A} \in \mathrm{Ex}_A$. Then $\mathfrak{N}''_\lambda(\tilde{A}) := (\tilde{A}-\lambda)^{-1}\mathfrak{N} \subset \mathfrak{N}_\lambda := \mathfrak{N}_\lambda(A)$ and the following equivalence holds: $\tilde{A}$ is an $\mathfrak{N}$-regular extension $\Longleftrightarrow \mathfrak{N}''_\lambda(\tilde{A})$ is a closed subspace.*

**Proof.** The inclusion $\mathfrak{N}''_\lambda(\tilde{A}) \subset \mathfrak{N}_\lambda$ is a consequence of the condition $\tilde{A} \in \mathrm{Ex}_A$. Clearly, $\tilde{A} + \hat{\mathfrak{N}} = \tilde{A}\dot{+}\hat{\mathfrak{N}}''$, where $\mathfrak{N}'' = \mathfrak{N}\ominus\tilde{A}(0)$. Assume that the linear manifold $\mathfrak{N}''_\lambda(\tilde{A})$ is closed, $n_k \in \mathfrak{N}''$, $\{f_k, f'_k\} \in \tilde{A}$, and the sequence $\{f_k, f'_k+n_k\}$ converges to $\hat{f} = \{f,g\}$ as $k\to\infty$. This implies that $f'_k - \lambda f_k + n_k$ converges to $g - \lambda f$, and by the equality $(A-\lambda)^{-1}(f'_k - \lambda f_k) = f_k$ there exists

$$\lim_{k\to\infty}(\tilde{A}-\lambda)^{-1}n_k = (\tilde{A}-\lambda)^{-1}(g-\lambda f) - f \in \mathfrak{N}''_\lambda(\tilde{A}). \tag{2.16}$$

Since $\mathfrak{N}''_\lambda(\tilde{A})$ is closed, we obtain from (2.16) that the sequence $n_k$ converges to some vector $n \in \mathfrak{N}''$ and the sequence $f'_k$ converges to vector $f' = g-n$. Therefore $g = f'+n$ and $\hat{f} = \{f, f'+n\} \in \tilde{A}\dot{+}\hat{\mathfrak{N}}'' = \tilde{A}+\hat{\mathfrak{N}}$, i.e., the linear manifold $\tilde{A}+\hat{\mathfrak{N}}$ is closed.

Conversely, assume that the linear manifold $\tilde{A}+\hat{\mathfrak{N}}$ is closed, $n''_k \in \mathfrak{N}''$, and the sequence $(\tilde{A}-\lambda)^{-1}n''_k$ converges to $h$ as $k\to\infty$. Then it follows from the relation

$$\{(\tilde{A}-\lambda)^{-1}n''_k, \lambda(\tilde{A}-\lambda)^{-1}n''_k\} = \{(\tilde{A}-\lambda)^{-1}n''_k, [I+\lambda(\tilde{A}-\lambda)^{-1}]n''_k\} - \{0, n''_k\} \in \tilde{A}\dot{+}\hat{\mathfrak{N}}''$$

that the sequence $n''_k \in \mathfrak{N}''$ also converges to some vector $n \in \mathfrak{N}''$. This implies that $h = (\tilde{A}-\lambda)^{-1}n$ and $\mathfrak{N}''_\lambda(\tilde{A})$ is closed. □

**Remark 2.6.** In the case $\tilde{A} = \tilde{A}^*$ we can choose a BVS $\Pi = \{\mathcal{H},\Gamma_1,\Gamma_2\}$ such that $\tilde{A} = \ker\Gamma_2 =: A_2$. Now Proposition 2.9 is implied by Corollary 2.4 and the relation

$$\Gamma_2\{(A_2-\lambda)^{-1}h, \lambda(A_2-\lambda)^{-1}h\} = \Gamma_2\{0,-h\} \Longrightarrow \Gamma_2\hat{\mathfrak{N}}_\lambda(A) = \Gamma_2\hat{\mathfrak{N}} =: \mathcal{H}_2$$

since the mapping $\Gamma_2\restriction\hat{\mathfrak{N}}_\lambda(A) : \hat{\mathfrak{N}}_\lambda(A) \to \mathcal{H}$ is a topological isomorphism and the linear manifolds $\mathfrak{N}_\lambda(A)$ and $\hat{\mathfrak{N}}_\lambda(A)$ are isomorphic.

## 3. FORMULA FOR GENERALIZED RESOLVENTS OF A NONDENSELY DEFINED HERMITIAN OPERATOR

**1.** An operator-valued $R$-function $Q(\lambda)$ ($\in R_{\mathcal{H}}$) with values in $[\mathcal{H}]$ is characterized by the following Nevanlinna integral representation (see [4–6])

$$Q(\lambda) = C_Q + B_Q\lambda + \int_{\mathbb{R}}\Big(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\Big)\,d\Sigma(t), \tag{3.1}$$

in which $C_Q = C_Q^* \in [\mathcal{H}]$, $B_Q \geq 0$, $\Sigma(t) = \Sigma(t-0)$ is a nondecreasing operator-valued function with values in $[\mathcal{H}]$ such that

$$\int_{\mathbb{R}} (1+t^2)^{-1}\, d\Sigma(t) \in [\mathcal{H}]. \tag{3.2}$$

Denote by $\mathcal{H}_2(Q)$ the linear manifold consisting of $h \in \mathcal{H}$, such that

$$\mathfrak{t}_Q^{(2)}[h] = \lim_{y\uparrow\infty} y(\operatorname{Im} Q(iy)h, h) < \infty \qquad (\mathfrak{D}(\mathfrak{t}_Q^{(2)}) = \mathcal{H}_2(Q)). \tag{3.3}$$

In view of the equality

$$y(\operatorname{Im} Q(iy)h, h) = y^2(B_Q h, h) + \int_{\mathbb{R}} \frac{y^2}{t^2+y^2}\, d(\Sigma(t)h, h) \tag{3.4}$$

[which is implied by (3.1)], the linear manifold $\mathcal{H}_2(Q)$ may be characterized differently:

$$\mathcal{H}_2(Q) = \tilde{\mathcal{H}}_2(Q) \cap \ker B_Q, \qquad \tilde{\mathcal{H}}_2 = \{h \in \mathcal{H} : \int_{\mathbb{R}} d(\Sigma(t)h, h) < \infty\}. \tag{3.5}$$

As a rule, the notations $C_Q$, $B_Q$, $\mathfrak{t}_Q$, $\mathcal{H}_2(Q)$, and others will be used without the subscript $Q$, which will be written explicitly only to avoid ambiguity from time to time.

**Proposition 3.1.** *Assume that $Q(\lambda) \in R_{\mathcal{H}}$ and takes values in $[\mathcal{H}]$. Then:*

(1) *for all $h \in \tilde{\mathcal{H}}_2 = \tilde{\mathcal{H}}_2(Q)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ the integrals*

$$Q_2(\lambda) := \int_{\mathbb{R}} \frac{d\Sigma(t)h}{t-\lambda}, \qquad \tilde{Q}_2(\lambda) := \int_{\mathbb{R}} \frac{d\Sigma(t)h}{|t|-\lambda} \tag{3.6}$$

*converge in the strong sense and for every $h \in \mathcal{H}_2$ the vector-valued function $Q(\lambda)h$ admits the representation*

$$Q(\lambda)h = Z_Q h + \int_{\mathbb{R}} \frac{d\Sigma(t)h}{t-\lambda}, \tag{3.7}$$

*in which $Z_Q$ is a Hermitian operator in $\mathcal{H}$, $\mathfrak{D}(Z_Q) = \mathcal{H}_2$;*

(2) *for every $h \in \mathcal{H}_2$ the relations*

$$s-\lim_{y\uparrow\infty} Q_2(iy)h = s-\lim_{y\uparrow\infty} \tilde{Q}_2(iy)h = 0, \qquad s-\lim_{y\uparrow\infty} Q(iy)h = Z_Q h \tag{3.8}$$

*are true and if the linear manifold $\mathcal{H}_2$ is closed, the operator $Z_Q$ is bounded;*

(3) *for every $y \in \mathbb{R}$ and $h \in \tilde{\mathcal{H}}_2$ the integral*

$$\tilde{Q}_{2R}(iy)h := \int_{-\infty}^{\infty} \frac{|t|\,d\Sigma(t)h}{t^2+y^2} \qquad (\mathfrak{D}(\tilde{Q}_{2R}(iy)) = \mathcal{H}_2) \tag{3.9}$$

*converges in the strong sense and the operator $\tilde{Q}_{2R}(iy)$ is closable;*

(4) *if $B_Q = 0$ and $Z_Q - C_Q \leq 0$, then the operator $Z_Q$ is closable.*

**Proof.** (1) Although the strong convergence of the integrals (3.6) can easily be proved with the help of the Cauchy–Bunyakovsky inequality, for our purposes it is more convenient to use the generalized Naimark lemma [5, 8]. According to this lemma we have

$$d\Sigma(t) = (1+t^2)K^*\, dE(t)K, \tag{3.10}$$

where $E(t) = E(t-0)$ is the orthogonal resolution of the identity for an operator $L = L^*$ acting in a separable Hilbert space $\mathfrak{h}$, $K \in [\mathcal{H}, \mathfrak{h}]$. Taking into account (3.10), we may rewrite the second condition in (3.5) in the form

$$\int_{\mathbb{R}} (1+t^2)\, d(E(t)Kh, Kh) < \infty \Longleftrightarrow Kh \in \mathfrak{D}(L), \tag{3.11}$$

that is, $\mathfrak{D}(LK) = \mathfrak{D}(K^*LK) = \tilde{\mathcal{H}}_2$. The representation

$$Q_2(\lambda)h = \int_{-\infty}^{\infty} \frac{d\Sigma(t)h}{t-\lambda} = K^* \int_{-\infty}^{\infty} \left(\frac{1+\lambda^2}{t-\lambda} + \lambda\right) dE(t)Kh + K^* \int_{-\infty}^{\infty} t\, dE(t)Kh \tag{3.12}$$

shows that the strong convergence of the integral on the left-hand side of (3.12) is equivalent to the convergence of the integral $\int_{\mathbb{R}} t\, dE(t)Kh$ in the strong sense, which holds true if condition (3.11) is satisfied [5]. Now for every $h \in \mathcal{H}_2$ the integral representation (3.1) of the vector-valued function $Q(\lambda)h$ may be written in the form in (3.7), where

$$Z_Q h = C_Q h - \int_{-\infty}^{\infty} \frac{t\, d\Sigma(t)h}{1+t^2} = C_Q h - K^* \int_{-\infty}^{\infty} t\, dE(t)Kh = (C_Q - K^*LK)h, \tag{3.13}$$

that is, $Z_Q = (C_Q - K^*LK)\restriction\mathcal{H}_2$ is the restriction of the operator $C_Q - K^*LK$ to $\mathcal{H}_2$.

(2) Since for all $h \in \mathcal{H}_2$ $Kh \in \mathfrak{D}(L) = \mathfrak{D}((I+L^2)^{1/2})$, we may put $h_1 = (I+L^2)^{1/2}Kh$ and rewrite equality (3.12) in the form

$$Q_2(\lambda)h = \int_{\mathbb{R}} \frac{d\Sigma(t)h}{t-\lambda} = K^*(I+L^2)^{1/2}(L-\lambda)^{-1}h_1. \tag{3.14}$$

Relations (3.8) are implied by (3.14) and by the evident equality $s-\lim_{y\uparrow\infty}(I+L^2)^{1/2}(L-iy)^{-1}f = 0$ $\forall f \in \mathfrak{h}$.

If the linear manifold $\mathcal{H}_2$ is closed, $Z_Q$ is a bounded linear operator, since it is the strong limit of a sequence of bounded operators [see (3.8)].

(3) The strong convergence of the integral (3.9) and, consequently, the correctness of the definition of the operator $\tilde{Q}_{2R}(iy)$ are implied by statement (1).

Setting $T = |L|^{1/2}(L^2+y^2)^{-1/2}(I+L^2)^{1/2}K$, we find that

$$T^*T \supset K^*|L| \cdot (I+L^2)(L^2+y^2)^{-1}K = \tilde{Q}_{2R}(iy).$$

The latter immediately leads to the closability of the operator $\tilde{Q}_{2R}(iy)$.

(4) Since $B_Q = 0$, we have $\tilde{\mathcal{H}}_2' = \mathcal{H}_2$ and $K^*LK = C_Q - Z_Q \geq 0$. Let $\mathfrak{h}_1 := \overline{K\mathcal{H}_2}$ and $L_1 := P_{\mathfrak{h}_1}L\restriction\mathfrak{h}_1$, $L_1 \geq 0$. Introducing the hard (Friedrichs) extension $L_{1F}$ of the operator $L_1 \geq 0$, we obtain

$$K^*LK = K^*L_1K \subset K^*L_{1F}K \subset (L_{1F}^{1/2}K)^*(L_{1F}^{1/2}K). \tag{3.15}$$

Hence follows statement (4).

**Corollary 3.1.** *Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS for a relation $A^*$ such that $A_2(0) = \{0\}$, let $M(\lambda)$ be the Weyl function corresponding to the BVS $\Pi$, and $\mathcal{F}_\Pi = \Gamma\hat{\mathfrak{N}}$ be the forbidden operator. Then $\mathcal{F}_\Pi = Z_M$.*

The result can be proved by a comparison of relation (3.8) with assertion 1 of Theorem 1.1.

**Remark 3.1.** We give an example showing that if the condition $Z_Q - C_Q \geq 0$ is not satisfied, the operator $Z_Q$ may be nonclosable.

Let operators $L_0$ and $K_0$ act in $\mathfrak{h} = l_2(1,\infty)$ with an orthogonal basis $\{e_n\}_1^\infty$ and let these operators be defined by the formulas $L_0 e_n = n^2 e_n$, $K_0^* e_n = \frac{1}{n} e_1$. Setting $\mathcal{H} = \mathfrak{h} \oplus \mathfrak{h}$, $L = \begin{pmatrix} 0 & L_0 \\ L_0 & 0 \end{pmatrix} \in \mathcal{C}(\mathcal{H})$, $K = \begin{pmatrix} I & 0 \\ 0 & K_0 \end{pmatrix} \in [\mathcal{H}]$, and taking $f_n = \begin{pmatrix} \frac{1}{n} e_n \\ 0 \end{pmatrix} \in \mathcal{H}$, we have $f_n \to 0$ as $n \to \infty$, although $K^*LKf_n = \begin{pmatrix} 0 \\ e_1 \end{pmatrix} \neq 0$. We define a function $Q(\lambda)$ by the formula

$$Q(\lambda) = \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) K^*\, dE_L(t)K.$$

Then $\mathcal{H}_2 = \mathfrak{D}(L_0) \oplus (\mathcal{H} \ominus e_1)$ and the operator $Z_Q = -K^*LK$ is nonclosable.

**Proposition 3.2.** *Assume that $Q(\lambda) \in R_{\mathcal{H}}$ and takes values in $[\mathcal{H}]$. Then the following statements hold true (the notation in Proposition 3.1 and in what follows is the same):*

(1) *for every $h \in \mathcal{H}_0 := \ker Z_Q (\subset \mathcal{H}_2)$ there exists a finite limit*

$$\mathfrak{t}_Q[h] := \lim_{y\uparrow\infty} iy(Q(-iy)h,h) < \infty \qquad \forall h \in \mathcal{H}_0 \tag{3.16}$$

*and the quadratic form $\mathfrak{t}_Q(\mathfrak{D}(\mathfrak{t}_Q) = \mathcal{H}_0(Q))$ is nonnegative and closed;*

(2) *the nonnegative self-adjoint operator $T_Q = T_Q^* \in \mathcal{C}(\mathcal{H}_0)$ associated with the form $\mathfrak{t}_Q$ (in accordance with the first representation theorem [29]) is bounded if and only if the linear manifold $\mathcal{H}_0$ is closed. In the former case the following strong limit exists:*

$$s - \lim_{y\uparrow\infty} iyQ_0(-iy) = T_Q(\in [\mathcal{H}_0]), \tag{3.17}$$

*where $Q_0(-iy) := P_{\mathcal{H}_0}Q(-iy)\lceil\mathcal{H}_0$;*

(3) *if $\mathcal{H}_2 = \mathcal{H}$ and $0 \in \rho(\mathrm{Im}\, Q(i))$, then the linear manifold $\mathcal{H}_0$ is closed, $T_Q = B_1^{-1}$ and the equalities*

$$\exists s - R - \lim_{y\uparrow\infty} iyQ_0(-iy) = T_Q = B_1^{-1}, \tag{3.18}$$

$$\mathfrak{t}_Q[h] = \lim_{y\uparrow\infty} iy(Q(-iy)h,h) = \|T_Q^{1/2}h\|^2 = \|B_1^{-1/2}h\|^2 \qquad \forall h \in \mathcal{H}_0,$$

*(in which $B_1 := B_{Q_1}\lceil\mathfrak{R}\,(B_{Q_1})$ and $B_{Q_1}$ is the coefficient at $\lambda$ in the integral representation (3.1) of the function $Q_1(\lambda) = -Q^{-1}(\lambda)$) hold true.*

**Proof.** It follows from equality (3.7) that the limit

$$\mathfrak{t}_Q[h] := \lim_{y\uparrow\infty} iy(Q(-iy)h,h) = \int_{-\infty}^{\infty} d(\Sigma(t)h,h) \qquad \forall h \in \mathcal{H}_0 \tag{3.19}$$

exists and the quadratic form $\mathfrak{t} = \mathfrak{t}_Q$ is nonnegative. In order to establish that this form is closed, we first show that the quadratic form $\mathfrak{t}^{(2)} := \mathfrak{t}_Q^{(2)}$ of the form in (3.3) is closed. We remark that the graph of the operator $T_2 := (I + L^2)^{1/2}K\lceil\mathcal{H}_2$ is closed since it is the intersection of two closed subspaces

$$\mathrm{gr}\, T_2 = \mathrm{gr}\,((I + L^2)^{1/2}K) \cap (\ker B_Q \oplus \mathfrak{h}).$$

Formulas (3.3), (3.10) and the equality

$$\mathfrak{t}^{(2)}[h] = \int_{-\infty}^{\infty} d(\Sigma(t)h,h) = \int_{-\infty}^{\infty} (1 + t^2)\, d(E(t)Kh,Kh) = \|T_2h\|^2 \tag{3.20}$$

imply that the quadratic form $\mathfrak{t}^{(2)} = \mathfrak{t}_q^{(2)}$ is closed. We show that the operator $T_0 := T_2\lceil\mathcal{H}_0$ is closed. If $h_n \in \mathfrak{D}(T_0)$, $h_n \to h$, and $T_0h_n \to g$ as $n \to \infty$, then $h \in \mathfrak{D}(T_2) = \mathcal{H}_2$ and $T_2h = g$, owing to the fact that $T_2$ is closed. Since $\mathfrak{D}(Z_Q) = \mathcal{H}_2 \subset \mathfrak{D}(LK)$, according to (3.13) we have

$$\|LK(h_n - h)\|^2 = \|T_2(h_n - h)\|^2 - \|K(h_n - h)\|^2.$$

From this we conclude that $0 = Z_Qh_n \to Z_Qh = 0$ as $n \to \infty$, that is, $h \in \ker Z_Q = \mathcal{H}_0$. The latter proves that $T_0$ is closed and consequently the quadratic form $\mathfrak{t}$ is closed by virtue of the equality

$$\mathfrak{t}[h] := \int_{-\infty}^{\infty} d(\Sigma(t)h,h) = \|T_0h\|^2 \qquad \forall h \in \mathcal{H}_0. \tag{3.21}$$

(2) Equality (3.21) shows that the operator $T = T_Q = T_0^* T_0$ associated with the form $\mathfrak{t}$ is self-adjoint. Since $\mathfrak{D}(T^{1/2}) = \mathfrak{D}(\mathfrak{t}) = \mathcal{H}_0$, the linear manifold $\mathcal{H}_0$ is closed if and only if the operator $T_Q^{1/2}$ is bounded (the Hellinger theorem), but the latter is equivalent to the boundedness of the operator $T_Q$. Since $\pm \int_0^{\pm\infty} d(\Sigma(t)h, h) < \infty$ for all $h \in \mathcal{H}_2$ we have nonnegative operators defined correctly for every $h \in \mathcal{H}_0$ by the formulas

$$R_1(-iy)h = P_{\mathcal{H}_0} \int_{-\infty}^{\infty} \frac{y^2 d\Sigma(t)h}{t^2 + y^2}, \qquad R_2^{\pm}(-iy)h = P_{\mathcal{H}_0} \int_0^{\pm\infty} \frac{y t d\Sigma(t)h}{t^2 + y^2}, \qquad \forall h \in \mathcal{H}_0.$$

The boundedness of the operators $R_1(-iy)$ is evident. The operators $R_2^{\pm}(-iy)$ are bounded in view of their closability (Proposition 3.1) and the fact that linear manifold $\mathcal{H}_0$ is closed. We can easily see that

$$\lim_{y\uparrow\infty} (R_1(-iy)h, h) = \int_{-\infty}^{\infty} d(\Sigma(t)h, h) = (Th, h), \qquad w - \lim_{y\uparrow\infty} R_2^{\pm}(-iy) = 0. \tag{3.22}$$

Since the weak convergence of $R_1(-iy)$ to $T$ is monotone and the operators $R_2^{\pm}(-iy)$ are nonnegative, the convergence in the weak sense in (3.22) implies that these operators converge also in the strong sense. Therefore, relation (3.17) is implied by the equality

$$iyQ_0(-iy) = R_1(-iy) + iR_2^+(-iy) - iR_2^-(-iy).$$

(3) If $\mathcal{H}_2 = \mathcal{H}$, the representation (3.7) holds true for all $h \in \mathcal{H}$ and $Z_Q = Z_Q^* \in [\mathcal{H}]$ (see Proposition 3.1). From this we obtain that $\mathcal{H}_0 = \ker Z_Q$ is closed. Let

$$Q(\lambda) = \begin{pmatrix} Q_{00}(\lambda) & Q_{01}(\lambda) \\ Q_{10}(\lambda) & Q_{11}(\lambda) \end{pmatrix}, Z_Q = \begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & Z_{11} \end{pmatrix}$$

be block-matrix representations of the operator-valued function $Q(\lambda)$ and of the operator $Z_Q$ corresponding to decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ ($\mathcal{H}_1 = \mathcal{H}_0^{\perp}$). Then we have

$$Q_{ij}(\lambda) = P_{\mathcal{H}_i} Q(\lambda) \upharpoonright \mathcal{H}_j = Z_{ij} + P_{\mathcal{H}_i} \int_{-\infty}^{\infty} \frac{d\Sigma(t)}{t - \lambda} \upharpoonright \mathcal{H}_j \qquad (i, j = 0, 1), \tag{3.23}$$

$Q_{00}(\lambda) = Q_0(\lambda)$, and $\ker Z_{11} = \{0\}$. It follows from the condition $0 \in \rho(\operatorname{Im} Q(i))$ that $0 \in \rho(Q_{00}(\lambda))$ for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and by the Frobenius formula, we have

$$Q^{-1}(\lambda) = \begin{pmatrix} Q_0^{-1} + Q_0^{-1} Q_{01} G^{-1} Q_{10} Q_0^{-1} & -Q_0^{-1} Q_{01} G^{-1} \\ -G^{-1} Q_{10} Q_0^{-1} & G^{-1} \end{pmatrix}, \tag{3.24}$$

where $G(\lambda) := Q_{11}(\lambda) - Q_{10}(\lambda) Q_0^{-1}(\lambda) Q_{01}(\lambda)$; since $Q_1(\lambda) := -Q^{-1}(\lambda) \in R_{\mathcal{H}}$ and takes values in $[\mathcal{H}]$, we can conclude that

$$\exists s - \lim_{y\uparrow\infty} \frac{Q_1(iy)}{iy} = s - \lim_{y\uparrow\infty} \frac{Q^{-1}(iy)}{-iy} = B_{Q_1} \geq 0, \qquad B_{Q_1} \in [\mathcal{H}].$$

By virtue of statement (2), the following limit exists in the strong sense: $s - \lim_{y\uparrow\infty} iyQ_0(-iy) = T_Q (\in [\mathcal{H}_0])$ since the linear manifold $\mathcal{H}_0$ is closed. Using this result and applying statement (2) to the function $Q(\lambda) - Z_Q$, we can see that the following strong limits exist:

$$\begin{aligned} C_{00} &:= s - \lim\nolimits_{y\uparrow\infty} (iy)^{-1} Q_0^{-1}(-iy) = T_Q^{-1}, & C_{01} &:= s - \lim\nolimits_{y\uparrow\infty} iyQ_{01}(-iy), \\ C_{10} &:= s - \lim\nolimits_{y\uparrow\infty} iyQ_{10}(-iy) = C_{01}^*, & C_{11} &:= s - \lim\nolimits_{y\uparrow\infty} iy[Q_{11}(-iy) - Z_{11}]. \end{aligned} \tag{3.25}$$

We have from (3.25)

$$\exists s - \lim_{y\uparrow\infty} iyQ_{10}(-iy)Q_{00}^{-1}(-iy)Q_{01}(-iy) = C_{01}^{*}C_{00}C_{01} \tag{3.26}$$

and consequently

$$\exists s - \lim_{y\uparrow\infty} Q_{10}(-iy)Q_{00}(-iy)Q_{01}(-iy) = 0. \tag{3.27}$$

Since $0 \in \rho(\text{Im } Q(i))$, we have $0 \in \rho(\text{Im } G(\lambda))$, i.e., $G(\lambda)$ is a Weyl function corresponding to some BVS (see Theorem 5.1). Using (3.23), (3.25), and (3.27), we can see that $B_G = 0$ and

$$Z_G := s - \lim_{y\uparrow\infty} G(iy) = s - \lim_{y\uparrow\infty} Q_{11}(iy) = Z_{11}.$$

Since $\ker Z_{11} = \ker Z_G = \{0\}$, according to Theorem 1.1 we have

$$B_{G^{-1}} = s - \lim_{y\uparrow\infty} \frac{G^{-1}(iy)}{iy} = 0. \tag{3.28}$$

We obtain from (3.25) and (3.28) that

$$s - \lim_{y\uparrow\infty} \frac{Q_{00}^{-1}(iy) + Q_{00}^{-1}(iy)Q_{01}(iy)G^{-1}(iy)Q_{10}(iy)Q_{00}^{-1}(iy)}{-iy}$$

$$= s - \lim_{y\uparrow\infty} (-iy)^{-1}Q_{00}^{-1}(iy) = C_{00} = T_Q^{-1}. \tag{3.29}$$

Consequently, (3.28) and (3.29) lead us to the relation

$$B_{Q_1} = s - \lim_{y\uparrow\infty} \frac{Q^{-1}(-iy)}{iy} = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_Q^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

proving statement (3). □

**Remark 3.2.** (1) It is noteworthy that a finite limit exists in (3.16) not only for all $h \in \mathcal{H}_0 = \ker Z_Q$. The following equivalence provides a sharper result:

$$\lim_{y\uparrow\infty} |iy(Q(-iy)h, h)| < \infty \iff (Z_Q h, h) = 0. \tag{3.30}$$

(2) Define an operator $S$ by the relation

$$Sh = \int_{-\infty}^{\infty} d\Sigma_0(t)h, \qquad \Sigma_0(t) = P_{\overline{\mathcal{H}}_0}\Sigma(t)\restriction\overline{\mathcal{H}}_0 \tag{3.31}$$

on the set $\mathfrak{D}(S)$ of the vectors for which the latter integral converges in the strong sense. If $\mathfrak{D}(S)$ is dense in $\overline{\mathcal{H}}_0$, then the operator $T_Q$ is the Friedrichs extension of the operator $S$ and the relation

$$s - R - \lim_{y\uparrow\infty} iyQ_0(-iy) = T_Q$$

holds true.

3) It must be mentioned that the invertibility of the operator $Q_{11}(i)$ in $[\mathcal{H}]$ [the fact has been used for proving assertion (3)] holds true provided that $0 \in \rho(\text{Im } Q(i))$, but it may not be true when $0 \in \rho(Q(i))$. The preresolvent matrix of a Hermitian operator $A$ gives a proper example (see §8).

In what follows we shall need

**Lemma 3.1.** *Assume that $S_n \in [\mathcal{H}]$, $T_n \in \tilde{C}(\mathcal{H})$. If*

$$\exists s - \lim_{n\to\infty} S_n = S, \qquad \exists s - R - \lim_{n\to\infty} T_n = T,$$

*and there exist numbers**) $\zeta_1 \in \Delta_b(S_n)$, $\zeta_2 \in \Delta_s(T_n)$ *such that* $\zeta_1 + \zeta_2 \in \Delta_b(S_n + T_n)$ *for all* $n \in \mathbb{Z}_+$, *then the following strong resolvent limit exists:*

$$s - R - \lim_{n\to\infty} (S_n + T_n) = S + T.$$

**Proof.** Since $S = s - \lim_{n\to\infty} S_n$, we have $S \in [\mathcal{H}]$. We derive from this and from the relation $\zeta_1 \in \Delta_b(S_n)$ that

$$\exists s - \lim_{n\to\infty} (S_n - \zeta_1)^{-1} = (S - \zeta_1)^{-1} \in [\mathcal{H}]. \tag{3.32}$$

Moreover, we have

$$s - \lim_{n\to\infty} (T_n - \zeta_2)^{-1} = (T - \zeta_2)^{-1}, \qquad \zeta_2 \in \Delta_s(T_n). \tag{3.33}$$

Now the condition $\zeta_1 + \zeta_2 \in \Delta_b(S_n + T_n)$ and the equality

$$I - (S_n - \zeta_n)(S_n + T_n - \zeta_1 - \zeta_2)^{-1} = [I + (S_n - \zeta_1)(T_n - \zeta_2)^{-1}]^{-1} \tag{3.34}$$

imply that $X_n^{-1}$ is a uniformly bounded sequence if $X_n := I + (S_n - \zeta_1)(T_n - \zeta_2)^{-1}$ by definition. It follows from this, (3.32), and (3.33) that the following two limits exist:

$$\begin{aligned} s - \lim_{n\to\infty} X_n &= I + (S - \zeta_1)(T - \zeta_2)^{-1}, \\ s - \lim_{n\to\infty} X_n^{-1} &= [I + (S - \zeta_1)(T - \zeta_2)^{-1}]^{-1}. \end{aligned} \tag{3.35}$$

We obtain from (3.35) and (3.34) that

$$\exists s - \lim_{n\to\infty} (S_n + T_n - \zeta_1 - \zeta_2)^{-1} = s - \lim_{n\to\infty} (S_n - \zeta_1)^{-1}[I - X_n^{-1}]$$

$$= -(S - \zeta_1)^{-1}\{[I + (S - \zeta_1)(T - \zeta_2)^{-1}]^{-1} - I\}$$

$$= -(S - \zeta_1)^{-1}[(T - \zeta_2)(T - \zeta_2 + S - \zeta_1)^{-1} - I] = (T + S - \zeta_1 - \zeta_2)^{-1}. \ \square$$

**Corollary 3.2.** *Assume that operators $S_n \in [\mathcal{H}]$ and linear relations $T_n \in \tilde{C}(H)$ are m-accretive. If the strong limit $s - \lim_{n\to\infty} S_n = S$ exists, then the following two relations are equivalent:*

$$\exists s - R - \lim_{n\to\infty} T_n = T \iff \exists s - R - \lim_{n\to\infty} (S_n + T_n) = S + T.$$

**Corollary 3.3.** *Assume that a sequence $T_n = T_n^* \in \tilde{C}(\mathcal{H})$ converges in the strong resolvent sense: $T = s - R - \lim_{n\to\infty} T_n$ and $K = K^* \in [\mathcal{H}]$. Then the limit*

$$s - R - \lim_{n\to\infty} (T_n + K) = T + K$$

*exists.*

**2.** We recall that the operator-valued function $\mathbf{R}_\lambda = P(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{h}$ holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$ is called the generalized pseudoresolvent of an operator $A$ and the notation $\mathbf{R}_\lambda \in P\Omega_A$ is used if $\tilde{A} (\in \tilde{C}(\tilde{\mathfrak{h}}))$ is the self-adjoint extension of $A$ going out into a larger Hilbert space $\tilde{\mathfrak{h}} \supset \mathfrak{h}$, $P = P_{\mathfrak{h}}$ is the orthogonal projection

*) Following [29], we denote the domains of boundedness and of the strong convergence of the sequence $(T_n - \zeta)^{-1}$ by $\Delta_b(T_n)$ and $\Delta_s(T_n)$ respectively.

of $\tilde{\mathfrak{h}}$ onto $\mathfrak{h}$. The set of generalized resolvents $\mathbf{R}_\lambda \in P\Omega_A$ for which $\tilde{A} \in \mathcal{C}(\tilde{\mathfrak{h}})$ (that is, $\tilde{A}$ is an operator) is denoted by $\Omega_A$.

The collection of extensions $\tilde{A} = \tilde{A}^* \in \tilde{\mathcal{C}}(\tilde{\mathfrak{h}})$ generating the resolvent $\mathbf{R}_\lambda \in P\Omega_A$ contains minimal extensions, i.e., such that

$$\tilde{\mathfrak{h}} \ominus \mathfrak{h} = c.l.s.\{(I-P)(\tilde{A}-\lambda)^{-1}\mathfrak{h} : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-\}. \tag{3.36}$$

A minimal extension is not unique although two arbitrary minimal extensions $\tilde{A}^{(i)} \in \tilde{\mathcal{C}}(\tilde{\mathfrak{h}}_i)$ $(i = 1,2)$ are unitary isomorphic. Moreover, there exists an isometric isomorphism $U$ from $\tilde{\mathfrak{h}}_1$ onto $\tilde{\mathfrak{h}}_2$ such that $Uf = f$ for all $f \in \mathfrak{h}$ and $\hat{U}\tilde{A}^{(1)} = \tilde{A}^{(2)}$, where $\hat{U}\{h,g\} = \{Uh, Ug\}$ for all $h, g \in \tilde{\mathfrak{h}}$.

In what follows we shall always assume that the resolvent $\mathbf{R}_\lambda \in P\Omega_A$ is generated by the minimal extension $\tilde{A}$.

According to a well-known Naimark theorem [5, 54] (and the generalization of this theorem relating to linear relations $\tilde{A}$ [77]), the generalized resolvent can be characterized by the integral representation

$$\mathbf{R}_\lambda = \int_{-\infty}^{\infty} \frac{d\Sigma(t)}{t-\lambda} = P_{\mathfrak{h}} \int_{-\infty}^{\infty} \frac{dE(t)}{t-\lambda} \restriction \mathfrak{h} = P_{\mathfrak{h}}(\tilde{A}-\lambda)^{-1} \restriction \mathfrak{h},$$

in which $\Sigma(t) = \Sigma(t-0)$ is a nondecreasing operator-valued function, $E(t)$ is an orthogonal spectral function $\Sigma(-\infty) = E(-\infty) - \lim_{t \downarrow -\infty} E(t) = 0$, $E(\infty) := s - \lim_{t \uparrow +\infty} E(t)$ is an orthogonal projection of $\tilde{\mathfrak{h}}$ onto $\overline{\mathfrak{D}(\tilde{A})}$, $\Sigma(\infty) = P_{\mathfrak{h}} E(\infty) \restriction \mathfrak{h} := s - \lim_{y \uparrow \infty} (-iy)\mathbf{R}(iy)$.

The following equivalences are evidently true:

$$\Sigma(+\infty) = I_{\mathfrak{h}} \iff E(+\infty) = I_{\tilde{\mathfrak{h}}} \iff \mathbf{R}_\lambda \in \Omega_A \iff \tilde{A} \in \mathcal{C}(\tilde{\mathfrak{h}}). \tag{3.37}$$

The following evident lemma will be useful for our further consideration.

**Lemma 3.2.** *Assume that* $h_n \in \mathcal{H}$, $\exists \lim_{n\to\infty} h_n = h$, $X_n \in [\mathcal{H}]$, *and* $\exists s - \lim_{n\to\infty} X_n = X$. *Then there exists the limit* $\lim_{n\to\infty} X_n h_n = Xh$.

**Lemma 3.3.** *Assume that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for a relation* $A^*$, *for which* $A_1(0) = \{0\}$, *that is* $A_1 \in \mathcal{C}(\mathfrak{h})$, $M(\lambda)$ *is the corresponding Weyl function,* $M_1(\lambda) = -M(\lambda)^{-1}$, $\mathcal{F}_\Pi$ *is the forbidden linear manifold. Then the limit*

$$\mathfrak{t}[h] := \mathfrak{t}_{M_1}[h] = \lim_{y\uparrow\infty} iy(M^{-1}(iy)h, h) \tag{3.38}$$

*exists for all* $h \in \mathcal{F}_\Pi(0)$.

**Proof.** The operator-valued function $M_1(\lambda) = -M(\lambda)^{-1}$ is the Weyl function corresponding to the BVS $\Pi_1 = \{\mathcal{H}, \Gamma_2, -\Gamma_1\}$, the forbidden linear manifold $\mathcal{F}_{\Pi_1}$ is an operator ($\mathcal{F}_{\Pi_1}(0) = \{0\} \iff A_1(0) = \{0\}$), and $\mathcal{F}_{\Pi_1} = -\mathcal{F}_\Pi^{-1}$. In view of Corollary 3.1 we have $\mathcal{F}_{\Pi_1} = M_1(\infty)$ and $\ker M_1(\infty) = \ker \mathcal{F}_{\Pi_1} \mathcal{F}_\Pi(0)$. Applying relation (3.16) to the equality $Q(\lambda) = M_1(\lambda)$, we obtain relation (3.38). □

**Theorem 3.1.** *Assume that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for a relation* $A^*$ *for which* $A_1(0) = \{0\}$, $M(\lambda)$ *is the corresponding Weyl function. Then:*

(1) *the equality*

$$(\tilde{A}_\theta - \lambda)^{-1} = (A_2 - \lambda)^{-1} + \gamma(\lambda)(\theta - M(\lambda))^{-1}\gamma^*(\bar{\lambda}) \tag{3.39}$$

*establishes a bijective correspondence between the resolvents* $R_\lambda = (\tilde{A}_\theta - \lambda)^{-1}$ *of the proper extensions* $\tilde{A}_\theta$ *of an operator* $A$ *and the closed linear relations* $\theta$ *in* $\mathcal{H}$;

(2) *the formula*

$$\mathbf{R}_\lambda = P(\tilde{A}-\lambda)^{-1} \restriction \mathfrak{h} = (A_2 - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1}\gamma^*(\bar{\lambda}) \tag{3.40}$$

*establishes a bijective correspondence between* $\mathbf{R}_\lambda \in P\Omega_A$ *and* $\tau(\lambda) \in \tilde{R}_{\mathcal{H}}$;

(3) $\mathbf{R}_\lambda = (\tilde{A}_{-\tau(\lambda)} - \lambda)^{-1}$, *that is, for every* $g \in \mathfrak{h}$, $\mathbf{R}_\lambda g$ *is the solution of the boundary-value problem with the spectral parameter* $\tau(\lambda)$ *in the boundary condition:*

$$\{f, g\} \in A^* - \lambda, \{\Gamma_2 \hat{f}, \Gamma_1 \hat{f}\} \in -\tau(\lambda), \tag{3.41}$$

*where* $\hat{f} = \{f, g + \lambda f\} \in A^*$;

(4) $\mathbf{R}_\lambda \in \Omega_A$ *in formula* (3.40) *if and only if* $\tau(\lambda)$ *satisfies the following conditions:*

(a) $\exists \lim_{y\to\infty} iy((\tau(iy) + M(iy))^{-1}h, h) = \mathfrak{t}_{M_1}[h] \qquad \forall h \in \mathcal{F}_\Pi(0)$; (3.42)

(b) $s - \lim_{y\to\infty} \frac{(\tau(iy)+M(iy))^{-1}}{y} = 0$, (3.43)

*where* $M_1(\lambda) = -M(\lambda)^{-1}$.

**Proof.** Assertions (1)–(3) have been proved in [53] (in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ they have been established in [20, 79]). We shall prove assertion (4).

*Necessity.* (a) Let $\tilde{A} = \tilde{A}^* \in \mathcal{C}(\tilde{\mathfrak{h}})$, that is, $\tilde{A}$ is an operator. Putting $\widehat{A_2(0)} = \{\{0, f\} : f \in A_2(0)\}$, $\mathfrak{h}_1 = A_2(0)^\perp$, and $\mathfrak{h} = \mathfrak{h}_1 \oplus A_2(0)$, we can easily see that

$$s - \lim_{y\to\infty} iy(A_2 - iy)^{-1} = -P_{A_2(0)^\perp} = -P_{\mathfrak{h}_1}. \tag{3.44}$$

If $\mathbf{R}_\lambda \in \Omega_A$, then it follows from (3.37) and (3.44) that

$$s - \lim_{y\uparrow\infty} iy[(A_2 - iy)^{-1} - \mathbf{R}_{iy}] = I - P_{A_2}(0)^\perp = P_{A_2}(0). \tag{3.45}$$

Further, we use the equality (see Proposition 1.5)

$$\gamma^*(\bar{\lambda})f = \Gamma_1\{(A_2 - \lambda)^{-1}f, f + \lambda(A_2 - \lambda)^{-1}f\} \qquad (f \in \mathfrak{h}). \tag{3.46}$$

We obtain from (3.46) that

$$\gamma^*(\bar{\lambda})f = \Gamma_1\{0, f\} \quad \forall f \in A_2(0), \qquad \forall \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{3.47}$$

In particular, $\gamma^*(\bar{\lambda})A_2(0) = \mathcal{F}_\Pi(0) := \Gamma_1\widehat{A_2(0)}\ \forall \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

It can also be seen from (3.47) that $\gamma^*(\bar{\lambda})f$ does not depend on $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ for every $f \in A_2(0)$. Setting $h = \gamma^*(i)f (= \gamma^*(\lambda)f\ \ \forall \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)$ for $f \in A_2(0)$ and taking into account the resolvent formula (3.40) and equality (3.45), we get the relation

$$\lim_{\lambda=iy\to\infty} \lambda((\tau(\lambda) + M(\lambda))^{-1}h, h) = \lim_{\lambda=iy\to\infty} \lambda((\tau(\lambda) + M(\lambda))^{-1}\gamma^*(\bar{\lambda})f, \gamma^*(\bar{\lambda})f)$$

$$= \lim_{\lambda=iy\to\infty} \lambda([(A_2 - \lambda)^{-1} - (\tilde{A}_{-\tau(\lambda)} - \lambda)^{-1}]f, f) = \|f\|^2 \qquad \forall h \in \mathcal{F}_\Pi(0). \tag{3.48}$$

In particular, relation (3.48) is true in the case $\tau(\lambda) = 0$ also (since $A_1 \in \mathcal{C}(\mathfrak{h})$), that is, for $h = \gamma^*(\lambda)f$, $f \in A_2(0)$ we have

$$\mathfrak{t}[h] := \mathfrak{t}_{M_1}[h] = \lim_{y\uparrow\infty} iy(M^{-1}(iy)h, h) = \|f\|^2. \tag{3.49}$$

Relations (3.48), (3.49), and (3.37) imply that condition (3.42) is satisfied.

(b) Being self-adjoint, the linear relation $A_2$ admits the canonical decomposition $A_2 = A_2' \oplus \widehat{A_2(0)}$, in which $A_2'$ is the operator part of the linear relation $A_2$ and $\mathfrak{D}(A_2) = \mathfrak{D}(A_2') \perp A_2(0)$. Therefore, it follows from (3.46) that $\forall f \in \mathfrak{D}(A_2')$

$$\lim_{y\uparrow\infty} iy\gamma^*(-iy)f = \lim_{y\uparrow\infty} \Gamma_1\{iy(A_2 - iy)^{-1}f, iy(A_2 - iy)^{-1}A_2'f\} = -\Gamma_1\{f, A_2'f\}. \tag{3.50}$$

Since $G(\lambda) := -(\tau(\lambda)+M(\lambda))^{-1} \in R_{\mathcal{H}}$ and takes values in $[\mathcal{H}]$, the integral representation (3.1) implies that the limit $B_G = s - \lim_{\lambda\to\infty} G(\lambda)/\lambda$ exists. If we observe that $P_{A_2(0)}f = 0\ \forall f \in \mathfrak{D}(A_2') \subset A_2(0)^{\perp}$ we can obtain from (3.45), (3.50), the resolvent formula (3.40), and from Lemma 3.2 that

$$0 = \|P_{A_2(0)}f\|^2 = \lim_{\lambda\to\infty} \lambda(\gamma(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\bar\lambda)f, f) = -\lim_{\lambda\to\infty}\left(\frac{G(\lambda)}{\lambda}\lambda\gamma^*(\bar\lambda)f, \bar\lambda\gamma^*(\lambda)f\right)$$

$$= -\|B_G^{1/2}\Gamma_1\{f, A_2'f\}\|^2, \tag{3.51}$$

where $\lambda = iy$. Consequently, we have

$$B_G^{1/2}\Gamma_1(\mathrm{gr}\ A_2') = 0 \Longrightarrow B_G\Gamma_1(\mathrm{gr}\ A_2') = 0. \tag{3.52}$$

On the other hand, (3.48) evidently implies that

$$(B_G h, h) = \lim_{\lambda\to\infty}\left(\frac{G(\lambda)}{\lambda}h, h\right) = 0 \qquad \forall h \in \mathcal{F}_\Pi(0). \tag{3.53}$$

It follows from (3.52), (3.53), and the equality

$$\mathcal{H} = \Gamma_1 A_2 = \Gamma_1 \mathrm{gr}\ A_2' \dotplus \Gamma_1\widehat{A_2(0)} = \Gamma_1\mathrm{gr}\ A_2' \dotplus \mathcal{F}_\Pi(0)$$

that condition (3.43) is satisfied.

*Sufficiency.* Suppose that $\mathbf{R}_\lambda \in P\Omega_A$ and conditions (a) and (b) are satisfied. Then

$$s - \lim_{y\to\infty} iy\mathbf{R}_{iy} = s\lim_{y\to\infty} P_{\mathfrak{h}}(\tilde A - iy)^{-1}\restriction\mathfrak{h} = -P_{\mathfrak{h}}Q, \tag{3.54}$$

where Q is the orthoprojection of $\tilde{\mathfrak{h}}$ onto $\tilde A(0)^{\perp} \subset \tilde{\mathfrak{h}}$. Since $\tilde A = \tilde A^*$ is a minimal extension, we have $\tilde A(0) \subset \mathfrak{h}$ (see [80]). Now the resolvent formula (3.40) and condition (a) imply that $\forall f \in A_2(0)$ and for $h = \gamma^*(i)f (= \gamma^*(\lambda)f)\quad \forall\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)$

$$((P_{\mathfrak{h}}Q - P_{A_2(0)^{\perp}})f, f) = \lim_{y\uparrow\infty} iy(((A_2 - iy)^{-1} - \mathbf{R}_{iy})f, f) = \lim_{y\uparrow\infty} iy((\tau(iy)+M(iy))^{-1}h, h)$$

$$= \lim_{y\uparrow\infty}(M^{-1}(iy)h, h) = \lim_{y\uparrow\infty} iy([(A_2 - iy)^{-1} - (A_1 - iy)^{-1}]f, f) = ([I - P_{A_2(0)^{\perp}}]f, f) = \|f\|^2$$

Thus we have

$$(P_{\mathfrak{h}}Qf, f) = \|f\|^2 \Longrightarrow (I - Q)f = 0 \qquad \forall f \in A_2(0). \tag{3.55}$$

Further, condition (c) means that $B_G = 0\ \ (G(\lambda) = -(\tau(\lambda)+M(\lambda))^{-1})$. Therefore, the resolvent formula (3.40) and equality (3.51) lead to the relation

$$\lim_{y\uparrow\infty} iy[(A_2 - iy)^{-1} - \mathbf{R}_{iy}]f = 0 \qquad \forall f \in A_2(0)^{\perp},$$

which [if we take into account (3.54) and (3.44)] means that

$$(P_{\mathfrak{h}}Qf, f) - \|f\|^2 = 0 \Longrightarrow (I_Q)f = 0 \qquad \forall f \in A_2(0)^{\perp}. \tag{3.56}$$

Since $\mathfrak{h} = A_2(0) \oplus A_2(0)^{\perp}$, we derive from (3.55) and (3.56) that $Qf = f\quad \forall f \in \mathfrak{h}$, that is, $\mathfrak{h} \subset \tilde A(0)^{\perp} \Longrightarrow \tilde A(0) \perp \mathfrak{h}$. The latter, together with condition (3.36) ($\tilde A$ is a minimal extension), mean that $\tilde A(0) = \{0\}$, that is, $\tilde A$ is an operator. □

**Definition 3.1.** *A holomorphic family of relations $\tau(\lambda) \in \tilde R_{\mathcal{H}}$ will be called M-admissible if the generalized resolvent $\mathbf{R}_\lambda \in \Omega_A$ corresponding to it in formula* (3.40) *is generated by an operator $\tilde A = \tilde A^*$.*

Consequently, if $A_1 = A_1^* \in \mathcal{C}(\mathfrak{h})$, then the family $\tau(\lambda)$ is $M$-admissible if and only if conditions (3.42) and (3.43) are satisfied.

**Corollary 3.4.** *Assume that under the conditions of Theorem* 3.1 *we have $A_1(0) \neq \{0\}$. Then the condition of $M$-admissibility* (3.43) *stays true, but condition* (3.42) *takes the following form :* $\forall h \in \mathcal{F}_\Pi(0)$

$$\exists \lim_{y\uparrow\infty}(iy[\tau(-iy)+M(-iy)]^{-1}h,h) = \lim_{y\uparrow\infty} iy([M(-iy)+K]^{-1}h,h), \tag{3.42$'$}$$

*where $K = K^*$ is an operator ($\in [\mathcal{H}]$) such that $\tilde{A}_K(0) = \{0\}$ (that is, $A_k$ is an operator).*

**Proof.** It is not difficult to see that an operator $K = K^* \in [\mathcal{H}]$ such that gr $K \cap \mathcal{F}_\Pi\{0\}$ exists and therefore $\tilde{A}_K(0) = \{0\}$. Let $\Pi' = \{\mathcal{H}, \Gamma_1', \Gamma_2\}$ be a BVS for which $\Gamma_1' = \Gamma_1 - K\Gamma_2$, $M'(\lambda)$ be the corresponding Weyl function. Then $\mathcal{F}_{\Pi'}(0) = \mathcal{F}_\Pi(0)$, $M'(\lambda) = M(\lambda) - K$, $A_1' = \ker \Gamma_1' = \tilde{A}_K$, and condition (3.41), which determines the generalized resolvent $\mathbf{R}_\lambda$, takes the form

$$\{\Gamma_2\hat{f}, -\Gamma_1'\hat{f}\} \in \tau'(\lambda) := \tau(\lambda) + K \quad (\hat{f} \in A^*).$$

Thus relation (3.42$'$) is implied by (3.42) and the relation $A_1'(0) = \{0\}$. On account of the equality $B_{M'} = B_M$ condition (3.43) remains true.

**3.** We should remark that the condition of $M$-admissibility for $\tau(\lambda)$ reduces to condition (3.43) or (3.42) if $A_2(0) = \{0\}$ or $A_2 = \tilde{A}_{\mathcal{F}_\Pi}(:= A \dotplus \hat{\mathfrak{N}})$ respectively. If $A_2(0) = \{0\}$, then we derive from Theorem 3.1 and Corollary 3.2 that the following holds true.

**Corollary 3.5** ([53]). *Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS for a relation $A^*$ for which $A_2(0) = \{0\}$, $M(\lambda)$ be the corresponding Weyl function. Then equality* (3.40) *establishes a bijective correspondence between $\mathbf{R}_\lambda \in \Omega_A$ and $\tau(\lambda) \in \tilde{R}_\mathcal{H}$ satisfying condition* (3.43). *It is a well-known fact that an indeterminate part $(\tau(\lambda))(0)$ of a family $\tau(\lambda)$ does not depend on $\lambda \in \mathbb{C}_+$ and $\tau(\lambda) =$ gr $\tau_1(\lambda) \oplus \{0, [\tau(\lambda)](0)\}$* ([36, 77]), *where $\tau_1(\lambda)$ is the operator part of the relation $\tau(\lambda)$, $\tau_1(\lambda) \in \mathcal{C}(\mathcal{H}^1)$, $\mathcal{H}^1 = \mathcal{H} \ominus (\tau(\lambda))(0)$. If $\tau_1(\lambda)$ takes values in $[\mathcal{H}^1]$, we define a Hermitian relation $\tau(\infty)$ by putting $\tau(\infty) = \tau_1(\infty) \oplus \{0, (\tau(\lambda))(0)\}$, where*

$$\tau_1(\infty)h = s - \lim_{y\uparrow\infty} \tau_1(iy)h \qquad \forall h \in \mathcal{H}_2(\tau_1).$$

We characterize the condition of $M$-admissibility (3.43) in terms of the relation $\tau(\infty)$.

**Proposition 3.3.** *Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS for a relation $A^*$ for which $A_2$ is an operator ($A_2(0) = \{0\}$), $M(\lambda)$ and $\mathcal{F}_\Pi$ are the corresponding Weyl function and the forbidden linear manifold respectively, $\tau(\lambda) \in \tilde{R}_\mathcal{H}$. If the operator-valued function $\tau_1(\lambda)$ takes values in $[\mathcal{H}^1]$, then condition* (3.43) *of $M$-admissibility of the family $\tau(\lambda)$ and the admissibility condition of the relation $-\tau(\infty) \oplus \{0, \mathfrak{R}(B_\tau^{1/2})\}$ are equivalent, that is, the following equivalence holds true:*

$$s - \lim_{\lambda\to\infty} \lambda^{-1}(\tau(\lambda) + M(\lambda))^{-1} = 0 \Longleftrightarrow \{-\tau(\infty) \oplus \{0, \mathfrak{R}(B_\tau^{1/2})\}\} \cap \mathcal{F}_\Pi = \{0\}.$$

**Proof.** Consider (for simplicity) the case $\tau_1(\lambda) = \tau(\lambda)$ (that is, $(\tau(\lambda))(0) = \{0\}$). Since $0 \in \rho(\mathrm{Im}\ M(i))$, we have $0 \in \rho(\mathrm{Im}\ (M(i) + \tau(i))$ and according to Theorem 5.1 $G(\lambda) := M(\lambda) + \tau(\lambda)$ is a Weyl function corresponding to a BVS $\Pi' = \{\mathcal{H}, \Gamma_1', \Gamma_2'\}$. Then we have $-G(\lambda)^{-1} = -(M(\lambda) + \tau(\lambda))^{-1}$ and it is the Weyl function corresponding to the BVS $\Pi'' = \{\mathcal{H}, \Gamma_2', -\Gamma_1'\}$; in view of Theorem 1.1 the condition $s - \lim\limits_{y\uparrow\infty} \dfrac{G(iy)^{-1}}{y} = 0$ means that $A_1' = \ker \Gamma_1'$ is an operator. Since the zero operator $\theta = \mathbb{O}_\mathcal{H} = \{\mathcal{H}, 0\}$ corresponds to the extension $A_1'$ in formula (1.4), that is, $A_1' = \tilde{A}_\mathbb{O}$, it follows from Proposition 1.4 that $\tilde{A}_1'(0) = \{0\} \Longleftrightarrow \mathbb{O}_\mathcal{H} \cap \mathcal{F}_{\Pi'} = \{0\}$, where the forbidden linear manifold $\mathcal{F}_{\Pi'}$ of the BVS $\Pi'$, on account of Proposition 2.8, is of the form $\mathcal{F}_{\Pi'} =$ gr $G(\infty) \dotplus \{0, \mathfrak{R}(B_G^{1/2})\}$. Since $A_2(0) = \{0\}$, it follows that $B_M = \mathbb{O}$, that is, $B_G = B_\tau$ and gr $M(\infty) = \mathcal{F}_\Pi$ (Theorem 1.1). Thus, the condition $\mathbb{O}_\mathcal{H} \cap \mathcal{F}_{\Pi'} = \{0\}$ is equivalent to the following one:

$$[-\tau(\infty) \oplus \{0, \mathfrak{R}(B_\tau^{1/2})\}] \cap \mathcal{F}_\Pi = \{0\}. \quad \square$$

**Corollary 3.5.** *Under the conditions of Proposition* 3.3 *the implication*

$$s - \lim_{\lambda = iy\uparrow\infty} \lambda^{-1}(\tau(\lambda) + M(\lambda))^{-1} = 0 \Longrightarrow -\tau(\infty) \cap \mathcal{F}_\Pi = \{0\}$$

*holds true. If, in addition,* $B_{\tau_1} = \mathbb{O}$, *then the implication must be replaced by the equivalence.*

**Corollary 3.6.** *Let the conditions of Proposition* 3.3 *be satisfied and the family* $\tau(\lambda) \in \tilde{R}_\mathcal{H}$ *be* $M$*-admissible. Then for every* $\lambda_0 (\in \mathbb{C}_+ \cup \mathbb{C}_-)$ *the relation* $-\tau(\lambda_0)$ *is admissible, that is,* $(-\tau(\lambda_0)) \cap \mathcal{F}_\Pi = \{0\}$.

**Proof.** If $(-\tau(\lambda_0)) \cap \mathcal{F}_\Pi \neq \{0\}$ for some $\lambda_0 \in \mathbb{C}$, then $\exists (0 \neq) h \in \ker(\tau(\lambda_0) + \mathcal{F}_\Pi)$. Consequently $k := (\tau(\lambda_0)h, h) = -(\mathcal{F}_\Pi h, h)$ is real and thus for every $\lambda \in \mathbb{C}_+$ we have $(\tau(\lambda)h, h) = k$. It follows that $h \in \mathcal{H}_2(\tau)$ and

$$\tau(\lambda)h = \tau(\lambda_0)h \quad \forall \lambda \in \mathbb{C}_+ \Longrightarrow \tau(\infty)h = \lim_{y\uparrow\infty} \tau(iy)h = \tau(\lambda_0)h = -\mathcal{F}_\Pi h.$$

In view of Corollary 3.5, the latter is in contradiction with the $M$-admissibility of the family $\tau(\lambda)$. □

**Corollary 3.7.** *All the families* $\tau(\lambda) \in \tilde{R}_\mathcal{H}$ *are* $M$*-admissible if and only if* $\overline{\mathfrak{D}(A)} = \mathfrak{h}$.

**Proof.** In the case $A_2(0) = \{0\}$ the statement follows from Proposition 3.3 and an evident equivalence, namely, $\overline{\mathfrak{D}(A)} = \mathfrak{h} \Longleftrightarrow \mathfrak{D}(\mathcal{F}_\Pi) = \{0\}$. □

**4. Remark 3.4** Assume that the conditions of Theorem 3.1 are satisfied, $\mathcal{H}_1 = \Gamma_1\hat{\mathfrak{N}} = \mathcal{H}$, and $\tau(i) \in [\mathcal{H}]$. Then, according to Proposition (3.2), the $M$-admissibility condition (3.42) can be made more precise:

$$\exists \lim_{y\uparrow\infty} iy([\tau(iy) + M(iy)]^{-1}h, h) = (B^{-1}h, h) \qquad \forall h \in \mathcal{F}_\Pi(0). \tag{3.57}$$

We denote by $B = B_M$ in (3.57) the coefficient of $\lambda$ in the integral representation (3.1) of the Weyl function $M(\lambda)$. The equality (3.57) is true, in particular, provided that $A_2 = A \dot{+} \hat{\mathfrak{N}}$, since

$$A_2 = A \dot{+} \hat{\mathfrak{N}} \Longleftrightarrow \mathcal{F}_\Pi = \{0, \mathcal{H}\} \Longleftrightarrow \mathcal{F}_\Pi(0) = \mathcal{H} \Longleftrightarrow \mathcal{H}_1 = \mathcal{H}, \ker \mathcal{F}_\Pi^{-1} = \mathcal{H}. \tag{3.58}$$

In the latter case, condition (3.42) can be rewritten in a more simple form.

**Corollary 3.8.** *Under the conditions of Theorem* 3.1 *assume that* $A_2 = \tilde{A}_{\mathcal{F}_\Pi} := A \dot{+} \hat{\mathfrak{N}}$. *Then the family* $\tau(\lambda) \in \tilde{R}_\mathcal{H}$ *is* $M$*-admissible if and only if the limit*

$$s - R - \lim_{y\uparrow\infty} \frac{\tau(iy)}{y} = 0 \tag{3.59}$$

*exists. In the latter case,* $\tau(\lambda) \in R_\mathcal{H}$ *(that is,* $(\tau(\lambda)(0) = \{0\}$ *for all* $\lambda \in \mathbb{C}_+$*). In particular, if* $\tau(\lambda)$ *takes values in* $[\mathcal{H}]$, *condition* (3.59) *takes the form*

$$B_\tau = s - \lim_{y\uparrow\infty} (iy)^{-1}\tau(iy) = 0 \tag{3.59}$$

**Proof.** Since $\mathcal{H}_0(-M^{-1}) = \mathcal{F}_\Pi(0) = \mathcal{H}$ [see (3.58)], the existence of the strong limit

$$s - \lim_{y\uparrow\infty} iyM^{-1}(iy) = B^{-1} (= B_M^{-1}) \in [\mathcal{H}] \tag{3.60}$$

can be established by applying Proposition 3.2 to the operator-valued function $Q(\lambda) = -M(\lambda)^{-1}$. Suppose that $\tau(\lambda)$ is $M$-admissible. Then for the operator-valued function $G(\lambda) := -(\tau(\lambda) + M(\lambda))^{-1}$ and for arbitrary $h \in \mathcal{H} = \mathcal{F}_\Pi(0)$ the weak limit in (3.42) exists and, on account of the equivalence (3.30), we see that $(Z_G h, h) = 0$ for all $h \in \mathcal{H}$. It follows from assertion (2) of Proposition 3.2 and relations (3.60) and (3.42) that the strong limit exists and the equality

$$s - \lim_{y\uparrow\infty} iyG(iy) = s - \lim_{y\uparrow\infty} \left(\frac{\tau(iy) + M(iy)}{iy}\right)^{-1} = B^{-1} \tag{3.61}$$

is true. From (3.60) and (3.61) we obtain the following relation:

$$\exists s - R - \lim_{y\uparrow\infty} \frac{\tau(iy) + M(iy)}{iy} = B = s - \lim_{y\uparrow\infty} \frac{M(iy)}{iy}. \tag{3.62}$$

Since operators $(iy)^{-1}M(iy)$ and relations $(iy)^{-1}\tau(iy)$ are $m$-accretive for $y > 0$, (3.59) follows from (3.62) and Corollary 3.2.

Conversely, equality (3.59) and (once again) Corollary 3.2 imply relations (3.62). Since $B^{-1} \in [\mathcal{H}]$ and $0 \in \rho(\tau(iy) + M(iy))$, we derive (3.61) and, consequently, (3.42) from (3.62). The latter proves the $M$-admissibility of $\tau(\lambda)$. The inclusion $\tau(\lambda) \in R_{\mathcal{H}}$ is a consequence of equality (3.59).

The converse of Corollary 3.8 holds.

**Proposition 3.4.** *Assume that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for a relation* $A^*$, $M(\lambda)$ *is the corresponding Weyl function,* $\mathcal{F}_\Pi$ *is the forbidden linear manifold. Then the condition of $M$-admissibility of the family* $\tau(\lambda)$ *is equivalent to condition* (3.59) *if and only if* $\mathcal{F}_\Pi(0) = \mathcal{H}$. *In this case the operator $A$ and its extensions* $A_1$, $A_2$ *are* $\mathfrak{N}$*-regular,* $A_2(0) = \mathfrak{N}$, *and* $n'_\pm(A) = 0$.

**Proof.** According to Corollary 3.8, the equivalence (3.42) $\Longleftrightarrow$ (3.59) is true if the condition $\mathcal{F}_\Pi(0) = \mathcal{H}(\Longleftrightarrow A_2 = A \dot{+} \hat{\mathfrak{N}})$ is satisfied. Taking into account Corollaries 2.3 and 2.4, we derive from (3.58) that the relations $A$, $A_1$, and $A_2$ are $\mathfrak{N}$-regular since the linear manifolds $\mathcal{F}_\Pi = \{0, \mathcal{H}\}$, $\mathcal{H}_1 = \Gamma_1\hat{\mathfrak{N}} = \mathcal{H}$, $\mathcal{H}_2 = \Gamma_2\hat{\mathfrak{N}} = \{0\}$ are closed. Proposition 1.8 implies the equalities $n'_\pm(A) = n_\pm(\mathcal{F}_\Pi) = 0$.

*Necessity.* If the condition $\mathcal{F}_\Pi(0) = \mathcal{H}$ is violated, then choose $h \notin \mathcal{F}_\Pi(0)$ and a relation $\theta = \theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$ with properties $\theta(0) = \{\mu h : \mu \in \mathbb{C}\}$ and $\theta \cap \mathcal{F}_\Pi = \{0\}$. Then the family $\tau(\lambda) = -\theta$ is $M$-admissible although condition (3.59) is not satisfied:

$$s - R - \lim_{y\uparrow\infty} \frac{\tau(\lambda)}{\lambda} \neq 0 \quad \text{since} \quad (\tau(\lambda)/\lambda + i)^{-1}h = 0 \qquad \forall\lambda \in \mathbb{C}_+. \ \Box$$

**Remark 3.5.** The formula of generalized resolvents (3.40) [but without formula (3.41) and the connection with a BVS] has been obtained in [82] and generalizes a well-known Krein formula [32, 33], which has been established by Krein in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ for operators with finite deficiency index $n_\pm(A) = n < \infty$ and by Saakyan [58] in the case $n_\pm(A) = \infty$ (see also [36, 37]). The following condition of $M$-admissibility has also been obtained in [82]:

$$\overset{\circ}{\lim_{y\uparrow\infty}} y^{-1}(Q_\tau(iy)h, h) = 0 \qquad \forall h \in \mathcal{H}, \tag{3.63}$$

where $Q_\tau(\lambda) := M(\lambda) - [M(\lambda) - M(\bar{\lambda}_0)][M(\lambda) + \tau(\lambda)]^{-1}[M(\lambda) - M(\lambda_0)]$.

Condition (3.63) does not depend on whether $A_2$ is an operator or a relation. Note, in addition, that formulas of generalized resolvents similar to (3.40) were obtained (in the case $A_2(0) = \{0\}$) from the Shtraus formula in [1, 2, 61]. The connection of the Shtraus formula with (3.40) was analyzed also in [53].

**5.** At this point we shall consider the case of a bounded Hermitian operator $A$ with $\overline{\mathfrak{D}(A)} = \mathfrak{h}_0 \subsetneqq \mathfrak{h}$.

**Proposition 3.5.** *Assume that $A$ is a bounded Hermitian operator in* $\mathfrak{h}$, $\overline{\mathfrak{D}(A)} = \mathfrak{h}_0$, $\mathfrak{N} = \mathfrak{h} \ominus \mathfrak{h}_0 = \mathfrak{h}_1$, *and* $\tilde{A} = \tilde{A}^*$ *is an extension of $A$ of the form*

$$\tilde{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \qquad (A_{ij} \in [\mathfrak{h}_j, \mathfrak{h}_i] \ (i, j = 0, 1)),$$

*Then:* (1) *the relation* $A^*$ *is of the form* $A^* = \{\{f, \tilde{A}f + n\} : f \in \mathfrak{h}_0, n \in \mathfrak{N}\}$;

(2) *the collection*

$$\mathcal{H} = \mathfrak{N}, \qquad \Gamma_1\{f, \tilde{A}f + n\} = n, \qquad \Gamma_2\{f, \tilde{A}f + n\} = P_{\mathfrak{N}}f \tag{3.64}$$

*forms the BVS for the relation* $A^*$;

(3) *the $\gamma$-field $\gamma(\lambda)$ and the Weyl function $M(\lambda)$ corresponding to the BVS* (3.64) *are of the form*

$$\gamma(\lambda) = I_{\mathfrak{N}} - (A_{00} - \lambda)^{-1} A_{01}, \qquad M(\lambda) = \lambda I_{\mathfrak{N}} - A_{11} + A_{10}(A_{00} - \lambda)^{-1} A_{01}. \tag{3.65}$$

**Proof.** The first and the second assertion are evident. Further, according to the Frobenius formula (3.24), we have

$$(\tilde{A} - \lambda)^{-1} = \begin{pmatrix} B_{00} & -(A_{00} - \lambda)^{-1} A_{01} B_{11} \\ B_{10} & B_{11} \end{pmatrix}, \qquad B_{11} := -(\lambda I_{\mathfrak{N}} - A_{11} + A_{10}(A_{00} - \lambda)^{-1} A_{01})^{-1}, \tag{3.66}$$

where the form of the submatrices $B_{00}$, $B_{10}$ of the operator matrix $(\tilde{A} - \lambda)^{-1}$ is not important. It is not difficult to see that $\mathfrak{N}_\lambda = (\tilde{A} - \lambda)^{-1} \mathfrak{N}$. Thus $f_\lambda \in \mathfrak{N}_\lambda \iff f_\lambda = (\tilde{A} - \lambda)^{-1} n$ $(n \in \mathfrak{N})$ and

$$\Gamma_2 \hat{f}_\lambda = \Gamma_2 \{f_\lambda, \lambda f_\lambda\} = \Gamma_2 \{(A - \lambda)^{-1} n, \lambda(\tilde{A} - \lambda)^{-1} n\} = P_{\mathfrak{N}} (\tilde{A} - \lambda)^{-1} n. \tag{3.67}$$

Putting $h := P_{\mathfrak{N}}(\tilde{A} - \lambda)^{-1} n = -[\lambda I_{\mathfrak{N}} - A_{11} + A_{10}(A_{00} - \lambda)^{-1} A_{01}]^{-1} n = B_{11} n$, we obtain from (3.66) and (3.67) that

$$\gamma(\lambda) h = \pi_1 (\Gamma_2 \upharpoonright \hat{\mathfrak{N}}_\lambda)^{-1} h = (\tilde{A} - \lambda)^{-1} n = \begin{pmatrix} -(A_{00} - \lambda)^{-1} A_{01} h \\ h \end{pmatrix}.$$

Expression (3.65) for $\gamma(\lambda)$ has been obtained. The formula $M(\lambda) = \Gamma_1 \hat{\gamma}(\lambda) = \{\gamma(\lambda), \lambda \gamma(\lambda)\}$ enables us to determine $M(\lambda)$. □

**Corollary 3.9.** *Under the condition of Proposition* 3.5 *suppose that $A_{11} = \mathbb{O}$, that is, $\tilde{A} = \tilde{A}_0 = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$. Then the Weyl function corresponding to the BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ of the form in* (3.64) *coincides with the spectral complement of the operator $A$ (in the sense of* [69]*); $M(\lambda) = \lambda I + A_{10}(A_{00} - \lambda)^{-1} A_{01}$.*

Corollary 3.4 together with formulas (3.65) for $\gamma(\lambda)$ and $M(\lambda)$ enables us to obtain a description of the class $\Omega_A$ of generalized resolvents of a bounded Hermitian operator $A$. Such a description has been obtained in a different way (and without the connection with a BVS and "boundary-value problems") by Shtraus [70].

Note, in addition, that the proper extension $\tilde{A}_B = \ker(\Gamma_1 - B\Gamma_2)$, $(B \in \mathcal{C}(\mathfrak{N}))$ of the operator $A$ in the BVS (3.64) is of the form

$$\tilde{A}_B = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} + B \end{pmatrix}, \qquad B \in \mathcal{C}(\mathfrak{N}). \tag{3.68}$$

According to Proposition 1.6 we obtain (for $\lambda \in \rho(A_{00})$) the equivalences

$$\lambda \in \rho(\tilde{A}_B) \iff 0 \in \rho(B - M(\lambda)), \qquad \lambda \in \sigma_i(\tilde{A}_B) \iff 0 \in \sigma_i(B - M(\lambda)) \qquad (i = p, c, r).$$

The first of these equivalences was proved differently in the case $A_{11} = \mathbb{O}$ in [70].

6. Let $A$ be a Hermitian contraction in $\mathfrak{h}$ with $\mathfrak{D}(A) =: \mathfrak{h}_0$. It is well known (see [5, 34]) that the collection $C_A(0)$ of all the self-adjoint contractive extensions (*sc*-extensions) of the operator $A$ forms an operator segment:

$$A_\mu := A_{\min} \le \tilde{A} \le A_M := A_{\max}, \qquad \tilde{A} \in C_A(0) \subset [\mathfrak{h}]. \tag{3.69}$$

If the operator $A$ is considered in $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{h}_1$ the extremal extensions $\tilde{A}_\mu$ and $\tilde{A}_M$ are of the form (see [40])

$$\tilde{A}_\mu = \begin{pmatrix} A_\mu & \mathbb{O} \\ \mathbb{O} & -I_{\mathfrak{h}_1} \end{pmatrix}, \qquad \tilde{A}_M = \begin{pmatrix} A_M & \mathbb{O} \\ \mathbb{O} & I_{\mathfrak{h}_1} \end{pmatrix}. \tag{3.70}$$

Denote by $\Omega_A(\mathbb{R}\setminus[-1,1])$ the collection of generalized resolvents $\mathbf{R}_\lambda = P_{\mathfrak{h}}(\tilde{A}-\lambda)^{-1}|_{\mathfrak{h}} \in \Omega_A$ which are generated by *sc*-extensions $\tilde{A} \in [\tilde{\mathfrak{h}}]$ of the operator $A$. It follows from (3.69) and (3.70) that the inequalities

$$(A_M-\lambda)^{-1} \le \mathbf{R}_\lambda = P_{\mathfrak{h}}(\tilde{A}-\lambda)^{-1}\restriction\mathfrak{h} \le (A_\mu-\lambda)^{-1} \qquad (\lambda\in\mathbb{R}\setminus[-1,1]) \tag{3.71}$$

are true.

**Proposition 3.6.** *Assume that the extensions $A_M$ and $A_\mu$ are disjoint, $C := (A_M - A_\mu)\restriction\mathfrak{N}$, $A_* = \{\{f, A_\mu f + C^{1/2}h\} : f\in\mathfrak{h}_0,\quad h\in\mathfrak{N}\}$. Then:* (1) *a collection $\Pi^\mu = \{\mathfrak{N},\Gamma_1^\mu,\Gamma_2^\mu\}$, in which*

$$\Gamma_1^\mu\hat{f} := C^{-1/2}(f'-A_Mf) \qquad \Gamma_2^\mu\hat{f} := C^{-1/2}(f'-A_\mu f), \qquad (\hat{f}=\{f,f'\}\in A_*), \tag{3.72}$$

*forms a generalized (in the sense of Definition* 6.1) *BVS of the relation $A_*$;* (2) *the $\gamma$-field and the Weyl function corresponding to the BVS $\Pi^\mu$ are of the form*

$$\gamma(\lambda) = -R^\mu(\lambda)C^{1/2} \qquad M_\mu(\lambda) = I + C^{1/2}R^\mu(\lambda)C^{1/2}, \quad (R^\mu(\lambda) = (A_\mu-\lambda)^{-1}); \tag{3.73}$$

(3) *if $A_\mu$ and $A_M$ are transversal, then $0\in\rho(C)$, $A_* = A^*$ and $\Pi^\mu$ is the ordinary BVS for the relation $A^*$.*

**Proof.** (1) The relations $\Gamma_j : A_* \to \mathfrak{N}$ $(j=1,2)$ are evidently surjective and closed. Further, $\forall\hat{f}=\{f,f'\}$, $\hat{g}=\{g,g'\}\in A_*$ we have

$$(\Gamma_1^\mu\hat{f},\Gamma_2^\mu\hat{g})_{\mathcal{H}} - (\Gamma_2^\mu\hat{f},\Gamma_1^\mu\hat{g})_{\mathcal{H}} = (C^{-1/2}(f'-A_Mf),C^{-1/2}(g'-A_\mu g)) - (C^{-1/2}(f'-A_\mu f),C^{-1/2}(g'-A_Mg))$$

$$= (C^{-1/2}(f'-A_\mu f - C_f f)C^{-1/2}(g'-A_\mu g))$$

$$-(C^{-1/2}(f'-A_\mu f),C^{-1/2}(g'-A_\mu g - C_g)) = (f'-A_\mu f,g) - (f,g'-A_\mu g) = (f',g)-(f,g').$$

(2) We can easily see that $\mathfrak{N}_\lambda = (A_\mu-\lambda)^{-1}\mathfrak{N}$, $\mathfrak{N}_\lambda^* := A_*\cap\mathfrak{N}_\lambda = (A_\mu-\lambda)^{-1}C^{1/2}\mathfrak{N}$ and for all $\lambda\in\mathbb{C}\setminus[-1,1]$, $\overline{\mathfrak{N}_\lambda^*} = \mathfrak{N}_\lambda$. It follows that $\forall\hat{f}_\lambda = \{f_\lambda,\lambda f_\lambda\}$, where $f_\lambda = (A_\mu-\lambda)^{-1}C^{1/2}n$, $n\in\mathfrak{N}$, we have

$$\Gamma_2^M\hat{f}_\lambda = C^{-1/2}(\lambda f_\lambda - A_\mu f_\lambda) = C^{-1/2}(\lambda-A_\mu)(A_\mu-\lambda)^{-1}C^{1/2}n = -n.$$

Consequently , $\hat{\gamma}(\lambda)n = -\hat{f}_\lambda$, $\gamma(\lambda)n = -f_\lambda = -(A_\mu-\lambda)^{-1}C^{1/2}n$. The first of equalities (3.73) has just been proved. The second equality is implied by the relations

$$M(\lambda)n = \Gamma_1^\mu\hat{\gamma}(\lambda)n = -C^{-1/2}(\lambda-A_M)R^\mu(\lambda)C^{1/2}n = [I + C^{1/2}R^\mu(\lambda)C^{1/2}]n.$$

(3) The extensions $A_\mu$ and $A_M$ are transversal if and only if $0\in\rho(C)$ (see Proposition 1.4). Thus, (3.72) implies that the mapping $\Gamma : A^*\to\mathfrak{N}\oplus\mathfrak{N}$ is transversal.

**Remark 3.6.** Since $\Gamma_1 A_* = \mathfrak{N}$ and $\ker\Gamma_1 = A_M$, the collection $\Pi^M = \{\mathfrak{N},\Gamma_1^M,\Gamma_2^M\} = \{\mathfrak{N},\Gamma_2^\mu,-\Gamma_1^\mu\}$ forms a generalized BVS such that the corresponding Weyl function $M_M(\lambda)$ is of the form

$$M_M(\lambda) = -M_\mu(\lambda)^{-1} = -I + C^{1/2}(A_M-\lambda)^{-1}C^{1/2}. \tag{3.74}$$

The functions $M_\mu(\lambda)$ and $M_M(\lambda)$ coincide with the functions $Q_\mu(\lambda)$ and $Q_M(\lambda)$ from [40].

**Proposition 3.7.** *Assume that $A$ is a Hermitian contraction, $\Pi^\mu = \{\mathfrak{N},\Gamma_1^\mu,\Gamma_2^\mu\}$ is a BVS of the form in* (3.72) *corresponding to the relation $A^*$. Then the equality*

$$\mathbf{R}_\lambda = R^\mu(\lambda) - R^\mu(\lambda)C^{1/2}(M_\mu(\lambda)+\tau(\lambda))^{-1}C^{1/2}R^\mu(\lambda) \tag{3.75}$$

*establishes a bijective correspondence between generalized sc-resolvents $\mathbf{R}_\lambda\in\Omega_A(\mathbb{R}\setminus[-1,1])$ and $\tau(\lambda)\in\tilde{S}_{\mathcal{H}}[-1,1]$.*

**Proof.** *Necessity.* Formula (3.75) is implied by Theorem 3.1 and Proposition 3.6. The operator-valued functions $\mathbf{R}_\lambda$, $R^\mu(\lambda)$, and $M_\mu(\lambda)$ are holomorphic in $\mathbb{R}\setminus[-1,1]$.

Consequently, (3.75) shows that the family $\tau(\lambda)+M_\mu(\lambda)$ and, therefore, $\tau(\lambda)$ is holomorphic in $\mathbb{R}\setminus[-1,1]$ since $M_\mu(\lambda)$ is boundedly holomorphic in $\mathbb{R}\setminus[-1,1]$.

Since $A_M$ and $A_\mu$ and defined in the BVS (3.72) by the equations $A_M = \ker\Gamma_1(\Longleftrightarrow \tau(\lambda)=\mathbb{O})$ and $A_\mu = \ker\Gamma_2(\Longleftrightarrow \tau(\lambda)=\{0,\mathcal{H}\})$, applying the extremal property (3.71) to $R^M(\lambda)=(A_M-\lambda)^{-1}$ and to the generalized resolvent $\mathbf{R}_\lambda = (\tilde{A}_{-\tau(\lambda)}-\lambda)^{-1}$, we obtain that for all $\lambda\in\mathbb{R}\setminus[-1,1]$

$$R^\mu(\lambda)-R^M(\lambda)=\gamma(\lambda)M_\mu^{-1}(\lambda)\gamma^*(\lambda)>0 \qquad (\tau(\lambda)+M_\mu(\lambda))^{-1}>0,$$

$$\gamma(\lambda)[M_\mu^{-1}(\lambda)-(\tau(\lambda)+M_\mu(\lambda))^{-1}]\gamma^*(\lambda)=(\tilde{A}_{-\tau(\lambda)}-\lambda)^{-1}-(A_M-\lambda)^{-1}>0. \tag{3.76}$$

It follows that

$$M_\mu(\lambda)^{-1}\ge(\tau(\lambda)+M_\mu(\lambda))^{-1}>0\Longrightarrow\tau(\lambda)+M_\mu(\lambda)\ge M_\mu(\lambda)\Longrightarrow\tau(\lambda)\ge 0 \quad (\forall\lambda\in\mathbb{R}\setminus[-1,1]).$$

Consequently, $\tau(\lambda)\in\tilde{S}_{\mathcal{H}}[-1,1]$.

Since $\mathbf{R}_\lambda = P(\tilde{A}-\lambda)^{-1}\restriction\mathfrak{h}$ is a generalized *sc*-resolvent, $\tilde{A}$ is a contraction, $\tilde{A}(0)=\{0\}$, and $\tau(\lambda)$ is $M$-admissible. Note that we can easily verify that the $M$-admissibility condition, which, in view of the equality $A_2(0)=A_\mu(0)=\{0\}$, is of the form in (3.43), is satisfied. By virtue of (3.74) we have $M_\mu(\lambda)>0$ $\forall\lambda\in\mathbb{R}\setminus[-1,1]$. Thus, for $\lambda>1$, $\forall\tau(\lambda)\in\tilde{S}_{\mathcal{H}}[-1,1]$ the following implication is true:

$$\lambda\tau(\lambda)+\lambda M_\mu(\lambda)\ge\lambda M_\mu(\lambda)>0\Longrightarrow\lambda^{-1}(\tau(\lambda)+M_\mu(\lambda))^{-1}\le\lambda^{-1}M_\mu^{-1}(\lambda).$$

Consequently, $s-\lim\limits_{\lambda\uparrow+\infty}\lambda^{-1}(\tau(\lambda)+M_\mu(\lambda))^{-1}=s-\lim\limits_{\lambda\uparrow+\infty}\lambda^{-1}M_\mu^{-1}(\lambda)=0$, and (3.43) is proved.

*Sufficiency.* Let $\tau(\lambda)\in\tilde{S}_{\mathcal{H}}[-1,1]$. Since $M_\mu(\lambda)\cdot M_M(\lambda)=-I$ $\forall\lambda\in\mathbb{C}\setminus[-1,1]$ and $M_\mu(x)>0$, $\tau(x)>0$ $\forall x\in\mathbb{R}\setminus[-1,1]$, we have $0\in\rho(\tau(x)+M_\mu(x))$ $\forall x\in\mathbb{R}\setminus[-1,1]$. In view of Proposition 1.6, $x\in\rho(\tilde{A}_{-\tau(x)})$ $\forall x\in\mathbb{R}\setminus[-1,1]$ and , consequently, $(\tilde{A}_{-\tau(\lambda)}-\lambda)^{-1}=\mathbf{R}_\lambda$ is regular in $\mathbb{C}\setminus[-1,1]$. Thus, the resolvent $(\tilde{A}-\lambda)^{-1}$ is regular in $\mathbb{C}\setminus[-1,1]$ and $\tilde{A}$ is a contraction.

**Remark 3.7.** Another proof of the implication $\|\tilde{A}\|\le 1\Longleftrightarrow\tau(\lambda)\in\tilde{S}_{\mathcal{H}}$ can be derived from the resolvent formula (3.75) and the following properties of the functions $M_\mu(\lambda)$, $M_M(\lambda)$:

$$M_\mu(+1):=s-\lim_{x\downarrow 1}M_\mu(x)=0, \qquad M_M(-1)=s-\lim_{x\uparrow -1}M_M(x)=0. \tag{3.77}$$

**Remark 3.8.** The formula

$$\mathbf{R}_\lambda=R^M(\lambda)-R^M(\lambda)C^{1/2}(M_M(\lambda)+\tau(\lambda))^{-1}C^{1/2}R^M(\lambda) \qquad (\lambda\in\rho(\tilde{A})) \tag{3.78}$$

is similar to (3.75) and establishes a bijective correspondence between generalized *sc*-resolvents $\mathbf{R}_\lambda$ and $\tau(\lambda)\in S_{\mathcal{H}}^{-}[-1,1]$, where $M_M(\lambda)$ is of the form in (3.74). Formula (3.78) can be obtained by application of Theorem 3.1 to the BVS $\Pi^M=\{\mathfrak{N},\Gamma_1^M,\Gamma_2^M\}=\{\mathfrak{N},\Gamma_2^\mu,-\Gamma_1^\mu\}$.

**Definition 3.2** [40]. *An operator-valued function $k(\lambda)$ with values in $[\mathfrak{N}]$ is related to the class $K_{\mathfrak{N}}[-1,1]$ if:*

*(1) $k(\lambda)\in R_{\mathfrak{N}}$;*

*(2) $k(\lambda)$ is holomorphic in $\mathbb{R}\setminus[-1,1]$ and $0\le k(\lambda)\le 1$.*

**Lemma 3.4** [40]. *If $\mathbb{O}\le K\le I_{\mathfrak{N}}$, then $0\in\rho(I-K+M_\mu(\lambda)K)$ $\forall\lambda\in\mathbb{R}\setminus[-1,1]$.*

**Proof.** Let $\lambda_0\in\mathbb{R}\setminus[-1,1]$. Then $\exists\varepsilon>0: M_\mu(\lambda_0)>\varepsilon I_{\mathfrak{N}}$. Therefore, we have the implications $I_{\mathfrak{N}}-K+K^{1/2}M_\mu(\lambda_0)K^{1/2}>I_{\mathfrak{N}}-(1-\varepsilon)K\Longrightarrow 0\in\rho(I_{\mathfrak{N}}-K+K^{1/2}M_\mu(\lambda_0)K^{1/2})\Longrightarrow 0\in\rho(I_{\mathfrak{N}}-K+M_\mu(\lambda_0)K)$. □

The transformation $\tau(\lambda)\to k(\lambda)=(I+\tau(\lambda))^{-1}$ establishes a bijective mapping from the class $\tilde{S}_{\mathfrak{N}}[-1,1]$ onto the class $K_{\mathfrak{N}}[-1,1]$. Consequently, by putting $k(\lambda)=(I+\tau(\lambda))^{-1}$ in (3.75) and taking into consideration Lemma 3.4, we get the main result of [40] (see also [82]).

**Theorem** [40]. *The equality*

$$\mathbf{R}_\lambda^c = R^\mu(\lambda) - R^\mu(\lambda)C^{1/2}k(\lambda)[I + (M_\mu(\lambda) - I)k(\lambda)]^{-1}C^{1/2}R^\mu(\lambda) \tag{3.79}$$

*establishes a bijective correspondence between the class of operator-valued functions $K_{\mathfrak{N}}[-1,1]$ and the collection $\{\mathbf{R}^c(\lambda)\}$ of all generalized sc-resolvents of operator $A$.*

Moreover, there is a one-to-one correspondence between canonical *sc*-resolvents and constants $k(\lambda) \equiv K \in [0, I_{\mathfrak{N}}]$ in (3.79).

## 4. EXTENSIONS OF A HERMITIAN OPERATOR WITH A GAP

**1.** Let $A$ be a nonnegative Hermitian operator in $\mathfrak{h}$: $(Af,f) \geq 0\ \forall f \in \mathfrak{D}(A)$. As is known [5, 34], in the class Ex $_A(-\infty,0)$ $(\overline{\mathfrak{D}(A)} = \mathfrak{h})$ of nonnegative self-adjoint extensions of densely defined operator $A$ there exists the greatest (the hard or Friedrichs) extension $A_F$ and the least (the soft or Krein) extension $A_K$, such that for all $\tilde{A} \in$ Ex $_A(-\infty,0)$, $\mathfrak{D}[A_F] \subseteq \mathfrak{D}[\tilde{A}] \subseteq \mathfrak{D}[A_K]$, and

$$A_K[f] \leq \tilde{A}[f] \leq A_F[f] \quad \forall f \in \mathfrak{D}[A_F]. \tag{4.1}$$

In the case of a nondensely defined operator $A$ the class Ex $_A(-\infty,0)$ is also nonempty, but sometimes it does not contain any operator. We define (after [73]) the extensions $A_F$ and $A_K$ as the strong resolvent limits of the linear relations $\tilde{A}_x = A \dot{+} \hat{\mathfrak{N}}_x$:

$$A_F = s - R - \lim_{x\downarrow-\infty} \tilde{A}_x, \qquad A_K = s - R - \lim_{x\uparrow 0} \tilde{A}_x. \tag{4.2}$$

In this case the extremal property (4.1) remains true and takes the following equivalent form:

$$(A_F - \lambda)^{-1} \leq (\tilde{A} - \lambda)^{-1} \leq (A_K - \lambda)^{-1} \qquad \forall \lambda \in (-\infty, 0), \quad \tilde{A} \in \text{Ex}\,_A(-\infty,0). \tag{4.3}$$

An extension $A_F$ is always a linear relation $A_F(0) = \mathfrak{N} = \mathfrak{h}_0^\perp$, while $A_K$ may be an operator. Moreover, an extension $A_K$ is an operator if and only if the class Ex $_A(-\infty,0)$ contains an operator. A criterion for this to be true was obtained in [73].

It is noteworthy that $(A_a)_F = A_F + a$ if $a > 0$ and hence the first inequality in (4.3) remains true if $\lambda < -a$ and $\tilde{A} \in$ Ex $_A(-\infty,-a)$.

Let $T = T^* \geq \beta$ be a self-adjoint semibounded below operator in $\mathfrak{h}$, $\overline{\mathfrak{D}(T)} = \mathfrak{h}_0$. As usual, $\mathfrak{D}[T]$ stands for the closure of $\mathfrak{D}(T)$ endowed with the norm $\|f\|_T^2 = (1-\beta)\|f\|^2 + (Tf,f)$, $f \in \mathfrak{D}(T)$. A closure of the form $(Tf,f)$ is denoted by either $\mathfrak{t}_T[f]$ or $T[f] = T[f,f]$. Clearly $\mathfrak{D}[T] = \mathfrak{D}((T-\beta)^{1/2})$. We put $\mathfrak{D}[\theta] = \mathfrak{D}[T]$ for a linear relation $\theta = \theta^* \geq \beta$ with the operator part $T = T^* \geq \beta$.

**Definition 4.1.** *Let $\mathfrak{t}$ be a closed semibounded below quadratic form with a nondense (generally speaking) domain $\mathfrak{D}(\mathfrak{t})$ in $\mathfrak{h}(\overline{\mathfrak{D}(\mathfrak{t})} = \mathfrak{h}_0 \subset \mathfrak{h})$. A semibounded below linear relation $\theta = \theta^*$ is said to be associated with the form $\mathfrak{t}$ and is written $\mathfrak{t} = \mathfrak{t}_\theta$ if $\theta(0) = \mathfrak{N} = \mathfrak{h}_0^\perp$ and the operator part $T = T^*$ of the linear relation $\theta$ is associated (in accordance with the first representation theorem [29]) with the form $\mathfrak{t}$ considered in $\mathfrak{h}_0$.*

Clearly, $\mathfrak{D}(\mathfrak{t}_\theta) = \mathfrak{D}[\theta] = \mathfrak{D}[T]$.

**Proposition 4.1.** *Let $\{\mathfrak{t}_n\}_1^\infty$ be a nondecreasing sequence of closed linear forms in $\mathfrak{h}$ semibounded below by $\gamma$,*

$$\gamma \leq \mathfrak{t}_1 \leq \mathfrak{t}_2 \leq \ldots \leq \mathfrak{t}_n \leq \ldots,$$

*and let $\theta_n = \theta_n^* \in \tilde{\mathcal{C}}(\mathfrak{h})$ be a linear relation associated with $\mathfrak{t}_n$. Then:*

(1) *the sequence $\theta_n$ converges in the strong resolvent sense to a linear relation $\theta = \theta^* \geq \gamma$ ($\theta := s - R - \lim_{n\to\infty}\theta_n$);*

(2) *if $\mathfrak{t} = \mathfrak{t}_\theta$ is a form associated with $\theta$, then*

$$\mathfrak{D}(\mathfrak{t}) = \mathfrak{D}[\theta] = \{f \in \bigcap_{n\geq 1} \mathfrak{D}(\mathfrak{t}_n) : \lim_{n\to\infty} \mathfrak{t}_n[f] < \infty\}. \tag{4.4}$$

The first statement is a simple generalization to the case of linear relations of the well-known result of Kato [29]; the second one was proved by the authors in the case $\overline{\mathfrak{D}(\mathfrak{t}_n)} = \mathfrak{h}$ [20, 79] and furnished the answer to the problem posed in [29, p. 570]. In the general case these statements can be proved in the same way.

**Proposition 4.2.** *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for a linear relation* $A^*$ *such that* $A_2 \geq 0$, $M(\lambda)$ *is the corresponding Weyl function,* $a > 0$. *Then*

(1) *there exists a strong resolvent limit*

$$M(0) := s - R - \lim_{x \uparrow 0} M(x) \qquad (M(-\infty) := s - R - \lim_{x \downarrow -\infty} M(x))$$

*which is a semibounded below (above) self-adjoint linear relation in* $\mathcal{H}$;

(2) *the linear relation* $M(0) = M(0)^*$ $(M(-\infty) = M(-\infty)^*)$ *is associated with the closed quadratic form*

$$\mathfrak{t}_0[f] = \lim_{x \uparrow 0} (M(x)f, f) \qquad (\mathfrak{t}_{-\infty}[f] = \lim_{x \downarrow -\infty} (M(x)f, f)) \tag{4.5}$$

*with the domain*

$$\mathfrak{D}(\mathfrak{t}_0) = \{f : \lim_{x \uparrow 0} (M(x)f, f) < \infty\} = \mathfrak{D}((M(0) - M(-a))^{1/2}), \tag{4.6}$$

$$\mathfrak{D}(\mathfrak{t}_{-\infty}) = \{f : \lim_{x \downarrow -\infty} (M(x)f, f) > -\infty\} = \mathfrak{D}((M(-a) - M(-\infty))^{1/2}); \tag{4.6'}$$

(3) *in this connection* $A_2$ *and* $A_K$ *are disjoint extensions if and only if* $M(0)$ *is an operator* $(M(0) \in \mathcal{C}(\mathcal{H}))$, $A_2$ *and* $A_K$ *are transversal extensions if and only if* $M(0) \in [\mathcal{H}]$;

(4) *one can determine* $A_F$ *and* $A_K$ *by the boundary conditions*

$$A_K = \{\hat{f} = \{f, f'\} \in A^* : \{\Gamma_2 \hat{f}, \Gamma_1 \hat{f}\} \in M(0)\} = \Gamma^{-1} M(0),$$

$$A_F = \{\hat{f} = \{f, f'\} \in A^* : \{\Gamma_2 \hat{f}, \Gamma_1 \hat{f}\} \in M(-\infty)\} = \Gamma^{-1} M(-\infty),$$

*which in the case* $M(0) \in \mathcal{C}(\mathcal{H})$, $M(-\infty) \in \mathcal{C}(\mathcal{H})$ *can be rewritten in the form*

$$A_K = \ker(\Gamma_1 - M(0)\Gamma_2), \qquad A_F = \ker(\Gamma_1 - M(-\infty)\Gamma_2). \tag{4.7}$$

**Proof.** (1) By virtue of the condition $A_2 \geq 0$, $M(\lambda)$ is holomorphic on $(-\infty, 0)$ and hence it is monotonically increasing there. Therefore, for some $\alpha > 0$ the operator-valued function $(M(x) + \alpha)^{-1}$ is nonnegative and monotonically decreasing on $(-1, 0)$. This implies that there exists a strong limit $T := s - \lim_{x \uparrow 0} (M(x) + \alpha)^{-1}$. Now it remains to put $M(0) = T^{-1} - \alpha$.

(2) The second statement is a consequence of Proposition 4.1, while relation (4.6) is implied by (4.4).

(3) Statement 3 is implied by Proposition 1.4 and the next statement (4).

(4) Let $\tilde{A}_x = A \dotplus \hat{\mathfrak{N}}_x$. Then $\tilde{A}_x = \ker(\Gamma_1 - M(x)\Gamma_2)$. By setting $\theta = \Gamma A_K = \{\Gamma_2, \Gamma_1\} A_K$ we apply the resolvent formula (3.39) to the extensions $\tilde{A}_x$ and $A_K$. We have

$$(\tilde{A}_x + I)^{-1} - (A_K + I)^{-1} = \gamma(-1)[(M(x) - M(-1))^{-1} - (\theta - M(-1))^{-1}]\gamma^*(-1). \tag{4.8}$$

Passing to the limit in (4.8) as $x \to 0$ and taking into account (4.2) and Corollary 3.3, we obtain $\theta = s - R - \lim_{x \uparrow 0} M(x) =: M(0)$.

**Corollary 4.1.** *Under the assumptions of Proposition* 4.2 *the following equivalences hold:*

$$A_2 = A_F \iff \lim_{x \downarrow -\infty} (M(x)h, h) = -\infty \qquad \forall h \in \mathcal{H} \setminus \{0\}, \tag{4.9}$$

$$A_2 = A_K \iff \lim_{x \uparrow 0} (M(x)h, h) = +\infty \qquad \forall h \in \mathcal{H} \setminus \{0\}. \tag{4.10}$$

**Proof.** If $A_2 = A_K$ is the Krein extension of the operator $A$, then $M(0) = \Gamma A_K = \{0, \mathcal{H}\}$ is a linear relation with zero operator part. Then the implication $\Rightarrow$ in (4.10) is implied by relations (4.5) and (4.6). Conversely, it follows from (4.10) that $\mathfrak{D}(\mathfrak{t}_0) = \{0\}$, where $\mathfrak{t}_0$ is defined by (4.5), (4.6). Therefore, the linear relation $M(0)$ associated with $\mathfrak{t}_0$ takes the form $M(0) = \{0, \mathcal{H}\}$, whence $A_K = \tilde{A}_{M(0)} = A_2$ and the equivalence (4.10) is proved. In complete analogy with this result we may prove the equivalence (4.9). □

**Corollary 4.2.** *Suppose that $M(\lambda)$ is the Weyl function corresponding to a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ such that $A_2 = A_F$. Then:*

(1) *there exists a unique nonnegative self-adjoint extension of $A \geq 0$ if and only if*

$$\lim_{x\uparrow 0}(M(x)h, h) = +\infty \qquad \forall h \in \mathcal{H} \setminus \{0\}; \tag{4.11}$$

(2) *there exists a nonnegative extension $\tilde{A} = \tilde{A}^* \in \mathcal{C}(\mathfrak{h})$ if and only if $M(0) \cap \mathcal{F}_\Pi = \{0\}$ (if, additionally, $n'_\pm(A) = 0$, then the latter condition may be replaced by the following one: $M(0) \in \mathcal{C}(\mathcal{H})$).*

**Proof.** The first statement is implied by (4.9) and (4.10), and the second one follows from Proposition 1.4 and the extremal property (4.2). Now it remains to note that under the assumption $n'_\pm(A) = 0$ the equivalence $M(0) \cap \mathcal{F}_\Pi = \{0\} \Longleftrightarrow M(0) \in \mathcal{C}(\mathcal{H})$ is a consequence of the relation $A_F(0) = \mathfrak{N} = \mathfrak{h}_0^\perp$.

**Remark 4.1.** (1) If, under the assumptions of Corollary 4.2, $A_2 = A_K$, then relation (4.11) must be replaced by the following one:

$$\lim_{x\downarrow -\infty}(M(x)h, h) = -\infty \qquad \forall h \in \mathcal{H} \setminus \{0\}, \tag{4.11'}$$

and the conditions $M(0) \cap \mathcal{F}_\Pi = \{0\}$ and $M(0) \in \mathcal{C}(\mathcal{H})$ must be replaced by the conditions $M(-\infty) \cap \mathcal{F}_\Pi = \{0\}$ and $M(-\infty) \in \mathcal{C}(\mathcal{H})$, respectively.

(2) Proposition 4.2 and Corollaries 4.1, 4.2 in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ were proved by the authors [20, 79]. If $\overline{\mathfrak{D}(A)} = \mathfrak{h}_0 \neq \mathfrak{h}$ and $A$ is a bounded operator, then Propositions 4.2 and 4.3 yield the result of Shtraus [69]. In this case $n'_\pm(A) = 0$ and it follows from (3.65), (4.5) that

$$\mathfrak{t}_0[h] = \lim_{x\uparrow 0}((xI_\mathfrak{h} + A_{10}(A_{00} - x)^{-1}A_{01})h, h) = \|A_{00}^{-1/2}A_{01}h\|^2 \quad \forall h \in \mathfrak{D}(\mathfrak{t}_0)$$

and the following equivalences hold:

(a) $A_K \in \mathcal{C}(\mathfrak{h}) \Longleftrightarrow M(0) \in \mathcal{C}(\mathcal{H})$;

(b) $A_K \in [\mathfrak{h}] \Longleftrightarrow M(0) \in [\mathcal{H}]$.

(3) The other criterion for a nonnegative operator $A$ to have a nonnegative self-adjoint extension $\tilde{A} = \tilde{A}^* \in \mathrm{Ex}_A(-\infty, 0)$ (positive closability of an operator $A$) was obtained in [73].

**Proposition 4.3.** *Suppose that $A_F$ and $A_K$ are disjoint extensions of operator $A \geq 0$, $C := 2[(I + A_K)^{-1} - (I + A_F)^{-1}]\mathfrak{N}_{-i}$, $\hat{C} = C \oplus C$, $A_* := A_F + \hat{C}^{1/2}\hat{\mathfrak{N}}_{-1}$. Then:*

(1) *the triple $\Pi^F = \{\mathfrak{N}_{-1}, \Gamma_1^F, \Gamma_2^F\}$ with*

$$\begin{gathered}\Gamma_1^F \hat{f} = -\sqrt{2}C^{-1/2}[f - (I + A_K)^{-1}(f' + f)],\\ \Gamma_2^F \hat{f} = \sqrt{2}C^{-1/2}[f - (I + A_F)^{-1}(f' + f)], \quad \hat{f} = \{f, f'\} \in A_*\end{gathered} \tag{4.12}$$

*is a generalized BVS (see §6) for the relation $A_*$, $\ker \Gamma_2^F = A_F$, $\ker \Gamma_1^F = A_K$;*

(2) *the $\gamma$-field and the Weyl function corresponding to the BVS $\Pi^F$ take the form*

$$\gamma_F(\lambda) = (\sqrt{2})^{-1}[I + (1 + \lambda)(A_F - \lambda)^{-1}]C^{1/2}, \tag{4.13}$$

$$M_F(\lambda) = -I + \frac{1}{2}(1 + \lambda)C^{1/2}[I + (1 + \lambda)(A_F - \lambda)^{-1}]C^{1/2}; \tag{4.14}$$

(3) *if the extensions $A_F$ and $A_K$ are transversal ones, then $0 \in \rho(C)$, $A_* = A^* = A_F \dot{+} \hat{\mathfrak{N}}_{-1}$, and $\Pi^F$ is a BVS for the relation $A^*$.*

**Proof.** (1) Let $g \in \mathfrak{N}_{-1}$ and $\hat{f}_F = \{f_F, f'_F\} \in A_F$. Setting $n = g/\sqrt{2}$, $f = f_F + C^{1/2}n$, $f' = f'_F - C^{1/2}n$ we obtain

$$\frac{1}{\sqrt{2}}\Gamma_2\{f, f'\} = C^{-1/2}[f - (I + A_F)^{-1}(f' + f)] = C^{-1/2}[f_F + C^{1/2}n - (I + A_F)^{-1}(f_F + f'_F)] = n.$$

Therefore the mapping $\Gamma_2^F : A_* \to \mathfrak{N}_{-1}$ is surjective. Further, for all $\hat{f} = \{f, f'\}$, $\hat{g} = \{g, g'\} \in A_*$ we have

$$\begin{aligned}
&(\Gamma_1\hat{f}, \Gamma_2\hat{g}) - (\Gamma_2\hat{f}, \Gamma_1\hat{g}) \\
&= 2(C^{-1/2}[f - (I + A_F)^{-1}(f' + f)], C^{-1/2}[g - (I + A_K)^{-1}(g' + g)]) \\
&\quad -2(C^{-1/2}[f - (I + A_K)^{-1}(f' + f)], C^{-1/2}[g - (I + A_F)^{-1}(g' + g)]) \\
&= (C^{-1/2}\{2f + [C - 2(I + A_K)^{-1}](f' + f)\}, C^{-1/2}[g - (I + A_K)^{-1}(g' + g)]) \\
&\quad -(C^{-1/2}[f - (I + A_K)^{-1}(f' + f)], C^{-1/2}\{2g + [C - 2(I + A_K)^{-1}](g' + g)) \\
&= (f', g) - (f, g').
\end{aligned}$$

(2) Let $\hat{\gamma}_F(\lambda) = \{\gamma_F(\lambda), \lambda\gamma_F(\lambda)\}$. Then it follows from (4.12), (4.13) that

$$\Gamma_2^F\hat{\gamma}_F(\lambda)h = C^{-1/2}[I + (\lambda + 1)(A_F - \lambda)^{-1} - (\lambda + 1)(A_F - \lambda)^{-1}]C^{1/2}h = h \qquad \forall h \in \mathfrak{N}_{-1},$$

and the $\gamma$-field of the extension $A_F$ takes the form in (4.13). Further, we have

$$\begin{aligned}
M_F(\lambda) = \Gamma_1^F\hat{\gamma}_F(\lambda) &= -C^{-1/2}\{I + (1 + \lambda)(A_F - \lambda)^{-1} - (1 + \lambda)(I + A_K)^{-1}[I + (1 + \lambda)(A_F - \lambda)^{-1}]\}C^{1/2} \\
&= -C^{-1/2}[I - (1 + \lambda)\Big(\frac{C}{2} + (I + A_F)^{-1}\Big)(I + (1 + \lambda)(A_F - \lambda)^{-1}) - (1 + \lambda)(A_F - \lambda)^{-1}]C^{1/2} \\
&= -I + \frac{1}{2}(1 + \lambda)C^{1/2}[I + (1 + \lambda)(A_F - \lambda)^{-1}]C^{1/2}.
\end{aligned}$$

Therefore, the Weyl function $M_F(\lambda)$ takes the form in (4.14).

(3) In accordance with Proposition 1.4 the transversality of $A_F$ and $A_K$ is equivalent to the condition $0 \in \rho(C)$. Corollary 6.1 implies that $\Pi^F$ is a BVS for the linear relation $A^* = A_* = A_F \dot{+} \hat{\mathfrak{N}}_{-1}$. Besides, setting

$$f = \sqrt{2}(I + A_F)^{-1}C^{-1/2}(\varphi_1 + \varphi_2) + \frac{1}{\sqrt{2}}C^{1/2}\varphi_2, \qquad f' = \sqrt{2}C^{-1/2}(\varphi_1 + \varphi_2) - \sqrt{2}f$$

for all $\varphi_1$, $\varphi_2 \in \mathfrak{N}_{-1}$ we obtain the equalities $\Gamma_i\hat{f} = \varphi_i$ $(i = 1, 2)$, which prove the surjectivity of the mapping $\Gamma$. □

**Remark 4.2.** (1) Note that $A_* = \mathfrak{D}[A_K] \cap A^*$ $\overline{\mathfrak{D}(A)} = \mathfrak{h}$. This relation is implied by the following formula given in [52]: $\mathfrak{D}[A_K] = \mathfrak{D}[A_F] + \gamma(-\varepsilon)\mathfrak{N}_{-\varepsilon}$ and the equality $\gamma(-1) = C^{1/2}$.

(2) The mapping $\Gamma_1 : A_* \to \mathfrak{N}_{-1}$ is surjective and $\ker\Gamma_1 = A_K$. Therefore, in accordance with Proposition 6.2, the function $M_K(\lambda) := -M_F(\lambda)^{-1}$ is the Weyl function corresponding to the generalized BVS $\Pi^K = \{\mathfrak{N}_{-1}, \Gamma_1^K, \Gamma_2^K\} = \{\mathfrak{N}_{-1}, \Gamma_2^F, -\Gamma_1^F\}$ and takes the form

$$M_K(\lambda) = -M_F(\lambda)^{-1} = I + 2^{-1}(1 + \lambda)C^{1/2}[I + (1 + \lambda)(A_K - \lambda)^{-1}]C^{1/2}. \tag{4.15}$$

**2.** Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS such that $A_2 = A_F$. Then the resolvent formula (3.39) and the extremal property (4.3) of the Friedrichs extension $A_F$ yield the following implication: $\tilde{A}_\theta = \tilde{A}_\theta^*$ is semibounded below $\Longrightarrow$ $\theta(\in \tilde{C}(\mathcal{H}))$ is semibounded below. Simple examples (see [20, 79]) show that the inverse implication does not hold in the general case. In order to formulate the corresponding criterion we introduce

**Definition 4.2.** *An operator-valued function $M(\lambda) \in R_{\mathcal{H}}$ holomorphic on $(-\infty, 0)$ is said to be uniformly convergent to $-\infty$ as $x \to -\infty$ if for all $N > 0$ there exists $x_N < 0$ such that $M(x_N) < -NI_{\mathcal{H}}$.*

It is clear that this condition is equivalent to the following one:

$$\lim_{x \downarrow -\infty} \|(M(x) - M(x_0)^{-1}\| = 0 \qquad (x_0 < 0). \tag{4.16}$$

In the case $\dim \mathcal{H} < \infty$ this condition is equivalent to condition (4.9), which characterizes the Weyl function of the Friedrichs extension $A_F$; in the general case (4.16) is stronger than (4.9).

**Proposition 4.4.** *Suppose that $A \geq 0$, $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$ such that $A_2 = A_F$, $M(\lambda)$ is the corresponding Weyl function. The following conditions are equivalent if and only if $M(\lambda)$ uniformly converges to $-\infty$ as $x \to -\infty$:*

(a) *the linear relation $\theta = \theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$ is semibounded below;*

(b) *the extension $\tilde{A}_\theta \in \mathrm{Ex}_A$ is semibounded below.*

**Corollary 4.3.** *Suppose that under the assumptions of Proposition 4.4 the extension $A_F$ has the discrete spectrum, that is, $(A_F + I)^{-1} \in \mathfrak{C}_\infty$. Then the equivalence* (a) $\Longleftrightarrow$ (b) *holds.*

Proposition 4.4 and Corollary 4.3 were proved in [20, 79] for the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}_0$. The proof in the general case is analogous. Corollary 4.3 in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ was obtained earlier in [12].

**Corollary 4.4.** *Suppose that under the assumptions of Proposition 4.4 the multivalued part $\mathcal{F}_\Pi(0)$ of the forbidden manifold $\mathcal{F}_\Pi$ is closed and $n'_+(A) = n'_-(A) < \infty$. Then the equivalence* (a) $\Longleftrightarrow$ (b) *holds.*

The proof is implied by the integral representation of $M(\lambda)$ and the equality $\mathfrak{R}\,(B_M^{1/2}) = \mathcal{F}_\Pi(0)$. □

In particular, the equivalence (a) $\Longleftrightarrow$ (b) holds if $A$ is an $\mathfrak{N}$-regular operator and $n'_\pm(A) = 0$ ($\Longleftrightarrow$ $A_F = A \dot{+} \hat{\mathfrak{N}}$).

**3. Definition 4.3.** *A Hermitian operator $A$ is said to have a gap $(\alpha, \beta)$ if*

$$\|(A - \frac{\alpha+\beta}{2} f\| \geq \frac{\beta-\alpha}{2}\|f\| \qquad \forall f \in \mathfrak{D}(A). \tag{4.17}$$

Inequality (4.17) is equivalent to the following one:

$$\|Af\|^2 - (\alpha+\beta)(Af, f) + \alpha\beta\|f\|^2 \geq 0 \qquad \forall f \in \mathfrak{D}(A) \tag{4.18}$$

and in the case $\alpha = -\infty$ it turns into the inequality $(Af, f) \geq \beta\|f\|^2$, which means that the operator $A$ is semibounded below.

$\mathrm{Ex}_A(\alpha, \beta)$ stands for the set of proper self-adjoint extensions $\tilde{A}$ of $A$ with the gap $(\alpha, \beta)$

$$\tilde{A} \in \mathrm{Ex}_A(\alpha,\beta) \Longleftrightarrow \tilde{A} = \tilde{A}^* \supset A, \|f' - \frac{\alpha+\beta}{2} f\| \geq \frac{\beta-\alpha}{2}\|f\| \qquad \forall\{f, f'\} \in \tilde{A}. \tag{4.19}$$

As in the case of nonnegative operators, there exist extremal extensions $A_\alpha$, $A_\beta \in \mathrm{Ex}_A(\alpha, \beta)$ defined by the equalities

$$\tilde{A}_\alpha := s - R - \lim_{x \downarrow \alpha} \tilde{A}_x, \qquad \tilde{A}_\beta := s - R - \lim_{x \uparrow \beta} \tilde{A}_x \quad x \in (\alpha, \beta), \tag{4.20}$$

where $\tilde{A}_x = A \dot{+} \hat{\mathfrak{N}}_x$.

**Proposition 4.5.** *The extensions $\tilde{A}_\alpha$, $\tilde{A}_\beta$ possess the following extremal property in the class $\tilde{A} \in$ $\mathrm{Ex}_A(\alpha, \beta)$:*

$$(\tilde{A}_\alpha - x)^{-1} \leq (\tilde{A} - x)^{-1} \leq (\tilde{A}_\beta - x)^{-1} \quad x \in (\alpha, \beta). \tag{4.21}$$

The proof is analogous to the one contained in [20, 79] for the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$. Inequalities (4.21) for $x = (\alpha+\beta)/2$ are implied by (3.69) and relations

$$\tilde{A}_\alpha = \frac{\beta-\alpha}{2}\tilde{C}_\mu^{-1} + \frac{\alpha+\beta}{2}, \qquad \tilde{A}_\beta = \frac{\beta-\alpha}{2}\tilde{C}_M^{-1} + \frac{\alpha+\beta}{2},$$

in which $\tilde{C}_\mu$ and $\tilde{C}_M$ are the minimum and maximum contractive extensions of Hermitian contraction $C := \frac{\beta-\alpha}{2}(A - \frac{\alpha+\beta}{2})^{-1}$. Let now $x \neq \frac{\alpha+\beta}{2}$. Since $\delta_r(z) = (z-r)(1-rz)^{-1} \in R$ for $r = (2x-(\alpha+\beta))/(\beta-\alpha)$ the transformation $\delta_r(T) = (T-r)(I-rT)^{-1}$ maps the operator segment $-I \leq T \leq I$ onto itself and keeps the order relation true. Therefore (3.69) yields

$$(\delta_r(C))_\mu = \delta_r(C_\mu) \leq \delta_r(\tilde{C}) \leq \delta_r(C_M) = (\delta_r(C))_M.$$

By virtue of the equality $\delta_r(\tilde{C}) = 2^{-1}(\beta-\alpha)(1-r^2)(\tilde{A}-x)^{-1} - r$ we obtain (4.21) for $x \neq (\alpha+\beta)/2$. □

**Proposition 4.6.** *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for* $A^*$, *such that* $A_2 \in \text{Ex}\,_A(\alpha,\beta)$ *and* $M(\lambda)$ *is the corresponding Weyl function. Then:*

(1) *there exists a strong resolvent limit*

$$M(\alpha) = M(\alpha)^* := s - R - \lim_{x\downarrow\alpha} M(x) \qquad (M(\beta) = M(\beta)^* := s - R - \lim_{x\uparrow\beta} M(x)), \tag{4.22}$$

(2) *the linear relation* $M(\alpha) = M(\alpha)^*$ $(M(\beta) = M(\beta)^*)$ *is associated with the semibounded above (below) form*

$$\mathfrak{t}_\alpha[f] = \lim_{x\downarrow\alpha}(M(x)f, f) \quad (\mathfrak{t}_\beta[f] = \lim_{x\uparrow\beta}(M(x)f, f)) \tag{4.23}$$

*with the domain (for* $x_0 \in (\alpha,\beta)$*)*

$$\mathfrak{D}(\mathfrak{t}_\alpha) = \{f : \lim_{x\downarrow\alpha} |(M(x)f,f)| < \infty\} = \mathfrak{D}((M(x_0) - M(\alpha))^{1/2}), \tag{4.24}$$

$$\mathfrak{D}(\mathfrak{t}_\beta) = \{f : \lim_{x\uparrow\beta} |(M(x)f,f)| < \infty\} = \mathfrak{D}((M(\beta) - M(x_0))^{1/2}),$$

*i.e.,* $\mathfrak{t}_\alpha = \mathfrak{t}_{M(\alpha)}$, $\mathfrak{t}_\beta = \mathfrak{t}_{M(\beta)}$;

(3) *the disjointness of the extensions* $A_2$ *and* $\tilde{A}_\alpha$ *(*$A_2$ *and* $\tilde{A}_\beta$*) is equivalent to the condition* $M(\alpha) \in \mathcal{C}(\mathcal{H})$ $(M(\beta) \in \mathcal{C}(\mathcal{H})$*), their transversality is equivalent to the inclusion* $M(\alpha) \in [\mathcal{H}]$ $(M(\beta) \in [\mathcal{H}])$;

(4) *the extensions* $\tilde{A}_\alpha$, $\tilde{A}_\beta$ *are defined by the boundary conditions*

$$\begin{aligned} \tilde{A}_\alpha &= \{\hat{f} = \{f, f'\} \in A^* : \{\Gamma_2\hat{f}, \Gamma_1\hat{f}\} \in M(\alpha)\} = \Gamma^{-1}M(\alpha); \\ \tilde{A}_\beta &= \{\hat{f} = \{f, f'\} \in A^* : \{\Gamma_2\hat{f}, \Gamma_1\hat{f}\} \in M(\beta)\} = \Gamma^{-1}M(\beta); \end{aligned} \tag{4.25}$$

*i.e.,* $\tilde{A}_\alpha = \tilde{A}_{M(\alpha)}$, $\tilde{A}_\beta = \tilde{A}_{M(\beta)}$. *Under the assumption* $M(\alpha) \in \mathcal{C}(\mathcal{H})$ *conditions* (4.25) *take the form*

$$\tilde{A}_\alpha = \ker(\Gamma_1 - M(\alpha)\Gamma_2), \qquad \tilde{A}_\beta = \ker(\Gamma_1 - M(\beta)\Gamma_2). \tag{4.26}$$

We omit the proof of Proposition 4.6, which is analogous to the proof of Proposition 4.2.

**Corollary 4.5.** *Under the assumptions of Proposition* 4.6, *the following relations hold:*

$$\begin{aligned} A_2 = \tilde{A}_\alpha &\Longleftrightarrow \lim_{x\downarrow\alpha}(M(x)h,h) = -\infty \quad \forall h \in \mathcal{H}\setminus\{0\}, \\ A_2 = \tilde{A}_\beta &\Longleftrightarrow \lim_{x\uparrow\beta}(M(x)h,h) = +\infty \quad \forall h \in \mathcal{H}\setminus\{0\}. \end{aligned} \tag{4.27}$$

**Corollary 4.6.** *Suppose that* $A$ *is a Hermitian operator with gap* $(\alpha,\beta)$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for* $A^*$ *such that* $A_2 = \tilde{A}_\alpha$, $M(\lambda)$ *is the corresponding Weyl function. Then the operator* $A$ *has the unique extension* $\tilde{A} = \tilde{A}^* \in \text{Ex}\,_A(\alpha,\beta)$ *(that is,* $\tilde{A}_\alpha = \tilde{A}_\beta$*) if and only if*

$$\lim_{x\uparrow\beta}(M(x)h,h) = +\infty \qquad \forall h \in \mathcal{H}\setminus\{0\}. \tag{4.28}$$

**4.** Let $\varkappa_-(\mathfrak{t})$ be the number of negative squares of the form $\mathfrak{t}$, i.e., the maximum dimension of negative linear manifolds contained in the cone $K_-(\mathfrak{t}) := \{f \in \mathfrak{D}(\mathfrak{t}) \setminus \{0\} : \mathfrak{t}[f] < 0\} \cup \{0\}$. We determine $\varkappa_-(T)$

for the operator $T = T^*$ by the equality $\varkappa_-(T) = \dim E_T(-\infty, 0)$. If $T$ is an operator associated with the closed form $\mathfrak{t}$, then in view of the mini-max principle $\varkappa_-(\mathfrak{t}) = \varkappa_-(T)$. We shall need the following two elementary lemmas.

**Lemma 4.1.** *Let $\mathfrak{t}_n$ be a monotonically decreasing sequence of symmetric semibounded above forms, which converges to the form $\mathfrak{t}$:*

$$\lim_{n\to\infty} \mathfrak{t}_n[f] = \mathfrak{t}[f] \qquad \forall f \in \mathfrak{D}(\mathfrak{t}) = \bigcap_{n\geq 1} \mathfrak{D}(\mathfrak{t}_n)$$

*Then $\varkappa_-(\mathfrak{t}) = \varkappa_-(\mathfrak{t}_n)$ for $n$ large enough. If the forms $\mathfrak{t}_n$ are closed and $T_n = T_n^*$ are operators associated with the forms $\mathfrak{t}_n$, then $\varkappa_-(\mathfrak{t}) = \varkappa_-(T_n)$ for $n$ large enough.*

**Lemma 4.2.** *Suppose that $\mathfrak{h}_1 \subset \mathfrak{h}$, $T_0 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} \in [\mathfrak{h}_1, \mathfrak{h}]$ is a nonnegative operator, $T = T^*(\in [\mathfrak{h}])$ is its extension which has the block-matrix representation*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad (T_{ij} \in [\mathfrak{h}_j, \mathfrak{h}_i]\,(i,j=1,2),\ T_{12} = T_{21}^*) \tag{4.29}$$

*which corresponds to the orthogonal decomposition $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. If there exists a nonnegative extension $T = T^* \in [\mathfrak{h}]$ of the form in (4.29), then*

(1) $\mathfrak{R}\,(T_{11}^{1/2}) \supset \mathfrak{R}\,(T_{21}^*)$ *and the operator $S := T_{11}^{-1/2} T_{21}^*$ is well defined and bounded;*

(2) *the operator $T = T_{\min}$ with $T_{22} = S^*S$ is the least one in the class $\mathrm{Ex}\ _{T_0}(-\infty, 0)$ of nonnegative extensions of the operator $T_0$;*

(3) $\varkappa_-(T) = \varkappa_-(T_{22} - S^*S)$, *if $T = T^*(\in [\mathfrak{h}])$ is an extension of the form in* (4.29).

The proof of the first two statements can be found in [61], the third statement is proved in [53].

**Theorem 4.1.** *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$ such that $A_2 \in \mathrm{Ex}\ _A(\alpha,\beta)$ and $A_2$, $\tilde{A}_\beta$ are disjoint, $M(\lambda)$ is the corresponding Weyl function, $\theta = \theta^*(\in \tilde{\mathcal{C}}(\mathcal{H}))$ is a semibounded below linear relation, $\mathfrak{D}(\mathfrak{t}_\theta) \subset \mathfrak{D}(\mathfrak{t}_{M(\beta)})$. Then*

(1) $\dim E_{\tilde{A}_\theta}(\alpha,\beta) \leq \varkappa_-(\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)})$;

(2) *if $A_2 = \tilde{A}_\alpha$, then the following equivalence holds:*

$$\dim E_{\tilde{A}_\theta}(\alpha,\beta) = n \iff \varkappa_-(\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)}) = n; \tag{4.30}$$

(3) *if, additionally, the form $\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)}$ is closable and $T_\beta = T_\beta^*$ is a linear relation associated with this form, then there exists*

$$s - R - \lim_{x\uparrow\beta}(\theta - M(x)) = T_\beta \tag{4.31}$$

*and the following equivalence holds:*

$$\dim E_{\tilde{A}_\theta}(\alpha,\beta) = n \iff \varkappa_-(T_\beta) = n, \tag{4.32}$$

*in which $T_\beta = \theta - M(\beta)$ if $\mathfrak{D}(\theta) \subset \mathfrak{D}(M(\beta))$.*

**Proof.** Let $g = (\tilde{A}_\theta - \lambda)f$. Then for all $\lambda \in \rho(\tilde{A}_\theta) \cap (\alpha,\beta)$ we have

$$((\tilde{A}_\theta - \alpha)(\tilde{A}_\theta - \lambda)^{-1}g, g) = \|\tilde{A}_\theta f\|^2 - (\alpha + \lambda)(\tilde{A}_\theta f, f) + \alpha\lambda\|f\|^2.$$

Hence for all $n \in \mathbb{Z}_+$ we obtain the equivalence

$$\dim E_{\tilde{A}_\theta}(\alpha,\beta) = n \iff \varkappa_-((\tilde{A}_\theta - \alpha)(\tilde{A}_\theta - \lambda)^{-1}) = n \qquad \forall\lambda \in (\beta_0, \beta), \tag{4.33}$$

where $\beta_0$ is close enough to $\beta$ and such that $(\beta_0,\beta) \subset \rho(\tilde{A}_\theta)$. In particular, in the case $n = 0$ the equivalence (4.33) takes the form

$$\tilde{A}_\theta = \mathrm{Ex}\ _A(\alpha,\beta) \iff (\tilde{A}_\theta - \alpha)(\tilde{A}_\theta - \lambda)^{-1} \geq 0 \qquad \forall\lambda \in (\beta_0, \beta).$$

By virtue of the extremal property (4.21) of the extension $\tilde{A}_\alpha$ the operator $T_{\min} = (\tilde{A}_\alpha - \alpha)(A_\alpha - \lambda)^{-1}$ is minimum in the class Ex $_{T_0}(-\infty, 0)$ of nonnegative extensions $T = T^*$ of the operator $T_0 = (A - \alpha)(A - \lambda)^{-1} \in [\mathfrak{M}_\lambda, \mathfrak{h}]$. Therefore, the operators $T_0$, $T = (\tilde{A}_\theta - \alpha)\tilde{A}_\theta - \lambda)^{-1}$, and $T_{\min}$ have the following block-matrix representations with respect to the decomposition $\mathfrak{h} = \mathfrak{M}_\lambda \oplus \mathfrak{N}_{\bar{\lambda}}$:

$$T_0 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{21}^* \\ T_{21} & T_{22} \end{pmatrix}, \quad T_{\min} = \begin{pmatrix} T_{11} & T_{21}^* \\ T_{21} & S^*S \end{pmatrix}, \tag{4.34}$$

where $S = T_{11}^{-1/2} T_{21}^*$. If $\tilde{A}_\theta = \tilde{A}_\theta^*$ is an extension of $A$ such that $\dim E_{\tilde{A}_\theta}(\alpha, \beta) = n$, then it follows from (4.33), (4.34), and Lemma 4.2 that for all $\lambda \in (\beta_0, \beta)$

$$n = \dim E_{\tilde{A}_\theta}(\alpha, \beta) = \varkappa_-((\tilde{A}_\theta - \alpha)(\tilde{A}_\theta - \lambda)^{-1})$$

$$= \varkappa_-((\tilde{A}_\theta - \alpha)(\tilde{A}_\theta - \lambda)^{-1} - (\tilde{A}_\alpha - \alpha)(\tilde{A}_\alpha - \lambda)^{-1}) = \varkappa_-((\tilde{A}_\theta - \lambda)^{-1} - (\tilde{A}_\alpha - \lambda)^{-1}). \tag{4.35}$$

According to Proposition 4.6 and (4.25), (4.26) we have $\tilde{A}_\alpha = \tilde{A}_{M(\alpha)}$. Since $M(\alpha) - M(\lambda) \leq 0 \ \forall \lambda \in (\beta_0, \beta)$, equality (4.35) and the resolvent formula (3.39) yield the estimate

$$n = \varkappa_-[(\theta - M(\lambda))^{-1} - (M(\alpha) - M(\lambda))^{-1}] \leq \varkappa_-((\theta - M(\lambda))^{-1}) \quad \forall \lambda \in (\beta_0, \beta). \tag{4.36}$$

The inclusion $\mathfrak{D}(\mathfrak{t}_\theta) \subset \mathfrak{D}(\mathfrak{t}_{M(\beta)})$ and Lemma 4.1 imply the equivalence

$$\varkappa_-(\theta - M(\lambda)) = n \quad \forall \lambda \in (\beta_0, \beta) \Longleftrightarrow \varkappa_-(\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)}) = n, \tag{4.37}$$

which, on account of (4.35), proves the first statement.

If $A_2 = \tilde{A}_\alpha$, then $M(\alpha) = \Gamma \tilde{A}_\alpha = \{0, \mathcal{H}\}$, whence $(M(\alpha) - M(\lambda))^{-1} = \mathbb{O}$ and inequality (4.36) turns into the equality. In this case relations (4.36) and (4.37) prove the implication

$$\dim E_{\tilde{A}_\theta}(\alpha, \beta) = n \Longrightarrow \varkappa_-(\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)}) = n.$$

These arguments carried out in reverse order yield the equivalence in the last implication.

If the form $\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)}$ is closable and $T_\beta = T_\beta^*$ is a linear relation associated with its closure, then the convergence theorem for sectorial forms [29, p. 563] implies that there exists a strong resolvent limit in (4.31), and (4.31) holds. Now the equivalence (4.32) is a consequence of (4.30).

**Corollary 4.7.** *Suppose that under the assumptions of Theorem 4.1 $A_2$ and $\tilde{A}_\beta$ are transversal extensions ($\Longleftrightarrow M(\beta) \in [\mathcal{H}]$). Then $\dim E_{\tilde{A}_\theta}(\alpha, \beta) \leq \varkappa_-(\theta - M(\beta))$. If, additionally, $A_2 = \tilde{A}_\alpha$, then*

$$\dim E_{\tilde{A}_\theta}(\alpha, \beta) = \varkappa_-(\theta - M(\beta)).$$

**Corollary 4.8.** *Suppose that under the assumptions of Theorem 4.1 $A_2 = \tilde{A}_\alpha$. Then*

$$\dim E_{\tilde{A}_\theta}(\alpha, x_0) = \varkappa_-(\theta - M(x_0)) \qquad \forall x_0 \in (\alpha, \beta). \tag{4.38}$$

*In the case $A_2 \neq \tilde{A}_\alpha$ we have $\dim E_{\tilde{A}_\theta}(\alpha, x_0) \leq \varkappa_-(\theta - M(x_0))$.*

**Corollary 4.9.** *Suppose that all conditions of Theorem 4.1, except for the last one, are fulfilled and $A_2 = \tilde{A}_\alpha (= \tilde{A}_{M(\alpha)})$. Then*

$$\tilde{A}_\theta \in \text{Ex}_A(\alpha, \beta) \Longleftrightarrow \mathfrak{D}(\mathfrak{t}_\theta) \subset \mathfrak{D}(\mathfrak{t}_{M(\beta)}), \qquad \mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)} \geq 0. \tag{4.39}$$

*In the case $M(\beta) \in [\mathcal{H}]$ (4.39) takes the form $\tilde{A}_\theta \in \text{Ex}_A(\alpha, \beta) \Longleftrightarrow \theta \geq M(\beta)$.*

**Proof.** It follows from Corollary 4.8 that the next implication holds: $\tilde{A}_\theta \in \mathrm{Ex}_A(\alpha,\beta) \Longrightarrow \theta - M(x) \geq 0\ \forall x \in (\alpha,\beta)$, whence $(M(x)f,f) \leq \mathfrak{t}_\theta[f]$ for all $f \in \mathfrak{D}(\mathfrak{t}_\theta)$. Now it remains to apply Lemma 4.1 and (4.23), (4.24). □

**Corollary 4.10.** *Let, under the assumptions of Theorem* 4.1, $(\alpha,\beta) = (-\infty,0)$ *and let* $A_2 = A_F$ *be the Friedrichs extension of the operator* $A \geq 0$. *Then the equivalences* (4.30), (4.32), (4.38), *and* (4.39) *take the form*

$$n = \dim E_{\tilde{A}_\theta}(-\infty,0) \Longleftrightarrow \varkappa_-(\mathfrak{t}_\theta - \mathfrak{t}_{M(0)}) = n; \tag{4.30'}$$

$$\dim E_{\tilde{A}_\theta}(-\infty,0) = n \Longleftrightarrow \varkappa_-(T_0) = n; \tag{4.32'}$$

$$\dim E_{\tilde{A}_\theta}(-\infty,x_0) = n \Longleftrightarrow \varkappa_-(\theta - M(x_0)) = n \quad (x_0 \in (-\infty,0)); \tag{4.38'}$$

$$\tilde{A}_\theta \geq 0 \Longleftrightarrow \mathfrak{D}(\mathfrak{t}_\theta) \subset \mathfrak{D}(\mathfrak{t}_{M(0)}), \quad \mathfrak{t}_\theta - \mathfrak{t}_{M(0)} \geq 0. \tag{4.39'}$$

**Remark 4.3.** (1) Generally speaking, it is impossible to characterize the spectrum of an extension $\tilde{A}_\theta$ in the gap $(\alpha,\beta)$ in terms of the linear relation $T_\beta$ in the case of a nonclosable form $\mathfrak{t}_\theta - \mathfrak{t}_{M(\beta)}$, as was done in Corollaries 4.7 and 4.9. In [20, 79] examples were constructed in which $\varkappa_-(T_\beta) = 0$ ($\Longleftrightarrow T_\beta \geq 0$) but $\dim E_{\tilde{A}_\theta}(\alpha,\beta) > 0$.

(2) As follows from the examples given in [79], the condition $\mathfrak{D}(\mathfrak{t}_\theta) \subset \mathfrak{D}(\mathfrak{t}_{M(\beta)})$ in Theorem 4.1 is essential. It is fulfilled automatically if the family $\theta - M(x)$ is uniformly semibounded below, i.e.,

$$\exists c \in \mathbb{R} : \theta - M(x) > cI_{\mathcal{H}} \quad \forall x \in (\alpha,\beta). \tag{4.40}$$

Indeed this implies

$$\lim_{x\uparrow\beta}(M(x)f,f) \leq \mathfrak{t}[f] - c\|f\|^2 \quad \forall f \in \mathfrak{D}(\mathfrak{t}_\theta),$$

and in view of Proposition 4.6 [see (4.24)] $f \in \mathfrak{D}(\mathfrak{t}_{M(\beta)})$. Note also that the inclusion $\mathfrak{D}(\mathfrak{t}_\theta) \subset \mathfrak{D}(\mathfrak{t}_{M(\beta)})$ does not imply (4.40).

(3) Let $\overline{\mathfrak{D}(A)} = \mathfrak{h}$. In this case another proof of Theorem 4.1 was given in [20, 79]. A more simple proof, similar to the one stated above, was obtained in [52]. If $A$ is a positively defined operator (i.e., $(-\infty,\varepsilon) \subset \rho(A_F)$ for some $\varepsilon > 0$), $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, and $\Pi = \{\mathcal{H},\Gamma_1,\Gamma_2\}$ is a BVS such that $A_2 = A_F$, $A_1 = A_K$, then $M(0) = 0$, $\mathfrak{t}_{M(0)} = 0$, and the equivalences (4.30′), (4.32′), (4.38′) coincide with the results of Birman [7] (see also [34] in the case $n_\pm(A) < \infty$).

**5.** Denote by $P\Omega^\varkappa(\alpha,\beta)$ ($\Omega^\varkappa(\alpha,\beta)$) the set of generalized pseudoresolvents (resolvents) $\mathbf{R}_\lambda = P(\tilde{A} - \lambda)^{-1}\lceil\mathfrak{h}$ of an operator $A$, generated by extensions $\tilde{A}$ acting in $\tilde{\mathfrak{h}} \subset \mathfrak{h}$, such that $\dim E_{\tilde{A}_\theta}(\alpha,\beta) = \varkappa$ and (3.36) holds.

**Theorem 4.2.** *Suppose that $A$ is a Hermitian operator with gap $(\alpha,\beta)$, $\Pi = \{\mathcal{H},\Gamma_1,\Gamma_2\}$ is a BVS for $A^*$ such that $A_2 = \tilde{A}_\alpha$ and $A_1 := \ker\Gamma_1$ is an operator, $M(\lambda)$ is the corresponding Weyl function and $M(\beta) \in [\mathcal{H}]$ (i.e., $\tilde{A}_\alpha$ and $\tilde{A}_\beta$ are transversal extensions). Then the formula*

$$\mathbf{R}_\lambda = (\tilde{A}_\alpha - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda) - M(\beta))^{-1}\gamma^*(\bar{\lambda}) \tag{4.41}$$

*establishes a one-to-one correspondence between $\mathbf{R}_\lambda \in P\Omega^\varkappa(\alpha,\beta)$ and $(\tau(\lambda) \in \tilde{S}_{\mathcal{H}}^{-\varkappa}(\alpha,\beta)$. Moreover, $\mathbf{R}_\lambda \in \Omega^\varkappa(\alpha,\beta)$ if and only if the function $\tau(\lambda) - M(\beta)$ is $M$-admissible.*

**Corollary 4.11.** *Suppose that $A$ is a nonnegative Hermitian operator, $n'_\pm(A) = 0$, $\Pi = \{\mathcal{H},\Gamma_1,\Gamma_2\}$ is a BVS for $A^*$ such that $A_2 = A_F$, $M(0) \in [\mathcal{H}]$ (i.e., $A_F$ and $A_K$ are transversal). Then formula* (4.41) *in which $\tilde{A}_\alpha = A_F$, $M(\beta) = M(0)$, establishes a one-to-one correspondence between $\mathbf{R}_\lambda \in P\Omega^\varkappa(-\infty,0)$ and $\tau(\lambda) \in \tilde{S}^{-\varkappa}(0,\infty)$. Moreover, $\mathbf{R}_\lambda \in \Omega^\varkappa(-\infty,0)$ if and only if*

$$s - R - \lim_{y\uparrow\infty} y^{-1}\tau(iy) = 0.$$

**Proof.** $A_1(= \ker\Gamma_1)$ is an operator since $A_2(0) = \mathfrak{N} = \mathfrak{h}_0^\perp$. Moreover, it follows from the condition $n'_\pm(A) = 0$ that $A_2 = A_F = A\dot{+}\hat{\mathfrak{N}}$ $(=: \tilde{A}_{\mathcal{F}_\Pi})$. Therefore, the desired statement is implied by Theorem 4.2 and Corollary 3.8. □

**Remark 4.4.** In the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ Theorem 4.2 was proved in [20, 79] by virtue of the theorem about zeros and "poles" of an operator-valued function from the class $S^{-\varkappa}(\mathbb{R}\setminus(\alpha,\beta))$ (see [22, 25, 79]). A more simple proof which is not based on the above-mentioned theorem was obtained in [17]. Note that a similar result was obtained earlier in [82] for the case $n_\pm(A) < \infty$, $(\alpha,\beta) = (-\infty,0)$. If, in addition, $A_2 = A_K$, then the generalized resolvents corresponding to the nonnegative extensions of $A$ (class $\Omega^0(-\infty,0)$) with an admissibility condition of the form in (3.63) were described in [83].

We omit the proof of Theorem 4.2 since in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}_0 \neq \mathfrak{h}$ one can deduce it from Theorem 3.1 in the same way as was done for the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ in [20, 79, 17]. We also omit descriptions of various classes of non-self-adjoint extensions. These problems will be considered in another paper (see [52] in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$).

## 5. FUNCTIONAL MODELS AND THE REALIZATION OF AN $R$-FUNCTION AS A WEYL FUNCTION

It was shown in [81, 83] that any $R$-function

$$Q(\lambda) = B\lambda + C + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\Sigma(t), \quad \int_{\mathbb{R}} \frac{d\Sigma(t)}{1+t^2} \in [\mathcal{H}], \tag{5.1}$$

satisfying the condition

$$0 \in \rho(\operatorname{Im} Q(i))(\Longleftrightarrow 0 \in \rho(\operatorname{Im} Q(\lambda)) \quad \forall\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-), \tag{5.2}$$

is a $Q$-function of some Hermitian operator. To this end a special functional model was used, constructed by the $R$-function $Q(\lambda)$. In the definite case these results can be derived also from other functional models (see, for example [71]). In this section for some functional models constructed by the $R$-function $Q(\lambda)$ we determine BVS's such that the corresponding Weyl functions coincide with $Q(\lambda)$.

**1.** Consider, following Krein and Langer [81, 83], the linear space $\mathfrak{L}$ of functions defined on $\mathbb{C}_+ \cup \mathbb{C}_-$ with finite support and values in $\mathcal{H}$ written in the form

$$f = \sum_\lambda \delta_\lambda \otimes f_\lambda \qquad (f_\lambda \in \mathcal{H},\ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-),$$

where $\delta_\lambda$ is a formal symbol (the delta-function) and only a finite number of $f_\lambda \in \mathcal{H}$ is distinct from zero. The space $\mathfrak{L}$ is endowed with the possibly degenerate nonnegative inner product ($f = \sum \delta_\lambda \otimes f_\lambda$, $g = \sum \delta_\mu \otimes g_\mu$)

$$(f,g)_{\mathfrak{L}} = \sum_{\lambda,\mu} \left(\frac{Q(\lambda) - Q^*(\mu)}{\lambda - \bar{\mu}} f_\lambda,\ g_\mu\right)_{\mathcal{H}}. \tag{5.3}$$

The quotient space $\mathfrak{L}/J$ ($J = \mathfrak{L} \cap \mathfrak{L}^\perp$) can be canonically extended to the Hilbert space $\mathfrak{K}(Q)$ (see [81]). Consider the linear manifold

$$\hat{G} = \left\{\hat{f} = \left\{\sum \delta_\lambda \otimes f_\lambda,\ \sum \lambda\delta_\lambda \otimes f_\lambda\right\} \in \mathfrak{L} \otimes \mathfrak{L}\right\} \tag{5.4}$$

in $\mathfrak{L} \oplus \mathfrak{L}$ and the corresponding subspace $G \subset \mathfrak{K} \oplus \mathfrak{K}$.

**Proposition 5.1.** *Let $Q(\lambda)$ be an operator-valued function of the form in* (5.1) *satisfying condition* (5.2), *and let $G$ be a linear relation in $\mathfrak{K}$ defined by* (5.4). *Then*

(1) *mappings $\chi_j : \hat{G} \to \mathcal{H}$ $(j = 1,2)$*

$$\chi_1(\hat{f}) = \sum_\lambda Q(\lambda) f_\lambda, \quad \chi_2(\hat{f}) = \sum_\lambda f_\lambda \quad \left(f = \sum_\lambda \delta_\lambda \otimes f_\lambda\right) \tag{5.5}$$

*can be extended to the continuous mappings $\Gamma_1, \Gamma_2 \in [G, \mathcal{H}]$ and the collection $\Pi' = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ forms a BVS for the relation $G$;*

(2) *$A := G^*$ is a simple Hermitian operator coinciding with the closure of the operator $A_0$*

$$A_0 = \{\hat{f} = \{\Sigma\delta_\lambda \oplus f_\lambda, \Sigma\lambda\delta_\lambda \oplus f_\lambda\} \in \mathfrak{L}\oplus\mathfrak{L} : \chi_1(\hat{f}) = \chi_2(\hat{f}) = 0\}; \tag{5.6}$$

(3) *the $\gamma$-field of the linear relation $A_2 := \ker\Gamma_2$ takes the form $\gamma(\lambda)h = \delta_\lambda \otimes h (h \in \mathcal{H})$, and the corresponding Weyl function $M(\lambda) = \Gamma_1\hat{\gamma}(\lambda)$ coincides with $Q(\lambda)$.*

**Proof.** (1) We show that the mappings $\chi_j : \hat{G} \to \mathcal{H} (j = 1, 2)$ can be correctly extended to the quotient space $\hat{G}/\hat{G} \cap (\mathfrak{J}\oplus\mathfrak{J})$. Indeed, if $\hat{f} = \{f, f'\} = \left\{\sum_\lambda \delta_\lambda \otimes f_\lambda, \sum_\lambda \lambda\delta_\lambda \otimes f_\lambda\right\} \in \mathfrak{J}\oplus\mathfrak{J}$, then $\forall\mu \in \mathbb{C}_+$, $h \in \mathcal{H}$

$$(f, \delta_\mu \otimes h)_\mathfrak{L} = \sum_\lambda \Big(\frac{Q(\lambda) - Q^*(\mu)}{\lambda - \bar{\mu}} f_\lambda, h\Big)_\mathcal{H}, \tag{5.7}$$

$$(f', \delta_\mu \otimes h)_\mathfrak{L} = \sum_\lambda \Big(\frac{Q(\lambda) - Q^*(\mu)}{\lambda - \bar{\mu}} \lambda f_\lambda, h\Big)_\mathcal{H}. \tag{5.8}$$

It follows from (5.7), (5.8) that

$$(f' - \mu f, \delta_\mu \otimes h)_\mathfrak{L} = \sum_\lambda ([Q(\lambda) - Q^*(\mu)]f_\lambda, h)_\mathcal{H} = 0. \tag{5.9}$$

By virtue of (5.9) and an analogous equality for $\bar{\mu}$ we have

$$\sum_\lambda ([Q(\mu) - Q^*(\mu)]f_\lambda, h)_\mathcal{H} = 2i(\mathrm{Im}\, Q(\mu)\chi_2\hat{f}, h)_\mathcal{H} = 0 \quad \forall h \in \mathcal{H}. \tag{5.10}$$

Since $0 \in \rho(\mathrm{Im}\, Q(\mu))$, we obtain the equality $\chi_2\hat{f} = \sum f_\lambda = 0\ \forall\hat{f} \in \hat{G} \cap (\mathfrak{J}\oplus\mathfrak{J})$. It also follows from (5.9) that $\chi_1\hat{f} = \sum Q(\lambda)f_\lambda = 0\ \forall\hat{f} \in \hat{G}\cap(\mathfrak{J}\oplus\mathfrak{J})$. Thus the mappings $\chi_j (j = 1, 2)$ induce the quotient mappings $\chi_j : \hat{G}/\hat{G} \cap (\mathfrak{J}\oplus\mathfrak{J}) \to \mathcal{H}$, which we denote by the same symbols.

Now we show that the mappings $\chi_j$ $(j = 1, 2)$ can be extended by continuity to $G$. The continuity of $\chi_1$ follows from the estimate ($\hat{f} = \{\sum \delta_\lambda \otimes f_\lambda, \sum \lambda\delta_\lambda \otimes f_\lambda\}$)

$$\|\hat{f}\|^2_{\mathfrak{K}\oplus\mathfrak{K}} = \sum_{\lambda,\mu} \Big(\Big[B + \int_\mathbb{R} \frac{1 + \lambda\bar{\mu}}{(t - \lambda)(t - \bar{\mu})} d\Sigma(t)\Big] f_\lambda, f_\mu\Big)_\mathcal{H}$$

$$= \sum_{\lambda,\mu} \Big\{(Bf_\lambda, f_\mu)_\mathcal{H} + \int_\mathbb{R} \Big[\frac{1 + t\lambda)(1 + t\bar{\mu})}{(t - \lambda)(t - \bar{\mu})} + 1\Big] \frac{d(\Sigma(t)f_\lambda, f_\mu)_\mathcal{H}}{1 + t^2}\Big\}$$

$$\geq (B\chi_2(\hat{f}), \chi_2(\hat{f}))_\mathcal{H} + \int_\mathbb{R} (1 + t^2)^{-1} d(\Sigma(t)\chi_2(\hat{f}), \chi_2(\hat{f}))_\mathcal{H}$$

$$= (\mathrm{Im}\, Q(i)\chi_2(\hat{f}), \chi_2(\hat{f}))_\mathcal{H} \geq c_1\|\chi_2(\hat{f})\|^2_\mathcal{H} \quad (c_1 > 0).$$

In order to prove the estimate $\|\hat{f}\|^2_{\mathfrak{L}\oplus\mathfrak{L}} \geq c_2\|\chi_1(\hat{f})\|^2_\mathcal{H}$ it is sufficient to use the relation

$$\|f\|^2_{\mathfrak{L}\oplus\mathfrak{L}} = -\sum_{\lambda,\mu} (1 + \lambda\bar{\mu})\Big(\frac{Q^{-1}(\lambda) - Q^{-1}(\bar{\mu})}{\lambda - \bar{\mu}} Q(\lambda)f_\lambda, Q(\mu)f_\mu\Big)_\mathcal{H}$$

and the integral representation of the function $-Q^{-1}(\lambda)$ (cf. [26]).

Let $\hat{f} = \{f, f'\} = \left\{\sum_\lambda \delta_\lambda \otimes f_\lambda, \sum_\lambda \lambda\delta_\lambda \otimes f_\lambda\right\}$, $g = \{g, g'\} = \left\{\sum_\mu \delta_\mu \otimes g_\mu, \sum_\mu \mu\delta_\mu \otimes g_\mu\right\}$. It follows from (5.3), (5.5) that the Green formula holds:

$$(f', g)_{\mathfrak{L}} - (f, g')_{\mathfrak{L}} = \sum_{\lambda,\mu} \left(\frac{Q(\lambda) - Q^*(\mu)}{\lambda - \bar{\mu}} \lambda f_\lambda, g_\mu\right)_{\mathcal{H}}$$

$$- \sum_{\lambda,\mu} \left(\frac{Q(\lambda) - Q^*(\mu)}{\lambda - \bar{\mu}} f_\lambda, \mu g_\mu\right)_{\mathcal{H}} = \sum_{\lambda,\mu} (Q(\lambda) f_\lambda, g_\mu)_{\mathcal{H}}$$

$$- \sum_{\lambda,\mu} (f_\lambda, Q(\mu) g_\mu)_{\mathcal{H}} = (\chi_1(\hat{f}), \chi_2(\hat{g}))_{\mathcal{H}} - (\chi_2(\hat{f}), \chi_1(\hat{g}))_{\mathcal{H}}. \tag{5.11}$$

Making use of the continuity of the mappings $\chi_j : \hat{G} \to \mathcal{H}$ $(j = 1, 2)$ we extend equality (5.11) to $G$ $(\hat{f}, \hat{g} \in G)$. To prove the surjectivity of the mapping $\Gamma = \{\Gamma_2, \Gamma_1\} : G \to \mathcal{H} \oplus \mathcal{H}$ we put for all $h_1, h_2 \in \mathcal{H}$ $f = \delta_i \otimes f_i - \delta_{-i} \otimes f_{-i}$, where

$$f_i = \frac{1}{2i} (\operatorname{Im} Q(i))^{-1} (h_1 - Q(-i) h_2), \quad f_{-i} = \frac{1}{2i} (\operatorname{Im} Q(i))^{-1} (h_1 - Q(i) h_2).$$

Then we have $\chi_1(\hat{f}) = h_1$, $\chi_2(\hat{f}) = h_2$.

(2) Let $A := G^*$. It follows from the Green formula (1.3) that the operator $A$ coincides with the closure of an operator $A_0$.

(3) Letting in (5.11) $g = \delta_\lambda \otimes h$, $\hat{f} \in A_0$ we have

$$((A - \bar{\lambda}) f, \delta_\lambda \otimes h)_{\mathfrak{L}} = (\Gamma_1 \hat{f}, h)_{\mathcal{H}} - (\Gamma_2 \hat{f}, Q(\lambda) h)_{\mathcal{H}} = 0,$$

that is, $\{\{\delta_\lambda \otimes h\} : h \in \mathcal{H}\} \subset \mathfrak{N}_\lambda$ $(\lambda \in \mathbb{C} \setminus \mathbb{R})$. In view of the equality $\Gamma_2(\delta_\lambda \otimes h) = h$ we obtain that the following equality holds:

$$\mathfrak{N}_\lambda = \{\delta_\lambda \otimes h : h \in \mathcal{H}\},$$

and the operator-valued function $\gamma(\lambda) : \gamma(\lambda) h = \delta_\lambda \otimes h$ is a $\gamma$-field of the extension $A_2$. Moreover, the Weyl function $M(\lambda)$, corresponding to the BVS $\Pi^1$, coincides with $Q(\lambda)$:

$$M(\lambda) h = \Gamma_1 \hat{\gamma}(\lambda) h = \chi_1 \{\delta_\lambda \otimes h, \lambda\delta_\lambda \otimes h\} = Q(\lambda) h. \qquad \square$$

**2.** In this section, we consider another realization of an $R$-function as a Weyl function, based on the representation of a self-adjoint operator as an operator of multiplication in $L_2(d\Sigma, \mathcal{H})$.

Let $Q(\lambda)$ $(\in R_{\mathcal{H}})$ have an integral representation (5.1) with an operator measured $\Sigma(t)$, and let $L_2(d\Sigma, \mathcal{H})$ be the space of vector-valued functions $f(t)$ on $\mathbb{R}$ with values in $\mathcal{H}$ such that

$$\|f\|_{L_2(d\Sigma, \mathcal{H})} := \int_{\mathbb{R}} (d\Sigma(t) f(t), f(t))_{\mathcal{H}} < \infty.$$

We define the space $\mathfrak{h}(Q) = \mathcal{H}_B \oplus L_2(d\Sigma, \mathcal{H})$, where $\mathcal{H}_B = \mathfrak{R}(B^{1/2})$ is a Hilbert space endowed with the scalar product

$$(h_1, h_2)_{\mathcal{H}_B} := (B^{-1/2} h_1, B^{-1/2} h_2)_{\mathcal{H}}, \tag{5.12}$$

and consider the self-adjoint relation $\tilde{A}$ in $\mathfrak{h}$:

$$\tilde{A} = \{\{\mathbb{O} \oplus f(t), \tilde{b} \oplus t f(t)\} : f(t), t f(t) \in L_2(d\Sigma, \mathcal{H}), \tilde{b} \in \mathcal{H}_B\}. \tag{5.13}$$

Let $A$ be the Hermitian operator defined by the relation

$$A = \{\{\mathbb{O} \oplus f(t), \tilde{b} \oplus t f(t)\} \in \tilde{A} : \int_{\mathbb{R}} d\Sigma(t) f(t) + \tilde{b} = 0\}. \tag{5.14}$$

It is easy to see that the operator $A$ is closed.

**Proposition 5.2.** *Let $Q(\lambda)$ ($\in R_{\mathcal{H}}$) be an operator-valued function of the form in* (5.1) *satisfying condition* (5.2), *and let $A$ be an operator of the form in* (5.14). *Then*

(1) *the defect subspace $\mathfrak{N}_\lambda$ of the operator $A$ takes the form*

$$\mathfrak{N}_\lambda = \{Bh \oplus \frac{h}{t-\lambda} : h \in \mathcal{H}\}; \tag{5.15}$$

(2) *the adjoint linear relation $A^*$ coincides with the relation $A_*$ of the form*

$$A_* = \{\{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in \mathfrak{h} \oplus \mathfrak{h} : \exists h \in \mathcal{H}\ \tilde{f}(t) - tf(t) = -h, b = Bh\}; \tag{5.16}$$

(3) *the triple $\Pi'' = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, where $\Gamma_1$, $\Gamma_2$ are defined by the equalities*

$$\Gamma_1 \hat{f} = \tilde{b} + Ch + \int_{\mathbb{R}} d\Sigma(t) \frac{t\tilde{f}(t) + f(t)}{1+t^2}, \tag{5.17}$$

$$\Gamma_2 \hat{f} = h \qquad (\hat{f} = \{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in A_*),$$

*is a BVS for $A^*$ and $A_2 := \ker \Gamma_2 = \tilde{A}$;*

(4) *an operator-valued function $\gamma(\lambda)$, defined by the equality*

$$\gamma(\lambda)h = Bh \oplus \frac{h}{t-\lambda}, \tag{5.18}$$

*is a $\gamma$-field of the extension $\tilde{A}_2$, and $Q(\lambda)$ coincides with the Weyl function $M(\lambda)$ corresponding to the BVS $\Pi''$.*

**Proof.** (1) It follows from (5.14) that $Bh \oplus \frac{h}{t-\lambda} \in \mathfrak{N}_\lambda\ \forall h \in \mathcal{H}$. By virtue of the relation $0 \in \rho(\operatorname{Im} Q(i))$ and identity

$$\|\gamma(\lambda)h\|^2_{\mathfrak{h}} = \left(\frac{\operatorname{Im} Q(\lambda)}{\operatorname{Im}\lambda}h, h\right)_{\mathcal{H}} = (Bh,h)_{\mathcal{H}} + \int_{\mathbb{R}} \frac{d(\Sigma(t)h,h)_{\mathcal{H}}}{|t-\lambda|^2}, \tag{5.19}$$

we obtain relation (5.15) and the surjectivity of the mapping $\gamma(\lambda) \in [\mathcal{H}, \mathfrak{N}_\lambda]$.

(2) Making use of (5.15) and the equality $A^* = \tilde{A} \dotplus \hat{\mathfrak{N}}_i$, we represent the vector $\hat{f} = \{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in A^*$ in the form

$$\hat{f} = \{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} = \hat{f}_0 + \hat{h}_i = \{Bh \oplus f(t), (iBh + \tilde{b}_0) \oplus (tf(t) - h)\}, \tag{5.20}$$

where $b_0 = b - iBh$, $\hat{f}_0 := \{0 \oplus f_0(t), \tilde{b}_0 \oplus tf_0(t)\} \in \tilde{A}$, $\hat{h}_i := \{Bh \oplus \frac{h}{t-i}, iBh \oplus \frac{ih}{t-i}\} \in \hat{\mathfrak{N}}_i$.

Thus $A^* \subset A_*$. In order to prove the inverse inclusion given $\hat{f} = \{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in \tilde{A}$ we put $f_0(t) = f(t) - h(t-i)^{-1}$, $\tilde{b}_0 = \tilde{b} - iBh$. By virtue of (5.16) we have

$$tf_0(t) = \tilde{f}(t) - ih(t-i)^{-1} \in L_2(d\Sigma, \mathcal{H}), \quad \tilde{b}_0 \in \mathcal{H}_B,$$

and hence $\{0 \oplus f_0(t), \tilde{b}_0 \oplus tf_0(t)\} \in \mathfrak{h}$. This implies that $\hat{f} = \hat{f}_0 + \hat{h}_i \in A^*$.

(3) Let $\hat{f}_j = \{f_j, f_j'\} = \{Bh_j \oplus f_j(t), \tilde{b}_j \oplus \tilde{f}_j(t)\} \in A^*$, $\tilde{f}_j(t) - tf_j(t) = -h_j \in \mathcal{H}$ ($j = 1,2$). Then we have

$$(\Gamma_1\hat{f}_1, \Gamma_2\hat{f}_2)_{\mathcal{H}} - (\Gamma_2\hat{f}_1, \Gamma_1\hat{f}_2)\mathcal{H} = (\tilde{b}_1, h_2)_{\mathcal{H}} - (b_1, \tilde{h}_2)_{\mathcal{H}}$$

$$+ \int_{\mathbb{R}} \left(\frac{d\Sigma(t)}{1+t^2}(t\tilde{f}_1(t) + f_1(t)), tf_2(t) - \tilde{f}_2(t)\right)_{\mathcal{H}} - \int_{\mathbb{R}} \left(\frac{d\Sigma(t)}{1+t^2}(tf_1(t) - \tilde{f}_1(t)), t\tilde{f}_2(t) + f_2(t)\right)_{\mathcal{H}}$$

$$= (\tilde{b}_1, h_2)_{\mathcal{H}} + (b_1, \tilde{h}_2)_{\mathcal{H}} + \int_{\mathbb{R}} (d\Sigma(t)\tilde{f}_1(t), f_2(t))_{\mathcal{H}} - \int_{\mathbb{R}} (d\Sigma(t) f_1(t), \tilde{f}_2(t))_{\mathcal{H}} = (\tilde{f}_1, f_2)_{\mathfrak{h}} - (f_1, \tilde{f}_2)_{\mathfrak{h}}.$$

This proves the Green formula.

To prove the surjectivity of the mapping $\Gamma : \hat{f} \to \{\Gamma_2\hat{f}, \Gamma_1\hat{f}\}$ we note that $\tilde{A} = \tilde{A}_2 := \ker\Gamma_2$ and $\hat{f} = \{0 \oplus (1+t^2)^{-1}h, Bh \oplus t(1+t^2)^{-1}h\} \in A_2\ \forall h \in \mathcal{H}$. After the application of the mapping $\Gamma_1$ to $\hat{f}$ we obtain the equality

$$\Gamma_1\hat{f} = Bh + \int_{\mathbb{R}} (1+t^2)^{-1} d\Sigma(t)h = [\operatorname{Im} Q(i)]h,$$

which, on account of the condition $0 \in \rho(\operatorname{Im} Q(i))$, implies that $\Gamma_1 A_2 = \mathcal{H}$. Now (5.15) and (5.17) yield that $\Gamma_2\hat{\mathfrak{N}}_i = \mathcal{H}$. The surjectivity of $\Gamma$ is a simple consequence of the relation $A^* = A_2 \dot{+} \hat{\mathfrak{N}}_i$.

(4) It follows from the definition of $\gamma(\lambda)$ that

$$\Gamma_2\hat{\gamma}(\lambda)h = \Gamma_2\Big\{Bh \oplus \frac{h}{t-\lambda}, \lambda Bh \oplus \frac{\lambda h}{t-\lambda}\Big\} = -\Big(\frac{\lambda h}{t-\lambda} - \frac{th}{t-\lambda}\Big) = h.$$

Therefore $\hat{\gamma}(\lambda) = (\Gamma_2 \restriction \hat{\mathfrak{N}}_\lambda)^{-1}$ and $\gamma(\lambda)$ is a $\gamma$-field of an extension $A_2 = \ker\Gamma_2$. Further, in accordance with (1.9), we obtain the equality

$$M(\lambda)h = \Gamma_1\hat{\gamma}(\lambda)h = \Gamma_1\Big\{Bh \oplus \frac{h}{t-\lambda}, \lambda Bh \oplus \frac{\lambda h}{t-\lambda}\Big\} = \Big(\lambda B + C + \int_{\mathbb{R}} \frac{(1+\lambda t)d\Sigma(t)}{(t-\lambda)(1+t^2)}\Big)h,$$

which shows that $M(\lambda)$ coincides with $Q(\lambda)$. □

**Corollary 5.1.** *If, under assumptions of Proposition* 5.2, *$Q(\lambda)$ satisfies the condition $\lim_{y\uparrow\infty}(Q(iy)/y) = 0$ (i.e., $B = 0$), then $\mathfrak{h} = L_2(d\Sigma, \mathcal{H})$, $\tilde{A} = A_2$ is an operator of multiplication in $L_2(d\Sigma,\mathcal{H})$, and the operator $A$ is defined by the usual condition:*

$$A = \tilde{A}\restriction_{\mathfrak{D}(A)}, \mathfrak{D}(A) = \{f \in \mathfrak{D}(\tilde{A}) : \int_{\mathbb{R}} d\Sigma(t)f(t) = 0\}. \tag{5.21}$$

**3.** Consider, after [75, 71], the Hilbert space $\mathfrak{B}(Q)$ of vector-valued functions $F(\lambda)$ on $\mathbb{C}_+ \cup \mathbb{C}_-$ with values in $\mathcal{H}$ of the form

$$F(\lambda) = b + \int_{\mathbb{R}} \frac{d\Sigma(t)f(t)}{t-\lambda} \qquad (b \in \mathcal{H}_B = \mathfrak{R}\,(B^{1/2}), f(t) \in L_2(d\Sigma,\mathcal{H})) \tag{5.22}$$

endowed with the scalar product

$$(F_1, F_2)_{\mathfrak{B}(Q)} = (b_1, b_2)_{\mathcal{H}_B} + (f_1, f_2)_{L_2(d\Sigma,\mathcal{H})} \qquad (F_j \in \mathfrak{B}(Q); j = 1,2). \tag{5.23}$$

The mapping $U : b \oplus f(t) \to F(\lambda)$ establishes an isometrical isomorphism from $\mathfrak{h}(Q)$ onto $\mathfrak{B}(Q)$ and with this isomorphism the operator $A$ in $\mathfrak{h}(Q)$ is isomorphic to the operator of multiplication in $\mathfrak{B}(Q)$, which we denote by the same symbol

$$A = \{\{F(\lambda), \lambda F(\lambda)\} : F(\lambda), \lambda F(\lambda) \in \mathfrak{B}(Q)\}. \tag{5.24}$$

**Proposition 5.3.** *Suppose that $Q(\lambda)(\in R_{\mathcal{H}})$ is an operator-valued function of the form in* (5.1), *satisfying condition* (5.2), *and $A$ is an operator of the form in* (5.24). *Then*

(1) *the defect subspace $\mathfrak{N}_\lambda$ of the operator $A$ takes the form*

$$\mathfrak{N}_\lambda = \Big\{h_\lambda = \frac{Q(\mu) - Q(\lambda)}{\mu - \lambda}h : h \in \mathcal{H}\Big\}; \tag{5.25}$$

(2) *the adjoint relation $A^*$ coincides with the relation*

$$A_* = \{\{F(\lambda), \tilde{F}(\lambda)\} \in \mathfrak{B}(Q)^2 : \exists h_1, h_2 \in \mathcal{H}, \tilde{F}(\lambda) - \lambda F(\lambda) = h_1 - Q(\lambda)h_2\}; \tag{5.26}$$

(3) *the triple* $\Pi''' = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, *where* $\Gamma_1$, $\Gamma_2$ *are defined by the equalities* $\Gamma_j \hat{F}(\lambda) = h_j$ $(j = 1,2; \hat{F}(\lambda) \in A^*)$ *is a BVS for* $A^*$;

(4) *the* $\gamma$*-field of the relation* $A_2 = \ker \Gamma_2$ *takes the form* $\gamma(\lambda)h = \frac{Q(\mu)-Q(\lambda)}{\mu-\lambda}h$, *and the corresponding Weyl function* $M(\lambda) = \Gamma_1\hat{\gamma}(\lambda)$ *coincides with* $Q(\lambda)$.

**Proof.** (1) After the application of the mapping $U$ we obtain from (5.18)

$$U\{Bh \oplus \frac{h}{t-\lambda}\} = Bh + \int_{\mathbb{R}} \frac{d\Sigma(t)}{(t-\mu)(t-\lambda)}h = \frac{Q(\mu)-Q(\lambda)}{\mu-\lambda}h \in \mathfrak{N}_\lambda.$$

(2, 3) Analogously, it follows from (5.16) that $\forall \hat{f} = \{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in A^*$

$$U\hat{f} = U\{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} = \{F(\lambda), \tilde{F}(\lambda)\} = \hat{F}(\lambda),$$

where

$$F(\lambda) = b + \int_{\mathbb{R}} \frac{d\Sigma(t)f(t)}{t-\lambda}, \qquad \tilde{F}(\lambda) = \tilde{b} + \int_{\mathbb{R}} \frac{d\Sigma(t)\tilde{f}(t)}{t-\lambda}. \tag{5.27}$$

Taking into account the relations $\tilde{f}(t) - tf(t) = -h$, $b = Bh$ we have

$$\tilde{F}(\lambda) - \lambda F(\lambda) = \tilde{b} - \lambda Bh + \int_{\mathbb{R}} \frac{d\Sigma(t)}{t-\lambda}(\tilde{f}(t) - \lambda f(t))$$

$$= \tilde{b} + Ch + \int_{\mathbb{R}} d\Sigma(t)\Big(f(t) - \frac{th}{1+t^2}\Big) - \Big[B\lambda + C + \int_{\mathbb{R}} d\Sigma(t)\Big(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\Big)\Big]h = h_1 - Q(\lambda)h_2,$$

where

$$h_1 = \tilde{b} + Ch + \int_{\mathbb{R}} d\Sigma(t)\frac{t\tilde{f}(t) + f(t)}{1+t^2} = \Gamma_1\hat{f}, \qquad h_2 = \Gamma_2\hat{f}. \tag{5.28}$$

This proves the inclusion $A^* \subset A_*$. Conversely, let $\hat{F} = \{F(\lambda), \tilde{F}(\lambda)\} \in A_*$. Then there exist $h_1$, $h_2 \in \mathcal{H}$ such that $\tilde{F}(\lambda) = \lambda F(\lambda) - h_1 + Q(\lambda)h_2$. By virtue of (5.27) we obtain the equality

$$\int_{\mathbb{R}} \frac{d\Sigma(t)\tilde{f}(t)}{(t-\lambda)(t-\mu)} = b - Bh_2 + \int_{\mathbb{R}} \frac{d\Sigma(t)(tf(t) - h_2)}{(t-\lambda)(t-\mu)} \tag{5.29}$$

which, on account of the uniqueness of the representation (5.22), gives $b = Bh_2$, $\tilde{f}(t) - tf(t) = -h_2$. This implies that $\{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in U^{-1}A^*U$ and $A_* = A^*$. Note that simultaneously it was shown that the triple $\Pi''' = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$.

(4) It follows from the equality

$$\hat{h}_\lambda = \Big\{\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h, \lambda\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h\Big\} = \Big\{\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h, \mu\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h + Q(\lambda)h - Q(\mu)h\Big\}$$

that $\Gamma_2\hat{h}_\lambda = h$, $\Gamma_1\hat{h}_\lambda = Q(\lambda)h$. Therefore $\gamma(\lambda)h = \frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h$ and $Q(\lambda)$ coincides with the Weyl function $M(\lambda)$ corresponding to the BVS $\Pi'''$. □

**4.** In this section we shall distinguish all objects connected with the BVS (5.4) by one prime and the others connected with the BVS (5.16) by two primes. In the proof of Proposition 5.3 an isomorphism of BVS's $\Pi''$ and $\Pi'''$ was established. In the next proposition the same is proved for BVS's $\Pi'$ and $\Pi''$.

**Proposition 5.4.** *Under the assumptions of Propositions* 5.1 *and* 5.2 *BVS's* $\Pi' = \{\mathcal{H}, \Gamma_1', \Gamma_2'\}$ *and* $\Pi'' = \{\mathcal{H}, \Gamma_1'', \Gamma_2''\}$ *are isomorphic, i.e., there exists an isometrical isomorphism* $U$ *from* $\mathfrak{K}(Q)$ *onto* $\mathfrak{h}(Q)$ *such that* $UA' = A''U$, $\Gamma_1' = \Gamma_1''\hat{U}$, $\Gamma_2' = \Gamma_2''\hat{U}$, *where* $\hat{U} = U \oplus U$.

**Proof.** We define an operator $U_0$ on the span of subspaces $\mathfrak{N}_\lambda$ $(\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-)$ by the equality

$$U_0\gamma'(\lambda)h = U_0(\delta_\lambda \otimes h) = \{Bh \oplus \frac{h}{t-\lambda}\} = \gamma''(\lambda)h \ (h \in \mathcal{H}). \tag{5.30}$$

It follows from (5.3), (5.19) that $U_0$ is an isometrical operator and can be extended to the isomorphism $U$ from $\mathfrak{K}(Q)$ into $\mathfrak{h}(Q)$. For $\hat{f} = \sum_{\lambda}\{\delta_\lambda \otimes h_\lambda, \lambda\delta_\lambda \otimes h_\lambda\}$ we have

$$\begin{gathered}\hat{U}\hat{f} = \sum_\lambda\{Bh_\lambda \oplus (t-\lambda)^{-1}h_\lambda, \qquad \lambda(Bh_\lambda \oplus (t-\lambda)^{-1}h_\lambda)\} = \sum_\lambda \hat{\gamma}(\lambda)h_\lambda, \\ \Gamma_1''\hat{U}\hat{f} = \sum_\lambda Q(\lambda)h_\lambda = \Gamma_1'\hat{f}, \quad \Gamma_2''\hat{U}\hat{f} = \sum_\lambda h_\lambda = \Gamma_2''\hat{f}.\end{gathered} \tag{5.31}$$

It follows from (5.31) that $UA' = A''U$. □

Each of the Propositions 5.1–5.3 contains, in particular, the following

**Theorem 5.1.** *An operator-valued function $M(\lambda)$ holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ with values in $[\mathcal{H}]$ is a Weyl function of some simple Hermitian operator corresponding to some BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ if and only if:*

(1) $M(\lambda) \in R_{\mathcal{H}}$; (2) $0 \in \rho(\text{Im } M(\lambda))$ $\quad \forall\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

As was shown in [51, 53] (in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ see [20, 79]) the Weyl function $M(\lambda)$, corresponding to a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, is a $Q$-function of a Hermitian operator $A$, corresponding to the extension $A_2 = \ker \Gamma_2$. The inverse assertion also holds: each $Q$-function of an operator $A$ is its Weyl function (see §1). Therefore, Theorem 5.1 is a consequence of the results obtained in [81, 83] on the inverse problem for a $Q$-function. However, we stated Propositions 5.1–5.3 since the constructions of the model operators will be used in what follows (see Secs. 7–9).

Note also that the first model was used in [81, 83] and the third one (in the case $n_+(A) = n_-(A) < \infty$) in [71, 72], where an isomorphism of the model operators was also established.

**Remark 5.1.** As follows from the proof of Propositions 5.1–5.3, under the assumptions of Theorem 5.1, $A_2$ is an operator if and only if

$$s - \lim_{y\uparrow\infty} y^{-1}M(iy) = 0, \tag{5.32}$$

and $A$ is a densely defined operator if, additionally,

$$\lim_{y\uparrow\infty} y\text{Im } (M(iy)h, h) = +\infty \quad \forall h \in \mathcal{H}. \tag{5.33}$$

Note also that if $M(\lambda)$ is a rational matrix-valued function ($\dim \mathcal{H} < \infty$) of the form

$$M(\lambda) = B\lambda + C + \sum_{j=1}^{k} \frac{A_j}{t_j - \lambda},$$

then $\dim \mathfrak{h}(M) < \infty$, and the following equality holds:

$$\dim \mathfrak{h}(M) = \text{rank } B + \sum_{j=1}^{k} \text{rank } A_j. \tag{5.34}$$

**5.** The next lemma is a generalization of Lemma 1.1 from [40].

**Lemma 5.1.** *Let $Q(x)$ $(a < x < b)$ be a nonnegative operator-valued function with values in $C(\mathcal{H})$ such that:*

*(1) $Q(x)$ is a monotonically increasing function on $(a, b)$, i.e., $\mathfrak{t}_x := \mathfrak{t}_{Q(x)}(\mathfrak{t}_{Q(x)}[f] = \|Q(x)^{1/2}f\|^2)$ is a monotonically increasing [on $(a, b)$] family of forms;*

*(2) $0 \in \rho(Q(x))$ $\quad \forall x \in (a, b)$;*

*(3) $\lim_{x\uparrow b} \mathfrak{t}_x[f] = +\infty$ $\quad \forall f \in \mathcal{H}^1 := \bigcap_{x<b} \mathfrak{D}(\mathfrak{t}_x)$, $f \neq 0$, $\overline{\mathcal{H}}_1^1 = \mathcal{H}$.*

*Then there exists $s - \lim_{x\uparrow b} Q(x)^{-1} = \mathbb{O}$. Conversely, conditions (1), (2), and the last equality imply that* $\lim_{x\uparrow\infty} \mathfrak{t}_x[f] = \infty$.

**Proof.** Let $g \in \mathcal{H}^1 \setminus \{0\}$. According to condition (3) we have

$$\forall M > 0 \quad \exists x_0 \in (a,b) : \|Q(x)^{1/2} g\|^2 \geq M\|g\|^2 \quad \forall x \in [x_0, b).$$

Letting $f := Q(x_0)^{1/2} g$ we obtain $M\|Q(x_0)^{-1/2} f\|^2 \leq \|f\|^2$. Making use of condition (2) and the well-known (see [29]) implication

$$\mathfrak{t}_{Q(x_1)} \geq \mathfrak{t}_{Q(x_2)} > 0 \Longrightarrow Q(x_1)^{-1} \leq Q(x_2)^{-1} \quad (x_1 > x_2),$$

we obtain the inequality $M\|Q(x)^{-1/2} f\|^2 \leq \|f\|^2 \; \forall x \in [x_0, b)$.

This implies that $\lim_{x \uparrow b} \|Q(x)^{-1/2} f\| = 0 \; \forall f \in \mathcal{H}$ and hence $s - \lim_{x \uparrow b} Q(x)^{-1/2} = 0$. Now, by virtue of the uniform boundedness principle, we have $s - \lim_{x \uparrow b} Q(x)^{-1} = 0$. □

**Lemma 5.2.** *Let* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$, $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_1, \tilde{\Gamma}_2\}$ *be BVS's for* $A^*$, *and let* $M(\lambda)$, $\tilde{M}(\lambda)$ *be the corresponding Weyl functions. If* $\ker \Gamma_2 = \ker \tilde{\Gamma}_2$, *then there exist operators* $K$, $K^* \in [\mathcal{H}]$; $X$, $X^{-1} \in [\mathcal{H}]$ *such that*

$$\begin{gathered} \tilde{\Gamma}_2 = X^{-1}\Gamma_2, \qquad \tilde{\Gamma}_1 = X^*(\Gamma_1 + K\Gamma_2), \\ \tilde{M}(\lambda) = X^* M(\lambda) X + X^* K X. \end{gathered} \tag{5.35}$$

*If, additionally,* $\ker \Gamma_1 = \ker \tilde{\Gamma}_1$, *then* $K + 0$ *and* $\tilde{M}(\lambda) = X^* M(\lambda) X$.

One can easily deduce Lemma 5.2 from formulas (1.12), (1.13) and the evident implication $\ker \Gamma_2 = \ker \tilde{\Gamma}_2 \Longrightarrow X_{21} = 0$.

In what follows we repeatedly apply Corollary 4.1. In view of its importance we give a simple proof which does not use the formula of resolvents.

**Proposition 5.5.** *Suppose that* $A \geq 0$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for* $A^*$ *such that* $A_2 \geq 0$, $M(\lambda)$ *is the corresponding Weyl function. Then the equality* $A_2 = A_F$ $(A_2 = A_K)$ *holds if and only if*

$$\lim_{x \downarrow -\infty} (M(x)h, h) = -\infty \qquad (\lim_{x \uparrow \infty} (M(x)h, h) = +\infty) \qquad \forall h \in \mathcal{H} \setminus \{0\}. \tag{5.36}$$

**Proof.** *Necessity.* Let $A_2 = A_F$. Then $\overline{\mathfrak{D}(A_2)} = \mathfrak{h}_0 := \overline{\mathfrak{D}(A)}$, $A_2(0) = \mathfrak{N} := \mathfrak{h} \ominus \mathfrak{h}_0$ (see [78]). By virtue of (1.7) and (1.10) we obtain the equality

$$M(\lambda) = C + (\lambda + 1)\gamma^*(-1)[I + (\lambda + 1)(A_2 - \lambda)^{-1}]P_0\gamma(-1) + \lambda\gamma^*(-1)P_1\gamma(-1), \tag{5.37}$$

where $P_0 := P_{\mathfrak{h}_0}$, $P_1 = I - P_0$, $C = M(-1) + \gamma^*(-1)P_1\gamma(-1)$. It follows from (5.37) that

$$(M(\lambda)h, h) = (Ch, h) + (\lambda + 1)\int_0^\infty \frac{t+1}{t-\lambda} d\|E_t P_0 \gamma(-1)h\|^2 + \lambda\|P_1\gamma(-1)h\|^2, \tag{5.37'}$$

where $E_t$ is the resolution of identity of the operator $A_2' \in \mathcal{C}(\mathfrak{h}_0)$. The operator part $A_2'$ of the linear relation $A_2$ is a Friedrichs extension of the operator $A' = P_0 A \in \mathcal{C}(\mathfrak{h}_0)$. In accordance with the extremal property of the Friedrichs extension [34] we have

$$\int_0^\infty t d(E_t f, f) = +\infty \qquad \forall f \in \mathfrak{N}'_{-1} := \mathfrak{N}_{-1}(A'). \tag{5.38}$$

Now relation (5.36) follows from (5.37') and (5.38).

*Sufficiency.* Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, such that $A_2 \geq 0$ and the corresponding Weyl function $M(\lambda)$ satisfies condition (5.36). Consider side by side with $\Pi$ the other BVS $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_1, \tilde{\Gamma}_2\}$ such that $\ker \tilde{\Gamma}_2 = A_F$. As shown above, its corresponding Weyl function $\tilde{M}(\lambda)$ satisfies condition (5.36) and is connected with $M(\lambda)$ by equality (1.13). Rewriting (1.13) in the form

$$\tilde{M}(\lambda)^{-1}[X_{12}M(\lambda)^{-1} + X_{11}] = X_{22}M(\lambda)^{-1} + X_{21}$$

we pass to the limit as $\lambda \to -\infty$. In accordance with Lemma 5.1 there exists

$$s - \lim_{\lambda \downarrow -\infty} \tilde{M}(\lambda)^{-1} = s - \lim_{\lambda \downarrow -\infty} M(\lambda)^{-1} = 0$$

and hence $X_{21} = 0$. This implies that $\exists X_{22}^{-1} = X_{11}^* \in [\mathcal{H}]$, $\tilde{\Gamma}_1 = X_{21}\Gamma_1 + X_{22}\Gamma_2 = X_{22}\Gamma_2$ and, therefore, $\ker \Gamma_2 = \ker \tilde{\Gamma}_2 = A_F$. □

We shall call (after [39]) the $Q_\mu$- and $Q_M$-functions of a nonnegative operator $A$ the operator-valued functions $X^* Q'_\mu(\lambda) X$ and $X^* Q'_M(\lambda) X$ with $X$, $X^{-1} \in [\mathcal{H}]$ and $Q'_\mu(\lambda)$, $Q'_M(\lambda)$ defined by equalities (4.14), (4.15). In Sec. 4 we have determined BVS's such that their corresponding Weyl functions coincide with $Q'_\mu(\lambda)$ and $Q'_M(\lambda)$. By virtue of (4.14), (4.15), and Proposition 5.5 any $Q_\mu(Q_M)$-function is a Weyl function, corresponding to some BVS, such that $A_2 = A_F$, $A_1 = A_K$ $(A_1 = A_F, A_2 = A_K)$.

**Proposition 5.6.** *Let $Q(\lambda) \in R_{\mathcal{H}}$ be an operator-valued function with values in $[\mathcal{H}]$ satisfying condition* (5.2). *Then for $Q(\lambda)$ to be a $Q_\mu$-function of a nonnegative operator it is necessary and sufficient that the following conditions hold:*

(1) *$Q(\lambda)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$;*

(2) $\lim_{x \downarrow -\infty}(Q(x)h, h) = -\infty \quad \forall h \in \mathcal{H} \setminus \{0\}$;

(3) $s - \lim_{x \uparrow 0} Q(x) = 0$.

**Proof.** Owing to Theorem 5.1 $Q(\lambda)$ is a Weyl function of some Hermitian operator $A$ corresponding to the BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$. By virtue of condition (1) the operator $A$ is nonnegative. It follows from Proposition 5.5 and condition (2) that $A_2 := \ker \Gamma_2 = A_F$. Analogously, making use of Lemma 5.1 and condition (3) we have $A_1 := \ker \Gamma_1 = A_K$. Now we obtain by Lemma 5.2 the equality $Q(\lambda) = X^* M_F(\lambda) X$ $(X \in [\mathcal{H}, \mathfrak{N}_{-i}], X^{-1} \in [\mathfrak{N}_{-i}, \mathcal{H}])$. □

**Corollary 5.2.** (1) *A $Q_\mu$-function belongs to the class $S_{\mathcal{H}}^-$ (that is, $Q(x) < 0$ $\forall x \in \mathbb{R}_-$);*

(2) *the following equivalence holds:*

$$\overline{\mathfrak{D}(A)} = \mathfrak{h} \iff s - \lim_{y \uparrow \infty} y^{-1} Q(iy) = 0. \tag{5.39}$$

**Proof.** (1) The inequality $Q(x) < 0$ $(x \in \mathbb{R}_-)$ follows from the monotonicity of $Q(x)$ on $\mathbb{R}_-$ and the condition $Q(0) = s - \lim_{x \uparrow 0} Q(x) = 0$. (2) The equivalence (5.39) is a consequence of Remark 5.1 and the relation $A_F(0) = \mathfrak{N} = \mathfrak{h}_0^\perp$.

Analogously one can prove

**Proposition 5.6′.** *Let $Q(\lambda)$ be an $R_{\mathcal{H}}$-function with values in $[\mathcal{H}]$ satisfying condition* (5.2). *Then $Q(\lambda)$ is a $Q_M$-function of a nonnegative operator if and only if the following conditions hold:*

(1) *$Q(\lambda)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$;*

(2) $\lim_{x \uparrow 0}(Q(x)h, h) = +\infty \ \forall h \in \mathcal{H} \setminus \{0\}$;

(3) $s - \lim_{x \downarrow -\infty} Q(x) = 0$.

**Corollary 5.2′.** *Let $Q(\lambda)$ be a $Q_M$-function of a nonnegative operator. Then:*

(1) *$Q(\lambda) \in S_{\mathcal{H}}^+ := S_{\mathcal{H}}(\mathbb{R}_+, 0)$ (that is, $Q(x) \geq 0$ $\forall x \in \mathbb{R}_-$);*

(2) *the following equivalence holds:*

$$\overline{\mathfrak{D}(A)} = \mathfrak{h} \iff \lim_{x \downarrow -\infty} x(Q(x)h, h) = -\infty \qquad \forall h \in \mathcal{H} \setminus \{0\}.$$

**Remark 5.2.** In the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, Propositions 5.6 and 5.6′ were presented in [39].

**6.** In Sec. 3 BVS's $\Pi^\mu = \{\mathcal{H}, \Gamma_1^\mu, \Gamma_2^\mu\}$ and $\Pi^M = \{\mathcal{H}, \Gamma_1^M, \Gamma_2^M\}$ were constructed, such that the corresponding Weyl functions take the form in (3.73), (3.74) and coincide with $Q_\mu$- and $Q_M$-functions of a Hermitian contraction from [40]. The full inner characterization of $Q_\mu$- and $Q_M$-functions of a Hermitian contraction was obtained in [40]. We now give another proof of this fact, based on Propositions 5.1 and 5.5.

**Proposition 5.7** ([40]). *Let $M(\lambda)$ be an $R_{\mathcal{H}}$-function with values in $[\mathcal{H}]$ satisfying condition* (5.2). *Then $M(\lambda)$ is a $Q_\mu$-function of a simple Hermitian contraction if and only if:*

(1) *$M(\lambda)$ is holomorphic on* $\mathbb{C}\setminus[-1,1]$;
(2) $\lim_{\lambda\uparrow -1}(M(\lambda)h,h)=\infty\ \forall h\in\mathcal{H}\setminus\{0\}$;
(3) $s-\lim\limits_{\lambda\downarrow 1}M(\lambda)=0$;
(4) $s-\lim\limits_{\lambda\uparrow\infty}M(\lambda)=I$.

**Proof.** *Necessity.* Suppose that $A$ is a Hermitian contraction, $\Pi^{\mu}=\{\mathcal{H},\Gamma_1^{\mu},\Gamma_2^{\mu}\}$ is a BVS for $A^*$ of the form in (3.72), and $M(\lambda)=Q_{\mu}(\lambda)$. The operator $\tilde{A}_{-1}$ defined [cf. (4.2)] by the equality $\tilde{A}_{-1}:=s-R-\lim\limits_{x\uparrow -1}\tilde{A}_x(\tilde{A}_x:=A\dotplus\tilde{\mathfrak{N}}_x)$ is the least one in the class Ex $_A(-\infty,-1)$ of all extensions $\tilde{A}=\tilde{A}^*$ such that $\tilde{A}\geq -I$. In particular, $\tilde{A}_{-1}\in C_A(0)$ is a minimal element in the class of *sc*-extensions of the operator $A$. Therefore, $\tilde{A}_{-1}=A_{\mu}=\ker\Gamma_2$. Owing to Proposition 5.5, condition (2) is fulfilled. Analogously, an extension $A_{+1}=s-R-\lim\limits_{x\downarrow 1}\tilde{A}_x$ is a maximal element in the classes Ex $_A(1,+\infty)$ and $C_A(0)(\subset$ Ex $_A(1,+\infty))$. This implies that $A_{+1}=A_M=\ker\Gamma_1$ and according to Proposition 5.5 we have $\lim\limits_{x\downarrow 1}(M^{-1}(x)h,h)=+\infty$. Condition (3) now follows from the monotonicity of $M(x)$ on $(1,\infty)$ and Lemma 5.1. Clearly, condition (4) for $M(\lambda)$ of the form in (3.73) is fulfilled.

*Sufficiency.* It follows from Theorem 5.1 that the function $M(\lambda)$ is a Weyl function of some simple Hermitian operator $A'$ corresponding to some BVS $\Pi'=\{\mathfrak{N},\Gamma_1',\Gamma_2'\}$ (without loss of generality we assume that $\mathcal{H}=\mathfrak{N}=\mathfrak{h}_0^{\perp}$). By virtue of condition (1) the extension

$$A_2'=\ker\Gamma_2'=\begin{pmatrix}A_{00} & A_{10}^*\\ A_{10} & A_{11}\end{pmatrix}\qquad (A_{ij}\in[\mathfrak{h}_j,\mathfrak{h}_i]\qquad (i,j=1,2))$$

is a contraction. It follows from conditions (2), (3), Proposition 5.5, and Lemma 5.1 that $A_2'=A_{\mu}$, $A_1'=A_M$. Thus we may apply Lemma 5.2 to conclude that the Weyl functions $M(\lambda)$ and $Q_{\mu}(\lambda)$ corresponding to BVS's $\Pi'=\{\mathfrak{N},\Gamma_1',\Gamma_2'\}$ and $\Pi^{\mu}=\{\mathfrak{N},\Gamma_1^{\mu},\Gamma_2^{\mu}\}$ of the form in (3.72) are connected with the equality

$$M(\lambda)=X^*Q_{\mu}(\lambda)X=X^*(I+C^{1/2}(A_{\mu}-\lambda)^{-1}C^{1/2})X, \tag{5.40}$$

where $X$, $X^{-1}\in[\mathfrak{N}]$, $C=(A_M-A_{\mu})\restriction_{\mathfrak{N}}$. It follows from (5.40) and condition (4) that $X^*X=I$, i.e., $X$ is a unitary operator in $\mathfrak{N}$.

We consider an isometric operator $U=I_{\mathfrak{h}_0}\oplus X\in[\mathfrak{h}]$ and put $\Gamma_j''=\Gamma_j'\hat{U}\ \ (j=1,2)$, $\hat{U}=U\oplus U$. We show that the Weyl function of the operator $A''=UA'$ corresponding to the BVS $\Pi''=\{\mathfrak{N},\Gamma_1'',\Gamma_2''\}$ for $(A'')^*=\hat{U}^*A^*\hat{U}$ is a $Q_{\mu}$-function of the Hermitian contraction $A''$. Indeed, $A_{\mu}''$ and $C''$ take the forms

$$A_{\mu}''=U^*A_{\mu}U=\begin{pmatrix}A_{00} & A_{10}^*X\\ X^*A_{10} & X^*A_{11}X\end{pmatrix},\qquad C''=(A_M''-A_{\mu}'')\restriction_{\mathfrak{N}}=X^*CX.$$

It follows now from relations (5.40), (3.73) that $M(\lambda)$ coincide with the Weyl function $M''(\lambda)$ of $A''$ corresponding to the BVS $\Pi''$:

$$M''(\lambda)=I+X^*C^{1/2}X(U^*(A_{\mu}-\lambda)^{-1}U)X^*C^{1/2}X=I+(C'')^{1/2}(A_{\mu}''-\lambda)^{-1}(C'')^{1/2}=M(\lambda).$$

7. Let $A=\begin{pmatrix}A_{00}\\ A_{10}\end{pmatrix}$ be a bounded Hermitian operator in $\mathfrak{h}$ with a nondense domain $\mathfrak{h}_0$ $(A_{i0}\in[\mathfrak{h}_0,\mathfrak{h}_i]$ $(i=0,1))$, $\mathfrak{h}_1=\mathfrak{h}_0^{\perp}$. In Proposition 3.5 a BVS for $A^*$ was constructed, such that the corresponding Weyl function coincides with the spectral complement of the operator $A$ [69]

$$M(\lambda)=\lambda I_{\mathfrak{N}}+A_{10}(A_{00}-\lambda)^{-1}A_{01}. \tag{5.41}$$

The next proposition contains a full inner description of the operator-functions of the form in (5.41) (i.e., spectral complements).

**Proposition 5.8.** *Let $M(\lambda)$ be an $R_{\mathcal{H}}$-function satisfying condition (5.2). Then for $M(\lambda)$ to be a spectral complement of some bounded Hermitian operator it is necessary and sufficient that the following conditions hold:*

(1) *$M(\lambda)$ is holomorphic exterior to some segment $[a,b]$;*

(2) $s-\lim_{\lambda\to\infty}\frac{M(\lambda)}{\lambda}=I_{\mathfrak{N}}$;

(3) $s-\lim_{\lambda\to\infty}(M(\lambda)-\lambda I_{\mathfrak{N}})=0$.

**Proof.** According to Theorem 5.1 there exist a Hermitian operator $A'(\overline{\mathfrak{D}(A')}=\mathfrak{h}_0)$ and a BVS $\Pi'=\{\mathfrak{N},\Gamma_1',\Gamma_2'\}$ for $(A')^*$ such that the corresponding Weyl function coincides with $M(\lambda)$. Making use of Theorem 1.1 and condition (2), we conclude that $A_2'=\ker\Gamma_1'$ is a linear relation and $A_2'(0)=\mathfrak{N}=\mathfrak{h}_0^{\perp}$. It follows from conditions (1), (2), and Proposition 1.6 that $A_1'=\ker\Gamma_1'$ is a bounded operator. In particular, the operator $A'=\begin{pmatrix}A_{00}\\A_{10}\end{pmatrix}$ $(A_{00}\in[\mathfrak{h}_0],A_{10}\in[\mathfrak{h}_0,\mathfrak{N}])$ is a bounded operator too.

Let $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ be a BVS of the form in (3.64). Since $\ker\Gamma_2'=\ker\Gamma_2=A\dot{+}\hat{\mathfrak{N}}$ we obtain from Lemma 5.2 the equality

$$M(\lambda)=X^*[\lambda I_{\mathfrak{N}}+A_{10}(A_{00}-\lambda)^{-1}A_{01}]X+Y,$$

where $Y=Y^*$, $X$, $X^{-1}\in[\mathfrak{N}]$. It now follows from conditions (2), (3) that $X^*X=I$ and $Y=0$ and, therefore,

$$M(\lambda)=\lambda I_{\mathfrak{N}}+X^*A_{10}(A_{00}-\lambda)^{-1}A_{01}X.$$

Consider the operator $A=\begin{pmatrix}A_{00}\\X^*A_{10}\end{pmatrix}$ and the BVS of the form in (3.64) for the linear relation $A^*$. Then by virtue of Proposition 3.5 we have that the corresponding Weyl function coincides with $M(\lambda)$. □

**8.** We give one simple application of Theorem 5.1.

Let $A$ be a Hermitian matrix in $\mathfrak{h}_0=\mathbb{C}^n$. Consider a bordered matrix $A_1=A_1^*=\begin{pmatrix}A&g\\g^*&a\end{pmatrix}$ $(a=\bar{a},g\in\mathbb{C}^n)$ as an extension of the operator $A_0=\begin{pmatrix}A\\g^*\end{pmatrix}\in[\mathbb{C}^n,\mathbb{C}^{n+1}]$ and a BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ of the form (3.64) for the linear relation $A_0^*=\{\{f,\tilde{A}_1f+c\}:f\in\mathbb{C}^{n+1},c\in\mathbb{C}\}$. It follows from (5.41) that the Weyl function $M(\lambda)$ corresponding to the BVS $\Pi$ takes the form

$$M(\lambda)=\lambda-a+g^*(A-\lambda)^{-1}g=-\frac{\det(A_1-\lambda)}{\det(A-\lambda)}\in R. \tag{5.42}$$

This implies the alternation of the eigenvalues $\{\lambda_k\}_1^n$ and $\{\lambda_k\}_1^{n+1}$ of matrices $A$ and $A_1$, which is usually derived from the Courant–Fisher principle [47].

Making use of Proposition 5.8, one can easily deduce the inverse assertion, which is also well known (see, for example, [47]).

**Proposition 5.9.** *Suppose that $\{\lambda_k\}_1^n$ and $\{\lambda_k^1\}_1^{n+1}$ are two alternating collections of numbers*

$$\lambda_1^1\le\lambda_1\le\lambda_2^1\le\lambda_2\le\ldots\le\lambda_n^1\le\lambda_n\le\lambda_{n+1}^1, \tag{5.43}$$

$A=\operatorname{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ *is a diagonal matrix. Then there exists a bordering $A_1=\begin{pmatrix}A&g\\g^*&a\end{pmatrix}$ of the matrix $A$ such that $\sigma(A_1)=\{\lambda_k^1\}_1^{n+1}$.*

**Proof.** Suppose that we have strict inequalities in (5.43) (one can easily deduce the general case to this one) and define a function $M(\lambda)$ and a number $a=\bar{a}$ by formulas

$$M(\lambda)=\prod_{j=1}^{n+1}(\lambda-\lambda_j^1)/\prod_{j+1}^{n}(\lambda-\lambda_j),\qquad a=\sum_{j=1}^{n+1}\lambda_j^1-\sum_{j+1}^{n}\lambda_j. \tag{5.44}$$

Clearly, $M(\lambda)+a$ satisfies the conditions of Proposition 5.8 and, therefore, $M(\lambda)$ is the Weyl function corresponding to some BVS of the form in (3.64) for $A^*$ and taking the form in (5.42). It follows from (5.34), (5.44) that $\dim \mathfrak{h}_0 = n$, $\dim \mathfrak{h} = n+1$, and $A_0 = \begin{pmatrix} A & \cdot \\ g^* & \cdot \end{pmatrix} \in [\mathbb{C}^n, \mathbb{C}^{n+1}]$, $A \in [\mathbb{C}^n]$. Note finally that the zeros of $M(\lambda)$ coincide with the eigenvalues of the matrix $A_1 = \begin{pmatrix} A & g \\ g^* & a \end{pmatrix}$ and the poles of $M(\lambda)$ coincide with the eigenvalues of the linear relation $A_2 := \ker \Gamma_2$ and, therefore, with the eigenvalues of the matrix $A$. □

## 6. GENERALIZED SPACES OF BOUNDARY VALUES

**1.** In this section we generalize the notion of a BVS to nonclosed linear relations, which enables us to realize an arbitrary $R_{\mathcal{H}}$-function [without condition (5.2)] as a Weyl function. Let $A$ be a closed operator in $\mathfrak{h}$, $A^*$ be an adjoint linear relation, and $A_*(\subset A)$ be a linear relation dense in $A^*$.

**Definition 6.1.** *A triple $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$, in which $\mathcal{H}$ is a Hilbert space and $\Gamma_j$ $(j=1,2)$ are closable mappings from $A_*$ to $\mathcal{H}$, will be called a generalized BVS for a linear relation $A_*$ if:*

(1) $\Gamma_2$ *is a surjective mapping;*

(2) $A_2 := \ker \Gamma_2$ *is a self-adjoint relation;*

(3) *for all $\hat{f} = \{f, f'\}$, $\hat{g} = \{g, g'\} \in A_*$ the Green formula holds:*

$$(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_2 \hat{g})_{\mathcal{H}} - (\Gamma_2 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{H}}. \tag{6.1}$$

One can easily deduce

**Proposition 6.1.** *Let $A_1$, $A_2$ be disjoint self-adjoint extensions of an operator $A$. Then there exists a generalized BVS $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ for the linear relation $A_* := A_1 + A_2$ such that $A_j = \ker \Gamma_j$ $(j=1,2)$.*

**Lemma 6.1.** *Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a generalized resolvent of a nonclosed linear relation $A_*$. Then the following assertions hold:*

(1) $A_* = A_2 \dot{+} \hat{\mathfrak{N}}^*$ $(\lambda \in \rho(A_2))$, *where $\mathfrak{N}_\lambda^* = A_* \cap \mathfrak{N}_\lambda$ is dense in $\mathfrak{N}_\lambda$,*

(2) $\overline{\Gamma_1 A_2} = \mathcal{H}$;

(3) $\ker \Gamma = A$, $\overline{\mathfrak{R}(\Gamma)} = \mathcal{H} \oplus H$ $(\Gamma = \{\Gamma_2, \Gamma_1\})$.

**Proof.** (1) Since $A_2 = A_2^*$, the following decomposition holds:

$$A^* = A_2 + \hat{\mathfrak{N}}_\lambda \qquad (\lambda \in \rho(A_2)). \tag{6.2}$$

It follows from the inclusion $A_* \supset A_2$ that $A_* = A_2 \dot{+} \hat{\mathfrak{N}}_\lambda^*$, where $\mathfrak{N}_\lambda^* = \mathfrak{N}_\lambda \cap A_*$. Since the angle between $A_2$ and $\hat{\mathfrak{N}}_\lambda$ is acute, the equivalence $\overline{A_*} = A^* \Longleftrightarrow \overline{\mathfrak{N}_\lambda^*} = \mathfrak{N}_\lambda$ holds.

(2) Assume that there exists $h \in \mathcal{H}$ such that $h \perp \Gamma_1 A_2$. It follows from condition (1) of Definition 6.1 that for some $\hat{f} = \{f, f'\} \in A_*$ we have $\Gamma_2 \hat{f} = h$. Making use of the Green formula (6.1) we obtain for all $\hat{g} = \{g, g'\} \in A_2$ the equality

$$(f', g) - (f, g') = -(\Gamma_2 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{H}} = -(h, \Gamma_1 \hat{g})_{\mathcal{H}} = 0.$$

The condition $A_2 = A_2^*$ implies that $\hat{f} = \{f, f'\} \in A_2$ and $h_2 = \Gamma_2 \hat{f} = 0$.

(3) Let $\hat{f} = \{f, f'\} \in \ker \Gamma$. Then it follows from formula (6.1) that $(f', g) = (f, g')$ for all $\hat{g} = \{g, g'\} \in A_*$ and hence $\{f, f'\} \in (A_*)^* = A$. Therefore, $\ker \Gamma \subset A$. Conversely, if $\hat{f} \in A \subset A_2 = \ker \Gamma_2$, then $(\Gamma_1 \hat{f}, \Gamma_2 \hat{g})_{\mathcal{H}} = 0$ for all $g \in A_*$. With regard to condition (1) this implies that $\hat{f} \in \ker \Gamma_1$.

Finally, we show that $\overline{\mathfrak{R}(\Gamma)} = \mathcal{H} \oplus \mathcal{H}$. For some $\{h_2, h_1\} \in \mathcal{H} \oplus \mathcal{H}$ we choose $\hat{f}_\lambda \in \hat{\mathfrak{N}}_\lambda^*$ such that $\Gamma_2 \hat{f}_\lambda = h_2$. In accordance with assertion (2) there exists a sequence $f_2^{(n)} \in A_2$ such that $\lim_{n\to\infty} \Gamma_1 f_2^{(n)} = h_1 - \Gamma_1 \hat{f}_\lambda$. Then it follows from the equalities

$$\Gamma_2(\hat{f}_\lambda + f_2^{(n)}) = h_2, \qquad \lim_{n\to\infty} \Gamma_1(\hat{f}_\lambda + f_2^{(n)}) = h_1$$

that $\{h_1, h_2\} \in \overline{\mathfrak{R}(\Gamma)}$. □

Let $\pi_1$ be an orthogonal projection onto the first component in $\hat{\mathfrak{N}}_\lambda = \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda\}$.

**Lemma 6.2.** *The equalities $\hat{\gamma}(\lambda) = (\Gamma_2 \restriction \hat{\mathfrak{N}}^*_\lambda)^{-1}$, $\gamma(\lambda) = \pi_1\hat{\gamma}(\lambda)$ define holomorphic on $\rho(A_2)$ operator-valued functions with values in $[\mathcal{H}, \hat{\mathfrak{N}}_\lambda]$ and $[\mathcal{H}, \mathfrak{N}_\lambda]$, respectively. The function $\gamma(\lambda)$ is a $\gamma$-field of extension $A_2$, i.e., the following relation holds*:

$$\gamma(\lambda) = \gamma(\zeta) + (\lambda - \zeta)(A_2 - \lambda)^{-1}\gamma(\zeta) \qquad \forall \lambda, \zeta \in \rho(A_2). \tag{6.3}$$

The proof follows immediately from (6.2) and Definition 6.1. In the same way as in [79, 53] one can obtain the relation

$$\gamma^*(\bar{\lambda})h = \Gamma_1\{(A_2 - \lambda)^{-1}h, h + \lambda(A_2 - \lambda)^{-1}h\} \qquad (h \in \mathcal{H}, \lambda \in \rho(A_2)). \tag{6.4}$$

**Definition 6.2.** *The operator-valued function $M(\lambda)$, defined for all $\lambda \in \rho(A_2)$ by the equality*

$$M(\lambda)\Gamma_2\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda \qquad \forall \hat{f}_\lambda \in \hat{\mathfrak{N}}^*_\lambda, \tag{6.5}$$

*will be called a Weyl function of the operator $A$ corresponding to the generalized BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ for the linear relation $A_*$.*

It is easy to see that the equality (6.5) is equivalent to the following one:

$$M(\lambda) = \Gamma_1\hat{\gamma}(\lambda). \tag{6.6}$$

Since the operator $\Gamma_1$ is closable, the function $M(\lambda) = \Gamma_1\hat{\gamma}(\lambda)$ is a well-defined, holomorphic on $\rho(A_2)$, operator-valued function with values in $[\mathcal{H}]$. It follows from (6.3), (6.4) that

$$M(\lambda) - M^*(\mu) = (\lambda - \bar{\mu})\gamma^*(\mu)\gamma(\lambda) \qquad (\forall \lambda, \mu \in \rho(A_2)). \tag{6.7}$$

**Proposition 6.2.** *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a generalized BVS for the linear relation $A_*$, $M(\lambda)$ is the corresponding Weyl function. Then*

(1) $0 \in \rho(\operatorname{Im} M(i)) \iff \mathfrak{R}(\Gamma) = \mathcal{H} \oplus \mathcal{H} \qquad (\iff A_* = A^*)$;

(2) $0 \in \rho(M(i)) \iff \mathfrak{R}(\Gamma_1) = \mathcal{H}$. *If, additionally, $A_1 = A_1^*$, then the operator-valued function $-M(\lambda)^{-1}$ is the Weyl function corresponding to the generalized BVS $\Pi_1 = \{\mathcal{H}, -\Gamma_2, \Gamma_1\}$.*

**Proof.** (1) Let $0 \in \rho(\operatorname{Im} M(i))$. Then it follows from (6.7) that the mapping $\gamma(\lambda) \in [\mathcal{H}, \mathfrak{N}_\lambda]$ is surjective, and hence $\mathfrak{N}^*_\lambda = \mathfrak{N}_\lambda$, $\mathfrak{R}(\gamma^*(\bar{\lambda})) = \mathcal{H}$. By virtue of (6.4) we have the equality $\Gamma_1 A_2 = \mathcal{H}$, which, with regard to the relation $\Gamma_2\hat{\mathfrak{N}}_\lambda = \mathcal{H}$, leads to the equality $\mathfrak{R}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$.

Conversely, if $\mathfrak{R}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$, then for all $h \in \mathcal{H}$ there exists $\hat{f} \in A_*$ such that $\Gamma_2\hat{f} = 0$, $\Gamma_1\hat{f} = h$. Thus $\hat{f} \in A_2$ and $\Gamma_1 A_2 = \mathcal{H}$. It follows from (6.4) that the mapping $\gamma^*(\lambda)$ as well as $\gamma(\lambda)$ is surjective and, owing to (6.7), we have $0 \in \rho(\operatorname{Im} M(\lambda))$ for all $\lambda \in \mathbb{C}_+$.

(2) The implication $0 \in \rho(M(i)) \Longrightarrow \mathfrak{R}(\Gamma_1) = \mathcal{H}$ is evident. Assume that $\mathfrak{R}(\Gamma_1) = \mathcal{H}$. Then it follows from the decomposition $A_* = A_j \dotplus \hat{\mathfrak{N}}^*_\lambda$ $(j = 1, 2)$ that the mappings $\Gamma_j : \hat{\mathfrak{N}}^*_\lambda \to \mathcal{H}$ are isomorphic and, therefore, $0 \in \rho(M(\lambda))$ $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$. □

**Corollary 6.1.** *If extensions $A_1$ and $A_2$ are transversal, then the BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is an ordinary one ($\iff \mathfrak{R}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$).*

Indeed, if $A_* = A^*$, then $\mathfrak{N}^*_\lambda = \mathfrak{N}_\lambda$ and the mapping $\gamma(\lambda) \in [\mathcal{H}, \mathfrak{N}_\lambda]$ is an isomorphism. It follows from (6.7) that $0 \in \rho(\operatorname{Im} M(i))$ and by virtue of Proposition 6.2 $\mathfrak{R}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$.

In the case $0 \in \rho(M(i))$ the following proposition holds.

**Proposition 6.3.** *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a generalized BVS and $M(\lambda)$ is the corresponding Weyl function. Then there exists a rigging $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ of the Hilbert space $\mathcal{H}$ ([6]) such that*

(1) *$\Gamma_2$ may be extended by continuity to the continuous mapping $\Gamma_2 \in [A^*, \mathcal{H}_-]$;*

(2) *$\gamma(\lambda)$ admits a continuation to $\gamma(\lambda) \in [\mathcal{H}_-, \mathfrak{N}_\lambda]$ such that $\gamma^*(\lambda) \in [\mathfrak{N}_\lambda, \mathcal{H}_+]$ $(\lambda \in \mathbb{C} \setminus \mathbb{R})$;*

(3) $\operatorname{Im} M(\lambda) \in [\mathcal{H}_-, \mathcal{H}_+]$, $(\operatorname{Im} M(\lambda))^{-1} \in [\mathcal{H}_+, \mathcal{H}_-]$ *for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

**Proof.** Let $\mathcal{H}_-$ be a completion of the space $\mathcal{H}$, endowed with the metric

$$\|h\|_- = (\operatorname{Im} M(i)h, h)^{1/2} \qquad (h \in \mathcal{H}), \tag{6.8}$$

and $\mathcal{H}_+ = \mathfrak{R}\left((\mathrm{Im}\, M(i))^{1/2}\right)$ be a Hilbert space with the norm

$$\|f\|_+^2 = \|(\mathrm{Im}\, M(i))^{-1/2} f\|^2 (= \|(\mathrm{Im}\, M(i))^{-1} f\|_-^2 \quad \forall f \in \mathfrak{R}\,(\mathrm{Im}\,(M(i))). \tag{6.9}$$

Then the triple of Hilbert spaces $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigging of $\mathcal{H}$ (see [6]), and the operator $\mathrm{Im}\, M(i)$ is an isometry from $\mathcal{H}_-$ onto $\mathcal{H}_+$ by virtue of (6.9).

It follows from (6.7), (6.8) that $\|\gamma(i)h\|^2 = \|h\|_-^2$ for all $h \in \mathcal{H}$. This implies that the operator $\gamma(i)$ may be continued to the isometrical operator $\gamma(i) \in [\mathcal{H}_-, \mathfrak{N}_i]$. By virtue of equality (6.3) we have $\gamma(\lambda) \in [\mathcal{H}_-, \mathfrak{N}_\lambda]$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Owing to the fact that the angle between $A_2$ and $\hat{\mathfrak{N}}_\lambda$ is acute, we obtain $\Gamma_2 \restriction \mathfrak{N}_\lambda = \hat{\gamma}(\lambda)^{-1} \in [\hat{\mathfrak{N}}_\lambda, \mathcal{H}_-]$ and, therefore, the operator $\Gamma_2$ may be extended to the continuous mapping $\Gamma_2 \in [A^*, \mathcal{H}_-]$. □

**Remark 6.1.** $\Gamma_1$ is a continuous mapping from $A^*$ into $\mathcal{H}_+$ if $M(i) \in [\mathcal{H}_-, \mathcal{H}_+]$. Indeed, making use of the decomposition $A_* = A_2 \dot{+} \hat{\mathfrak{N}}_{\bar\lambda}^*$ and the equalities $\Gamma_1 \hat{f} = \gamma^*(\bar\lambda)(f' - \lambda f)$, $\Gamma_1 \hat{f}_\lambda = M(\lambda)\gamma^{-1}(\lambda) f_\lambda$ ($\hat{f} = \{f, f'\} \in A_2$, $\hat{f}_\lambda \in \hat{\mathfrak{N}}_{\bar\lambda}^*$), we obtain that the restrictions of $\Gamma_1$ onto $A_2$ and $\hat{\mathfrak{N}}_{\bar\lambda}^*$ are continuous mappings from $A^*$ into $\mathcal{H}_+$. The desired statement now follows from the fact that the angle between $A_2$ and $\hat{\mathfrak{N}}_\lambda$ is acute.

**Theorem 6.1.** *For a holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ operator-valued function $Q(\lambda)$ with values in $[\mathcal{H}]$ to be a Weyl function of a simple Hermitian operator corresponding to a generalized BVS it is necessary and sufficient that $Q(\lambda) \in R_{\mathcal{H}}$.*

**Proof.** Consider a space $\mathfrak{B}(Q)$ and a Hermitian operator $A$ of the form (5.24). Let $A_*$ be a linear relation defined by equalities (5.26) and $\Gamma_1$, $\Gamma_2$ be the mappings from $A_*$ to $\mathcal{H}$ defined by the formulas

$$\Gamma_j \hat{F} = f_j \quad (j = 1, 2); \qquad \hat{F} = \{F, \tilde{F}\} \in A_*, \quad \tilde{F}(\mu) - \mu F(\mu) = f_1 - Q(\mu) f_2.$$

The mapping $\Gamma$ is closed. Indeed, assume that $\hat{F}_n = \{F_n(\mu), \tilde{F}_n(\mu)\}$ converges to $\hat{F} = \{F(\mu), \tilde{F}(\mu)\}$ in the space $\mathfrak{B}(Q)^2$ as $n \to \infty$ and $\Gamma \hat{F}_n = \{f_2^{(n)}, f_1^{(n)}\}$ converges to $\{f_2, f_1\}$ in $\mathcal{H} \oplus \mathcal{H}$. Then the pointwise convergence also takes place, and it follows from the equalities $\tilde{F}_n(\mu) - \mu F_n(\mu) = f_1^{(n)} - Q(\mu) f_2^{(n)}$ that $\tilde{F}(\mu) - \mu F(\mu) = f_1 - Q(\mu) f_2$. Therefore $\hat{F} = \{F, \tilde{F}\} \in A^*$ and $\Gamma \hat{F} = \{f_2, f_1\}$.

The surjectivity of the mapping $\Gamma$ follows from the relation

$$\Gamma_2\{h_\lambda(\mu), \lambda h_\lambda(\mu)\} = h, \qquad h_\lambda(\mu) = \frac{Q(\lambda) - Q(\mu)}{\lambda - \mu} h, \qquad h \in \mathcal{H}. \tag{6.10}$$

In order to prove the Green formula we put for $\hat{F} = \{F, \tilde{F}\} \in A_*$, $\hat{G} = \{G, \tilde{G}\} \in A_*$

$$\Phi(\mu) = \tilde{F}(\mu) - \lambda F(\mu) \in \mathfrak{B}(Q), \qquad \Psi(\mu) = \tilde{G}(\mu) - \bar{\lambda} G(\mu) \in \mathfrak{B}(Q), \tag{6.11}$$

With regard to (6.11) we obtain the evident equality

$$(\tilde{F}, G)_{\mathfrak{B}(Q)} - (F, \tilde{G})_{\mathfrak{B}(Q)} = (\Phi, G)_{\mathfrak{B}(Q)} - (F, \Psi)_{\mathfrak{B}(Q)}. \tag{6.12}$$

Let $f_j = \Gamma_j \hat{F}$, $g_j = \Gamma_j \hat{G}$ $(j = 1, 2)$. Then it follows from (6.11) and the relations

$$\tilde{F}(\mu) - \mu F(\mu) = f_1 - Q(\mu) f_2, \qquad \tilde{G}(\mu) - \mu G(\mu) = g_1 - Q(\mu) g_2 \tag{6.13}$$

that the vector-functions $F(\mu)$ and $G(\mu)$ can be represented in the form

$$F(\mu) = \frac{\Phi(\mu) - \Phi(\lambda)}{\mu - \lambda} + \frac{Q(\mu) - Q(\lambda)}{\mu - \lambda} f_2, \qquad G(\mu) = \frac{\Psi(\mu) - \Psi(\lambda)}{\mu - \lambda} + \frac{Q(\mu) - Q(\lambda)}{\mu - \lambda} g_2.$$

Taking into account the equalities

$$\left(\Phi(\mu), \frac{\Psi(\mu) - \Psi(\bar{\lambda})}{\mu - \bar{\lambda}}\right)_{\mathfrak{B}(Q)} = \left(\frac{\Phi(\mu) - \Phi(\lambda)}{\mu - \lambda}, \Psi(\mu)\right)_{\mathfrak{B}(Q)};$$

$$(\Phi(\mu),(\mu-\bar{\lambda})^{-1}(Q(\mu)-Q(\bar{\lambda}))g_2)_{\mathfrak{B}(Q)}=(\Phi(\lambda),g_2)_{\mathcal{H}};$$
$$((\mu-\lambda)^{-1}(Q(\mu)-Q(\lambda))f_2,\Psi(\mu))_{\mathfrak{B}(Q)}=(f_2,\Psi(\bar{\lambda}))_{\mathcal{H}},$$

we obtain after the substitution $F(\mu)$ and $G(\mu)$ in (6.12):

$$(\tilde{F},G)_{\mathfrak{B}(Q)}-(F,\tilde{G})_{\mathfrak{B}(Q)}=(\Phi(\lambda),g_2)_{\mathcal{H}}-(f_2,\Psi(\bar{\lambda}))_{\mathcal{H}}. \tag{6.14}$$

Equality (6.1) is a consequence of (6.14), relations

$$\Gamma_1\hat{F}=\Phi(\lambda)+Q(\lambda)f_2, \qquad \Gamma_1\hat{G}=\Psi(\bar{\lambda})+Q(\bar{\lambda})g_2, \qquad \Gamma_2\hat{F}=f_2, \qquad \Gamma_2\hat{G}=g_2,$$

and equality $Q(\bar{\lambda})=Q^*(\lambda)$.

Since $\mathfrak{N}_\lambda^*=\left\{\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h, h\in\mathcal{H}\right\}$, we have

$$\Gamma_1\left\{\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h,\lambda\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu}h\right\}=Q(\lambda)h, \qquad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{6.15}$$

This implies that the Weyl function $M(\lambda)$ corresponding to the generalized BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ coincides with $Q(\lambda)$. □

**Remark 6.2.** As in 5.2, a function $M(\lambda)\in R_{\mathcal{H}}$ can be realized as a Weyl function of an operator $A$ in the space $\mathfrak{h}(M)=\mathcal{H}_B\oplus L_2(d\Sigma,\mathcal{H})$ with $B$ and $d\Sigma$ defined by (5.1). In this case the operator $A$ and the linear relation $A_*$ can be defined by equalities (5.14), (5.16), but unlike in Proposition 5.2 $A_*\neq A^*$. The generalized BVS for $A_*$ should be defined by (5.17).

## 7. CHARACTERISTIC FUNCTIONS OF LINEAR RELATIONS

**1. Definition 7.1.** *A closed linear relation $T\in\widetilde{\mathcal{C}}(\mathfrak{h})$ is said to be almost solvable if there exists a Hermitian relation $A$ and its self-adjoint extension $\tilde{A}$ such that $T\in\mathrm{Ex}_A$ and $\tilde{A}\dot{+}T=A^*$. We denote by $\mathcal{A}s$ the set of almost solvable linear relations, and write $\mathcal{A}s(A):=\mathcal{A}s\cap\mathrm{Ex}_A$ for the set of almost solvable extensions of $A$.*

**Proposition 7.1.** *A proper extension $T$ of $A$ belongs to the class $\mathcal{A}s(A)$ if and only if there exist a BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ and an operator $B\in[\mathcal{H}]$ such that $T=\tilde{A}_B=\ker(\Gamma_1-B\Gamma_2)$.*

**Proof.** Assume that $T\in\mathcal{A}s(A)$, $\tilde{A}=\tilde{A}^*\in\mathrm{Ex}_A$, and $\tilde{A}\dot{+}T=A^*$. We choose a BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ such that $\ker\Gamma_2=\tilde{A}$. By virtue of Proposition 1.4 the transversality of $\tilde{A}$ and $T$ is equivalent to the condition $B:=\Gamma T\in[\mathcal{H}]$. □

To prove the next proposition we need

**Lemma 7.1** (cf. [26]). *Suppose that $\tilde{\mathcal{H}}$ is a Krein space and $\mathfrak{M}_\pm$ are maximal uniformly positive and uniformly negative subspaces in $\tilde{\mathcal{H}}$, $\theta$ is a subspace in $\tilde{\mathcal{H}}$ such that*

$$\overline{\theta\dot{+}\mathfrak{M}_+}=\overline{\theta\dot{+}\mathfrak{M}_-}=\tilde{\mathcal{H}}. \tag{7.1}$$

*Then there exists a hypermaximal neutral subspace $\mathfrak{M}_0$ transversal to $\theta$.*

**Proof.** Let $\tilde{\mathcal{H}}=\mathfrak{M}_+[+]\mathfrak{M}_+^{[\perp]}$ be a canonical decomposition of the Krein space $\tilde{\mathcal{H}}$, and let $\|\cdot\|_1$ be the corresponding Hilbert norm in $\tilde{\mathcal{H}}$. It follows from Proposition 1.4 that the angular operators $B$ and $K$ of the subspaces $\theta$ and $\mathfrak{M}_-$ with respect to the decomposition $\tilde{\mathcal{H}}=\mathfrak{M}_+[+]\mathfrak{M}_+^{[\perp]}$,

$$\theta=\mathrm{gr}\,B=\{\{\varphi,B\varphi\}:\varphi\in\mathfrak{D}(B)\subset\mathfrak{D}(B)=\mathfrak{M}_+^{[\perp]}\}, \qquad \mathfrak{M}_-=\mathrm{gr}\,K=\{\{\varphi,K\varphi\}:\varphi\in\mathfrak{M}_+^{[\perp]}\},$$

satisfy the conditions $B\in\mathcal{C}(\mathfrak{M}_+^{[\perp]},\mathfrak{M}_+)$, $K\in[\mathfrak{M}_+^{[\perp]},\mathfrak{M}_+]$. Moreover, we have $\|K\|_1\leq 1-\varepsilon$ $(\varepsilon>0)$ since $\mathfrak{M}_-$ is a uniformly negative subspace. In accordance with Proposition 1.4 the condition $\overline{\theta\dot{+}\mathfrak{M}_-}=\tilde{\mathcal{H}}$ ensures the invertibility of the operator $B-K$. Therefore, the operator $U$ in the polar decomposition of $B-K$ is an isometric operator from $\mathfrak{M}_+^{[\perp]}$ onto $\mathfrak{M}_+$, and $R>0$. Since $U^*B+I=R+I+U^*K$, we have

$\mathrm{Re}(U^*B) \geq \varepsilon I$ and $0 \in \rho(U^*B + I)$. Thus $(U + B)^{-1} \in [\mathcal{H}]$ and by virtue of Proposition 1.4 the subspace $\theta$ is transversal to the hypermaximal neutral subspace

$$\mathfrak{M}_0 = (I - U)\mathfrak{M}_+^{[\perp]} = \mathrm{gr}\,(-U) = \{\{\varphi, -U\varphi\} : \varphi \in \mathfrak{M}_+^{[\perp]}\}.\square$$

**Proposition 7.2.** *Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS for $A^*$, $\theta \in \tilde{\mathcal{C}}(\mathcal{H})$. Then each of the following conditions is sufficient for $\tilde{A}_\theta$ to belong to the class $\mathcal{A}s\,(A)$:*

(1) $\exists \lambda_1, \lambda_2 \in \rho(\tilde{A}_\theta) \cup \sigma_c(\tilde{A}_\theta)$, $\quad \mathrm{Im}\,\lambda_1 \cdot \mathrm{Im}\,\lambda_2 < 0$;

(2) $\exists \lambda_1, \lambda_2 \in \rho(\theta) \cup \sigma_c(\theta)$, $\quad \mathrm{Im}\,\lambda_1 \cdot \lambda_2 < 0$;

(3) $\theta \in \mathcal{C}(\mathcal{H})$, $\mathfrak{D}(\theta) = \mathfrak{D}(\theta^*)$, $\mathrm{Im}\,\theta = [\mathcal{H}]$.

**Proof.** (1) Suppose that the first condition is fulfilled. Then it follows from Proposition 1.6 that $0 \in \rho(\theta - M(\lambda_j)) \cup \sigma_c(\theta - M(\lambda_j))$ $(j = 1, 2)$. In accordance with Proposition 1.4 the last conditions are equivalent to relations (7.1), where $\mathfrak{M}_j = \mathrm{gr}\,M(\lambda_j) = \{\{h, M(\lambda_j)h\} : h \in \mathcal{H}\}$. By the condition $M(\lambda) \in R_{\mathcal{H}}$, we obtain that $\mathfrak{M}_1$ is uniformly positive if $\mathrm{Im}\,\lambda_1 > 0$ and $\mathfrak{M}_2$ is uniformly negative if $\mathrm{Im}\,\lambda_2 < 0$. Now it remains to apply Lemma 7.1.

(2) The second assertion can be proved in the same way if we put $\mathfrak{M}_j = \mathrm{gr}\,(\lambda_j I_{\mathcal{H}}) = \{\{h, \lambda_j h\} : h \in \mathcal{H}\}$ $(j = 1, 2)$ and notice that $\mathfrak{M}_j$ $(j = 1, 2)$ and $\theta$ satisfy the conditions of Lemma 7.1, provided that $\mathrm{Im}\,\lambda_1 > 0$, $\mathrm{Im}\,\lambda_2 < 0$.

(3) The third assertion follows from item (2). $\square$

**2.** Suppose that $T \in \mathcal{A}s\,(A)$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$ such that $T = \ker(\Gamma_1 - B\Gamma_2)$, $B \in [\mathcal{H}]$; $R_T(\lambda) = (T - \lambda)^{-1}$ is the resolvent of $T$. We define the operator-valued functions $\hat{R}_T(\lambda) \in [\mathfrak{h}, A^*]$ and $\Gamma_2(\lambda) \in [\mathfrak{N}_\lambda, \mathcal{H}]$ by the equalities

$$\hat{R}_T(\lambda)f := \{\{R_T(\lambda)f, f + \lambda R_T(\lambda)f\} : f \in \mathfrak{h}\}, \qquad \lambda \in \rho(T); \tag{7.2}$$

$$\Gamma_2(\lambda)f_\lambda := \Gamma_2\hat{f}_\lambda = \Gamma_2\{f_\lambda, \lambda f_\lambda\} \qquad (f_\lambda \in \mathfrak{N}_\lambda, \lambda \in \rho(A_2)). \tag{7.3}$$

**Definition 7.2.** *Let $\varphi = (B, \mathcal{H}; K, J, E)$ be a colligation (see [8]) (i.e., $E$ is a Hilbert space and $K \in [E, \mathcal{H}]$, $J \in [E]$ are linear operators such that $J = J^* = J^{-1}$, $\mathrm{Im}\,B = KJK^*$. An operator-valued function defined by the equality*

$$W_T(\lambda) = I + 2iK^*\Gamma_2\hat{R}_{T^*}(\lambda)\Gamma_2^*(\bar{\lambda})KJ \qquad (\lambda \in \rho(T^*)) \tag{7.4}$$

*will be called a characteristic function (CF) of $T$, and $W_T(\lambda)$ will be said to belong to the class $\Lambda_J$. We shall write $W_T(\lambda) \in \Lambda_J^0$ if additionally $\ker K = \{0\}$.*

**Theorem 7.1.** *Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for the linear relation $A^*$, $M(\lambda)$ is the corresponding Weyl function, $B \in [\mathcal{H}]$, $\varphi = (B, \mathcal{H}; K, J, E)$ is a colligation. Then the corresponding CF of the relation $T := \tilde{A}_B$ takes the form*

$$W_T(\lambda) = I + 2iK^*(B^* - M(\lambda))^{-1}KJ. \tag{7.5}$$

**Proof.** Making use of the relations $\Gamma_2\hat{R}_{A_2}(\lambda) = 0$, $\Gamma_2\hat{\gamma}(\lambda) = I_{\mathcal{H}}$, $\gamma^*(\bar{\lambda})\Gamma_2(\bar{\lambda}) = I_{\mathcal{H}}$, we obtain from (3.39), (7.4) for $\lambda \in \rho(T^*)$

$$\Gamma_2\hat{R}_{T^*}(\lambda) = \Gamma_2\{\hat{R}_{A_2}(\lambda) + \hat{\gamma}(\lambda)(B^* - M(\lambda))^{-1}\gamma^*(\bar{\lambda})\} = (B^* - M(\lambda))^{-1}\gamma^*(\bar{\lambda}), \tag{7.6}$$

$$\begin{aligned} W_T(\lambda) = I + 2iK^*\Gamma_2\hat{R}_{T^*}(\lambda)\Gamma_2^*(\bar{\lambda})KJ &= I + 2iK^*\Gamma_2\{\hat{R}_{A_2}(\lambda) + \hat{\gamma}(\lambda)(B^* - M(\lambda))^{-1}\gamma^*(\bar{\lambda})\}\Gamma_2^*(\bar{\lambda})KJ \\ &= I + 2iK^*(B^* - M(\lambda))^{-1}KJ. \ \square \end{aligned} \tag{7.7}$$

**Corollary 7.1.** *In the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ formula (7.5) was obtained by the authors in [18, 26]. In this case definition (7.4) of the CF $W_T(\lambda)$ takes the form*

$$W_T(\lambda) = I + 2iK^*\Gamma_2(T^* - \lambda)^{-1}\Gamma_2^*(\bar{\lambda})KJ, \tag{7.8}$$

*where* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *is a BVS for* $A^*$, $\Gamma_2 \in [\mathfrak{D}(A^*), \mathcal{H}]$, $\Gamma_2(\lambda) = \Gamma_2 \restriction \mathfrak{N}_\lambda$.

**Remark 7.2.** If boundary spaces $\mathfrak{L}$, $\mathfrak{L}'$ and boundary operators $\Gamma$, $\Gamma'$ for $T$ and $-T^*$ (in the sense of [67]) are defined by the equalities $\mathfrak{L} = \mathfrak{L}' = K^*\mathcal{H}$, $\Gamma = K^*\Gamma_2 \restriction T$, $\Gamma' = K^*\Gamma_2 \restriction T^*$, then one can easily show (see [18]) that $W_T(\lambda)$ coincides with the characteristic function of Shtraus [67] in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ and with its generalization [10] to a linear relation in the case $\overline{\mathfrak{D}(A)} \neq \mathfrak{h}$.

**Remark 7.3.** Suppose that $T \in [\mathfrak{h}]$, $A = \begin{pmatrix} A_{00} & \cdot \\ A_{10} & \cdot \end{pmatrix}$ is its Hermitian part (i.e., $A = T \restriction \mathfrak{D}(A)$, where $\mathfrak{D}(A) = \mathfrak{h}_0 = \ker T_I$), $\Pi = \{\mathfrak{N}, \Gamma_1, \Gamma_2\}$ is a BVS for the linear relation $A^*$ defined by equalities (3.64), $\mathfrak{N} = \mathfrak{h} \ominus \mathfrak{h}_0$, $T_{11} = P_{\mathfrak{N}} T \restriction \mathfrak{N}$. Then the operator $T$ admits the representation $T = \begin{pmatrix} A_{00} & A_{10}^* \\ A_{10} & T_{11} \end{pmatrix}$ and can be defined in the BVS (3.64) by the equality $\operatorname{gr} T = \ker(\Gamma_1 - B\Gamma_2)$ with $B = T_{11} - A_{11} \in [\mathfrak{N}]$. In accordance with Theorem 7.1 a CF of the operator $T$ takes the form in (7.5), where $M(\lambda)$ is defined by formula (3.65). Recall that $M(\lambda)$ coincides with the spectral complement of the operator $A$ if $A_{11} = \mathbb{O}$.

If we consider an operator $T$ as an extension of the operator $A = \{0\}$ ($\mathfrak{h}_0 = \mathfrak{D}(A) = \{0\}$), then $T = B$, $M(\lambda) = \lambda I_{\mathcal{H}}$ (for $A_{11} = \mathbb{O}$). In this case formula (7.5) for the calculation of a CF takes the form

$$W(\lambda) I + 2iK^*(T^* - \lambda)^{-1} KJ \tag{7.9}$$

and coincides with the definition of Livšic [48, 8, 9].

The utility of a colligation $\varphi = (B, \mathcal{H}; K, J, E)$ with $\ker K \neq \{0\}$ can be illustrated, in particular, by the following example.

**Example.** Suppose that $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, $B = B^* \in [\mathcal{H}]$, $T = T^* = \tilde{A}_B$, $\varphi = (B, \mathcal{H}; K, J, E)$ is a colligation with $E = \mathcal{H} \oplus \mathcal{H}$, $J = \begin{pmatrix} I_{\mathcal{H}} & \mathbb{O} \\ \mathbb{O} & -I_{\mathcal{H}} \end{pmatrix}$, $K = (I_{\mathcal{H}}, I_{\mathcal{H}})$. Then according to (7.5) we have

$$W_T(\lambda) = I_{\mathcal{H}\oplus\mathcal{H}} + 2i(B^* - M(\lambda))^{-1} \begin{pmatrix} I_{\mathcal{H}} & -I_{\mathcal{H}} \\ I_{\mathcal{H}} & -I_{\mathcal{H}} \end{pmatrix} \tag{7.10}$$

**3. Proposition 7.3.** *Let* $T \in \operatorname{Ex}_A$ *be the maximal dissipative extension of* $A$, $C = (T - i)(T + i)^{-1}$, $\zeta = (\lambda - i)(\lambda + i)^{-1}$. *Then the CF* $W_T(\lambda)$ *coincides with the CF of Nagy and Foias (see* [59]*).*

**Proof.** (1) The defect operator $D_C = (I - C^*C)^{1/2}$ of the contraction $C$ admits the representation

$$D_C^2 = 2iR_T(-i) - 2iR_T^*(-i) - 4R_T^*(-i)R_T(-i). \tag{7.11}$$

After the substitution $\hat{f} = \{f, f'\} = \hat{R}_T(-i)h$, $\hat{g} = \{g, g'\} = \hat{R}_T(-i)h'$ $(h, h' \in \mathfrak{h})$ we obtain from (7.2) and the Green identity (1.3)

$$2(B_I\Gamma_2\hat{R}_T(-i)h, \Gamma_2\hat{R}_T(-i)h')_{\mathcal{H}} = i\{(f, g') - (f', g)\}$$

$$= i\{(R_T(-i)h, h' - iR_T(-i)h') - (h - iR_T(-i)h, R_T(-i)h')\} = 2^{-1}(D_C^2 h, h').$$

This implies that the mapping $V : D_C h \to 2B_I^{1/2}\Gamma_2\hat{R}_T(-i)h$ is isometric. In the same way one can show that the mapping $V_* : D_{C^*}h \to 2B_I^{1/2}\Gamma_2\hat{R}_{T^*}(-i)h$ is an isometry.

(2) Consider an operator-valued function $\tilde{W}(\lambda) = V_*\theta_C(\zeta)V^{-1}$ with $\theta_C(\zeta)$ defined by the equality

$$\theta_C(\zeta)D_C = D_{C^*}(I - \zeta C^*)^{-1}(\zeta - C) \qquad (\zeta^{-1} \in \rho(C^*)).$$

Then we have

$$\tilde{W}(\lambda)VD_C = V_*\theta_C(\zeta)D_C = V_*D_{C^*}(I - \zeta C^*)^{-1}(\zeta - C) = 2B_I^{1/2}\Gamma_2\hat{R}_{T^*}(i)[-I + (\lambda + i)R_T(-i)].$$

Making use of (7.6) and the relation $g_{\bar\lambda} := 2(\Gamma_2\hat{R}_{T^*}(\lambda))^* B_I^{1/2}\varphi \in \mathfrak{N}_{\bar\lambda}$, $\varphi \in \mathcal{H}$, we obtain from the Green identity applied to the vectors $\hat{f} = \hat{R}_T(-i)h$, $\hat{g}_{\bar\lambda} = \{g_{\bar\lambda}, \bar\lambda g_{\bar\lambda}\} \in \hat{\mathfrak{N}}_{\bar\lambda}$

$$-(\tilde{W}(\lambda)VD_C h, \varphi)_{\mathcal{H}} = ([I - (\lambda + i)R_T(-i)]h, g_{\bar\lambda})_{\mathfrak{h}} = ((I - iR_T(-i))h, g_{\bar\lambda})_{\mathfrak{h}} - (R_T(-i)h, \bar\lambda g_{\bar\lambda})_{\mathfrak{h}}$$

$$= (\Gamma_1\hat{f},\Gamma_2\hat{g}_{\bar\lambda})_{\mathcal{H}} - (\Gamma_2\hat{f},\Gamma_1\hat{g}_{\bar\lambda})_{\mathcal{H}} = ((B - M(\lambda))\Gamma_2\hat{f},\Gamma_2\hat{g}_{\bar\lambda})_{\mathcal{H}}$$
$$= 2(B_I^{1/2}\Gamma_2\hat{R}_{T^*}(\lambda)\Gamma_2^*(\bar\lambda)(B - M(\lambda))\Gamma_2\hat{R}_T(-i)h,\varphi)_{\mathcal{H}}$$
$$= 2(B_I^{1/2}(B^* - M(\lambda))^{-1}(B - M(\lambda))\Gamma_2\hat{R}_T(-i)h,\varphi)_{\mathcal{H}}$$
$$= 2((I + 2iB_I^{1/2}(B^* - M(\lambda))^{-1}B_I^{1/2})B_I^{1/2}\hat{R}_T(-i)h,\varphi)_{\mathcal{H}} = (W_T(\lambda)VD_Ch,\varphi)_{\mathcal{H}}.$$

This implies that $W_T(\lambda) = -\tilde{W}(\lambda) = -V_*\theta_C(\zeta)V^{-1}$.

**4.** Let $W(\lambda) \in \Lambda_J$ be a CF of a linear relation $T = \tilde{A}_B \in \mathcal{A}s\,(A)$. Put

$$\mathfrak{h}_W = \operatorname{span}\{\gamma(\lambda)(B^* - M(\lambda))^{-1}Kh, \gamma(\bar\lambda)(B - M(\bar\lambda))^{-1}Kh' : \lambda \in \rho(T^*); h, h' \in \mathcal{H}\}.$$

Notice that $\mathfrak{h}_W$ is a reducing subspace for the relation $T$ and on $\mathfrak{h} \ominus \mathfrak{h}_W$ $T$ induces a self-adjoint linear relation.

**Theorem 7.2.** *Suppose that $A_j$ are Hermitian operators in $\mathfrak{h}_j$, $J = J^* = J^{-1} \in [\mathcal{H}]$, $W_j(\lambda) \in \Lambda_J$ are CF's of relations $T_j = \tilde{A}_{B_j} \in \mathcal{A}s\,(A_j)$ $(j = 1,2)$. If on some neighborhood of the point $\lambda_0 \in \rho(T_1^*) \cap \rho(T_2^*)$ $W_1(\lambda) \equiv W_2(\lambda)$ and $\mathfrak{h}_{W_1} = \mathfrak{h}_{W_2}$, then the linear relations $T_1$ and $T_2$ are unitarily equivalent.*

We omit the proof, which is straightforward (see [8, 67]). The following corollary results from (7.9), (7.10), and Theorem 7.2.

**Corollary 7.1.** *A Weyl function $M(\lambda)$ corresponding to a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ defines the simple part of a Hermitian operator up to unitary equivalence.*

Since $M(\lambda)$ is a $Q$-function of a Hermitian operator corresponding to an extension $A_2$ (see [79, 53]), by virtue of Corollary 7.1 a $Q$-function is a unitary invariant of a simple Hermitian operator $A$ (see [81, 83]). Further, from Corollary 7.1 and formulas (3.65), (3.73) follow these well-known assertions: a bounded simple Hermitian operator (as well as a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$) is defined uniquely (up to unitary equivalence) either by a spectral complement of a Hermitian operator $A$ [69] or by its $Q_\mu$-function $Q_\mu(\lambda)$ [40].

**5.** Let $W(\lambda) \in \Lambda_J$. Define an operator-valued function $V(\lambda)$ holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ by the equality

$$V(\lambda) = K^*(B_R - M(\lambda))^{-1}K \qquad (\lambda \in \rho(\tilde{A}_{B_R})). \tag{7.12}$$

**Lemma 7.2.** *For all $\lambda \in \rho(\tilde{A}_B^*) \cap (\mathbb{C}_+ \cup \mathbb{C}_-)$ there exists $(I + W(\lambda))^{-1} \in [E]$ and*

$$V(\lambda) = i(I - W(\lambda)(I + W(\lambda))^{-1}J. \tag{7.13}$$

**Proof** (cf. [26]). It follows from the relation

$$(B^* - M(\lambda))^{-1} - (B_R - M(\lambda))^{-1} = i(B^* - M(\lambda))^{-1}KJK^*(B_R - M(\lambda))^{-1}$$

multiplied from the left by $2iK^*$ and from the right by $KJ$ that

$$W(\lambda) - I - 2iV(\lambda)J = i(W(\lambda) - I)V(\lambda)J.$$

This implies that $(W(\lambda) + I)(I - iV(\lambda)J) = 2I$. Equality (7.13) is a consequence of the last relation and the next one, $(I - iV(\lambda)J)(W(\lambda) + I) = 2I$, which can be established in the same way.

We mention the following properties of $W(\lambda) \in \Lambda_J$ and $V(\lambda)$.

**Lemma 7.3.** (a) $V(\lambda) \in R_E$; (b) $\operatorname{Im}\lambda(J - W^*(\lambda)JW(\lambda)) \geq 0$,

$$\operatorname{Im}\lambda(J - W(\lambda)JW^*(\lambda)) \geq 0 \qquad \forall\lambda \in \rho(T^*); \tag{7.14}$$

(c) $J - W(\lambda)JW^*(\bar\lambda) = J - W^*(\lambda)JW(\bar\lambda) = 0$ $\forall\lambda \in \rho(T^*) \cap \rho(T)$; (d) *for all $\lambda \in \rho(T) \cap \rho(T^*)$ there exists $W^{-1}(\lambda)$, and the following equalities hold:*

$$W_T(\lambda)^{-1} = W_{T^*}(\lambda) = JW_T^*(\bar\lambda)J. \tag{7.15}$$

**Proof.** Making use of (7.12) and the condition $\text{Im}\,\lambda \cdot \text{Im}\,M(\lambda) \geq 0$ we obtain the inequality

$$\text{Im}\,V(\lambda) = K^*(B_R - M(\lambda))^{-1}\text{Im}\,M(\lambda)(B_R - M^*(\lambda))^{-1}K \geq 0 \qquad (\lambda \in \mathbb{C}_+). \tag{7.16}$$

Assertions (b) and (c) follow from the identity

$$V(\lambda) - V^*(\mu) = 2i(I + W(\lambda))^{-1}(J - W(\lambda)JW^*(\mu))(I + W^*(\mu))^{-1}. \tag{7.17}$$

Equalities (7.15) are the consequences of (7.9) and the identity

$$(B^* - M(\lambda))^{-1} - (B - M(\lambda))^{-1} = 2i(B^* - M(\lambda))^{-1}KJK^*(B - M(\lambda))^{-1}. \ \square$$

**Proposition 7.4.** *Let $W(\lambda) \in \Lambda_J^0$. The triple $\Pi' = \{\mathcal{H}, \Gamma_1', \Gamma_2'\}$, where*

$$\Gamma_1'\hat{f} = K^*\Gamma_2\hat{f} \qquad \Gamma_2'\hat{f} = K^{-1}(B_R\Gamma_2 - \Gamma_1)\hat{f} \quad (\hat{f} \in A_*, B_R = \frac{B + B^*}{2}),$$

*is a generalized BVS for the relation $A_* = \{\hat{f} \in A^* : (B_R\Gamma_2 - \Gamma_1)\hat{f} \in \mathfrak{R}\,(K)\}$. The Weyl function of the operator $A$ corresponding to the BVS $\Pi'$ coincides with $V(\lambda)$. If, additionally, $0 \in \rho(\text{Im}\,V(i))$, then the triple $\Pi'$ is a BVS of the relation $A_* = A^*$.*

**Proof.** Indeed, it follows from the condition $\Gamma_1 A_2 = \mathcal{H}$ that $\Gamma_2'(A_* \cap A_2) = \mathcal{H}$. The relation $\ker \Gamma_2 = \tilde{A}_{B_R} = (\tilde{A}_{B_R})^*$ and the Green formula (6.1) are evident. We prove that the mapping $\Gamma_2'$ is closable. Suppose that $\hat{f}_n$ converges to 0 in $A^*$ and $\Gamma_2'\hat{f}_n = K^{-1}(\Gamma_1 - B_R\Gamma_2)\hat{f}_n$ converges to $h$ in $\mathcal{H}$ as $n \to \infty$. Letting $h_n = (\Gamma_1 - B_R\Gamma_2)\hat{f}_n$ we have $h_n \to 0$, $K^{-1}h_n \to h$ as $n \to \infty$. This implies $h = 0$.

It follows from Definition 1.6 and the relations

$$\Gamma_1'\hat{f}_\lambda = K^*\Gamma_2\hat{f}_\lambda, \qquad \Gamma_2'\hat{f}_\lambda = K^{-1}(B_R - M(\lambda))\Gamma_2\hat{f}_\lambda, \qquad \hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda$$

that the Weyl function $M_1(\lambda)$ corresponding to the BVS $\Pi'$ takes the form $M_1(\lambda) = K^*(B_R - M(\lambda))^{-1}K$ and coincides with $V(\lambda)$.

According to Propositions 6.3 and 7.4, given an operator-valued function $W(\lambda) \in \Lambda_J^0$, we can define the rigging $E_+^W \subset E \subset E_-^W$ of the Hilbert space $E$ (see [6]) with the help of formulas (6.14), (6.15), letting here $M(\lambda) = V(\lambda)$. Then by virtue of (7.12)

$$E_+^W = \mathfrak{R}\,((\text{Im}\,V(i))^{1/2}) = \mathfrak{R}\,(K^*), \quad \|h\|_+^2 = \|(\text{Im}\,V(i))^{-1/2}h\|^2 = \|(K^*)^{-1}h\|^2, \tag{7.18}$$

and the space $E_-^W$ is a completion of $E$ endowed with the norm

$$\|h\|_-^2 = (\text{Im}\,V(i)h, h) = \|K^*h\|^2 \qquad (h \in E). \tag{7.19}$$

Notice also that $\text{Im}\,V(i)$ is an isometry from $E_-^W$ onto $E_+^W$ and $K^* \in [E, E_+^W]$, $K \in [E_-^W, E]$ are isomorphisms.

**Lemma 7.4.** *Let $W(\lambda) \in \Lambda_J^0$. Then we have*

(a) $V(\lambda) \in [E_-^W, E_+^W]$, $V^{-1}(\lambda) \in [E_+^W, E_-^W]$ $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$;

(b) $[I - iJV(\lambda)] \in [E_-^W]$ $\forall \lambda \in \rho(\tilde{A}_B^*) \setminus \mathbb{R}$.

**Proof.** Assertion (a) is a consequence of (7.12). As follows from Lemma 7.2, for all $\lambda \in \rho(A_B^*) \setminus \mathbb{R}$ there exists $(I - iV(\lambda)J)^{-1} \in [E]$. Note that $I - iJV(\lambda) \in [E_-]$ and $(I - iJV(\lambda))E_- \subset E_-$ since for all $g \in E_-$ the equation $(I - iJV(\lambda))h = g$ has the solution $h = g + i(I - iJV(\lambda))JV(\lambda)g \in E_-$. Assertion (b) is now a consequence of the Banach theorem. $\square$

We give a criterion for a holomorphic operator-valued function $W(\lambda)$ to belong to the class $\Lambda_J^0$.

**Theorem 7.3.** *Let $W(\lambda)$ be an operator-valued function with values in $[E]$ holomorphic on a domain $G \subset \mathbb{C}_+$, $J = J^* = J^{-1} \in [E]$. Then for the condition $W(\lambda) \in \Lambda_J^0$ to hold it is necessary and sufficient* *that $-1 \in \rho(W(\lambda))$ for all $\lambda \in G$ and the operator-valued function $V(\lambda)$ with values in $[E]$ and defined by equality* (7.13) *has the holomorphic continuation on $\mathbb{C}_+$ such that:*

(1) $\operatorname{Im} V(\lambda) > 0$, $\ker(\operatorname{Im} V(\lambda)) = \{0\}$ $\forall\lambda \in \mathbb{C}_+$;

(2) $V(\lambda) \in [E_-^W, E_+^W]$ ($\Longleftrightarrow (W(\lambda) - I)J \in [E_-^W, E_+^W]$) $\forall\lambda \in \mathbb{C}_+$.

**Proof.** The necessity of the conditions of Theorem 7.3 follows from Lemmas 7.2–7.4.

*Sufficiency.* Let $K(\in [E_-^W, E])$ be a closure of an operator $(\operatorname{Im} V(i))^{1/2}$; $K^* \in [E, E_+^W]$, $M_0(\lambda) = (K^*)^{-1}V(\lambda)K$. It follows from conditions (1) and (2) that $M_0(\lambda) \in R_E$ and takes values in $[E]$. Moreover, $0 \in \rho(\operatorname{Im} M_0(\lambda))$ for all $\lambda\in \mathbb{C}\setminus\mathbb{R}$ since $\operatorname{Im} M(i) = I$. This implies that the operator-valued function $M(\lambda) = -M_0(\lambda)^{-1} = -KV(\lambda)^{-1}K^* \in R_E$ also satisfies the conditions of Proposition 5.1 and therefore there exists a Hermitian operator $A$ and a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ for $A^*$ such that $M(\lambda)$ is the Weyl function of the operator $A$ corresponding to the BVS $\Pi$.

Setting $B = iKJK^*$ we define an extension $T = \tilde{A}_B \in \mathcal{A}s\,(A)$ by the condition $\tilde{A}_B = \ker(\Gamma_1 - B\Gamma_2)$. By conditions

$$(I - iJV(\lambda))^{-1} \in [E_-^W] \qquad \forall\lambda \in G,$$

$$(B^* - M(\lambda))^{-1} = (K^*)^{-1}V(\lambda)(I - iJV(\lambda))^{-1}K^{-1} \in [E] \quad \forall\lambda \in G$$

we obtain from (7.9), (7.13) that

$$W_T(\lambda) = I + 2iK^*(B^* - M(\lambda))^{-1}KJ = I + 2iV(\lambda)(I - iJV(\lambda))^{-1}J =$$

$$= J(I + iJV(\lambda)(I - iJV(\lambda))^{-1}J = W(\lambda). \qquad \square$$

**Remark 7.4.** The description of CF's of operators with densely defined Hermitian parts was obtained in [18, 26]. Under the assumptions of Theorem 7.3 an operator-valued function $W(\lambda)$ is a CF of some almost solvable extension of a densely defined Hermitian operator if in addition to conditions (1) and (2) of Theorem 7.3 the following relations hold:

$$(3)\ \lim_{y\uparrow\infty} \frac{V(iy)}{y} = 0, \qquad (4)\ \lim_{y\uparrow\infty} y(\operatorname{Im} V(iy)h, h) = \infty \qquad \forall h \in E_-^W \setminus \{0\}.$$

**Proposition 7.5.** *Let $W(\lambda)$ be an operator-valued function with values in $[E]$, holomorphic on the domain $G(\subset \mathbb{C})$, $J = J^* = J^{-1} \in [E]$. For the condition $W(\lambda) \in \Lambda_J^0$ to hold it is necessary that the following conditions hold:*

$$(1)\ J = W^*(\lambda)JW(\lambda) > 0, \quad J - W(\lambda)JW^*(\lambda) > 0 \quad \forall\lambda \in \mathbb{C}_+ \cap G,$$

*and it is sufficient that the first of condition* (1) *and the following conditions* (2) *hold:*

$$(2)\ 0 \in \rho(J - W^*(\lambda)JW(\lambda)), \qquad 0 \in \rho(J - W(\lambda)JW^*(\lambda)) \qquad \forall\lambda \in \mathbb{C}_+ \cap G.$$

**Proof.** The necessity of conditions (1) in the case $W(\lambda) \in \Lambda_J^0$ was proved in Lemma 7.3.

*Sufficiency.* Let us prove the implications

$$0 \in \rho(J - W^*(\lambda)JW(\lambda)) \Longrightarrow -1 \in \hat{\rho}(W(\lambda)), \qquad 0 \in \rho(J - W(\lambda)JW^*(\lambda)) \Longrightarrow -1 \in \hat{\rho}(W^*(\lambda)).$$

If $-1 \notin \hat{\rho}(W(\lambda))$, then there exists a sequence $\{f_n\}$ ($\|f_n\| = 1$) such that $W(\lambda)f_n = -f_n + h_n$, $h_n \to 0$. In view of the relation

$$(Jf_n, f_n) - (JW(\lambda)f_n, W(\lambda)f_n) = -2\operatorname{Re}(Jf_n, h_n) - (Jh_n, h_n) \to 0,$$

this contradicts (2). Therefore, $-1 \in \hat{\rho}(W(\lambda))$ and the operator-function $V(\lambda) = i(I - W(\lambda))(I + W(\lambda))^{-1}$ is well defined. Further, each of the hypotheses (1) on $W(\lambda)$ is equivalent to condition (1) of Theorem 7.3

on $V(\lambda)$. By virtue of (7.17) the hypothesis (2) is equivalent to the condition $0 \in \rho(\operatorname{Im} V(i))$. This implies $E_-^W = E_+^W$ [see (7.18), (7.19)], and condition (2) of Theorem 7.3 is fulfilled. $\square$

**Corollary 7.2.** *Assume that $J = J^* = J^{-1} \in [E]$, $\dim E = n < \infty$, and $W(\lambda)$ is an operator-valued function with values in $[E]$ holomorphic on some domain $G_W \subset \mathbb{C}_+$. For $W(\lambda)$ to be a CF of the class $\Lambda_J^0$ of some linear relation $T \in \mathcal{A}s$ it is necessary and sufficient that the first of conditions (1) from Proposition 7.5. holds.*

**Proof.** Since $\dim E = n < \infty$, we have $0 \in \rho(J - W^*(\lambda)JW(\lambda))$ and consequently $0 \in \rho(J - W(\lambda)JW^*(\lambda))$. Corollary 7.2 now follows from Proposition 7.5. $\square$

**6. Definition 7.3.** *Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and let $P_j$ be an orthogonal projection from $\mathfrak{h}$ onto $\mathfrak{h}_j$ $(j = 1, 2)$. A linear relation $T \in \tilde{\mathcal{C}}(\mathfrak{h})$ is the coupling of the linear relations $T_j \in \tilde{\mathcal{C}}(\mathfrak{h}_j)$ $(j = 1, 2)$ and is written $T = T_1 V T_2$ if*

$$T_1 = T \cap (\mathfrak{h}_1 \oplus \mathfrak{h}_1), \qquad T_2 = \{\{P_2 f, P_2 f'\} : \{f, f'\} \in T\}. \tag{7.20}$$

**Proposition 7.6.** *Let $T \in \tilde{\mathcal{C}}(\mathfrak{h})$ and $T = T_1 V T_2$. Then:*

*(1) $T^* = T_2^* V T_1^*$; (2) $T^{-1} = T_1^{-1} V T_2^{-1}$; (3) if $T_j$ $(j = 1, 2)$ and $T = T_1 V T_2$ are operators in $\mathfrak{h}_j$ $(j = 1, 2)$ and $\mathfrak{h}$, then the subspace $\mathfrak{h}_1$ is invariant for $T$, and*

$$Tf = T_1 f \quad \forall f \in \mathfrak{D}(T) \cap \mathfrak{h}_1, \qquad T_2 P_2 f = P_2 T f \quad \forall f \in \mathfrak{D}(T).$$

**Proof.** (1) Assume that $\{g, g'\} \in T * \cap (\mathfrak{h}_2 \oplus \mathfrak{h}_2)$. Then for all $\{f, f'\} \in T$ we have $\{P_2 f, P_2 f'\} \in T_2$,

$$(g, P_2 f') = (g, f') = (g', f) = (g', P_2 f),$$

and hence $\{g, g'\} \in T_2^*$. Analogously, if $\{g, g'\} \in T^*$, then $(P_1 g, f') = (g, f') = (P_1 g', f)$ $\forall \{f, f'\} \in T_1 \subset T$, and therefore $\{P_1 g, P_1 g'\} \in T_1^*$. Statement (2) is evident.

(3) Assume that $T$, $T_1$, $T_2$ are operators and $f \in \mathfrak{D}(T) \cap \mathfrak{h}_1$. Then by virtue of (7.20) we have $\{P_2 f, P_2 T f\} = \{0, P_2 T f\} \in \operatorname{gr} T_2$ and, consequently, $P_2 T f = 0$, that is, $Tf \in \mathfrak{h}_1$. The first relation in (7.20) yields $\{f, Tf\} \in \operatorname{gr} T_1$ and hence $T_1 f = Tf$ for all $f \in \mathfrak{D}(T) \cap \mathfrak{h}_1$. $\square$

**Remark 7.6.** One can conclude from the following example that all the conditions $T_j \in \mathcal{C}(\mathfrak{h}_j)$ $(j = 1, 2)$ in statement (3) from Proposition 7.6 are essential. Let $\mathfrak{h} = l_2(-\infty, \infty)$, $\mathfrak{h}_1 = l_2[1, \infty)$, $\mathfrak{h}_2 = \mathfrak{h} \ominus \mathfrak{h}_1$, and let $U$ be a bilateral shift in $l_2(-\infty, \infty)$; $T_1 = U \restriction \mathfrak{h}_1$, $T_2 = (U^* \restriction \mathfrak{h}_2)^*$. Then the equality $U = T_1 V T_2$ holds and by virtue of item (2) of Proposition 7.6 we have $U^{-1} = T_1^{-1} V T_2^{-1}$. Here $U^{-1}$, $T_1^{-1}$ are operators in $\mathfrak{h}$, $\mathfrak{h}_1$; $T_2^{-1}$ is a linear relation in $\mathfrak{h}_2$ and $\mathfrak{h}_1$ is not invariant subspace for $U^{-1}$.

Recall (see [8]) that a colligation $\varphi = (B, \mathcal{H}; K, J, E)$ is a product $\varphi = \varphi_1 \varphi_2$ of the colligation $\varphi_j = (B_j, \mathcal{H}_j; K, J, E)$ $(j = 1, 2)$ if

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad B = B_1 P_{\mathcal{H}_1} + B_2 P_{\mathcal{H}_2} + 2i K_1 J K_2^* P_{\mathcal{H}_2}, \qquad K = K_1 + K_2. \tag{7.21}$$

**Theorem 7.4.** *Suppose that $A_j$ are Hermitian operators in $\mathfrak{h}_j$, $\{\mathcal{H}_j, \Gamma_1^j, \Gamma_2^j\}$ are BVS's for $A_j^*$; $B_j \in [\mathcal{H}_j]$; $\varphi_j = (B_j, \mathcal{H}_j; K_j, J, E)$ are colligations; $W_j(\lambda)$ are the corresponding CF of the extensions $\tilde{A}_{B_j}$ $(j = 1, 2)$. Define a BVS for the relation $A^* = A_1^* \oplus A_2^*$ by the equalities $\Gamma_k = \Gamma_K^{(1)} \oplus \Gamma_K^{(2)}$ $(k = 1, 2)$ and put $\varphi = \varphi_1 \varphi_2$. Then $\tilde{A}_B = \tilde{A}_{B_1} V \tilde{A}_{B_2}$, $\rho(\tilde{A}_{B_1}) \cap \rho(\tilde{A}_{B_2}) \subset \rho(\tilde{A}_B)$, and the corresponding Weyl function $W(\lambda)$ of the extension $\tilde{A}_B$ takes the form*

$$W(\lambda) = W_1(\lambda) \cdot W_2(\lambda) \qquad \forall \lambda \in \rho(\tilde{A}_{B_1}) \cap \rho(\tilde{A}_{B_2}). \tag{7.22}$$

**Proof** (cf. [8]). Let $\hat{f} = \{f, f'\} \in \tilde{A}_B \cap (\mathfrak{h}_1 \oplus \mathfrak{h}_2)$. Then $\Gamma_1 \hat{f} = B \Gamma_2 \hat{f}$ and it follows from (7.21) that $\Gamma_1^{(1)} \hat{f} = B_1 \Gamma_2^{(1)} \hat{f}$ and hence $\hat{f} \in \tilde{A}_{B_1}$. Making use of the relation $\hat{f} = \{f, f'\} \in \tilde{A}_B$ we obtain from (7.21) that $\Gamma_1^{(2)}\{P_2 f, P_2 f'\} = B_2\Gamma_2^{(2)}\{P_2 f, P_2 f'\}$, that is, $\{P_2 f, P_2 f'\} \in \tilde{A}_{B_2}$. Thus $\tilde{A}_B = \tilde{A}_{B_1} V \tilde{A}_{B_2}$. By virtue of Proposition 1.6 $(B_j - M_j(\lambda))^{-1} \in [\mathcal{H}_j]$ for all $\lambda \in \rho(\tilde{A}_{B_1}) \cap \rho(\tilde{A}_{B_2})$, and the following equality holds:

$$(B^* - M(\lambda))^{-1} = (B_1^* - M_1(\lambda))^{-1} P_{\mathcal{H}_1} + (B_2^* - M_2(\lambda))^{-1} P_{\mathcal{H}_2} + 2i(B_2^* - M_2(\lambda))^{-1} K_2 J K_1^* (B_1^* - M_1(\lambda))^{-1} P_{\mathcal{H}_1}. \tag{7.23}$$

Theorem 7.1 and (7.23) yield equality (7.22). □

**Remark 7.7.** Theorems 7.2–7.4 are generalizations of the corresponding statements of Brodskii–Livšic [8, 9, 48], and coincide with them in the case $\mathfrak{h}_0 = \{0\}$ ($\Longrightarrow A = \{0\}$, $M(\lambda) = \lambda$). Definition 7.2 of the class $\mathcal{A}s\,(A)$ enables us to prove Theorems 7.2–7.4 in the same way as was done in [8, 9] (in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, see also [18, 26]).

The CF of various classes of unbounded operators have been investigated from different points of view by many authors (see [46, 60, 67] and the bibliography in [60]). The inverse problem for the CF defined by means of biextension theory was considered in [60], and the multiplication theorem was obtained in [46, 66].

In the framework of Shtraus' approach, the multiplication theorem was obtained for the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ (for $\overline{\mathfrak{D}(A)} \neq \mathfrak{h}$, see [10]) provided that the coupling is regular. This assumption corresponds to the relations $\ker K_1 = \{0\}$, $\mathfrak{R}\,(K_1^*) = \mathfrak{R}\,(K_2^*)$ in our approach.

## 8. PRERESOLVENT AND RESOLVENT MATRICES OF A HERMITIAN OPERATOR

**1.** Let $\mathfrak{L}$ be a subspace of $\mathfrak{h}$. A point $\lambda \in \mathbb{C}$ is said to be $\mathfrak{L}$-regular for a Hermitian operator $A$ [42] if $\lambda \in \hat{\rho}(A)$ and

$$\mathfrak{h} = \mathfrak{M}_\lambda \dotplus \mathfrak{L} \qquad (\mathfrak{M}_\lambda = (A - \lambda)\mathfrak{D}(A)). \tag{8.1}$$

Let $\rho(A, \mathfrak{L})$ be the set of $\mathfrak{L}$-regular points of $A$, $\rho_s(A; \mathfrak{L}) := \rho(A; \mathfrak{L}) \cap \overline{\rho(A; \mathfrak{L})}$. We define (after [42]) two holomorphic on $\rho(A; \mathfrak{L})$ operator-valued functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$: $\mathcal{P}(\lambda)$ is a skew projection onto $\mathfrak{L}$ parallel to $\mathfrak{M}_\lambda$, $\mathcal{Q}(\lambda) = P_{\mathfrak{L}}(A - \lambda)^{-1}(I - \mathcal{P}(\lambda))$.

Operator-valued functions $\mathcal{P}(\lambda), \mathcal{Q}(\lambda) \in [\mathfrak{h}, \mathfrak{L}]$ and $\mathcal{P}^*(\lambda)$, $\mathcal{Q}^*(\lambda)$ with values in $[\mathfrak{L}, \mathfrak{h}]$ have the following properties:

$$\mathcal{P}(\lambda) A f = \lambda \mathcal{P}(\lambda) f \qquad \forall f \in \mathfrak{h}; \tag{8.2}$$

$$\mathcal{Q}(\lambda) A f = \lambda \mathcal{Q}(\lambda) f + P_{\mathfrak{L}} f \qquad \forall f \in \mathfrak{h}; \tag{8.3}$$

$$\hat{\mathcal{P}}^*(\lambda) l := \{\mathcal{P}^*(\lambda) l, \bar{\lambda} \mathcal{P}^*(\lambda) l\} \in A^* \qquad \forall l \in \mathfrak{L}; \tag{8.4}$$

$$\hat{\mathcal{Q}}^*(\lambda) l := \{\mathcal{Q}^*(\lambda) l, \bar{\lambda} \mathcal{Q}^*(\lambda) l + l\} \in A^* \qquad \forall l \in \mathfrak{L}; \tag{8.5}$$

$$\mathcal{P}(\lambda) l = l, \qquad P_{\mathfrak{L}} \mathcal{P}^*(\lambda) = I_{\mathfrak{L}}, \qquad \mathcal{P}^*(\lambda) P_{\mathfrak{L}} = \mathcal{P}^*(\lambda); \tag{8.6}$$

$$\mathcal{Q}(\lambda) l = 0, \qquad P_{\mathfrak{L}} \mathcal{Q}^*(\lambda) = 0, \qquad \mathcal{Q}^*(\lambda) P_{\mathfrak{L}} = \mathcal{Q}^*(\lambda). \tag{8.7}$$

It is easy to see that $\mathcal{P}^*(\lambda)$ isomorphically maps $\mathfrak{L}$ onto $\mathfrak{N}_{\bar{\lambda}}$ and $P_{\mathfrak{L}} \upharpoonright \mathfrak{N}_{\bar{\lambda}} = (\mathcal{P}^*(\lambda))^{-1} \in [\mathfrak{N}_{\bar{\lambda}}, \mathfrak{L}]$.

The next proposition is an analog of the first Neumann formula.

**Proposition 8.1.** *Let $\mathfrak{L}$ be a subspace in $\mathfrak{h}$, $\lambda \in \rho_s(A; \mathfrak{L})$. Then the following direct decomposition holds:*

$$A^* = A \dotplus \hat{\mathcal{P}}^*(\lambda)\mathfrak{L} \dotplus \hat{\mathcal{Q}}^*(\lambda)\mathfrak{L}, \tag{8.8}$$

*while the decomposition $\mathfrak{D}(A^*) = \mathfrak{D}(A) + \mathcal{P}^*(\lambda)\mathfrak{L} + \mathcal{Q}^*(\lambda)\mathfrak{L}$ is not direct if and only if $\overline{\mathfrak{D}(A)} \neq \mathfrak{h}$. The equality*

$$f_A + \mathcal{P}^*(\lambda) k + \mathcal{Q}^*(\lambda) l = 0 \qquad (f_A \in \mathfrak{D}(A);\ l, k \in \mathfrak{L})$$

*is fulfilled if and only if there exists $n \in \mathfrak{N} := \mathfrak{h}_0^{\perp}$ such that*

$$l = \mathcal{P}(\bar{\lambda}) n, \qquad k = -\mathcal{Q}(\bar{\lambda}) n, \qquad f_A = (A - \bar{\lambda})^{-1}(I - \mathcal{P}(\bar{\lambda})) n. \tag{8.9}$$

*Conversely, if $\lambda \in \rho(A; \mathfrak{L})$ and formula* (8.8) *holds, then $\lambda \in \rho_s(A; \mathfrak{L})$.*

**Proof.** The inclusion $A\dot{+}\,\hat{\mathcal{P}}^*(\lambda)\mathfrak{L}\dot{+}\,\hat{\mathcal{Q}}^*(\lambda)\mathfrak{L}\subset A^*$ is obvious. Let $\{f,f'\}\in A^*$, $\lambda\in\rho_s(A;\mathfrak{L})$. Then there exist $l\in\mathfrak{L}$, $f_A\in\mathfrak{D}(A)$ such that

$$f'-\bar\lambda f=(A-\bar\lambda)f_A+l. \tag{8.10}$$

It follows from (8.5), (8.10) that $\{f-f_A-\mathcal{Q}^*(\lambda)l,0\}\in A^*-\bar\lambda$, that is, $f-f_A-\mathcal{Q}^*(\lambda)l\in\mathfrak{N}_{\bar\lambda}$. By virtue of the equality $\mathcal{P}^*(\lambda)\mathfrak{L}=\mathfrak{N}_{\bar\lambda}$ there exists $k\in\mathfrak{L}$ such that $f-f_A-\mathcal{Q}^*(\lambda)l=\mathcal{P}^*(\lambda)k$ and hence

$$f=f_A+\mathcal{P}^*(\lambda)k+\mathcal{Q}^*(\lambda)l. \tag{8.11}$$

Relations (8.10), (8.11) imply

$$f'=Af_A+\bar\lambda\mathcal{P}^*(\lambda)k+\bar\lambda\mathcal{Q}^*(\lambda)l+l. \tag{8.11'}$$

Therefore, $\{f,f'\}\in A\dot{+}\hat{\mathcal{P}}^*(\lambda)\mathfrak{L}\dot{+}\hat{\mathcal{Q}}^*(\lambda)\mathfrak{L}$; moreover the decomposition (8.8) is a direct sum since $l$, $k$, $f_A$ are defined uniquely by (8.10), (8.11):

$$l=\mathcal{P}(\bar\lambda)(f'-\bar\lambda f),\qquad f_A=(A-\lambda)^{-1}(I-\mathcal{P}(\bar\lambda))(f'-\bar\lambda f),\qquad k=P_{\mathfrak{L}}(f-f_A)=P_{\mathfrak{L}}f-\mathcal{Q}(\bar\lambda)(f'-\lambda f). \tag{8.12}$$

If $\{0,f'\}\in A^*$, then $n:=f'\in\mathfrak{N}$ and formulas (8.12) take the form (8.9).

Conversely, vectors $l$, $k$, $f_A$ defined by (8.9) are connected by relation (8.11), in which $f=0$.

Assume now that $\lambda\in\rho(A;\mathfrak{L})$ and relation (8.8) is fulfilled. Then for all $g\in\mathfrak{h}$ there exists $\{f,f'\}\in A^*$ such that $f'-\bar\lambda f=g$ since $\mathfrak{R}(A^*-\bar\lambda)=\mathfrak{h}$. Formula (8.8) implies that $f'-\bar\lambda f=(A-\bar\lambda)f_A+l$ for some $f_A\in\mathfrak{D}(A)$, $l\in\mathfrak{L}$ and therefore $\mathfrak{h}=\mathfrak{M}_{\bar\lambda}+\mathfrak{L}$. It is easy to see that $\mathfrak{M}_{\bar\lambda}\cap\mathfrak{L}=\{0\}$. Indeed, if $(A-\bar\lambda)f_A+l=0$, then $f_A+\mathcal{Q}^*(\lambda)l\in\mathfrak{N}_{\bar\lambda}$ and by virtue of the relation $\mathcal{P}^*(\lambda)\mathfrak{L}=\mathfrak{N}_{\bar\lambda}$ there exists $k\in\mathfrak{L}$ such that $f_A+\mathcal{Q}^*(\lambda)l+\mathcal{P}^*(\lambda)k=0$. Making use of the equality $(A-\bar\lambda)f_A+l=0$ we obtain $\hat f_A+\hat{\mathcal{P}}^*(\lambda)k+\hat{\mathcal{Q}}^*(\lambda)l=0$ and hence $f_A=l=k=0$. So $\mathfrak{h}=(A-\bar\lambda)\mathfrak{D}(A)\dot{+}\mathfrak{L}$ and $\bar\lambda\in\rho(A;\mathfrak{L})$. □

**2.** Suppose that $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ is a BVS for $A^*$, $M(\lambda)$ is the corresponding Weyl function, $\gamma(\lambda)$ is a $\gamma$-field of the extension $A_2\in\operatorname{Ex}_A$. A block operator-valued function $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ of the form

$$\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)=\begin{pmatrix}a_{11}(\lambda)&a_{12}(\lambda)\\a_{21}(\lambda)&a_{22}(\lambda)\end{pmatrix}:=\begin{pmatrix}M(\lambda)&\gamma^*(\bar\lambda)\restriction\mathfrak{L}\\P_{\mathfrak{L}}\gamma(\lambda)&P_{\mathfrak{L}}(A_2-\lambda)^{-1}\restriction\mathfrak{L}\end{pmatrix}\qquad(\lambda\in\rho(A_2)) \tag{8.13}$$

is the preresolvent matrix of the operator $A$, corresponding to the BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$.

The next proposition is well known.

**Proposition 8.2.** *Suppose that* $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ *is a BVS for* $A^*$, $\mathfrak{L}$ *is a subspace of* $\mathfrak{h}$. *Then the following hold:* (1) $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)\in R_{\mathcal{H}\oplus\mathfrak{L}}$ ; (2) $a_{12}(\lambda_0)^{-1}\in[\mathcal{H},\mathfrak{L}]\iff\lambda_0\in\rho(A;\mathfrak{L})$.

**Proof.** (1) It follows from (8.14), (1.7), (1.10) that

$$\frac{\mathfrak{A}(\lambda)-\mathfrak{A}^*(\lambda)}{\lambda-\bar\lambda}=\frac{1}{\lambda-\bar\lambda}\begin{pmatrix}M(\lambda)-M^*(\lambda)&[\gamma^*(\bar\lambda)-\gamma^*(\lambda)]\restriction\mathfrak{L}\\P_{\mathfrak{L}}(\gamma(\lambda)-\gamma(\bar\lambda))&(\lambda-\bar\lambda)P_{\mathfrak{L}}(A_2-\bar\lambda)^{-1}(A_2-\lambda)^{-1}\restriction\mathfrak{L}\end{pmatrix}$$

$$=\begin{pmatrix}\gamma^*(\lambda)\gamma(\lambda)&\gamma^*(\lambda)(A_2-\lambda)^{-1}\restriction\mathfrak{L}\\P_{\mathfrak{L}}(A_2-\lambda)^{-1}\gamma(\lambda)&P_{\mathfrak{L}}(A_2-\bar\lambda)^{-1}(A_2-\lambda)^{-1}\restriction\mathfrak{L}\end{pmatrix}=:T^*(\lambda)T(\lambda), \tag{8.14}$$

where $T(\lambda)=\{\gamma(\lambda),(A_2-\lambda)^{-1}\restriction\mathfrak{L}\}\in[\mathcal{H}\oplus\mathfrak{L},\mathfrak{h}]$. This implies $(\operatorname{Im}\mathfrak{A}(\lambda)h,h)=\operatorname{Im}\lambda\|T(\lambda)h\|^2\geq0$ $\forall\lambda\in\mathbb{C}_+\iff\mathfrak{A}(\lambda)\in R_{\mathcal{H}\oplus\mathfrak{L}}$.

(2) If $\lambda\in\rho(A;\mathfrak{L})$, then the operator $P_{\mathfrak{L}}\restriction\mathfrak{N}_{\bar\lambda}\in[\mathfrak{N}_{\bar\lambda},\mathfrak{L}]$ is invertible and, therefore, $a_{21}(\bar\lambda)=P_{\mathfrak{L}}\gamma(\bar\lambda)$ is an isomorphism from $\mathcal{H}$ onto $\mathfrak{L}$.

Conversely, assume that the operator $a_{12}(\lambda)$ is invertible. Then $P_{\mathfrak{L}}\mathfrak{N}_{\bar\lambda}=\mathfrak{L}$ and by virtue of Proposition 2.2 the following direct decomposition holds: $\mathfrak{h}=\mathfrak{L}\dot{+}\mathfrak{M}_\lambda$ i.e., $\lambda\in\rho(A;\mathfrak{L})$. □

**Remark 8.1.** The inclusion $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)\in R_{\mathcal{H}\oplus\mathfrak{L}}$ is a consequence of the Silvester identity, the relation $M(\lambda)\in R_{\mathcal{H}}$, and the following equalities:

$$\operatorname{Im}a_{22}-(a_{21}-a_{12}^*)(\operatorname{Im}a_{11})^{-1}(a_{12}-a_{21}^*)=P_{\mathfrak{L}}(A_2-\bar\lambda)^{-1}P_{\mathfrak{M}_{\bar\lambda}}(A_2-\lambda)^{-1}\restriction\mathfrak{L}, \tag{8.15}$$

$$(P_{\mathfrak{L}}(A_2-\bar{\lambda})^{-1}P_{\mathfrak{M}_{\bar{\lambda}}}(A_2-\lambda)^{-1}l,l)=\|P_{\mathfrak{M}_{\bar{\lambda}}}(A_2-\lambda)^{-1}l\|^2\geq 0. \tag{8.16}$$

**3.** If $\rho(A;\mathfrak{L})\neq\varnothing$, then the matrix

$$W_{\Pi\lambda}^{-}(\lambda):=\begin{pmatrix} a_{22}(\lambda)a_{12}(\lambda)^{-1} & a_{22}(\lambda)a_{12}(\lambda)^{-1}a_{11}(\lambda)-a_{21}(\lambda)\\ a_{12}(\lambda)^{-1} & a_{12}(\lambda)^{-1}a_{11}(\lambda)\end{pmatrix} \tag{8.17}$$

is defined and holomorphic on the set $\rho(A;\mathfrak{L})$. It is said to be the $\Pi\mathfrak{L}$-resolvent matrix of the operator $A$, corresponding to the BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$.

**Proposition 8.3.** *Let $W_{\Pi\mathfrak{L}}(\lambda)$ be a $\Pi\mathfrak{L}$-resolvent matrix of $A$, corresponding to the BVS $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$. Then*

$$\frac{W^*(\lambda)JW(\lambda)-J}{i(\lambda-\bar{\lambda})}\geq 0,\qquad \frac{W(\lambda)JW^*(\lambda)-J}{i(\lambda-\bar{\lambda})}\geq 0,\qquad J=\begin{pmatrix}\mathbb{O} & -iI\\ iI & \mathbb{O}\end{pmatrix}. \tag{8.18}$$

**Proof.** The following identities hold:

$$\begin{aligned} W(\lambda)&=\left[\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ \mathbb{O}&I\end{pmatrix}+\begin{pmatrix}\mathbb{O}&I\\ \mathbb{O}&\mathbb{O}\end{pmatrix}\mathfrak{A}(\lambda)\right]\left[\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ -I&\mathbb{O}\end{pmatrix}+\begin{pmatrix}I&\mathbb{O}\\ \mathbb{O}&\mathbb{O}\end{pmatrix}\mathfrak{A}(\lambda)\right]^{-1},\\ W(\lambda)&=\left[\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ -I&\mathbb{O}\end{pmatrix}+\mathfrak{A}(\lambda)\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ \mathbb{O}&I\end{pmatrix}\right]^{-1}\left[\mathfrak{A}(\lambda)\begin{pmatrix}\mathbb{O}&I\\ \mathbb{O}&\mathbb{O}\end{pmatrix}+\begin{pmatrix}I&\mathbb{O}\\ \mathbb{O}&\mathbb{O}\end{pmatrix}\right].\end{aligned} \tag{8.19}$$

Straightforward calculations give the equalities

$$\begin{aligned} J-W^*(\lambda)JW(\lambda)&=i(Y_1^{-1}(\lambda)^*(\mathfrak{A}^*(\lambda)-\mathfrak{A}(\lambda))Y_1(\lambda)^{-1},\\ J-W(\lambda)JW^*(\lambda)&=i(Y_2(\lambda)^{-1}(\mathfrak{A}^*(\lambda)-\mathfrak{A}(\lambda))(Y_2(\lambda)^{-1})^*,\end{aligned} \tag{8.20}$$

in which

$$Y_1(\lambda)=\left[\begin{pmatrix}I&\mathbb{O}\\ \mathbb{O}&\mathbb{O}\end{pmatrix}\mathfrak{A}(\lambda)+\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ -I&\mathbb{O}\end{pmatrix}\right]=\begin{pmatrix}a_{11}(\lambda)&a_{12}(\lambda)\\ -I&\mathbb{O}\end{pmatrix},$$

$$Y_1(\lambda)^{-1}=\begin{pmatrix}\mathbb{O}&-I\\ a_{12}^{-1}(\lambda)&a_{12}^{-1}(\lambda)a_{11}(\lambda)\end{pmatrix},$$

$$Y_2(\lambda)=\left[\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ -I&\mathbb{O}\end{pmatrix}+\mathfrak{A}(\lambda)\begin{pmatrix}\mathbb{O}&\mathbb{O}\\ \mathbb{O}&I\end{pmatrix}\right]=\begin{pmatrix}\mathbb{O}&a_{12}(\lambda)\\ -I&a_{22}(\lambda)\end{pmatrix},$$

$$Y_2(\lambda)^{-1}=\begin{pmatrix}a_{22}(\lambda)a_{12}^{-1}(\lambda)&-I\\ a_{12}^{-1}(\lambda)&\mathbb{O}\end{pmatrix}.$$

The invertibility of the operators $Y_1(\lambda)$ and $Y_2(\lambda)$ for all $\lambda\in\rho(A;\mathfrak{L})$ is a consequence of the evident equivalences:

$$0\in\rho(Y_i(\lambda))\quad(i=1,2)\iff 0\in\rho(a_{12}(\lambda))\iff\lambda\in\rho(A;\mathfrak{L}).$$

Relations (8.18) are implied by (8.20) and Proposition 8.2. □

**Theorem 8.1.** *Suppose that $A$ is a Hermitian operator $\overline{\mathfrak{D}(A)}=\mathfrak{h}_0\subseteq\mathfrak{h}$, $\mathfrak{L}$ is a subspace of $\mathfrak{h}$, $\rho(A;\mathfrak{L})\neq\varnothing$, $\Pi=\{\mathcal{H},\Gamma_1,\Gamma_2\}$ is a BVS for $A^*$*

$$G(\lambda)=\begin{pmatrix}-\mathcal{Q}(\lambda)\\ \mathcal{P}(\lambda)\end{pmatrix}\in[\mathfrak{h},\mathfrak{L}\oplus\mathfrak{L}],\qquad \Gamma=\begin{pmatrix}\Gamma_2\\ \Gamma_1\end{pmatrix},\qquad J=i\begin{pmatrix}\mathbb{O}&-I\\ I&\mathbb{O}\end{pmatrix}. \tag{8.21}$$

*Then the operator-valued function*

$$\tilde{W}_{\Pi\mathfrak{L}}(\lambda):=(\Gamma\hat{G}^*(\lambda))^*=\begin{pmatrix}-\Gamma_2\hat{\mathcal{Q}}^*(\lambda)&\Gamma_2\hat{\mathcal{P}}^*(\lambda)\\ -\Gamma_1\hat{\mathcal{Q}}^*(\lambda)&\Gamma_1\hat{\mathcal{P}}^*(\lambda)\end{pmatrix}^*,\qquad (\hat{G}^*(\lambda)=(-\hat{\mathcal{Q}}^*(\lambda),\hat{\mathcal{P}}^*(\lambda)) \tag{8.22}$$

*satisfies the identity*

$$\tilde{W}_{\Pi\mathfrak{L}}(\lambda)J\tilde{W}_{\Pi\mathfrak{L}}(\mu) = J + i(\lambda - \bar{\mu})G(\lambda)G^*(\mu) \qquad (\lambda, \mu \in \rho(A;\mathfrak{L}). \tag{8.23}$$

*If, additionally, $\rho_s(A;\mathfrak{L}) \neq \varnothing$, then the matrix $\tilde{W}_{\Pi\mathfrak{L}}(\lambda) = (\Gamma\hat{G}^*(\lambda))^*$ coincides with the $\Pi\mathfrak{L}$-resolvent matrix $W_{\Pi\mathfrak{L}}(\lambda)$:*

$$\tilde{W}_{\Pi\mathfrak{L}}(\lambda) = W_{\Pi\mathfrak{L}}(\lambda) \quad \textit{for all} \quad \lambda \in \rho(A;\mathfrak{L}).$$

**Proof.** (1) Let $l = \{l_1, l_2\}$, $k = \{k_1, k_2\} \in \mathfrak{L}\oplus\mathfrak{L}$, $\hat{f} = \{f, f'\} = \hat{G}^*(\mu)l$, $\hat{g} = \{g, g'\} = \hat{G}^*(\lambda)k$. Then it follows from (8.4)–(8.7) that

$$(f', g) - (f, g') = (\bar{\mu}f - l_1, g) - (f, \bar{\lambda}g - k_1) = (\bar{\mu} - \lambda)(f, g) + (f, k_1) - (l_1, g) = (\bar{\mu} - \lambda)(f, g) - (l_1, k_2) + (l_2, k_1)$$

$$= i(Jl, k) + (\bar{\mu} - \lambda)(G^*(\mu)l, G^*(\lambda)k). \tag{8.24}$$

On the other hand, the Green identity (1.2) and (8.22) yield

$$(f', g) - (f, g') = (\Gamma_1\hat{f}, \Gamma_2\hat{g}) - (\Gamma_2\hat{f}, \Gamma_1\hat{g}) = i(J\Gamma\hat{f}, \Gamma\hat{g}) = i(J\tilde{W}^*_{\Pi\mathfrak{L}}(\mu)l, \tilde{W}^*_{\Pi\mathfrak{L}}(\lambda)k). \tag{8.25}$$

After the comparison of (8.24) and (8.25) we obtain (8.23).

(2) Let $\rho_s(A;\mathfrak{L}) \neq \varnothing$ and $\lambda \in \rho_s(A, \mathfrak{L})$. Show that $0 \in \rho(\tilde{W}_{\Pi\mathfrak{L}}(\lambda))$. Indeed if $l \in \ker \tilde{W}_{\Pi\mathfrak{L}}(\lambda)$, then we have

$$-\hat{\mathcal{Q}}^*(\lambda)l_1 + \hat{\mathcal{P}}^*(\lambda)l_2 \in \ker\Gamma_1 \cap \ker\Gamma_2 = A.$$

Taking account of (8.8) we obtain $\hat{\mathcal{Q}}^*(\lambda)l_1 = \hat{\mathcal{P}}^*(\lambda)l_2 = 0$ and, by virtue of (8.5), (8.6), $l_1 = l_2 = 0$. Moreover, it follows from (8.8) that $\mathfrak{R}(\tilde{W}^*_{\Pi\mathfrak{L}}(\lambda)) = \Gamma A^* = \mathfrak{L}\oplus\mathfrak{L}$ and, therefore, $0 \in \rho(\tilde{W}^*_{\Pi\mathfrak{L}}(\lambda)) \iff 0 \in \rho(\tilde{W}_{\Pi\mathfrak{L}}(\lambda))$. Now (8.23) implies (for $\mu = \bar{\lambda}$)

$$\tilde{W}_{\Pi\mathfrak{L}}(\lambda)J\tilde{W}^*_{\Pi\mathfrak{L}}(\bar{\lambda}) = J, \qquad \tilde{W}^*_{\Pi\mathfrak{L}}(\bar{\lambda})J\tilde{W}_{\Pi\mathfrak{L}}(\lambda) = J. \tag{8.26}$$

The latter equality is a consequence of the former and the condition $0 \in \rho(\tilde{W}_{\Pi\mathfrak{L}}(\lambda))$.

(3) Setting for all $\lambda \in \rho(A;\mathfrak{L})$

$$\begin{aligned} w^*_{11}(\lambda) &= -\Gamma_2\hat{\mathcal{Q}}^*(\lambda), \quad w^*_{21}(\lambda) = \Gamma_2\hat{\mathcal{P}}^*(\lambda), \\ w^*_{12}(\lambda) &= -\Gamma_1\hat{\mathcal{Q}}^*(\lambda), \quad w^*_{22}(\lambda) = \Gamma_1\hat{\mathcal{P}}^*(\lambda) \end{aligned} \tag{8.27}$$

we obtain explicit formulas for the components of the matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ by means of $w^*_{ij}(\lambda)$:

$$a_{11}(\lambda) = M(\lambda) = w^*_{22}(\bar{\lambda})w^*_{21}(\bar{\lambda})^{-1}, \tag{8.28}$$

since $0 \in \rho(w_{21}(\lambda))$ for all $\lambda \in \rho(A;\mathfrak{L})$. Further, it follows from (8.6) and (8.27) that for all $l \in \mathfrak{L}$

$$P_{\mathfrak{L}}\gamma(\lambda)w^*_{21}(\bar{\lambda})l = P_{\mathfrak{L}}\gamma(\lambda)\Gamma_2\hat{\mathcal{P}}^*(\bar{\lambda})l = P_{\mathfrak{L}}\mathcal{P}^*(\bar{\lambda})l = l$$

$$\Longrightarrow a_{21}(\lambda) = P_{\mathfrak{L}}\gamma(\lambda) = w^*_{21}(\bar{\lambda})^{-1}, \qquad a_{12}(\lambda) = a^*_{21}(\bar{\lambda}) = w_{21}(\lambda)^{-1}. \tag{8.29}$$

To find the expression of $a_{22}(\lambda) = P_{\mathfrak{L}}(A_2 - \lambda)^{-1} \upharpoonright \mathfrak{L}$ we consider the problem

$$(g, l) \in (A^* - \lambda), \qquad \Gamma_2\hat{g} = \Gamma_2\{g, l + \lambda g\} = 0 \qquad (\hat{g} = \{g, l + \lambda g\}). \tag{8.30}$$

It follows from (8.8) that this problem has a solution of the form $g = \mathcal{Q}^*(\bar{\lambda})k_1 - \mathcal{P}^*(\bar{\lambda})k_2$ with $k_1, k_2 \in \mathfrak{L}$. Taking account of (8.27), (8.4), and (8.5) we obtain

$$\Gamma_2\hat{g} = \Gamma_2\hat{\mathcal{Q}}^*(\bar{\lambda})k_1 - \Gamma_2\hat{\mathcal{P}}^*(\bar{\lambda})k_2 = -w^*_{11}(\bar{\lambda})k_1 - w^*_{21}(\bar{\lambda})k_2 = 0,$$

and hence $k_2 = (w_{21}^*(\bar\lambda))^{-1} w_{11}^*(\bar\lambda) k_1$. Making use of the relation $\{g, k_1\} = \hat{\mathcal{Q}}^*(\bar\lambda)k_1 - \hat{\mathcal{P}}^*(\bar\lambda)k_2 \in A^* - \lambda$ and setting $k_1 = l$ we have

$$g = [\mathcal{Q}^*(\bar\lambda) + \mathcal{P}^*(\bar\lambda) w_{21}^*(\bar\lambda)^{-1} w_{11}^*(\bar\lambda)] l = (A_2 - \lambda)^{-1} l, \tag{8.31}$$

$$a_{22}(\lambda) = P_{\mathfrak{L}}[\mathcal{Q}^*(\bar\lambda) + \mathcal{P}^*(\bar\lambda) w_{21}^*(\bar\lambda)^{-1} w_{11}^*(\bar\lambda)] = w_{21}^*(\lambda)^{-1} w_{11}^*(\bar\lambda). \tag{8.32}$$

Thus, the preresolvent matrix has the form (for $\lambda \in \rho_s(A; \mathfrak{L})$)

$$\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} w_{22}^*(\bar\lambda) & w_{21}^*(\bar\lambda)^{-1} w_{21}(\lambda)^{-1} \\ w_{21}^*(\bar\lambda)^{-1} & w_{21}^*(\bar\lambda)^{-1} w_{11}^*(\bar\lambda) \end{pmatrix} \tag{8.33}$$

(4) Now it follows from (8.26) and (8.29)–(8.32) that for all $\lambda \in \rho_s(A; \mathfrak{L})$

$$\begin{gathered} a_{22}(\lambda) a_{12}(\lambda)^{-1} = w_{21}^*(\bar\lambda)^{-1} w_{11}^*(\bar\lambda) w_{21}(\lambda) = w_{11}(\lambda), \qquad a_{12}(\lambda)^{-1} = w_{21}(\lambda), \\ a_{22}(\lambda) a_{12}(\lambda)^{-1} a_{11}(\lambda) - a_{21}(\lambda) = w_{11}(\lambda) w_{22}^*(\bar\lambda) w_{21}^*(\bar\lambda)^{-1} \end{gathered} \tag{8.34}$$

$$-w_{21}^*(\bar\lambda)^{-1} = w_{12}(\lambda) w_{21}^*(\bar\lambda) w_{21}^*(\bar\lambda)^{-1} = w_{12}(\lambda), \tag{8.35}$$

$$a_{12}(\lambda)^{-1} a_{11}(\lambda) = w_{21}(\lambda) w_{22}^*(\bar\lambda) w_{21}^*(\bar\lambda)^{-1} = w_{22}(\lambda). \tag{8.36}$$

These equalities yield that $\tilde{W}_{\Pi\mathfrak{L}}(\lambda) = W_{\Pi\mathfrak{L}}(\lambda)$ for all $\lambda \in \rho_s(A; \mathfrak{L})$. □

**Remark 8.2.** Identity (8.23) means that the operator-valued function $\tilde{W}_{\Pi\mathfrak{L}}(\lambda)$ given by (8.22) is an $\mathfrak{L}$-resolvent matrix of the operator $A$ in the sense of [42, 43]. Hence Theorem 8.1, which was proved by the authors in [21, 79] for the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, shows how to calculate the $\Pi\mathfrak{L}$-resolvent matrix of $A$ on the one hand, and establishes the equivalence of the definitions in [42] and (8.17) on the other hand.

**Remark 8.3.** Every $\mathfrak{L}$-resolvent matrix $W(\lambda)$ of $A$ in the sense of [42] (i.e., which satisfies (8.23)) can be expressed in the form (8.22) for a suitable choice of the BVS. Indeed $W(\lambda)$ may be connected with some $\Pi_1\mathfrak{L}$-resolvent matrix $W_1(\lambda)$ corresponding to the BVS $\Pi_1 = \{\mathfrak{L}, \Gamma_1^1, \Gamma_2^1\}$ by the formula $W(\lambda) = W_1(\lambda)U$, where $U$ is a $J$-unitary operator in $\mathfrak{L} \oplus \mathfrak{L}$. Setting $\{\Gamma_2, \Gamma_1\} = U^*\{\Gamma_2^1, \Gamma_1^1\}$ we obtain from (8.22) for the BVS $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$

$$W_{\Pi\mathfrak{L}}^*(\lambda) = \Gamma\hat{G}^*(\lambda) = U^*\Gamma^1\hat{G}^*(\bar\lambda) = U^* W_{\Pi_1\mathfrak{L}}^*(\lambda)$$

and, therefore, $W_{\Pi\mathfrak{L}}(\lambda) = W_{\Pi_1\mathfrak{L}}(\lambda)U = W(\lambda)$.

**4.** Now we obtain the explicit form of the BVS $\Pi$ for the latter equality to hold.

**Proposition 8.4.** *Suppose that $A$ is a Hermitian operator in $\mathfrak{h}$, $\mathfrak{L}$ is a subspace of $\mathfrak{h}$ such that $\rho_s(A; \mathfrak{L}) \neq \varnothing$, $\pi_k$ is an orthogonal projection onto the $k$th component of $\mathfrak{L} \oplus \mathfrak{L}$ $(k = 1, 2)$. If $W(\lambda)$ is an operator-valued function with values in $[\mathfrak{L} \oplus \mathfrak{L}]$ such that (8.23) holds and $0 \in \rho(W(\mu))$ for some $\mu \in \rho_s(A; \mathfrak{L})$, then the triple $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ of the form*

$$\begin{gathered} \Gamma_1 \hat{f} = \pi_2 W^*(\mu) l, \qquad \Gamma_2 \hat{f} = \pi_1 W^*(\mu) l, \\ \forall \hat{f} = \hat{f}_A + \hat{G}^*(\mu) l, \qquad \hat{f}_A \in A, \qquad l = \operatorname{col}(l_1, l_2) \in \mathfrak{L} \oplus \mathfrak{L} \end{gathered} \tag{8.37}$$

*is a BVS for $A^*$ and $W_{\Pi\mathfrak{L}}(\lambda) = W(\lambda)$ for all $\lambda \in \rho_s(A; \mathfrak{L})$. Equalities (8.23), (8.37) take the following more simple forms if $\mu = \bar\mu = a$ and $W(a) = I$:*

$$W(\lambda) = I + i(\lambda - a) G(\lambda) G^*(a) J, \tag{8.38}$$

$$\Gamma_2 \hat{f} = -\mathcal{P}(a)(f' - af), \qquad \Gamma_1 \hat{f} = P_{\mathfrak{L}} f - \mathcal{Q}(a)(f' - af), \qquad \hat{f} = \{f, f'\} \in A^*. \tag{8.39}$$

**Proof.** Let $\hat{f} = \hat{f}_A + \hat{G}^*(\mu) l$, $\hat{g} = \hat{g}_A + \hat{G}^*(\mu) k \in A^*$, $\hat{g}_A = \{g_A, A g_A\}$, $k = \operatorname{col}(k_1, k_2) \in \mathfrak{L} \oplus \mathfrak{L}$, and $\lambda = \mu \in \rho_s(A; \mathfrak{L})$. Then (8.23) implies

$$(f', g) - (f, g') = i(Jl, k) + (\bar\mu - \mu)(G^*(\mu) l, G^*(\mu) k) = i([J + i(\mu - \bar\mu) G(\mu) G^*(\mu)] l, k) = i(W(\mu) J W^*(\mu) l, k)$$

$$= i(J W^*(\mu) l, W^*(\mu) k) = (\pi_2 W^*(\mu) l, \pi_1 W^*(\mu) k)_{\mathfrak{L}} - (\pi_1 W^*(\mu) l, \pi_2 W^*(\mu) k)_{\mathfrak{L}} = (\Gamma_1 \hat{f}, \Gamma_2 \hat{g})_{\mathfrak{L}} - (\Gamma_2 \hat{f}, \Gamma_1 \hat{g})_{\mathfrak{L}},$$

which proves the Green formula (1.3). Making use of (8.8) and the condition $0 \in \rho(W^*(\mu))$ we conclude that the mapping $\Gamma = \{\Gamma_2, \Gamma_1\}$: $A^* \to \mathfrak{L} \oplus \mathfrak{L}$ is surjective since $\Gamma A^* = \Gamma \hat{G}^*(\mu)(\mathfrak{L} \oplus \mathfrak{L}) = W^*(\mu)(\mathfrak{L} \oplus \mathfrak{L})$. Thus the triple $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$. The equality $W_{\Pi\mathfrak{L}}(\mu) = W(\mu)$ is implied by (8.22). Taking account of Remark 8.3 we obtain the equality $W_{\Pi\mathfrak{L}}(\lambda) = W(\lambda)$ for all $\lambda \in \rho_s(A; \mathfrak{L})$.

In the case $\mu = a$ and $W(a) = I$ one can easily deduce from (8.37), (8.3) the equalities

$$\Gamma_1 \hat{f} = \pi_2 W(a) l = \pi_2 \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = l_2 = P_{\mathfrak{L}} f - \mathcal{Q}(a)(f' - af),$$

$$\Gamma_2 \hat{f} = \pi_1 W(a) l = \pi_1 l = l_1 = -\mathcal{P}(a)(f' - af) \qquad \forall \hat{f} = \{f, f'\} \in A^*,$$

which coincide with (8.39). □

The next corollary is an analog of the second Neumann formula.

**Corollary 8.1.** *Suppose that $\mu \in \rho_s(A; \mathfrak{L}) \neq \varnothing$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, and $W_{\Pi\mathfrak{L}}(\lambda)$ is the corresponding $\Pi\mathfrak{L}$-resolvent matrix of the form in* (8.22). *Then every extension $\tilde{A} = \tilde{A}^* \in \operatorname{Ex}_A$ takes the form*

$$\tilde{A} = A \dotplus \hat{G}^*(\mu) W^*(\mu)^{-1} \theta =: A \dotplus \hat{G}^*(\mu) \mathfrak{L}(\theta), \tag{8.40}$$

*where $\theta = \theta^* \in \tilde{\mathcal{C}}(\mathfrak{L})$ and $\mathfrak{L}(\theta) := W^*(\mu)^{-1}\theta = \{l \in \mathfrak{L} \oplus \mathfrak{L}\colon W^*(\mu) l \in \theta\}$. Relation* (8.40) *takes the more simple form $\tilde{A} = A \dotplus \hat{G}^*(a)\theta$ if $\mu = a = \bar{a} \in \rho_s(A; \mathfrak{L})$ and $W(a) = I$.*

**Proof.** By virtue of (1.4) the mapping $\Gamma = \{\Gamma_2, \Gamma_1\}$ defines a bijective correspondence (cf. [53]):

$$\tilde{A} = (\tilde{A})^* \in \operatorname{Ex}_A \Longleftrightarrow \theta := \Gamma \tilde{A} \in \tilde{\mathcal{C}}(\mathcal{H}), \tag{8.41}$$

On the other hand, in view of formula (8.8), for all $\hat{f} \in A^*$ there exists $l \in \mathfrak{L} \oplus \mathfrak{L}$ such that $\hat{f} = \hat{f}_A + \hat{G}^*(\mu) l$. In accordance with Theorem 8.1, $\Gamma \hat{f} = W^*(\mu) l$ and the equivalence (8.41) taking the form

$$\hat{f} = \hat{f}_A + \hat{G}^*(\mu) l \in \tilde{A} \Longleftrightarrow \Gamma \hat{f} = W^*(\mu) l \in \theta \Longleftrightarrow l \in W^*(\mu)^{-1}\theta$$

proves relation (8.40). □

**Remark 8.4.** (a) The triple (8.37) is a generalized BVS of the linear relation

$$A_* = A + \hat{\mathcal{P}}^*(\lambda)\mathfrak{L} + \hat{\mathcal{Q}}^*(\lambda)\mathfrak{L} (\subset A^*)$$

if, under the assumptions of Proposition 8.4, $0 \in \sigma_c(W(\lambda))$.

(b) Corollary 8.1 holds true also for maximally dissipative extensions $\tilde{A} \in \operatorname{Ex}_A$.

**5.** In the next theorem we show that a preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ of an operator $A$ with finite defect numbers is a Weyl function of the operator $A \restriction \mathfrak{L}^\perp$ and find the criterion for this to be true in the case $\dim \mathfrak{L} = n_\pm(A) = \infty$.

**Theorem 8.2.** *Suppose that $A$ is a Hermitian operator in $\mathfrak{h}$, $\mathfrak{h}_0 = \overline{\mathfrak{D}(A)}$, $\mathfrak{L}$ is a subspace of $\mathfrak{h}$ such that $\mathfrak{L} \cap \mathfrak{N} = \{0\}$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ and $W_{\Pi\mathfrak{L}}(\lambda)$ are the corresponding preresolvent and resolvent matrices. Then the following assertions are equivalent:* (1) $P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}$; (2) *the linear relation $A^*$ is $\mathfrak{L}$-regular;* (3) *the domain of the operator $A_0 = A \restriction \mathfrak{L}^\perp = A \cap (\mathfrak{L}^\perp \oplus \mathfrak{h})$ is dense in $(\mathfrak{N} \dotplus \mathfrak{L})^\perp$ and $A_0^* = A^* \dotplus \hat{\mathfrak{L}} = \{\{f, f' + l\} : \{f, f'\} \in A^*, l \in \mathfrak{L}\}$;* (4) *the operator $A$ is $\mathfrak{L}$-regular, and linear relations $T : A \dotplus \hat{\mathfrak{L}}$ and $T^* = A^* \restriction \mathfrak{L}^\perp = A^* \cap (\mathfrak{L}^\perp \oplus \mathfrak{h})$ are transversal;*

(5) $0 \in \rho(\mathcal{Q}(\lambda)\mathcal{Q}^*(\lambda))$ $\forall \lambda \in \rho(A; \mathfrak{L})$;

(6) $0 \in \rho(J - W^*_{\Pi\mathfrak{L}}(\lambda) J W_{\Pi\mathfrak{L}}(\lambda))$ $\forall \lambda \in \rho(A; \mathfrak{L})$;

(7) $0 \in \rho(\operatorname{Im} \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda))$ $\forall \lambda \in \rho(A; \mathfrak{L})$.

*In this case the triple $\Pi_1 = \{\mathcal{H} \oplus \mathfrak{L}, \Gamma_1', \Gamma_2'\}$, in which*

$$\Gamma_1' \hat{f} = \{\Gamma_1\{f, f'\}, P_{\mathfrak{L}} f\}, \qquad \Gamma_2' \hat{f} = \{\Gamma_2\{f, f'\}, -l\} \qquad (\hat{f} = \{f, f' + l\} \in A_0^*), \tag{8.42}$$

*is a BVS for the linear relation $A_0^*$, and the $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ coincides with the Weyl function of the operator $A_0$, corresponding to the BVS $\Pi_1$.*

**Proof.** (1) $\iff$ (2). The relation $\mathfrak{L} \cap \mathfrak{N} = \{0\}$ implies that $P_{\mathfrak{L}}\overline{\mathfrak{D}(A)} = \mathfrak{L}$. Therefore the equivalence of statement (1) and (2) follows from Proposition 2.5.

(2) $\iff$ (3). By virtue of the relation $A_0^* = [A \cap (\mathfrak{L}^\perp \oplus \mathfrak{h})]^* = \overline{A^* \dotplus \hat{\mathfrak{L}}}$, the linear manifold $A^* \dotplus \hat{\mathfrak{L}}$ is closed if and only if $A_0^* = A^* \dotplus \hat{\mathfrak{L}}$. In this case $\mathfrak{L} \dotplus \mathfrak{N} = (A^* \dotplus \hat{\mathfrak{L}})(0)$ is a closed subspace and, therefore, $\mathfrak{D}(A_0)^\perp = \mathfrak{L} \dotplus \mathfrak{N}$, i.e., $\overline{\mathfrak{D}(A_0)} = (\mathfrak{L} \dotplus \mathfrak{N})^\perp$.

(1) $\iff$ (4). Let $P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}$. Then $P_{\mathfrak{L}}\mathfrak{D}(A^*) = \mathfrak{L}$ and, owing to Proposition 2.5, the operator $A$ is $\mathfrak{L}$-regular. Hence the linear manifold $A \dotplus \hat{\mathfrak{L}}$ is closed and the following equalities hold:

$$T^* = (A \dotplus \hat{\mathfrak{L}})^* = j(A \dotplus \hat{\mathfrak{L}})^\perp = j(A^\perp \cap \hat{\mathfrak{L}}^\perp) = A^* \cap (\mathfrak{L}^\perp \oplus \mathfrak{h}) = A^* \restriction \mathfrak{L}^\perp,$$

where $j\{h_1, h_2\} = \{h_2, -h_1\}$. The disjointness of the extensions $T$, $T^* \in \operatorname{Ex}_{A_0}$ is implied by the condition $\mathfrak{N} \cap \mathfrak{L} = \{0\}$. Indeed, let $\hat{f} = \{f, f' + l\} \in T = A \dotplus \hat{\mathfrak{L}}^\perp(\{f, f'\} \in A, l \in \mathfrak{L})$ and $\hat{f} \in T^* = A^* \restriction \mathfrak{L}^\perp$. Then $\{0, l\} \in A^*$ and hence $l \in \mathfrak{N}$. Therefore $l = 0$ since $\mathfrak{L} \cap \mathfrak{N} = \{0\}$ and

$$\hat{f} = \{f, f'\} \in A^* \restriction \mathfrak{L}^\perp \Longrightarrow f \in \mathfrak{L}^\perp \Longrightarrow \hat{f} \in A \restriction \mathfrak{L}^\perp = A_0.$$

Now it remains to prove the inclusion $T + T^* \supset A_0^*$. Let $\{f, f' + l\} \in A^* \dotplus \hat{\mathfrak{L}} = A_0^*$, where $\hat{f} = \{f, f'\} \in A^*$, $l \in \mathfrak{L}$. In view of the relation $P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}$ there exists $\hat{g} = \{g, g'\} \in A$ such that $P_{\mathfrak{L}} f = P_{\mathfrak{L}} g$. Then we have

$$\{f, f' + l\} = \{g, g' + l\} + \{f - g, f' - g'\} \in T + T^*$$

since $\{g, g' + l\} \in T := A \dotplus \hat{\mathfrak{L}}$, $\{f - g, f' - g'\} \in A \cap (\mathfrak{L}^\perp \oplus \mathfrak{h}) = T^*$.

(4) $\iff$ (1). Assume that $A$ is an $\mathfrak{L}$-regular operator and $T$, $T^*$ are transversal extensions of $A_0$. Then we have

$$A^* = T^* + T = (A^* \cap (\mathfrak{L}^\perp \oplus \mathfrak{h})) + (A \dotplus \hat{\mathfrak{L}}) \tag{8.43}$$

and, therefore, $\mathfrak{D}(A^*) = (\mathfrak{D}(A^*) \cap \mathfrak{L}^\perp) \dotplus \mathfrak{D}(A)$. This implies the equality $P_{\mathfrak{L}}\mathfrak{D}(A^*) = P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}$ and according to Proposition 2.5 the linear relation $A^*$ is $\mathfrak{L}$-regular.

(1) $\iff$ (5). It follows from the definition of $Q(\lambda)$ that

$$\mathfrak{R}(Q(\lambda)) = P_{\mathfrak{L}}(A - \lambda)^{-1}(I - \mathcal{P}(\lambda))\mathfrak{h} = P_{\mathfrak{L}}(A_{-\lambda})^{-1}\mathfrak{M}_\lambda = P_{\mathfrak{L}}\mathfrak{D}(A). \tag{8.44}$$

This yields that statement (1) is equivalent to the condition $\mathfrak{R}(Q(\lambda)) = \mathfrak{L}$ [for all $\lambda \in \rho(A; \mathfrak{L})$] which, in turn, is equivalent to statement (5).

(1) $\iff$ (6). The stated equivalence is a consequence of identity (8.21) and the equivalence (1) $\iff$ (5).

(1) $\iff$ (7). This equivalence is implied by the equivalence (1) $\iff$ (6) and identity (8.19).

We show now that the triple $\Pi_1 = \{\mathcal{H} \oplus \mathfrak{L}, \Gamma_1', \Gamma_2'\}$ with $\Gamma_1'$, $\Gamma_2'$ defined by (8.42) forms a BVS of the linear relation $A^*$. Indeed, with regard to (8.43), the mapping $\Gamma' = \{\Gamma_2', \Gamma_1'\}$ is surjective since

$$\Gamma'(A^* \cap (\mathfrak{L}^\perp \oplus \mathfrak{h})) = (\mathcal{H} \oplus \{0\}) \oplus (\mathcal{H} \oplus \{0\}), \qquad \Gamma'(A \dotplus \hat{\mathfrak{L}}) = (\{0\} \oplus \mathfrak{L}) \oplus (\{0\} \oplus \mathfrak{L}).$$

The proof of the Green identity (1.3) is straightforward. Let $\hat{f} = \{f, f' + l\}$, $\hat{g} = \{g, g' + k\} \in A_0^*(\{f, f'\}, \{g, g'\} \in A^*; l, k \in \mathfrak{L})$. Then we have

$$\begin{aligned}(f' + l, g) - (f, g' + k) &= (f', g) - (f, g') + (f, -k) - (-l, g)\\ &= (\Gamma_1\{f, f'\}, \Gamma_2\{g, g'\})_{\mathcal{H}} - (\Gamma_2\{f, f'\}, \Gamma_1\{g, g'\})_{\mathcal{H}} + (P_{\mathfrak{L}} f, -k)_{\mathfrak{L}} - (-l, P_{\mathfrak{L}} g)_{\mathfrak{L}}\\ &= (\Gamma_1'\hat{f}, \Gamma_2'\hat{g})_{\mathcal{H}\oplus\mathfrak{L}} - (\Gamma_2'\hat{f}, \Gamma_1'\hat{g})_{\mathcal{H}\oplus\mathfrak{L}}.\end{aligned}$$

Consider the linear manifold

$$\mathfrak{N}_\lambda^*(A_0) := \mathfrak{N}_\lambda(A) \dotplus (A_2 - \lambda)^{-1}\mathfrak{L} \subset \mathfrak{N}_\lambda(A_0). \tag{8.45}$$

By virtue of the condition $\mathfrak{L} \cap \mathfrak{N} = \{0\}$ this is a direct sum and $\overline{\mathfrak{N}_\lambda^*(A_0)} = \mathfrak{N}_\lambda(A_0)$. It follows from (1.6), (1.8), (1.9) that for all

$$\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} = \{f_\lambda, f' - l\} \in \hat{\mathfrak{N}}_\lambda^*(A_0) \qquad (f_\lambda = \gamma(\lambda)h + (A_2 - \lambda)^{-1}l, f' = \lambda f_\lambda + l)$$

we have

$$\Gamma_2' \hat{f}_\lambda = \{\Gamma_2\{\gamma(\lambda)h + (A_2 - \lambda)^{-1}l, \lambda\gamma(\lambda)h + l + \lambda(A_2 - \lambda)^{-1}l\}, l\} = \{h, l\}, \tag{8.46}$$

$$\begin{aligned}\Gamma_1' \hat{f}_\lambda &= \{\Gamma_1\hat{\gamma}(\lambda)h + \Gamma_1\{(A_2 - \lambda)^{-1}l, l + \lambda(A_2 - \lambda)^{-1}l\}, \quad P_{\mathfrak{L}}\gamma(\lambda)h + P_{\mathfrak{L}}(A_2 - \lambda)^{-1}l\} \\ &= \{M(\lambda)h + \gamma^*(\bar{\lambda})l, P_{\mathfrak{L}}\gamma(\lambda)h + P_{\mathfrak{L}}(A_2 - \lambda)^{-1}l\} = \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)\Gamma_2'\hat{f}_\lambda.\end{aligned} \tag{8.47}$$

By virtue of (8.46) $\Gamma_2'\hat{\mathfrak{N}}_\lambda^* = \mathcal{H} \oplus \mathfrak{L}$ and the inclusion in (8.45) may be replaced by the equality $\mathfrak{N}_\lambda^*(A_0) = \mathfrak{N}_\lambda(A_0)$. Now it follows from (8.47) that $M(\lambda) = \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$. □

Here we provide another proof of the latter statement.

**Proposition 8.5.** *Let, under the assumptions of Theorem* 8.2, *one of the equivalent conditions* (1)–(7) *be fulfilled. Then the Weyl function $M(\lambda)$ corresponding to a BVS $\Pi_1$ of the form in* (8.42) *coincides with the $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$.*

The proof is based on the analog of the Neumann formula (8.8) and, therefore, may be applied only if $\rho_s(A;\mathfrak{L}) \neq \varnothing$. Setting $f_\lambda = G^*(\bar{\lambda})l$, $l = \text{col}\,(l_1, l_2) \in \mathfrak{L} \oplus \mathfrak{L}$ and making use of the relation

$$\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} = \hat{G}^*(\bar{\lambda})l + \{0, l_1\} \in \hat{\mathfrak{N}}_\lambda(A_0) \subset A_0^* = A^* \dotplus \hat{\mathfrak{L}}, \tag{8.48}$$

we obtain from (8.48), (8.42), and (8.27):

$$\Gamma_1'\hat{f}_\lambda = \begin{pmatrix} \Gamma_1(-\hat{\mathcal{Q}}^*(\bar{\lambda})l_1 + \hat{\mathcal{P}}^*(\bar{\lambda})l_2) \\ l_2 \end{pmatrix} = \begin{pmatrix} w_{12}^*(\bar{\lambda}) & w_{22}^*(\bar{\lambda}) \\ \mathbb{O} & \mathbb{I} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix};$$

$$\Gamma_2'\hat{f}_\lambda = \begin{pmatrix} \Gamma_2(-\hat{\mathcal{Q}}^*(\bar{\lambda})l_1 + \hat{\mathcal{P}}^*(\bar{\lambda})l_2) \\ -l_1 \end{pmatrix} = \begin{pmatrix} w_{11}^*(\bar{\lambda}) & w_{21}^*(\bar{\lambda}) \\ -\mathbb{I} & \mathbb{O} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.$$

By virtue of the equality $w_{22}^*(\bar{\lambda})w_{21}^*(\bar{\lambda})^{-1}w_{11}^*(\bar{\lambda}) - w_{12}^*(\bar{\lambda}) = w_{21}(\lambda)^{-1}$ we obtain the following expression for the Weyl function:

$$\begin{aligned} M(\lambda) &= \begin{pmatrix} w_{12}^*(\bar{\lambda}) & w_{22}^*(\bar{\lambda}) \\ \mathbb{O} & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{O} & -\mathbb{I} \\ w_{21}^*(\bar{\lambda})^{-1} & w_{21}^*(\bar{\lambda})^{-1}w_{11}^*(\bar{\lambda}) \end{pmatrix} \\ &= \begin{pmatrix} w_{22}^*(\bar{\lambda})w_{21}^*(\bar{\lambda})^{-1} & w_{21}(\lambda)^{-1} \\ w_{21}^*(\bar{\lambda})^{-1} & w_{22}^*(\bar{\lambda})^{-1}w_{11}^*(\bar{\lambda}) \end{pmatrix}. \end{aligned} \tag{8.49}$$

After the comparison of (8.49) and (8.33) we obtain the desired equality $M(\lambda) = \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$. □

In the next proposition we give a direct proof of the $\mathfrak{L}$-regularity criterion of the linear relation $A^*$.

**Proposition 8.6.** *Under the assumptions of Theorem* 8.2 *the following assertions are equivalent:*

(1) *the linear relation $A^*$ is $\mathfrak{L}$-regular;*

(2) $0 \in \rho(\text{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda))$;

(3) *there exists $\varepsilon > 0$ such that for all $h \in \mathcal{H}$, $l \in \mathfrak{L}$, the following inequality of the acute angle holds:*

$$\|\gamma(\lambda)h + (A_2 - \lambda)^{-1}l\| \geq \varepsilon(\|f\| + \|l\|); \tag{8.50}$$

(4) $\mathfrak{N}_\lambda(A_0) = \mathfrak{N}_\lambda(A) \dotplus (A_2 - \lambda)^{-1}\mathfrak{L}$.

**Proof.**

(2) $\Longleftrightarrow$ (3). This equivalence follows from (8.14) and (8.16).

(3) $\Longleftrightarrow$ (4). By virtue of (8.50) the linear manifold $(A_2-\lambda)^{-1}\mathfrak{L}$ is closed. Since $0\in\rho(\gamma^*(\lambda)\gamma(\lambda))$ and $|\mathrm{Im}\,\lambda|\cdot\|l\|\geq\|(A_2-\lambda)^{-1}l\|$ we obtain from (8.50) that the angle between $\mathfrak{N}_\lambda(A)$ and $(A_2-\lambda)^{-1}\mathfrak{L}$ is acute.

Conversely, in accordance with Theorem 2.4 of [45] it follows from decomposition (4) that the linear manifold $(A_2-\lambda)^{-1}\mathfrak{L}$ is closed and, therefore, inequality (8.50) holds.

(3) $\Longrightarrow$ (1). Let us show using inequality (8.50) that the linear manifold $A^*+\hat{\mathfrak{L}}$ is closed. Let the sequence $\{\gamma(\lambda)f_n+(A_2-\lambda)^{-1}h_n, h_n+l_n\}(\in(A^*-\lambda)\dotplus\hat{\mathfrak{L}})$ converge to $\{f,f'\}$ as $n\to\infty$ $(f_n\in\mathfrak{h}, l_n\in\mathfrak{L})$. Then $P_0h_n=P_0(h_n+l_n)\to P_0f'=:h''$ as $n\to\infty$ and, therefore,

$$\lim_{n\to\infty}(A_2-\lambda)^{-1}P_0h_n=(A_2-\lambda)^{-1}h'',\qquad \lim_{n\to\infty}[\gamma(\lambda)f_n+(A_2-\lambda)^{-1}P_{\mathfrak{L}}h_n]=f-(A_2-\lambda)^{-1}P_0f',$$

where $P_0=P_{\mathfrak{h}_0}$. It follows from inequality (8.50) that $\lim_{n\to\infty}f_n=g$ and $\lim_{n\to\infty}P_{\mathfrak{L}}h_n=h'$. Therefore

$$\exists\lim_{n\to\infty}h_n=h'+h''=h,\qquad \exists\lim_{n\to\infty}l_n=f'-h=:l\in\mathfrak{L}\Longrightarrow\lim_{n\to\infty}(A_2-\lambda)^{-1}h_n=(A_2-\lambda)^{-1}h,$$

$$\lim_{n\to\infty}\gamma(\lambda)f_n=\gamma(\lambda)f'\Longrightarrow\{f,f'\}=\{\gamma(\lambda)g+(A_2-\lambda)^{-1}h,h+l\}\in(A^*-\lambda)\dotplus\hat{\mathfrak{L}}.$$

So, the linear manifold $(A^*-\lambda)\dotplus\hat{\mathfrak{L}}$ is closed and, therefore, the same is true for the linear manifold $A^*\dotplus\hat{\mathfrak{L}}$.

**Proposition 8.7.** *Let the conditions of Theorem 8.2 be satisfied and $0\in\sigma(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(i))$. Then*

(1) $\dim\ker(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(i))=\dim\mathfrak{L}_0$, *where* $\mathfrak{L}_0=\mathfrak{L}\cap\mathfrak{N}$ *and*

$$\ker(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda))=\{\{\Gamma_2\{0,-l\},l\}:l\in\mathfrak{L}_0\};\tag{8.51}$$

(2) *if $0\in\sigma(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(i))$, then the triple $\Pi_1=\{\mathcal{H}\oplus\mathfrak{L},\Gamma_1',\Gamma_2'\}$ forms a generalized BVS for the linear relation $A_{0*}=A^*\dotplus\hat{\mathfrak{L}}$;*

(3) *if $0\notin\sigma_p(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(i))$, then the triple $\Pi_1'=\{\mathcal{H}\oplus\mathfrak{L},\Gamma_2',-\Gamma_1'\}$ forms a generalized BVS for the relation $A_{0*}(\Longleftrightarrow 0\in\rho(\mathfrak{A}_{\Pi\mathfrak{L}}(i))$ if and only if $A_1'=\ker\Gamma_1'$ is $\mathfrak{L}$-regular.*

*In this case the Weyl function corresponding to the BVS $\Pi_1'$ coincides with the function $-\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)^{-1}$*

**Proof.** (1) It follows from (8.13), (8.15), (8.16), and the condition $0\in\rho(\mathrm{Im}\,M(i))$ that

$$0\in\sigma_p(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(i))\Longleftrightarrow\ker P_{\mathfrak{M}_{\bar\lambda}}(A_2-\lambda)^{-1}\restriction\mathfrak{L}\neq\{0\}.$$

Equality $(A_2-\lambda)^{-1}\mathfrak{L}\cap\mathfrak{N}_\lambda=(A_2-\lambda)^{-1}\mathfrak{L}_0$ implies relation (8.51). Therefore $\dim\mathfrak{L}_0=\dim\ker(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda))$.

(2) It is easy to verify that the mapping $\Gamma_2$: $A_{0*}\to\mathcal{H}\oplus\mathfrak{L}$ is surjective, the mapping $\Gamma=\{\Gamma_2,\Gamma_1\}$: $A_{0*}\to(\mathcal{H}\oplus\mathfrak{L})^2$ is closed, and $A_2(=\ker\Gamma_2)=A_2^*$. The Green identity is verified above [see (8.44)].

(3) The extension $A_1'=A_1\cap(\mathfrak{L}^\perp\oplus\mathfrak{h})+\hat{\mathfrak{L}}=\ker\Gamma_1'$ is self-adjoint in essence. Indeed

$$(A_1')^*=j\{(A_1\cap(\mathfrak{L}^\perp\oplus\mathfrak{h})+\hat{\mathfrak{L}})^\perp\}=j((A_1^\perp\oplus(\mathfrak{L}\oplus\{0\}))\cap(\mathfrak{h}\oplus\mathfrak{L}^\perp))=\overline{(A_1^*+\hat{\mathfrak{L}})\cap(\mathfrak{L}^\perp\oplus\mathfrak{h})}=\bar{A}_1'.$$

It follows that $(A_1')^*=A_1'$ iff $A_1$ is $\mathfrak{L}$-regular. Further, if $\ker(\mathrm{Im}\,\mathfrak{A}_{\Pi\mathfrak{L}}(i))=\{0\}$, then $\mathfrak{L}\cap\mathfrak{N}=\{0\}$ and the linear manifold $P_{\mathfrak{L}}\mathfrak{D}(A_1)$ is dense in $\mathfrak{L}$. Since $\Gamma_1A^*=\mathcal{H}$, the equivalence $\mathfrak{R}(\Gamma_1')=\mathcal{H}\oplus\mathfrak{L}\Longleftrightarrow P_{\mathfrak{L}}\mathfrak{D}(A)=\mathfrak{L}$ holds. By Proposition 2.5 the relation $P_{\mathfrak{L}}\mathfrak{D}(A_1)=\mathfrak{L}$ means that the extension $A_1=A_1^*$ is $\mathfrak{L}$-regular. It remains to note that by Proposition 6.2 the mapping $\Gamma_1'$: $A^*\to\mathcal{H}$ is surjective iff $0\in\rho(\mathfrak{A}_{\Pi\mathfrak{L}}(i))$. In this case the function $-\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)^{-1}$ is the Weyl function corresponding to the generalized BVS $\Pi_1'$. □

**Remark 8.5.** Assertion (3) can be proved in another way. Indeed, as follows from the Frobenius formula (3.66) and the condition $0\in\rho(M(\lambda))$ $\forall\lambda\in\mathbb{C}_+$, the matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ is invertible iff $0\in\rho(a_{22}-a_{21}a_{11}^{-1}a_{12})$. But by the resolvent formula (3.39)

$$a_{22}-a_{21}a_{11}^{-1}a_{12}=P_{\mathfrak{L}}[(A_2-\lambda)^{-1}-\gamma(\lambda)M(\lambda)^{-1}\gamma^*(\bar\lambda)]\restriction\mathfrak{L}=P_{\mathfrak{L}}(A_1-\lambda)^{-1}\restriction\mathfrak{L}.$$

Therefore, $0 \in \rho(\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)) \Longleftrightarrow 0 \in \rho(P_{\mathfrak{L}}(A_1 - \lambda)^{-1} \upharpoonright \mathfrak{L})$. It remains to note that by Proposition 2.9 $0 \in \rho(P_{\mathfrak{L}}(A_1 - \lambda)^{-1} \upharpoonright \mathfrak{L}$ if and only if the extension $A_1 = \ker \Gamma_1 \supset A_0$ is $\mathfrak{L}$-regular ($\mathfrak{L} \subset \mathfrak{D}(A_0)^{\perp}$).

**6.** In this section we shall show that the $\Pi\mathfrak{L}$-resolvent matrix $W_{\Pi\mathfrak{L}}(\lambda)$ coincides with the characteristic operator-valued function of some almost solvable linear relation. We shall also obtain from this fact a new proof of formula (8.22).

**Theorem 8.3.** *Let $A$ be a Hermitian operator in $\mathfrak{h}$, $\mathfrak{h}_0 = \overline{\mathfrak{D}(A)}$, $\mathfrak{L}$ be a subspace of $\mathfrak{h}$ such that $\dim \mathfrak{L} = n_{\pm}(A)$ and $\mathfrak{L} \cap \mathfrak{N} = \{0\}$. Let also $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ be a BVS for $A^*$, $W_{\Pi\mathfrak{L}}(\lambda)$ be the corresponding Weyl function, $A_0 = A \upharpoonright \mathfrak{L}^{\perp} := A \cap (\mathfrak{L}^{\perp} \oplus \mathfrak{h})$. If the linear relation $A^*$ is $\mathfrak{L}$-regular, then:* (1) $P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}$; (2) $A_0^* = A^* \dotplus \hat{\mathfrak{L}}$; (3) *the triple* $\Pi_2 = \{\mathfrak{L} \oplus \mathfrak{L}, \Gamma_1'', \Gamma_2''\}$, *where*

$$\Gamma_1''\hat{f} = \frac{1}{\sqrt{2}}\{\Gamma_2\{f,f'\} + l, \Gamma_1\{f,f'\} + P_{\mathfrak{L}}f\}, \qquad \Gamma_2''\hat{f} = \frac{1}{\sqrt{2}}\{-\Gamma_1\{f,f'\} + P_{\mathfrak{L}}f, \Gamma_2\{f,f'\} - l\}, \tag{8.52}$$

*forms a BVS for $A_0^*$ (here $\{f,f'\} \in A^*$, $l \in \mathfrak{L}$, $\hat{f} = \{f, f'+l\}$);* (4) *the linear relation $T := A \dotplus \hat{\mathfrak{L}}$ is closed, almost solvable ($T \in \mathcal{A}s\,(A_0)$), and $\rho(T) = \rho(A;\mathfrak{L})$. The extensions $T$ and $T^* = A^* \upharpoonright \mathfrak{L}^{\perp} := A^* \cap (\mathfrak{L}^{\perp} \oplus \mathfrak{h})$ of a Hermitian operator $A$ are transversal and can be defined by the equalities*

$$T = \ker(\Gamma_1'' + iJ\Gamma_2''), \qquad T^* = \ker(\Gamma_1'' - iJ\Gamma_2''),$$

*that is, $T = (\tilde{A}_0)_{-iJ}$, $T^* = (\tilde{A}_0)_{iJ}$, where $J = i\begin{pmatrix} 0 & -I_{\mathfrak{L}} \\ I_{\mathfrak{L}} & 0 \end{pmatrix}$;* (5) *the $\Pi\mathfrak{L}$-resolvent matrix $W_{\Pi\mathfrak{L}}(\lambda)$ coincides with the characteristic operator-function $W_{T^*}(\lambda)$ of the linear relation $T^* = (\tilde{A}_0)_{iJ} (\in \mathcal{A}s\,(A_0))$ corresponding to the colligation $\varphi = (iJ, \mathfrak{L}; I, J, \mathfrak{L})$ and has the form*

$$W_{\Pi\mathfrak{L}}(\lambda) = J(M_2(\lambda) - iJ)(M_2(\lambda) + iJ)^{-1}J = W_{T^*}(\lambda) \qquad (\lambda \in \rho(T)), \tag{8.53}$$

*where $M_2(\lambda)$ is the Weyl function of the operator $A_0$ corresponding to the BVS $\Pi_2$;* (6) *formula* (8.22) *for $W_{T^*}(\lambda) = W_{\Pi\mathfrak{L}}(\lambda)$ holds.*

**Proof.** We present the proof under several headings. Assertions (1) and (2) have been proved earlier (see the proof of Theorem 8.2).

(3) The Green identity is simply verified. Let us show that the mapping $\Gamma'' = \{\Gamma_2'', \Gamma_1''\}$ maps $A_0^*$ onto $\mathfrak{L}^4$. Put $l = (l_1 - l_4)/2$, and choose $\hat{f}_1 = \{f, f'\} \in A^*$ such that $\Gamma_1\hat{f}_1 = (l_2 - l_3)/2$, $\Gamma_2\hat{f}_1 = (l_1 + l_4)/2$. Since $P_{\mathfrak{L}}\mathfrak{D}(A) = \mathfrak{L}$ we can choose $f_A \in \mathfrak{D}(A)$ such that $P_{\mathfrak{L}}f_A = (l_2 + l_3)/2 - k$, where $k = P_{\mathfrak{L}}f$. Putting $\hat{f} = \hat{f}_1 + \{f_A, Af_A\} + \{0, l\}$ we obtain the required equalities: $\Gamma_1''\hat{f} = \{l_1, l_2\}$, $\Gamma_2''\hat{f} = \{l_3, l_4\}$.

(4) When $\hat{f} = \{f, f'+l\} \in A^* + \hat{\mathfrak{L}}$, the inclusion $\hat{f} \in T = A + \hat{\mathfrak{L}}$ is characterized by the equalities

$$\Gamma_1''\hat{f} = \frac{1}{\sqrt{2}}\mathrm{col}\,\{l, P_{\mathfrak{L}}f\}, \qquad \Gamma_2''\hat{f} = \frac{1}{\sqrt{2}}\mathrm{col}\,\{P_{\mathfrak{L}}f, -l\},$$

that is, $\hat{f} \in T \Longleftrightarrow \hat{f} \in \ker(\Gamma_1'' + iJ\Gamma_2'')$. In other words $-iJ = \Gamma''T$ and $iJ = \Gamma''T^*$, that is, $T = (\tilde{A}_0)_{-iJ}$, $T^* = (\tilde{A}_0)_{iJ}$. Since $0 \in \rho(J)$ and $J = \mathrm{Im}\,(iJ)$, it follows from Proposition 1.4 that the extensions $T$ and $T^*$ are transversal. Since $T - \lambda = \{\{f, (A-\lambda)f + l\} : f \in \mathfrak{D}(A), l \in \mathfrak{L}\}$, we have

$$\mathfrak{R}(T - \lambda) = (A-\lambda)\mathfrak{D}(A) \dotplus \mathfrak{L} = \mathfrak{M}_\lambda \dotplus \mathfrak{L}, \qquad \ker(T - \lambda) = \{f \in \mathfrak{D}(A) : (A-\lambda)f \in \mathfrak{L}\}. \tag{8.54}$$

Hence the equivalence $\lambda \in \rho(T) \Longleftrightarrow \lambda \in \rho(A;\mathfrak{L})$.

(5) Let $M_2(\lambda)$ be a Weyl function of an operator $A_0$, corresponding to the BVS $\Pi_2$ of the form in (8.52). Let us include the operator $B = iJ$ in the colligation $\varphi = (B, \mathfrak{L}, K; J, \mathfrak{L})$, where $K = |B_I|^{1/2} = |J| = I_{\mathfrak{L}}$, $J = \mathrm{sgn}\,B_I$. In view of Theorem 7.1, the characteristic operator-function has the form

$$W_{T^*}(\lambda) = I + 2i((iJ)^* - M_2(\lambda))^{-1}J = J(M_2(\lambda) - iJ)(M_2(\lambda) + iJ)^{-1}J. \tag{8.55}$$

By Theorem 8.2 the $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ coincides with the Weyl function $M_1(\lambda)$ of the operator $A_0$ corresponding to the BVS $\Pi = \{\mathfrak{L}\oplus\mathfrak{L}, \Gamma'_1, \Gamma'_2\}$ of the form in (8.42).

But the BVS $\Pi_1$ and $\Pi_2$ of the form in (8.42) and (8.52) are related as

$$\begin{pmatrix}\Gamma''_1\\ \Gamma''_2\end{pmatrix} = \begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}\begin{pmatrix}\Gamma'_1\\ \Gamma'_2\end{pmatrix} =: X\begin{pmatrix}\Gamma'_1\\ \Gamma'_2\end{pmatrix}, \tag{8.56}$$

where $I = I_{\mathfrak{L}}$,

$$X_{11} = \begin{pmatrix}\mathbb{O} & \mathbb{O}\\ I & I\end{pmatrix}, \quad X_{12} = \begin{pmatrix}I & -I\\ \mathbb{O} & \mathbb{O}\end{pmatrix}, \quad X_{21} = \begin{pmatrix}-I & I\\ \mathbb{O} & \mathbb{O}\end{pmatrix}, \quad X_{22} = \begin{pmatrix}\mathbb{O} & \mathbb{O}\\ I & I\end{pmatrix}, \tag{8.57}$$

and the operator $X = \begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}$ is $J_2$-unitary in $\mathfrak{L}\oplus\mathfrak{L}$ (that is, $X^*J_2X = XJ_2X^* = J_2$), $J_2 = i\begin{pmatrix}\mathbb{O} & -I_{\mathfrak{L}\oplus\mathfrak{L}}\\ I_{\mathfrak{L}\oplus\mathfrak{L}} & \mathbb{O}\end{pmatrix}$. Therefore, it follows from (8.56), (8.57), and Proposition 1.7 that

$$\begin{aligned} M_2(\lambda) &= (X_{11}\mathfrak{A}(\lambda) + X_{12})(X_{21}\mathfrak{A}(\lambda) + X_{22})^{-1}\\ &= \begin{pmatrix}I & -I\\ a_{11}+a_{21} & a_{12}+a_{22}\end{pmatrix}\begin{pmatrix}a_{21}-a_{11} & a_{22}-a_{12}\\ I & I\end{pmatrix}^{-1}. \end{aligned} \tag{8.58}$$

Let us denote by $Y_1(\lambda)$ and $Y_2(\lambda)^{-1}$ the first and the second factor on the right-hand side of equality (8.58). It follows from (8.55) and (8.58) that

$$\begin{aligned} W_{T^*}(\lambda) &= J(Y_1Y_2^{-1} - iJ)(Y_1Y_2^{-1} + iJ)^{-1}J = (JY_1 - iY_2)(JY_1 + iY_2)^{-1}\\ &= 2i\begin{pmatrix}-a_{21} & -a_{22}\\ \mathbb{O} & -I\end{pmatrix}\frac{1}{2i}\begin{pmatrix}-a_{11} & -a_{12}\\ I & \mathbb{O}\end{pmatrix}^{-1}\\ &= \begin{pmatrix}-a_{21} & -a_{22}\\ \mathbb{O} & -I\end{pmatrix}\begin{pmatrix}\mathbb{O} & I\\ -a_{12}^{-1} & -a_{12}^{-1}a_{11}\end{pmatrix} = \begin{pmatrix}a_{22}a_{12}^{-1} & a_{22}a_{12}^{-1}a_{11} - a_{21}\\ a_{12}^{-1} & a_{12}^{-1}a_{11}\end{pmatrix} = W_{\Pi\mathfrak{L}}(\lambda). \end{aligned} \tag{8.59}$$

(6) Using another expression for the Weyl function, we shall obtain a new proof of formula (8.22) for $M_2(\lambda)$. Let us put $f_\lambda = G^*(\bar\lambda)l = -\mathcal{Q}^*(\bar\lambda)l_1 + \mathcal{P}^*(\bar\lambda)l_2$; $l_1, l_2 \in \mathfrak{L}$, $\hat f_\lambda = \{f_\lambda, \lambda f_\lambda\}$, and note that

$$\hat f_\lambda = -\hat{\mathcal{Q}}^*(\bar\lambda)l_1 + \hat{\mathcal{P}}^*(\bar\lambda)l_2 + \{0, l_1\} \in \hat{\mathfrak{N}}_\lambda(A_0) \subset A^* \overset{\circ}{+} \hat{\mathfrak{L}}. \tag{8.60}$$

It follows from (8.52), (8.60), and (8.27) that

$$\Gamma''_1\hat f_\lambda = \begin{pmatrix}\Gamma_2(-\hat{\mathcal{Q}}^*(\bar\lambda)l_1 + \hat{\mathcal{P}}^*(\bar\lambda)l_2) + l_1\\ \Gamma_1(-\hat{\mathcal{Q}}^*(\bar\lambda)l_1 + \hat{\mathcal{P}}^*(\bar\lambda)l_2) + l_2\end{pmatrix} = \begin{pmatrix}w_{11}^*(\bar\lambda) + I & w_{21}^*(\bar\lambda)\\ w_{12}^*(\bar\lambda) & w_{22}^*(\bar\lambda) + I\end{pmatrix}\begin{pmatrix}l_1\\ l_2\end{pmatrix},$$

$$\Gamma''_2\hat f_\lambda = \begin{pmatrix}-\Gamma_1(-\hat{\mathcal{Q}}^*(\bar\lambda)l_1 + \hat{\mathcal{P}}^*(\bar\lambda)l_2 + l_2\\ \Gamma_2(-\hat{\mathcal{Q}}^*(\bar\lambda)l_1 + \hat{\mathcal{P}}^*(\bar\lambda)l_2) - l_1\end{pmatrix} = \begin{pmatrix}-w_{12}^*(\bar\lambda) & I - w_{22}^*(\bar\lambda)\\ w_{11}^*(\bar\lambda) - I & w_{21}^*(\bar\lambda)\end{pmatrix}\begin{pmatrix}l_1\\ l_2\end{pmatrix}.$$

Hence

$$M_2(\lambda) = \begin{pmatrix}w_{11}(\bar\lambda)^* + I & w_{21}(\bar\lambda)^*\\ w_{12}(\bar\lambda)^* & w_{22}(\bar\lambda)^* + I\end{pmatrix}\begin{pmatrix}-w_{12}(\bar\lambda)^* & I - w_{22}(\bar\lambda)^*\\ w_{11}(\bar\lambda)^* - I & w_{21}(\bar\lambda)^*\end{pmatrix}^{-1} \tag{8.61}$$

Denoting by $Z_1(\lambda)$ and $Z_2(\lambda)^{-1}$ the first and the second factor in (8.61) respectively we obtain from (8.55) and (8.61) the following expression for $W_{T^*}(\lambda)^*$:

$$(W_{T^*}(\lambda))^* = I + 2iJ(M_2(\lambda)^* - iJ)^{-1} = (M_2(\bar\lambda) + iJ)(M_2(\bar\lambda) - iJ)^{-1}$$

$$= (Z_1(\bar\lambda) + iJZ_2(\bar\lambda))(Z_1(\bar\lambda) - iJZ_2(\bar\lambda))^{-1} = 2(Z_1(\bar\lambda) - I)\frac{1}{2}I = (W_{\Pi\mathfrak{L}}(\lambda))^*.$$

(7) Now we are going to prove identity (8.23). It follows from (8.55) and the equality $W_{\Pi\mathfrak{L}}(\lambda) = W_{T^*}(\lambda)$ (recall that $T^* = (\tilde A_0)_{iJ}$) that

$$\frac{W_{\Pi\mathfrak{L}}(\lambda)JW_{\Pi\mathfrak{L}}(\bar\mu)^* - J}{2i(\lambda-\mu)} = (iJ + M_2(\lambda))^{-1}\gamma^*(\bar\lambda)\gamma(\mu)(-iJ + M_2(\mu))^{-1}. \tag{8.62}$$

Since $\Gamma_2''\hat f_\lambda = \frac{1}{\sqrt2}Z_2(\lambda)\binom{l_1}{l_2}$ we have $\hat\gamma(\lambda) = (\Gamma_2'' \upharpoonright \hat{\mathfrak{N}}_\lambda)^{-1}$:

$$\hat\gamma(\lambda) = \sqrt2\hat G^*(\bar\lambda)Z_2(\lambda)^{-1}. \tag{8.63}$$

Now using (8.63), (8.60), and (8.61) we obtain

$$(iJ + M_2(\lambda))^{-1}\hat\gamma^*(\bar\lambda) = \sqrt2 Z_2(\lambda)(Z_1(\lambda) + iJZ_2(\lambda))^{-1}(Z_2(\bar\lambda)^*)^{-1}\hat G(\lambda)$$

$$= \sqrt2 Z_2(\lambda)[2Z_2(\bar\lambda)^*(Z_1(\lambda) - I)]^{-1}\hat G(\lambda) = \frac{1}{\sqrt2}\hat G(\lambda). \tag{8.64}$$

Relations (8.26), which are correct because $W_{\Pi\mathfrak{L}}(\lambda)$ is a characteristic function and $\lambda \in \rho_s(A;\mathfrak{L})$, are used in the proof of equality (8.64). We obtain identity (8.23), comparing (8.64) and (8.62). □

**Remark 8.6.** (1) The equalities $\rho(A;\mathfrak{L}) = \rho(T)$ and $W_{T^*}(\lambda) = W_{\Pi\mathfrak{L}}(\lambda)$ are correct without the condition of $\mathfrak{L}$-regularity of the relation $A^*$. It can be proved just as it was done in [79] for the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$.

(2) In the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, Krein and Saakyan (see [43]) have established a connection between an $\mathfrak{L}$-resolvent matrix $W(\lambda)$ of the operator $A$ on the one hand, and the characteristic function of some $\mathcal{Y}$-colligation on the other hand. They have also obtained (see [42]) identity (8.23) for $W_{\Pi\mathfrak{L}}(\lambda)$, which is an abstract analog of the well-known Christoffel identities in the moment problem [4, 6].

It is worth mentioning that, as follows from the proof of Theorem 8.3, identity (8.23) is a corollary of the simple relation

$$\frac{W_T(\lambda)JW_T(\bar\mu)^*}{2i(\lambda-\mu)} = K^*(B^* - M(\lambda))^{-1}\gamma^*(\bar\lambda)\gamma(\mu)(B - M(\mu))^{-1}K. \tag{8.65}$$

Here $W_T(\lambda)$ is a characteristic operator-valued function of the extension $T = \tilde A_B (\in \mathcal{A}s\ (A))$. Formula (8.62) is a particular case of the previous one.

**7.** Let $A$ be a Hermitian operator in $\mathfrak{h}$ such that the linear relation $A^*$ is $\mathfrak{L}$-regular. Let us denote by $G(J_2)$ the group of $J_2$-unitary operators $Z = (Z_{jk})_{j,k=1}^2$ in $\mathfrak{L}^4$, where $J_2 = \begin{pmatrix} \mathbb{O} & -I_{\mathfrak{L}\oplus\mathfrak{L}} \\ I_{\mathfrak{L}\oplus\mathfrak{L}} & \mathbb{O}\end{pmatrix}$, and by $\mathfrak{M}(A_0)$ the set of Weyl functions of the operator $A_0 = A \upharpoonright \mathfrak{L}^\perp := A \cap (\mathfrak{L}^\perp \oplus \mathfrak{h})$. The group $G(J_2)$ acts transitively and effectively on the set $\mathfrak{M}(A_0)$ as a group of fractional-linear mappings by the formula

$$Z \circ M(\lambda) := (Z_{11}M(\lambda) + Z_{12})(Z_{21}M(\lambda) + Z_{22})^{-1} \qquad M(\lambda) \in \mathfrak{M}(A_0). \tag{8.66}$$

The subset of Weyl functions $M(\lambda) \in \mathfrak{M}(A_0)$, which are the $\Pi\mathfrak{L}$-preresolvent matrices of some operator $A'$ such that $A' \upharpoonright \mathfrak{L}^\perp = A_0$, will be denoted by $\mathfrak{A}(A_0;\mathfrak{L})$.

Two subgroups of the group $G(J_2)$ acting on the set $\mathfrak{A}(A_0;\mathfrak{L})$ will be found in the two next propositions.

**Proposition 8.8.** *Let $X = (X_{jk})_{j,k=1}^2$ be a $J$-unitary operator in $\mathfrak{L}^2 = \mathfrak{L}\oplus\mathfrak{L}$, $J = i\begin{pmatrix} \mathbb{O} & -I_{\mathfrak{L}} \\ I_{\mathfrak{L}} & \mathbb{O}\end{pmatrix}$, $Z = (Z_{jk})_{j,k=1}^2$, where*

$$Z_{11} = \begin{pmatrix} X_{11} & \mathbb{O} \\ \mathbb{O} & I\end{pmatrix}, \quad Z_{12} = \begin{pmatrix} X_{1O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}\end{pmatrix}, \quad Z_{21} = \begin{pmatrix} X_{21} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}\end{pmatrix}, \quad Z_{22} = \begin{pmatrix} X_{22} & \mathbb{O} \\ \mathbb{O} & I\end{pmatrix}. \tag{8.67}$$

*Then the matrices $Z$ form the subgroup $G_r(J_2)$ of the group $G(J_2)$. The element $Z$ acts on the set $\mathfrak{M}(A_0)$ and its action is equivalent to the following replacement of the BVS $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$:*

$$\Gamma_1' = \begin{pmatrix} \Gamma_1' \\ \Gamma_2' \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \tag{8.68}$$

**Proof.** Let $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ be a preresolvent matrix of the form in (8.13), corresponding to the BVS $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$. Further, let $\Pi_1 = \{\mathfrak{L}, \Gamma_1', \Gamma_2'\}$ be the BVS defined by equality (8.68), $M_1(\lambda)$ and $\gamma_1(\lambda)$ be the corresponding Weyl function and $\gamma$-field respectively, $A_2' = \ker(X_{21}\Gamma_1 + X_{22}\Gamma_2) = \ker \Gamma_2'$. Putting $\mathfrak{A}'(\lambda) = Z \circ \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ we obtain from (8.66), (8.67)

$$\begin{aligned}
\mathfrak{A}'(\lambda) &= (Z_{11}\mathfrak{A}(\lambda) + Z_{12})(Z_{21}\mathfrak{A}(\lambda) + Z_{22})^{-1} \\
&= \begin{pmatrix} X_{11}M(\lambda) + X_{12} & X_{11}\gamma^*(\bar\lambda) \upharpoonright \mathfrak{L} \\ P_{\mathfrak{L}}\gamma(\lambda) & P_{\mathfrak{L}}(A_2 - \lambda)^{-1} \upharpoonright \mathfrak{L} \end{pmatrix} \times \\
&\times \begin{pmatrix} (X_{21}M(\lambda) + X_{22})^{-1} & -(X_{21}M(\lambda) + X_{22})^{-1}X_{21}\gamma(\bar\lambda)^* \upharpoonright \mathfrak{L} \\ \mathbb{O} & I_{\mathfrak{L}} \end{pmatrix} \\
&= \begin{pmatrix} (X_{11}M + X_{12})(X_{21}M + X_{22})^{-1} & [X_{11} - (X_{11}M + X_{12})(X_{21}M + X_{22})^{-1}X_{21}]\gamma(\bar\lambda)^* \upharpoonright \mathfrak{L} \\ P_{\mathfrak{L}}\gamma(\lambda)(X_{21}M + X_{22})^{-1} & P_{\mathfrak{L}}(A_2 - \lambda)^{-1} \upharpoonright \mathfrak{L} - P_{\mathfrak{L}}\gamma(\lambda)(X_{21}M + X_{22})^{-1}X_{21}\gamma(\bar\lambda)^* \upharpoonright \mathfrak{L} \end{pmatrix} \\
&=: \begin{pmatrix} a_{11}'(\lambda) & a_{12}'(\lambda) \\ a_{21}'(\lambda) & a_{22}'(\lambda) \end{pmatrix} = \begin{pmatrix} M_1(\lambda) & \gamma_1(\bar\lambda)^* \upharpoonright \mathfrak{L} \\ P_{\mathfrak{L}}\gamma_1(\lambda) & P_{\mathfrak{L}}(A_2' - \lambda) \upharpoonright \mathfrak{L} \end{pmatrix} = \mathfrak{A}_{\Pi_1\mathfrak{L}}(\lambda).
\end{aligned} \tag{8.69}$$

To prove equality (8.69) we have used the following formulas:

$$a_{11}'(\lambda) = (X_{11}M(\lambda) + X_{12})(X_{21}M(\lambda) + X_{22})^{-1} = M_1(\lambda), \qquad \gamma_1(\lambda) = \gamma(\lambda)(X_{21}M(\lambda) + X_{22})^{-1}$$

(see Proposition 1.7), and the equality $a_{22}'(\lambda) = P_{\mathfrak{L}}(A_2' - \lambda)^{-1} \upharpoonright \mathfrak{L}$, which easily follows from the resolvent formula (3.39) for $\tilde{A}_\theta = A_2' = \ker \Gamma_2'$. The equality $a_{12}'(\lambda) = \gamma_1^*(\bar\lambda) \upharpoonright \mathfrak{L}$ follows from the identity

$$X_{11} - M_1(\lambda)X_{21} = X_{11} - (X_{11}M(\lambda) + X_{12})(X_{21}M(\lambda) + X_{22})^{-1} = (M(\lambda)X_{21}^* + X_{22}^*)^{-1}. \tag{8.70}$$

To prove it we multiply the equality $M_1(\lambda)(X_{21}M(\lambda) + X_{22}) = X_{11}M(\lambda) + X_{12}$ from the left by $X_{21}^*$. Making use of the identities $X_{21}^*X_{11} = X_{11}^*X_{21}$, $X_{21}^*X_{12} = X_{11}^*X_{22} - I$ [see (1.15)] we obtain the equality $(X_{11}^* - X_{21}^*M_1(\lambda))(X_{21}M(\lambda) + X_{22}) = I$, which is equivalent to (8.70).

**Remark 8.6.** Let the BVS $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ and $\tilde\Pi = \{\mathfrak{L}, \tilde\Gamma_1, \tilde\Gamma_2\}$ be connected by equality (8.68), and $\Pi_1 = \{\mathfrak{L} \oplus \mathfrak{L}, \Gamma_1', \Gamma_2'\}$ and $\tilde\Pi_1 = \{\mathfrak{L} \oplus \mathfrak{L}, \tilde\Gamma_1', \tilde\Gamma_2'\}$ be the corresponding BVS of the form in (8.42). It is easy to see that $\Pi_1$ and $\tilde\Pi_1$ are connected by the relation

$$\begin{pmatrix} \tilde\Gamma_1' \\ \tilde\Gamma_2' \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \Gamma_1' \\ \Gamma_2' \end{pmatrix} =: Z \begin{pmatrix} \Gamma_1' \\ \Gamma_2' \end{pmatrix}, \tag{8.71}$$

where $Z = (Z_{ij})_{i,j=1}^2$ is a $J_2$-unitary matrix in $\mathfrak{L}^4$ with elements of the form in (8.67). The following connection exists between the preresolvent matrices $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ and $\mathfrak{A}_{\tilde\Pi\mathfrak{L}}(\lambda)$, corresponding to the BVS $\Pi$ and $\tilde\Pi$:

$$\mathfrak{A}_{\tilde\Pi\mathfrak{L}}(\lambda) = Z \circ \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) = (Z_{11}\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) + Z_{12})(Z_{21}\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) + Z_{22})^{-1}. \tag{8.72}$$

One can deduce (8.72) either from Propositions 1.7, 8.5, or from the equality $W_{\tilde\Pi\mathfrak{L}}(\lambda) = W_{\Pi\mathfrak{L}}(\lambda)X$, which connects the resolvent matrices $W_{\tilde\Pi\mathfrak{L}}(\lambda)$ and $W_{\Pi\mathfrak{L}}(\lambda)$ corresponding to the BVS $\tilde\Pi$ and $\Pi$ respectively. It follows from (8.72) that

$$P_{\mathfrak{L}}(A_2' - \lambda)^{-1} \upharpoonright \mathfrak{L} = P_{\mathfrak{L}}((A_2 - \lambda)^{-1} + \gamma(\lambda)(X_{21}M(\lambda) + X_{22})^{-1}X_{21}\gamma(\bar\lambda)^*) \upharpoonright \mathfrak{L}, \tag{8.73}$$

where $A_2' = \ker \tilde{\Gamma}_2' = \ker(X_{21}\Gamma_1 + X_{22}\Gamma_2)$, $A_2 = \ker \Gamma_2$, and consequently, owing to the fact that $\mathfrak{L}(\dim \mathfrak{L} = n_\pm(A))$ is arbitrary, formula (3.69) holds. Therefore, the resolvent formula (3.69) is a simple corollary of Propositions 1.7 and 8.5.

The action of an element $Z = Z_X \in G_r(J_2)$ on the set $\mathfrak{A}(A_0; \mathfrak{L})$ is equivalent to multiplication from the right of the $\Pi\mathfrak{L}$-resolvent matrix: $W_{\Pi\mathfrak{L}}(\lambda) \to W_{\Pi\mathfrak{L}}(\lambda)X$. In the next proposition we describe matrices $Y = (Y_{ij})_{i,j=1}^2$ such that the property of $W_{\Pi\mathfrak{L}}(\lambda)$ to be an $\mathfrak{L}$-resolvent matrix of some operator remains true under the transformation $W_{\Pi\mathfrak{L}}(\lambda) \to YW_{\Pi\mathfrak{L}}(\lambda)$.

**Proposition 8.9.** *Let $A$ be a Hermitian operator in $\mathfrak{h}$, $\overline{\mathfrak{D}(A)} = \mathfrak{h}$, $W_{\Pi\mathfrak{L}}(\lambda)$ be a resolvent matrix, $X = (X_{ij})_{i,j=1}^2$ be a $J$-unitary operator in $\mathfrak{L} \oplus \mathfrak{L}$. The matrix-function $XW_{\Pi\mathfrak{L}}(\lambda)$ is an $\mathfrak{L}$-resolvent matrix of some Hermitian operator $A'$ if and only if $X_{21} = 0$, $X_{11} = X_{22} = V = (V^*)^{-1}$.*

**Proof.** *Necessity.* It follows from (8.19) that

$$\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) = \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} W_{\Pi\mathfrak{L}}(\lambda)\right]\left[\begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} W_{\Pi\mathfrak{L}}(\lambda)\right]^{-1}. \tag{8.74}$$

Putting $\tilde{W}(\lambda) = X^* W_{\Pi\mathfrak{L}}(\lambda)$ and substituting it in (8.7) instead of $W_{\Pi\mathfrak{L}}(\lambda)$, we obtain the matrix $\tilde{\mathfrak{A}}(\lambda)$. To prove the proposition, it is sufficient to find the conditions on $X$ in order for $\tilde{\mathfrak{A}}(\lambda)$ to be an $\mathfrak{L}$-preresolvent matrix. We obtain the following from (8.19) and (8.74):

$$\begin{aligned}\tilde{\mathfrak{A}}(\lambda) &= \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} X^* W_{\Pi\mathfrak{L}}(\lambda)\right] \\ &\times \left[\begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} X^* W_{\Pi\mathfrak{L}}(\lambda)\right]^{-1} \\ &= \left[\begin{pmatrix} I & 0 \\ 0 & X_{11}^* \end{pmatrix} \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) + \begin{pmatrix} 0 & 0 \\ 0 & X_{21}^* \end{pmatrix}\right]\left[\begin{pmatrix} 0 & 0 \\ 0 & X_{12}^* \end{pmatrix} \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) + \begin{pmatrix} I & 0 \\ 0 & X_{22}^* \end{pmatrix}\right]^{-1} \\ &= \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ X_{11}^* a_{21}(\lambda) & X_{11}^* a_{22}(\lambda) + X_{21}^* \end{pmatrix}\begin{pmatrix} I & 0 \\ X_{12}^* a_{21}(\lambda) & X_{12}^* a_{22}(\lambda) + X_{22}^* \end{pmatrix}^{-1} \\ &= \begin{pmatrix} a_{11}(\lambda) - a_{12}(\lambda)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} X_{12}^* a_{12}^*(\bar{\lambda}) & a_{12}(\lambda)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} \\ X_{11}^* a_{21}(\lambda) - \tilde{a}_{22}(\lambda) & \tilde{a}_{22}(\lambda) \end{pmatrix},\end{aligned} \tag{8.75}$$

where

$$\tilde{a}_{22}(\lambda) := (X_{11}^* a_{22}(\lambda) + X_{21}^*)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1}. \tag{8.76}$$

Since $s - \lim_{\lambda\to\infty} a_{22}(\lambda) = s - \lim_{\lambda\to\infty} \tilde{a}_{22}(\lambda) = 0$, we obtain from equality (8.76), rewritten as

$$\tilde{a}_{22}(\lambda)[X_{12}^* a_{22}(\lambda) + X_{22}^*] = X_{11}^* a_{22}(\lambda) + X_{21}^*, \tag{8.77}$$

that $X_{21}^* = 0$ and consequently $X_{11}^* X_{22} = I$. It follows from (8.77) and the equalities

$$s - \lim_{\lambda = iy\to\infty} \lambda a_{22}(\lambda) = s - \lim_{\lambda = iy \to\infty} \lambda \tilde{a}_{22}(\lambda) = -I$$

that $X_{22} = X_{11}$. So the operators $X_{11} = X_{22} =: V$ are unitary, $V = (V^*)^{-1}$, and $X = \begin{pmatrix} V & X_{12} \\ 0 & V \end{pmatrix}$.

*Sufficiency.* Let $X_{21} = 0$, $X_{11} = X_{22} = V = (V^*)^{-1}$, $X = (X_{ij})_{i,j=1}^2$, and $\tilde{W}(\lambda) = XW(\lambda) = (\tilde{W}_{ij}(\lambda))_{i,j=1}^2$. Then $\tilde{w}_{12}(\lambda) = X_{11}^* w_{12}(\lambda)$ and $0 \in \rho(\tilde{w}_{12}(\lambda))$. Consequently the following matrix-function [cf (8.17)] is well defined:

$$\tilde{\mathfrak{A}}(\lambda) = \begin{pmatrix} \tilde{a}_{11}(\lambda) & \tilde{a}_{12}(\lambda) \\ \tilde{a}_{21}(\lambda) & \tilde{a}_{22}(\lambda) \end{pmatrix} := \begin{pmatrix} \tilde{w}_{21}(\lambda)^{-1}\tilde{w}_{22}(\lambda) & \tilde{w}_{21}^{-1}(\lambda) \\ \tilde{w}_{11}(\lambda)\tilde{w}_{21}(\lambda)^{-1}\tilde{w}_{22}(\lambda) - \tilde{w}_{12}(\lambda) & \tilde{w}_{11}(\lambda)\tilde{w}_{21}^{-1}(\lambda) \end{pmatrix}. \tag{8.78}$$

As follows from (8.19), the functions $\tilde{\mathfrak{A}}(\lambda)$ and $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ are connected by the equality

$$\tilde{\mathfrak{A}}(\lambda) = \begin{pmatrix} a_{11}(\lambda) - a_{12}(\lambda)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} X_{12}^* a_{21}(\lambda) & a_{12}(\lambda)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} \\ (a_{22}(\lambda)X_{12} + X_{22})^{-1} a_{21}(\lambda) & X_{11}^* a_{22}(\lambda)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} \end{pmatrix}. \tag{8.79}$$

By virtue of (8.20) $\tilde{\mathfrak{A}}(\lambda) \in R_{\mathfrak{L}\oplus\mathfrak{L}}$ and $0 \in \rho(\operatorname{Im} \tilde{a}_{11}(\lambda))$. To complete the proof it is enough to use the obvious relations

$$s - \lim_{y\uparrow\infty} (a_{22}(iy)X_{12} + X_{22})^{-1} = X_{22}^{-1} \Longrightarrow s - \lim_{y\uparrow\infty} \tilde{a}_{21}(iy) = 0;$$

$$s - \lim_{y\uparrow\infty} (X_{12}^* a_{22}(iy) + X_{22}^*)^{-1} = X_{22} \Longrightarrow s - \lim_{y\uparrow\infty} (iy)\tilde{a}_{22}(iy) = -I,$$

and to apply Theorem 9.1.

However, we give a direct proof of the sufficiency by showing that $\tilde{\mathfrak{A}}(\lambda)$ is an $\mathfrak{L}$-preresolvent matrix of the operator

$$A' = V_1 A V_1^* + X_{12}^* X_{22} P_{\mathfrak{L}}, \tag{8.80}$$

where $V_1 (\in [\mathfrak{h}])$ is a unitary continuation of the operator $V^* = X_{22}^*$. Since the subspace $\mathfrak{L}(\subset \mathfrak{h})$ reduces $V_1$, the operator $X_{12}^* X_{22} P_{\mathfrak{L}}$ is self-adjoint and $(A')^* = V_1 A^* V_1^* + X_{12}^* X_{22} P_{\mathfrak{L}}$. This implies that the Green identity for $(A')^*$

$$((A')^* f, g) - (f, (A')^* g) = (A^* V_1^{-1} f, V_1^{-1} g) - (V_1^{-1} f, A^* V_1^{-1} g) = (\Gamma_1 V_1^{-1} f, \Gamma_2 V_1^{-1} g) - (\Gamma_2 V_1^{-1} f, \Gamma_1 V_1^{-1} g)$$

holds, and the triple $\Pi' = \{\mathfrak{L}, \Gamma_1', \Gamma_2'\}$ with $\Gamma_i' = \Gamma_i V_1^{-1}$ $(i = 1, 2)$ forms a BVS for the relation $(A')^*$. The $\gamma$-field $\gamma'(\lambda) = (\Gamma_2' \restriction \mathfrak{N}_\lambda(A'))^{-1}$ corresponding to the BVS $\Pi'$ can be represented as

$$\gamma'(\lambda) = [I - (X_{12}^* X_{22} P_{\mathfrak{L}} + V_1 A_2 V_1^* - \lambda)^{-1} X_{12}^* X_{22} P_{\mathfrak{L}}] V_1 \gamma(\lambda). \tag{8.81}$$

It follows that the Weyl function corresponding to the BVS $\Pi'$ has the form

$$M'(\lambda) = \Gamma_1' \gamma'(\lambda) = M(\lambda) - \Gamma_1 V_1^{-1} (X_{12}^* X_{22} P_{\mathfrak{L}} + V_1 A_2 V_1^* - \lambda)^{-1} X_{12}^* P_{\mathfrak{L}} \gamma(\lambda) =$$

$$= M(\lambda) - \Gamma_1 (X_{22} X_{12}^* P_{\mathfrak{L}} + A_2 - \lambda)^{-1} V_1^{-1} X_{12}^* P_{\mathfrak{L}} \gamma(\lambda)$$

$$= M(\lambda) - \Gamma_1 (A_2 - \lambda)^{-1} [X_{22} X_{12}^* P_{\mathfrak{L}} (A_2 - \lambda)^{-1} + I]^{-1} V^{-1} X_{12}^* P_{\mathfrak{L}} \gamma(\lambda)$$

$$= a_{11}(\lambda) - \gamma^*(\bar{\lambda})(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} X_{12}^* a_{21}(\lambda)$$

$$= a_{11}(\lambda) - a_{21}(\lambda)(X_{12}^* a_{22}(\lambda) + X_{22}^*)^{-1} X_{12}^* a_{21}(\lambda).$$

The following identity can be proved analogously:

$$P_{\mathfrak{L}}(A_2' - \lambda)^{-1} \restriction \mathfrak{L} = P_{\mathfrak{L}}(V_1 A_2 V_1^* - \lambda + X_{12}^* X_{22} P_{\mathfrak{L}})^{-1} \restriction \mathfrak{L}$$

$$= V_1 P_{\mathfrak{L}} (A_2 - \lambda)^{-1} [V_1 + X_{12}^* P_{\mathfrak{L}} (A_2 - \lambda)^{-1}] \restriction \mathfrak{L} = X_{11}^* a_{22}(\lambda)[X_{12}^* a_{22}(\lambda) + X_{22}^*]^{-1},$$

$$P_{\mathfrak{L}} \gamma'(\lambda) = P_{\mathfrak{L}} V_1 \{I - [X_{12} P_{\mathfrak{L}} V_1 + (A_2 - \lambda)^{-1}]^{-1} X_{12} P_{\mathfrak{L}} V_1\} \gamma(\lambda)$$

$$= P_{\mathfrak{L}} V_1 \{I - [(A_2 - \lambda)^{-1} X_{12} P_{\mathfrak{L}} V_1 + I]^{-1} (A_2 - \lambda)^{-1} X_{12} P_{\mathfrak{L}} V_1\} \gamma(\lambda)$$

$$= P_{\mathfrak{L}} V_1 \{P_{\mathfrak{L}} (A_2 - \lambda)^{-1} X_{12} P_{\mathfrak{L}} V_1 + I]^{-1} \gamma(\lambda) = (a_{22}(\lambda) X_{12} + X_{22})^{-1} a_{21}(\lambda). \quad \square$$

**8.** Let $\mathfrak{L}$ be a subspace of $\mathfrak{h}$. Recall that an operator-valued function $P_{\mathfrak{L}} \mathbf{R}_\lambda \restriction \mathfrak{L}$ is said to be an $\mathfrak{L}$-pseudoresolvent ($\mathfrak{L}$-resolvent) of an operator $A$ if $\mathbf{R}_\lambda \in P\Omega_A$ $(\mathbf{R}_\lambda \in \Omega_A)$. The set of $\mathfrak{L}$-pseudoresolvents ($\mathfrak{L}$-resolvents) of $A$ is denoted by $P\Omega_A^{\mathfrak{L}}$ $(\Omega_A^{\mathfrak{L}})$. A nondecreasing operator-valued function $\Sigma(t) := P_{\mathfrak{L}} E(t) \restriction \mathfrak{L} = \Sigma(t-0)$ is called an $\mathfrak{L}$-spectral function of $A$ if $E(t)$ is a generalized (extended to $\tilde{\mathfrak{h}} \supset \mathfrak{h}$) spectral function of the operator $A$. A function $\Sigma(t)$ is said to be orthogonal if $E(t)$ is orthogonal. $\mathfrak{L}$-Pseudoresolvents and $\mathfrak{L}$-spectral functions are related as

$$\int_{-\infty}^{\infty} \frac{d\Sigma(t)}{t-\lambda} = P_{\mathfrak{L}} \mathbf{R}_\lambda \upharpoonright \mathfrak{L} (\mathbf{R}_\lambda := P_{\mathfrak{h}} (\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{h} = P_{\mathfrak{h}} \int_{-\infty}^{\infty} \frac{dE(t)}{t-\lambda} \upharpoonright \mathfrak{h}).$$

If $P_1$ is the orthogonal projection of $\tilde{\mathfrak{h}}$ onto $\tilde{\mathfrak{h}} \ominus \tilde{A}(0)$, then

$$E(\infty) := s - \lim_{t\uparrow\infty} E(t) = P_1, \qquad \Sigma(\infty) := s - \lim_{t\uparrow\infty} \Sigma(t) = P_{\mathfrak{L}} P_1 \upharpoonright \mathfrak{L}.$$

In the case $\mathfrak{L} \cap \mathfrak{N} = \{0\}$ the following equivalence holds:

$$\Sigma(\infty) = I_{\mathfrak{L}} \Longleftrightarrow \mathbf{R}_\lambda \in \Omega_A.$$

A full description of the set $P\Omega_A^{\mathfrak{L}}$ of $\mathfrak{L}$-pseudoresolvents and, therefore, a description of the set of $\mathfrak{L}$-spectral functions of the operator $A$ is given in

**Proposition 8.10.** *Suppose that* $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ *is a BVS for a linear relation* $A^*$, $W_{\Pi\mathfrak{L}}(\lambda) = (w_{ij}(\lambda))_{i,j=1,2}$ *is the corresponding* $\Pi\mathfrak{L}$*-resolvent matrix* (8.17). *Then the formula*

$$\int_{-\infty}^{\infty} \frac{d\Sigma(t)}{t-\lambda} = P_{\mathfrak{L}} \mathbf{R}_\lambda \upharpoonright \mathfrak{L} = [w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda)][w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda)]^{-1} \tag{8.82}$$

*establishes a one-to-one correspondence between* $\mathfrak{L}$*-pseudoresolvents* $P_{\mathfrak{L}} \mathbf{R}_\lambda \upharpoonright \mathfrak{L} \in P\Omega_A^{\mathfrak{L}}$ *and* $\tau(\lambda) \in \tilde{R}_{\mathfrak{L}}$. *Further, the following equivalence holds:*

$$P_{\mathfrak{L}} \mathbf{R}_\lambda \upharpoonright \mathfrak{L} \in \Omega_A^{\mathfrak{L}} \Longleftrightarrow \tau(\lambda) \quad \textit{is } M\textit{-admissible.} \tag{8.83}$$

**Proof.** $a_{12}(\lambda)^{-1} \in [\mathfrak{L}]$ for all $\lambda \in \rho(A;\mathfrak{L})$ since $P_{\mathfrak{L}}$ maps isomorphically $\mathfrak{N}_\lambda$ onto $\mathfrak{L}$ for all $\lambda \in \rho(A;\mathfrak{L})$. Therefore (3.69) and (8.13) yield

$$\begin{aligned} P_{\mathfrak{L}} \mathbf{R}_\lambda \upharpoonright \mathfrak{L} &= P_{\mathfrak{L}} (\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{L} = P_{\mathfrak{L}} (A_2 - \lambda)^{-1} \upharpoonright \mathfrak{L} - P_{\mathfrak{L}} \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1} \gamma^*(\bar{\lambda}) \upharpoonright \mathfrak{L} \\ &= a_{22}(\lambda) - a_{21}(\lambda)[a_{12}(\lambda)^{-1}(\tau(\lambda) + a_{11}(\lambda))]^{-1} \\ &= \{a_{22}(\lambda)[a_{12}^{-1}(\lambda)\tau(\lambda) + a_{12}^{-1}(\lambda)a_{11}(\lambda)] - a_{21}(\lambda)\}[a_{12}^{-1}(\lambda)\tau(\lambda) + a_{12}^{-1}(\lambda)a_{11}(\lambda)]^{-1} \\ &= [w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda)][w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda)]^{-1}. \ \square \end{aligned}$$

**Corollary 8.2.** *Suppose that under the assumptions of Proposition* 8.10 $A_2 (:= \ker \Gamma_2) = A \dot{+} \hat{\mathfrak{N}}$. *Then the equivalence* (8.83) *takes the form*

$$P_{\mathfrak{L}} \mathbf{R}_\lambda \upharpoonright \mathfrak{L} \in \Omega_A^{\mathfrak{L}} \Longleftrightarrow s - R - \lim_{y\uparrow\infty} y^{-1}\tau(iy) = 0. \tag{8.84}$$

The proof follows from a comparison of Proposition 8.10 and Corollary 3.8.

## 9. INVERSE PROBLEMS FOR PRERESOLVENT AND AND RESOLVENT MATRICES OF A HERMITIAN OPERATOR

**1.** In this section, the inverse problem for a $\Pi\mathfrak{L}$-preresolvent matrix is solved in the framework of each of the three models considered in Sec. 5.

**Theorem 9.1.** *Let* $\mathcal{H}$, $\mathfrak{L}$ *be Hilbert spaces,* $\dim \mathcal{H} = \dim \mathfrak{L}$. *For an operator-valued function* $\mathfrak{A}(\lambda) = (a_{jk}(\lambda))_{j,k=1}^2$ *(holomorphic on* $\mathbb{C}_+ \cup \mathbb{C}_-$ *with values in* $[\mathcal{H} \oplus \mathfrak{L}]$*) to be the* $\Pi\mathfrak{L}$*-preresolvent matrix corresponding* *to some BVS of a linear relation $A^*$ such that $\mathfrak{L} \perp A_2(0)$, it is necessary and sufficient that the following conditions hold:*

(1) $\mathfrak{A}(\lambda) \in R_{\mathcal{H}\oplus\mathfrak{L}}$;

(2) $0 \in \rho(\operatorname{Im} a_{11}(i))$;

(3) $\lim_{y\uparrow\infty} iy(a_{22}(iy)l, l) = -\|l\|^2 \quad \forall l \in \mathfrak{L}$;

(4) $s - \lim_{y\uparrow\infty} a_{12}(iy) = 0$.

**Proof.** *Necessity* of these conditions is implied by formula (8.13) for the $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$.

*Sufficiency.* (1) Suppose that conditions (1)–(4) are fulfilled. Since $\mathfrak{A}(\lambda) \in R_{\mathcal{H}\oplus\mathfrak{L}}$, it follows from (5.1) that

$$\mathfrak{A}(\lambda) = B\lambda + C + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\Sigma(t), \qquad \int_{\mathbb{R}} \frac{d\Sigma(t)}{1+t^2} \in [\mathcal{H} \oplus \mathfrak{L}], \tag{9.1}$$

where $\Sigma(t) = (\Sigma_{jk}(t))_{j,k=1}^2 = \Sigma(t-0)$ is a nondecreasing operator-valued function. As in Sec. 5, $\mathfrak{B}(\mathfrak{A})$ stands for the Hilbert space of vector-valued functions $F(\lambda)$ of the form in (5.22)

$$F(\lambda) = b + \int_{\mathbb{R}} \frac{d\Sigma(t) f(t)}{t-\lambda}, \qquad b \in \mathcal{H}_B = \mathfrak{R}(B^{1/2}), \qquad f(t) \in L_2(d\Sigma, \mathcal{H} \oplus \mathfrak{L}) \tag{9.2}$$

endowed with the inner product (5.23), $A_0$ is a Hermitian operator of multiplication by $\lambda$ in $\mathfrak{B}(\mathfrak{A})$,

$$A_{0*} = \{\hat{F}(\mu) = \{F(\mu), \tilde{F}(\mu)\} \in \mathfrak{B}(\mathfrak{A})^2 : \exists h_1, h_2 \in \mathcal{H} \oplus \mathfrak{L}, \tilde{F}(\mu) - \mu F(\mu) = h_1 - \mathfrak{A}(\mu) h_2\}.$$

In accordance with Theorem 6.1, $\mathfrak{A}(\lambda)$ is the Weyl function corresponding to the generalized BVS $\Pi_3 = \{\mathcal{H} \oplus \mathfrak{L}, \chi_1, \chi_2\}$ for the linear relation $A_{0*}$, where $\chi_j \hat{F} = h_j$ $(j = 1, 2)$. Because of condition (3), the measure $d\Sigma_{22}(t)$ is bounded: $\int_{\mathbb{R}} d(\Sigma_{22}(t)l, l) < \infty$ for all $l \in \mathfrak{L}$. Therefore,

$$\left| \int_{\mathbb{R}} \frac{d(\Sigma_{12}(t)h, l)}{t-\lambda} \right|^2 \leq \int_{\mathbb{R}} \frac{d(\Sigma_{11}(t)h, h)}{|t-\lambda|^2} \cdot \int_{\mathbb{R}} d(\Sigma_{22}(t)l, l).$$

This inequality and condition (4) imply that

$$\mathfrak{A}(\lambda)l = \int_{\mathbb{R}} \frac{d\Sigma(t)}{t-\lambda} l \left( \Longleftrightarrow a_{12}(\lambda)l = \int_{\mathbb{R}} \frac{d\Sigma_{12}(t)}{t-\lambda} l, \qquad a_{22}(\lambda)l = \int_{\mathbb{R}} \frac{d\Sigma_{22}(t)}{t-\lambda} l \right) \tag{9.3}$$

and the integral in (9.3) converges in the strong sense. This enables us to embed the subspace $\mathfrak{L}$ into $\mathfrak{B}(\mathfrak{A})$ ($i_{\mathfrak{A}}$: $l \to \mathfrak{A}(\mu)l$). In view of definition (5.23) the linear manifold

$$\mathfrak{L}_\infty = \{l_\infty(\mu) := \mathfrak{A}(\mu)l : l \in \mathfrak{L}\}$$

is closed. It follows from condition (3) that the mapping $i_{\mathfrak{A}}$ is isometric:

$$(l_\infty(\mu), k_\infty(\mu))_{\mathfrak{B}(\mathfrak{A})} = (l, k)_{L_2(d\Sigma, \mathcal{H}\oplus\mathfrak{L})} = \int_{\mathbb{R}} d(\Sigma(t)l, k) = (l, k)_{\mathfrak{L}}.$$

Setting for all $h \in \mathcal{H} \oplus \mathfrak{L}$

$$h_\lambda(\mu) := \frac{\mathfrak{A}(\lambda) - \mathfrak{A}(\mu)}{\lambda - \mu} h = \int_{\mathbb{R}} \frac{d\Sigma(t)}{(t-\lambda)(t-\mu)} h + B_{\mathfrak{A}} h \in \mathfrak{N}_\lambda^*(A_0), \tag{9.4}$$

we obtain from (9.4), (9.3), and (5.23) the main identity:

$$(l_\infty(\mu), h_\lambda(\mu))_{\mathfrak{B}(\mathfrak{A})} = \int_{\mathbb{R}} \frac{d(\Sigma(t)l, h)}{t - \bar{\lambda}} = (\mathfrak{A}(\bar{\lambda})l, h)_{\mathcal{H}\oplus\mathfrak{L}} = (l, \mathfrak{A})\lambda)l)_{\mathcal{H}\oplus\mathfrak{L}}. \tag{9.5}$$

We denote by $\pi_{\mathfrak{L}}$, $\pi_{\mathcal{H}}$ the orthogonal projections from $\mathcal{H} \oplus \mathfrak{L}$ onto $\mathfrak{L}$ and $\mathcal{H}$ respectively and consider the linear relation

$$A_* := \{\hat{F}(\mu) \in A_{0*} : \pi_2 \chi_2 \hat{F} = 0\}. \tag{9.6}$$

$A_*$ is a closed subspace. Indeed suppose that $F_n(\lambda) \to F(\lambda)$ and $\tilde{F}_n(\lambda) \to \tilde{F}(\lambda)$ in $\mathfrak{B}(\mathfrak{A})$ as $n \to \infty$. Then for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ the sequence $\hat{F}_n(\lambda)$ strongly converges to $\hat{F}(\lambda) = \{F(\lambda), \tilde{F}(\lambda)\}$. It follows from (9.2) (for $\lambda = \pm i$) that the sequence $(\operatorname{Im} \mathfrak{A}(i))\chi_2 \hat{F}_n$ is convergent. Now condition (2) yields that the sequences $\chi_2 \hat{F}_n$, $\chi_1 \hat{F}_n$ are convergent. Setting $h_j := \lim_{n\to\infty} \chi_j \hat{F}_n$ $(j = 1, 2)$ we obtain from (9.2) that $\tilde{F}(\lambda) - \lambda F(\lambda) = h_1 - \mathfrak{A}(\lambda) h_2$, that is, $\hat{F}(\lambda) \in A_*$.

Define a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ of the linear relation $A_*$ by the equality

$$\Gamma_j \hat{F} = \pi_{\mathcal{H}} \chi_j \hat{F} \qquad \forall \hat{F} \in A_* \qquad (j = 1, 2). \tag{9.7}$$

The Green identity for the BVS $\Pi$ is implied by the Green identity for the BVS $\Pi'$ of the linear relation $A_{0*}$. The surjectivity of the mapping $\Gamma = \{\Gamma_2, \Gamma_1\}$ is implied by the equality $\Gamma_2 \hat{\mathfrak{N}}_\lambda(A) = \{\chi_2 \hat{h}_\lambda(\mu) = h : h \in \mathcal{H}\} = \mathcal{H}$ and the relation $\Gamma_1 A_2 = \mathcal{H}$. To prove the last relation we first note that

$$\hat{f}(\mu) = \{f(\mu), \tilde{f}(\mu)\} := \{\frac{h_{-i}(\mu) - h_{-i}(i)}{\mu - i}, \frac{\mu h_{-i}(\mu) - i h_{-i}(i)}{\mu - i}\} \in A_2 := \ker \Gamma_2.$$

Taking account of (9.4) we have

$$\tilde{f}(\mu) - \mu f(\mu) = h_{-i}(i) = \operatorname{Im} \mathfrak{A}(i) h \Longrightarrow \chi_1 \hat{f} = h_{-i}(i) = \operatorname{Im} \mathfrak{A}(i) h \Longrightarrow \Gamma_1 \hat{f} = \pi_1 (\operatorname{Im} \mathfrak{A}(i)) h = \operatorname{Im} a_{11}(i) h. \tag{9.8}$$

The equality $\Gamma_1 A_2 = \mathcal{H}$ is a consequence of relation (9.8) and condition (2). Therefore, the triple $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ forms a BVS for $A_*$, and $a_{11}(\lambda)$ coincides with the corresponding Weyl function $M(\lambda)$. Note that by virtue of (9.7) the operator $A$ and the linear relation $A_2 = \ker \Gamma_2$ take the form

$$A = \{\hat{F} \in A_{0*} : \chi_2 \hat{F} = \pi_{\mathcal{H}} \chi_1 \hat{F} = 0\}, \qquad A_2 = \{\hat{F} \in A_{0*} : \chi_2 \hat{F} = 0\} = \ker \chi_2,$$

$A^* = A_*$ and $\mathfrak{N}_\lambda(A) = \{h_\lambda(\mu) : h \in \mathcal{H}\}$.

It follows from (9.5) and the relation

$$\Gamma_2 \hat{h}_\lambda(\mu) = \Gamma_2 \{h_\lambda(\mu), \lambda h_\lambda(\mu)\} = h \qquad \forall h \in \mathcal{H}, \quad \lambda \in \rho(A_2)$$

that

$$\gamma(\lambda) h = h_\lambda(\mu), \qquad P_{\mathfrak{L}_\infty} \gamma(\lambda) = P_{\mathfrak{L}_\infty} h_\lambda(\mu) = \pi_{\mathfrak{L}} \mathfrak{A}(\lambda) h = a_{21}(\lambda) h.$$

Putting for some $\lambda \in \rho(A_2)$ $\hat{F} = \{F(\mu), \tilde{F}(\mu)\} = \{l_\lambda(\mu), \lambda l_\lambda(\mu) + l_\infty(\mu)\}$ (where $l_\infty(\mu) = \mathfrak{A}(\mu) l$, $l \in \mathfrak{L}$) we find that

$$\tilde{F}(\mu) - \mu F(\mu) = l_\infty(\lambda) = \mathfrak{A}(\lambda) l \to \chi_2 \hat{F} = 0,$$

that is, $\hat{F} \in A_2$. This implies $\{l_\lambda(\mu), l_\infty(\mu)\} \in A_2 - \lambda$; hence $(A_2 - \lambda)^{-1} l = l_\lambda(\mu)$. We derive from (9.5) the following equality:

$$(P_{\mathfrak{L}} (A_2 - \lambda)^{-1} l, k)_{\mathfrak{B}(\mathfrak{A})} = (l_\lambda(\mu), k_\infty)_{\mathfrak{B}(\mathfrak{A})} = (a_{22}(\lambda) l, k)_{\mathfrak{L}},$$

which yields $P_{\mathfrak{L}} (A_2 - \lambda)^{-1} \upharpoonright \mathfrak{L} = a_{22}(\lambda)$ for all $\lambda \in \rho(A_2)$.

To complete the proof it remains to note that the condition $\mathfrak{L}_\infty \perp A_2(0)$ is a consequence of condition (3).

(2) Starting from the integral representation (9.1) of the operator-valued function $\mathfrak{A}(\lambda)$ we consider the Hilbert space $\mathfrak{h} = \mathcal{H}_B \oplus L_2(d\Sigma, \mathcal{H})$ and the linear relation $A_{0*}$ in it [cf. (5.16)]:

$$A_{0*} = \{\hat{f} = \{b \oplus f(t), \tilde{b} \oplus \tilde{f}(t)\} \in \mathfrak{h}^2 : \exists h \in \mathcal{H} \oplus \mathfrak{L}, t f(t) - \tilde{f}(t) = h, b = Bh\}. \tag{9.9}$$

In this case $\mathfrak{A}(\lambda)$ is the Weyl function (see Remark 6.2) of the operator

$$A_0 = \{\{0 \oplus f(t), \tilde{b} \oplus tf(t)\} \in A_{0*} : \int_{\mathbb{R}} d\Sigma(t) f(t) + \tilde{b} = 0\}, \tag{9.10}$$

corresponding to a generalized BVS $\Pi_2 = \{\mathcal{H} \oplus \mathfrak{L}, \chi_1, \chi_2\}$ of the form in (5.17) with

$$\chi_1 \hat{f} = \tilde{b} + Ch + \int_{\mathbb{R}} d\Sigma(t) \frac{t\tilde{f}(t) + f(t)}{1+t^2}, \qquad \chi_2 \hat{f} = h. \tag{9.11}$$

Define the operator $A$ by the equality

$$A = \{\hat{f} \in A_{0*} : \pi_{\mathcal{H}} \left[\tilde{b} + \int_{\mathbb{R}} d\Sigma(t) f(t)\right] = \pi \mathcal{H} \tilde{b} + \int_{\mathbb{R}} (d\Sigma_{11}(t) f_1(t) + d\Sigma_{12}(t) f_2(t)) = 0\},$$

in which $f(t) = f_1(t) \oplus f_2(t)$; $f_1(t) \in \mathcal{H}$, $f_2(t) \in \mathfrak{L}$, $\forall t \in \mathbb{R}$. It easy to see that $A = \ker \chi_2 \cap \ker(\pi_{\mathcal{H}} \chi_1)$. Further, the triple $\Pi_2' = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ in which $\Gamma_j = \pi_{\mathcal{H}} \chi_j$ $(j = 1, 2)$, forms a generalized BVS of the linear relation

$$A_* = \{\hat{f} \in A_{0*} : \pi_{\mathfrak{L}} \chi_2 \hat{f} = 0\}. \tag{9.12}$$

Indeed, the Green formula for $A_*$ is implied from the definition of the BVS $\Pi_2'$ and the Green formula for $A_{0*}$. It follows from the equality

$$\ker \Gamma_2 = \ker \chi_2 =: A_2 = \{\hat{f} = \{0 \oplus f(t), \tilde{b} \oplus tf(t)\} \in \mathfrak{h}^2\} \tag{9.13}$$

that the linear relation $A_2 := \ker \Gamma_2$ is self-adjoint. The closability of the mappings $\Gamma_1, \Gamma_2$ is a consequence of (9.11). Making use of relation (5.15) and the definition of the operator $A$, we find an explicit form of the linear manifold $\mathfrak{N}_\lambda^*(A)$:

$$\mathfrak{N}_\lambda^*(A) := \mathfrak{N}_\lambda(A) \cap A_* = \{h_\lambda = \{Bh \oplus \frac{h}{t-\lambda}\} : h \in \mathcal{H}\}. \tag{9.14}$$

We derive from (9.14) and the equalities $\chi_2 \hat{h}_\lambda = h$, $\pi_1 \chi_1 \hat{h}_\lambda = \pi_1 \mathfrak{A}(\lambda) \chi_2 \hat{h}_\lambda$ $(h \in \mathcal{H})$ that $a_{11}(\lambda)$ coincides with the Weyl function $M(\lambda)$ corresponding to the BVS $\Pi_2' : a_{11}(\lambda) = M(\lambda)$. By Proposition 6.2, the generalized BVS $\Pi_2'$ is an ordinary one, that is, $A_* = A^*$, $\mathfrak{N}_\lambda^*(A) = \mathfrak{N}_\lambda(A)$, and the mapping $\Gamma$: $A_* \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Define now the embedding of the space $\mathfrak{L}$ into $\mathfrak{h}$, identifying the vector $l \in \mathfrak{L}$ with the constant vector-valued function $l(t) \equiv l \in \mathfrak{h}$. It follows from condition (3) that this embedding is isometric:

$$(l_1, l_2)_{\mathfrak{h}} := \int_{\mathbb{R}} d(\Sigma(t) l_1, l_2)_{\mathfrak{L}} = \int_{\mathbb{R}} d(\Sigma_{22}(t) l_1, l_2)_{\mathfrak{L}} = (l_1, l_2)_{\mathfrak{L}}.$$

Conditions (3) and (4) also yield that $\mathfrak{L} \subset \ker B$ and the representation (9.3) for the vector-valued functions $a_{12}(\lambda) l$, $a_{22}(\lambda) l$ $(l \in \mathfrak{L})$ hold. Making use of (9.13) and the formula for the $\gamma$-field, $\gamma(\lambda) h = (\Gamma_2 \restriction \mathfrak{N}_\lambda(A))^{-1} h = \hat{h}_\lambda = \{h_\lambda, \lambda h_\lambda\}$ implied by (9.11), (9.14), we find that for all $l, k \in \mathfrak{L}$, $h \in \mathcal{H}$,

$$(\gamma^*(\bar{\lambda}) l, h)_{\mathfrak{h}} = (l, \gamma(\bar{\lambda}) h)_{\mathfrak{h}} = \left(l, \frac{h}{t-\bar{\lambda}}\right)_{\mathfrak{h}} = \int_{\mathbb{R}} \frac{d(\Sigma_{12}(t) l, h)}{t-\lambda} = (a_{12}(\lambda) l, h)_{\mathcal{H}};$$

$$(P_{\mathfrak{L}}(A_2 - \lambda)^{-1} l, k)_{\mathfrak{h}} = \left(P_{\mathfrak{L}} \frac{l}{t-\lambda}, k\right)_{\mathfrak{h}} = \int_{\mathbb{R}} \frac{d(\Sigma_{22}(t) l, k)}{t-\lambda} = (a_{22}(\lambda) l, k)_{\mathfrak{L}}.$$

Thus, the coincidence of $\mathfrak{A}(\lambda)$ with the $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ is proved.

(3) We give one more sketch of the proof of Theorem 9.1 in the framework of the Krein–Langer model [81]. To this end we introduce, as in Sec. 5, the linear manifold $G = \Phi \otimes (\mathcal{H} \oplus \mathfrak{L})$, which is the

algebraic tensor product of the space $\mathcal{H} \oplus \mathfrak{L}$ by the linear space $\Phi$ of functions on $\mathbb{C}_+$ with finite support. Further we associate with the operator-valued function $\mathfrak{A}(\lambda)$ a Hilbert space $\mathfrak{K}(\mathfrak{A})$, which is obtained in a canonical way from the linear manifold $G$ metrized by means of the kernel $(\lambda - \bar{\mu})^{-1}(\mathfrak{A}(\lambda) - \mathfrak{A}^*(\mu))$ [see (5.3)].

Consider the linear relation

$$A_{0*} = \left\{\hat{f} = \left\{\sum_\lambda f_\lambda \otimes \delta_\lambda, \sum \lambda f_\lambda \otimes \delta_\lambda\right\} \in (\mathfrak{K}(\mathfrak{A})^2\right\}$$

in $\mathfrak{K}(\mathfrak{A})$ and the generalized BVS $\Pi_1 = \{\mathcal{H} \oplus \mathfrak{L}, \chi_1, \chi_2\}$ for $A_{0*}$ with

$$\chi_1 \hat{f} = \sum_\lambda \mathfrak{A}(\lambda) f_\lambda, \qquad \chi_2 \hat{f} = \sum_\lambda f_\lambda.$$

Define a linear relation $A^*$ as a closure of the linear manifold $A_* = \{\hat{f} \in A_{0*} : \pi_{\mathfrak{L}} \chi_2 \hat{f} = 0\}$. As in Proposition 5.1 (see also [26]), one can check that the mappings $\Gamma_j := \pi_{\mathcal{H}} \chi_j$: $A^* \to \mathcal{H}$ $(j = 1, 2)$ are continuous and the triple $\Pi_1' = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ forms (an ordinary) BVS for $A^*$. Define the embedding of $\mathfrak{L}$ into $\mathfrak{K}(\mathfrak{A})$ by the equality

$$\mathfrak{L} \ni l \to l_\infty := \lim_{y \uparrow \infty} (-iy)(l \otimes \delta_{iy}) \in \mathfrak{K}(\mathfrak{A}). \tag{9.15}$$

The existence of a limit in (9.15) is implied by condition (3). Further, it follows from conditions (3) and (4) and equality (9.3) that the following equalities hold:

$$\begin{aligned} (l_\infty, k_\infty)_{\mathfrak{B}(\mathfrak{A}) = \lim_{y\uparrow\infty}} y(\operatorname{Im} \mathfrak{A}(iy) l, k)_{\mathfrak{L}} &= \lim_{y\uparrow\infty} y(\operatorname{Im} a_{22}(iy) l, k)_{\mathfrak{L}} = (l, k)_{\mathfrak{L}}; \\ (l_\infty, \delta_\lambda \otimes h)_{\mathfrak{B}(\mathfrak{A}) = \lim_{y\uparrow\infty}} iy\left(\tfrac{\mathfrak{A}^*(\lambda) - \mathfrak{A}(iy)}{iy - \lambda} l, h\right)_{\mathcal{H}\oplus\mathfrak{L}} &= (l, P_{\mathfrak{L}} \mathfrak{A}(\lambda) h)_{\mathfrak{L}}, \end{aligned} \tag{9.16}$$

the former one of which means that the embedding (9.15) is isometric.

For all $l \in \mathfrak{L}$, $\lambda \in \rho(A_2)$ we have $(A_2 - \lambda)^{-1} l = \delta_\lambda \otimes l$, where $A_2 := \ker \Gamma_2 = \ker \chi_2$. Indeed, passing to the limit as $n \to \infty$ in the relation

$$\hat{g}_n = \{\delta_\lambda \otimes l - \delta_{\lambda_n} \otimes l, \lambda \delta_\lambda \otimes l - \lambda_n \delta_{\lambda_n} \otimes l\} \in A_2 \qquad (\lambda_n = in)$$

and making use of the equality $\|\delta_{iy} \otimes l\|^2_{\mathfrak{K}(\mathfrak{A})} = y^{-1}(\operatorname{Im} \mathfrak{A}(iy) l, l)_{\mathcal{H}\oplus\mathfrak{L}}$, we obtain $\{\delta_\lambda \oplus l, l_\infty\} \in A_2 - \lambda$, that is, $(A_2 - \lambda)^{-1} l_\infty = \delta_\lambda \otimes l$. Now the last equality in (9.16) yields $a_{22}(\lambda) = P_{\mathfrak{L}}(A_2 - \lambda)^{-1} \restriction \mathfrak{L}$. Further, it follows from (9.16) and the equality $\Gamma_2\{\delta_\lambda \otimes h, \lambda \delta_\lambda \otimes h\} = h$ that $\gamma(\lambda) h = \delta_\lambda \otimes h$ for all $h \in \mathcal{H}$ and $a_{12}(\lambda) = \gamma^*(\bar{\lambda})$. □

**2.** Let $A$ be a Hermitian operator with gap $(\alpha, \beta)$ and $A_\alpha$, $A_\beta$ be its extremal extensions defined by equalities (4.20). We characterize the $\Pi\mathfrak{L}$-preresolvent matrices which correspond to the BVS's $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ with $A_2 \in \operatorname{Ex}_A(\alpha, \beta)$.

**Lemma 9.1.** *Let $-\infty \le \alpha < \beta \le \infty$, $A$ be a simple Hermitian operator with gap $(\alpha, \beta)$, $E = \mathbb{R} \setminus (\alpha, \beta)$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS for $A^*$. Then the following equivalences hold:*

*(1) $M(\lambda) \in S_{\mathcal{H}}(E) \iff A_1, A_2 \in \operatorname{Ex}_A(\alpha, \beta)$ and $(A_1 - x)^{-1} \le (A_2 - x)^{-1}$ $\forall x \in (\alpha, \beta)$;*

*(2) $M(\lambda) \in S^-_{\mathcal{H}}(E) \iff A_1, A_2 \in \operatorname{Ex}_A(\alpha, \beta)$ and $(A_1 - x)^{-1} \ge (A_2 - x)^{-1}$ $\forall x \in (\alpha, \beta)$.*

**Proof.** Suppose that $M(\lambda) \in S_{\mathcal{H}}(E)$. Then $(\alpha, \beta) \subset \rho(A_2)$ since $M(\lambda)$ is holomorphic on $(\alpha, \beta)$. Owing to the fact that $M(\lambda)$ is nonnegative in the gap $(\alpha, \beta)$ we obtain $0 \in \rho(M(\lambda))$ for all $\lambda \in (\alpha, \beta)$ and in view of Proposition 1.6 we have $(\alpha, \beta) \subset \rho(A_1)$. The inequality $(A_1 - x)^{-1} \le (A_2 - x)^{-1}$ for all $x \in (\alpha, \beta)$ and, therefore, the implication $\Longrightarrow$ in statement (1) are implied by formula (3.39). The inverse implication follows also from formula (3.39). □

**Corollary 9.1.** *Assume that $A$ is a Hermitian operator with gap $(\alpha, \beta)$, $E = \mathbb{R} \setminus (\alpha, \beta)$, $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ is a BVS for $A^*$, $M(\lambda)$ is the corresponding Weyl function. Then we have $A_2 \in \operatorname{Ex}_A(\alpha, \beta)$ and the extension $A_2$ is transversal to $A_\alpha$ ($A_\beta$) if and only if there exists an operator $K = K^* \in [\mathcal{H}]$ such that $M(\lambda) - K \in S^+(E)$ ($S^-(E)$).*

**Proof.** Assume that $A_2 \in \text{Ex}_A(\alpha,\beta)$ and the extensions $A_2$, $A_\alpha$ are transversal. Then there exists a BVS $\Pi_1 = \{\mathcal{H}, \Gamma_1', \Gamma_2'\}$ such that $A_\alpha = \ker \Gamma_1'$, $A_2 = \ker \Gamma_2'$. It follows from Lemma 9.1 and the extremal property (4.21) of the extension $A_\alpha$ that the Weyl function $M_1(\lambda)$ corresponding to the BVS $\Pi_1$ satisfies the relation $M_1(\lambda) \in S^+_{\mathcal{H}}(E)$. In view of the equalities $\ker \Gamma_2 = \ker \Gamma_2' = A_2$ we obtain from Lemma 5.2 that $M(\lambda) = M_1(\lambda) + K$ for some $K = K^* \in [\mathcal{H}]$.

Conversely, assume that $M_1(\lambda) := M(\lambda) - K \in S_{\mathcal{H}}(E)$ for some $K = K^* \subset [\mathcal{H}]$. Then $M_1(\lambda)$ is the Weyl function corresponding to the BVS $\Pi_2 = \{\mathcal{H}, \Gamma_1'', \Gamma_2''\}$, where $\Gamma_2'' = \Gamma_1 - K\Gamma_2$, and Lemma 9.1 yields that $A_1'' := \ker(\Gamma_1 - K\Gamma_2) \in \text{Ex}_A(\alpha,\beta)$ and $A_2$, $A_1''$ are transversal extensions of $A$. In accordance with Proposition 1.4 we have $0 \in \rho[(A_2-\lambda)^{-1} - (A_1''-\lambda)^{-1}]$ for all $\lambda \in (\alpha,\beta)$. In view of (4.21) $(A_1''-\lambda)^{-1} \geq (A_\alpha-\lambda)^{-1}$ for all $\lambda \in (\alpha,\beta)$. Therefore, $0 \in \rho[(A_2-\lambda)^{-1} - (A_\alpha-\lambda)^{-1}]$, i.e., $A_2$ and $A_\alpha$ are transversal extensions. □

**Definition 9.1** [38]. An operator-valued function $F(\lambda)$ ($\in R_{\mathcal{H}}$) is said to be from the class $R_{\mathcal{H}}[a,b]$ if $F(\lambda)$ is holomorphic and nonnegative on $(-\infty, a)$ and $F(\lambda)$ is holomorphic and nonpositive on $(b, +\infty)$.

It is easy to see that $R_{\mathcal{H}}[a,b] = S^+_{\mathcal{H}}[a,+\infty) \cap S^-_{\mathcal{H}}(-\infty, b]$.

**Theorem 9.2.** *For the operator-valued function* $\mathfrak{A}(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix}$ *to be the* $\Pi\mathfrak{L}$*-preresolvent matrix of a Hermitian operator* $A$ *with gap* $(\alpha,\beta)$*, corresponding to the BVS* $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ *such that* $A_2 \in \text{Ex}_A(\alpha,\beta)$*,* $A_2(0) \perp \mathfrak{L}$*, it is necessary and sufficient that conditions* (1)-(4) *from Theorem* 9.1 *hold and* $a_{11}(\lambda)$ *be holomorphic on* $(\alpha,\beta)$*.*

*In this case the following equivalences hold:*

(1) *the equality* $A_2 = A_\alpha$ $(A_2 = A_\beta)$ *is equivalent to the first (second) condition from* (4.27) *for* $M(\lambda) := a_{11}(\lambda)$*;*

(2) *in the case of a simple semibounded below operator* $A \geq a$ *the following equivalences hold:*

$$\mathfrak{A}(\lambda) \in S_{\mathcal{H}\oplus\mathfrak{L}}(a,+\infty) \Longleftrightarrow a_{11}(\lambda) \in S_{\mathcal{H}}(a,+\infty) \Longleftrightarrow A_1 \geq A_2 \geq a; \tag{9.17}$$

(3) *in the case of a simple semibounded above operator* $A \leq b$ *the following equivalences hold:*

$$\mathfrak{A}(\lambda) \in S^-_{\mathcal{H}\oplus\mathfrak{L}}(-\infty,b) \Longleftrightarrow a_{11}(\lambda) \in S^-_{\mathcal{H}}(-\infty,b) \Longleftrightarrow A_1 \leq A_2 \leq b; \tag{9.18}$$

(4) *if* $A$ *is a bounded operator* $(a \leq A \leq b)$ *the following equivalences hold:*

$$\mathfrak{A}(\lambda) \in R_{\mathcal{H}\oplus\mathfrak{L}}[a,b] \Longleftrightarrow a_{11}(\lambda) \in R_{\mathcal{H}}[a,b] \Longleftrightarrow A_1 = A + \hat{\mathfrak{N}}, a \leq A_2 \leq b. \tag{9.19}$$

**Proof.** The first part of the theorem is a consequence of Theorem 9.1. Statement (1) is implied by Theorem 9.1 and Corollary 4.5. Implication $\mathfrak{A}(\lambda) \in S_{\mathcal{H}\oplus\mathfrak{L}}(a,+\infty) \Longrightarrow a_{11}(\lambda) \in S_{\mathcal{H}}(a,+\infty)$ in statement (2) is evident. Conversely, if $a_{11}(\lambda) \in S_{\mathcal{H}}(a,+\infty)$, then by Lemma 9.1 we have $(-\infty,a) \subset \rho(A_1) \cap \rho(A_2)$. Now it remains to note that according to (3.39) we have for all $\lambda < a$

$$a_{22}(\lambda) - a_{21}(\lambda)a_{11}^{-1}(\lambda)a_{12}(\lambda) = P_{\mathfrak{L}}[(A_2-\lambda)^{-1} - \gamma(\lambda)M(\lambda)^{-1}\gamma^*(\bar{\lambda})] \upharpoonright \mathfrak{L} = P_{\mathfrak{L}}(A_1-\lambda)^{-1} \upharpoonright \mathfrak{L} \geq 0.$$

The proof of statement (3) is analogous.

(4) Let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be a BVS whose preresolvent matrix coincides with $\mathfrak{A}(\lambda)$ and let $M(\lambda) = a_{11}(\lambda)$ be the corresponding Weyl function. Combining statements (2) and (3) we obtain the equivalence

$$\mathfrak{A}(\lambda) \in R_{\mathcal{H}}[a,b] \Longleftrightarrow a_{11}(\lambda) \in R_{\mathcal{H}}[a,b]$$

and the inequalities

$$a\|f\|^2 \leq \mathfrak{t}_{A_2}[f] \leq \mathfrak{t}_{A_1}[f] \leq \mathfrak{t}_{A_2}[f] \leq b\|f\|^2 \qquad \forall f \in \mathfrak{D}(\mathfrak{t}_{A_1}) \subset \mathfrak{D}(\mathfrak{t}_{A_2}). \tag{9.20}$$

By virtue of the relation $a_{11}(\lambda) = M(\lambda) \in R_{\mathcal{H}}[a,b]$ we have $A_2 \in [\mathcal{H}]$, $a \leq A_2 \leq b$, $a \leq A_1' \leq b$, where $A_1'$ is an operator part of the linear relation $A_1$. Inequalities (9.18) yield

$$(A_1' f, f) = (A_2 f, f) \qquad \forall f \in \mathfrak{D}(A_1) = \overline{\mathfrak{D}(A_1)}.$$

Hence, taking into account the equality $A_1(0) = \mathfrak{D}(A_1)^{\perp}$, we obtain that for all $f \in \mathfrak{D}(A_1)$ there exists $h \in \mathfrak{D}(A_1)^{\perp}$ such that

$$A_2 f = A_1' f + h \Longrightarrow \{f, A_2 f\} \in A_2 \cap A_1 = A \Longrightarrow f \in \mathfrak{D}(A) \Longrightarrow A_1 = A + \hat{\mathfrak{N}}. \ \square$$

**Remark 9.1.** It is worth mentioning that inequalities (9.17), (9.18) [in the sense of (9.20)] are fulfilled if and only if $A_1(0) \neq \{0\}$, i.e., $A_1$ is a linear relation, and they turn into the evident equalities provided that $A_1, A_2 \in \mathcal{C}(\mathfrak{h})$.

A $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ corresponding to the BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ will be called (after [41]) an $M\mathfrak{L}$-preresolvent matrix if $\ker\Gamma_1 = A_F$, $\ker\Gamma_2 = A_K$.

**Corollary 9.2.** *For an operator-valued function $\mathfrak{A}(\lambda) = (a_{jk}(\lambda))_{j,k=1}^2$ (holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$ with values in $[\mathcal{H} \oplus \mathfrak{L}]$) to be an $M\mathfrak{L}$-preresolvent matrix of a nonnegative operator $A \geq 0$ such that $\mathfrak{L} \perp A_K(0)$, it is necessary and sufficient that*

(1) $\mathfrak{A}(\lambda) \in R_{\mathcal{H}\oplus\mathfrak{L}}$;
(2) $s - \lim_{x\downarrow -\infty} x a_{22}(x) = -I_{\mathfrak{L}}$;
(3) $s - \lim_{x\downarrow -\infty} a_{21}(x) = 0$;
(4) $0 \in \rho(\operatorname{Im} a_{11}(i))$;
(5) $s - \lim_{x\downarrow -\infty} a_{11}(x) = 0$;
(6) $\lim_{x\uparrow 0}(a_{11}(x)h, h) = +\infty \quad \forall h \in \mathcal{H} \setminus \{0\}$.

**Proof.** According to Theorem 9.2, $\mathfrak{A}(\lambda)$ is a $\Pi\mathfrak{L}$-preresolvent matrix of an operator $A \geq 0$. Making use of Proposition 5.6′ and hypotheses (4)–(6) we conclude that $a_{11}(\lambda)$ is a $Q_M$-function of an operator $A \geq 0$. To complete the proof it remains to apply the implication

$$a_{11}(-\infty) = s - \lim_{x\downarrow -\infty} a_{11}(x) = 0 \Longrightarrow a_{11}(x) \geq 0 \qquad \forall x < 0 \Longleftrightarrow a_{11}(\lambda) \in S(0,\infty). \ \square$$

**Definition 9.2.** Let $A$ be a Hermitian contraction in $\mathfrak{h}$, $\mathfrak{D}(A) = \mathfrak{h}_0$, $\mathfrak{L}$ be a subspace of $\mathfrak{h}$. A $\Pi\mathfrak{L}$-preresolvent matrix $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ corresponding to a BVS $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_2\}$ will be called a $\Pi_{\pm 1}\mathfrak{L}$-preresolvent matrix of the contraction $A$ if $\ker\Gamma_1 = A \dot{+} \hat{\mathfrak{N}}$, $\ker\Gamma_2 = A_{\pm 1}$.

Recall that $A_{+1} = A_M$ and $A_{-1} = A_\mu$ are extreme extensions (in the sense of Krein [34, 5]) of the operator $A$. The general form of BVS's with $\ker\Gamma_1 = A \dot{+} \hat{\mathfrak{N}}$, $\ker\Gamma_2 = A_{\pm 1}$ is given by the formula $\Pi_{\pm 1} = \{\mathcal{H}, X^*\Gamma_1^{\pm}, X^{-1}\Gamma_2^{\pm}\}$, where

$$\Gamma_1^{\pm}\hat{f} = P_{\mathfrak{N}} f, \qquad \Gamma_2^{\pm}\hat{f} = -n \qquad \forall \hat{f} = \{f, A_{\pm 1} f + n\} \in A^* = A_{\pm 1} + \hat{\mathfrak{N}} \tag{9.21}$$

and $X^*, X^{-1} \in [\mathfrak{N}, \mathcal{H}]$. The Weyl functions corresponding to the BVS (9.21) take the form

$$M_+(\lambda) = X^*[M_M(\lambda) + I]X, \qquad M_-(\lambda) = X^*[M_\mu(\lambda) - I]X, \tag{9.22}$$

where $M_M(\lambda)$ and $M_\mu(\lambda)$ are $Q_M$- and $Q_\mu$-functions of the forms in (3.73), (3.74).

**Corollary 9.3.** *For the operator-valued function $\mathfrak{A}(\lambda) = (a_{jk}(\lambda))_{j,k=1}^2$ to be a $\Pi_{-1}\mathfrak{L}$ ($\Pi_{+1}\mathfrak{L}$)-preresolvent matrix of a Hermitian contraction, it is necessary and sufficient that conditions* (1)–(4) *of Corollary* 9.2 *as well as the following conditions hold:*

(5) $a_{11}(\lambda) \in R_{\mathcal{H}}[-1,1]$;
(6) $\lim_{x\uparrow -1}(a_{11}(x)h, h) = +\infty (\lim_{x\downarrow +1}(a_{11}(x)h, h) = -\infty) \quad \forall h \in \mathcal{H} \setminus \{0\}$.

**Proof.** *Necessity.* Conditions (5) and (6) are implied by equalities (9.22) and Corollary 4.5.

*Sufficiency.* By Theorem 9.2, $\mathfrak{A}(\lambda)$ is a $\Pi\mathfrak{L}$-preresolvent matrix of a Hermitian contraction $A$. It follows from Corollary 4.5 and hypothesis (6) that $\ker\Gamma_2 = A_{-1} = A_\mu$. Owing to hypotheses (4) and (5) there exists $s - \lim_{\lambda\to\infty} \lambda a_{11}(\lambda) = B_1$ and $0 \in \rho(B_1)$. Therefore $B_1^{-1} = s - \lim_{\lambda\to\infty} a_{11}^{-1}(\lambda)/\lambda \in [\mathcal{H}]$ and in accordance with Theorem 1.1 we have $\ker\Gamma_1 = A \dotplus \hat{\mathfrak{N}}$. □

**Remark 9.2.** A statement close to Theorem 9.1 was obtained in [63]. A somewhat weaker version of Corollary 9.2 in the case $\overline{\mathfrak{D}(A)} = \mathfrak{h}$ was obtained in [41] (with the substitution of the condition $a_{11}(\lambda) \in S_{\mathcal{H}}(0,\infty)$ by a stronger one: $\mathfrak{A}(\lambda) \in S_{\mathcal{H}\oplus\mathfrak{L}}(0,\infty)$).

Note also that in [41] hypothesis (3) is omitted.

**3.** Here we give an inner description of the set of $\Pi\mathfrak{L}$-resolvent matrices of Hermitian operators. Everywhere in this section $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda)\end{pmatrix}$ and $W_{\Pi\mathfrak{L}}(\lambda) = \begin{pmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda)\end{pmatrix}$ stand for $\Pi\mathfrak{L}$-preresolvent and $\Pi\mathfrak{L}$-resolvent matrices of the form in (8.13) and (8.17).

Let

$$K = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{O} & \sqrt{2}I \\ -I & \mathbb{O}\end{pmatrix}, \qquad C = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{O} & I \\ I & \mathbb{O}\end{pmatrix}. \tag{9.23}$$

**Proposition 9.1.** *Suppose that $A$ is a Hermitian operator in $\mathfrak{h}$, $\mathfrak{L}$ is a subspace of $\mathfrak{h}$ such that $\rho(A;\mathfrak{L}) \neq \varnothing$, and the linear relation $A^*$ is $\mathfrak{L}$-regular, $\Pi = \{\mathfrak{L},\Gamma_1,\Gamma_2\}$ is a BVS for $A^*$, $W_{\Pi\mathfrak{L}}(\lambda)$ is a $\Pi\mathfrak{L}$-resolvent matrix,*

$$V_{\Pi\mathfrak{L}}(\lambda) = (v_{jk}(\lambda))_{j,k=1}^2 = (I - W_{\Pi\mathfrak{L}}(\lambda))(I + W_{\Pi\mathfrak{L}}(\lambda))^{-1}J. \tag{9.24}$$

*Then* (1) *the operator-valued function*

$$\mathfrak{A}_1(\lambda) = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22}\end{pmatrix} = KV_{\Pi\mathfrak{L}}(\lambda)K^* + C = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}V_{22}(\lambda) & I - V_{21}(\lambda) \\ I - V_{12}(\lambda) & \frac{1}{\sqrt{2}}V_{11}(\lambda)\end{pmatrix} \tag{9.25}$$

*is the Weyl function of the operator $A_0 = A \restriction \mathfrak{L}^\perp$ corresponding to the BVS $\tilde{\Pi}^{\mathfrak{A}_1} = \{\mathfrak{L}\oplus\mathfrak{L}; \tilde{\Gamma}_1^{\mathfrak{A}_1}, \tilde{\Gamma}_2^{\mathfrak{A}_1}\}$ ($\hat{f} = \{f, f'+l\}, \{f,f'\} \in A^*, l \in \mathfrak{L}$)*

$$\tilde{\Gamma}_1^{\mathfrak{A}_1}\hat{f} = \frac{1}{\sqrt{2}}\begin{pmatrix} -2\Gamma_2\{f,f'\} \\ \sqrt{2}P_{\mathfrak{L}}f\end{pmatrix}, \qquad \tilde{\Gamma}_2^{\mathfrak{A}_1}\hat{f} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Gamma_1\{f,f'\} + P_{\mathfrak{L}}f \\ -\sqrt{2}\Gamma_2\{f,f'\} - \sqrt{2}l\end{pmatrix} \tag{9.26}$$

(2) *the operator-valued function $\mathfrak{A}_1(\lambda)$ is the $\Pi^{\mathfrak{A}_1}\mathfrak{L}$-preresolvent matrix of the operator*

$$S = \{\{f,f'\} \in A_2 = \ker\Gamma_2 : \Gamma_1\{f,f'\} + P_{\mathfrak{L}}f = 0\} = \ker\Gamma_2 \cap \ker \pi_1\tilde{\Gamma}_1^{\mathfrak{A}_1}, \tag{9.27}$$

*corresponding to the BVS $\Pi^{\mathfrak{A}_1} = \{\mathfrak{L}, \Gamma_1^{\mathfrak{A}_1}, \Gamma_2^{\mathfrak{A}_1}\}$ of the linear relation*

$$S^* = \{\hat{f} = \{f,f'\} + \{0,l\} : \{f,f'\} \in A^*, l \in \mathfrak{L}, \Gamma_2\{f,f'\} + l = 0\}, \tag{9.28}$$

*where*

$$\Gamma_1^{\mathfrak{A}_1}\hat{f} = \pi_1\tilde{\Gamma}_1^{\mathfrak{A}_1}\hat{f} = -\sqrt{2}\Gamma_2\{f,f'\}, \qquad \Gamma_2^{\mathfrak{A}_1}\hat{f} = \frac{1}{\sqrt{2}}[\Gamma_1\{f,f'\} + P_{\mathfrak{L}}f]. \tag{9.29}$$

**Proof.** (1) Owing to the fact that the linear relation $A^*$ is $\mathfrak{L}$-regular, we have, by Theorem 8.2,

$$0 \in \rho(J - W_{\Pi\mathfrak{L}}^*(\lambda)JW_{\Pi\mathfrak{L}}(\lambda)) \cap \rho(J - W_{\Pi\mathfrak{L}}(\lambda)JW_{\Pi\mathfrak{L}}^*(\lambda))$$

and hence $-1 \in \rho(W_{\Pi\mathfrak{L}}(\lambda))$ $\forall\lambda \in \rho(A;\mathfrak{L})$ (Proposition 7.5). Therefore an operator-valued function $V(\lambda)$ is well defined by (9.24) and because of Theorem 8.2 $V(\lambda)$ is the Weyl function of the operator $A_0$, corresponding to the BVS $\tilde{\Pi}^V = \{\mathfrak{L}\oplus\mathfrak{L}, \tilde{\Gamma}''_1, \tilde{\Gamma}''_2\} = \{\mathfrak{L}\oplus\mathfrak{L}, -\Gamma'_2, \Gamma'_1\}$. Since the BVS's $\tilde{\Pi}^V$ and $\tilde{\Pi}^{\mathfrak{A}_1}$ are related as

$$\begin{pmatrix} \tilde{\Gamma}_1^{\mathfrak{A}_1} \\ \tilde{\Gamma}_2^{\mathfrak{A}_1}\end{pmatrix} = \begin{pmatrix} K & C(K^*)^{-1} \\ \mathbb{O} & (K^*)^{-1}\end{pmatrix}\begin{pmatrix} \tilde{\Gamma}_1^V \\ \tilde{\Gamma}_2^V\end{pmatrix}, \tag{9.30}$$

the corresponding Weyl functions are connected by equality (9.25).

(2) Since $0 \in \rho(\operatorname{Im} V(i))$, a defect subspace $\mathfrak{N}_\lambda(A_0)$ of the operator $A_0 = A \upharpoonright \mathfrak{L}^\perp$ takes the form [cf. (8.45)]

$$\mathfrak{N}_\lambda(A_0) = \mathfrak{N}_\lambda(A) \dotplus (A_2 - \lambda)^{-1}\mathfrak{L} = \gamma(\lambda)\mathfrak{L} \dotplus (A_2 - \lambda)^{-1}\mathfrak{L}. \tag{9.31}$$

Setting here

$$f_\lambda := \gamma(\lambda)l_1 + (A_2 - \lambda)^{-1}l_2 (\in \mathfrak{N}_\lambda(A_0)), \qquad \hat{R}_{A_2}(\lambda) = \{(A_2 - \lambda)^{-1}, I + \lambda(A_2 - \lambda)^{-1}\},$$

we obtain

$$\hat{f}_\lambda := \{f_\lambda, \lambda f_\lambda\} = \hat{\gamma}(\lambda)l_1 + \hat{R}_{A_2}(\lambda)l_2 - \{0, l_2\} \in \hat{\mathfrak{N}}_\lambda(A_0). \tag{9.32}$$

Making use of (9.26), (9.32) we have

$$\tilde{\Gamma}_1^{\mathfrak{A}_1}\hat{f}_\lambda = \begin{pmatrix} -\sqrt{2}\Gamma_2\hat{\gamma}(\lambda)l_1 \\ P_\mathfrak{L} f_\lambda \end{pmatrix} = \begin{pmatrix} -\sqrt{2}l_1 \\ P_\mathfrak{L}\gamma(\lambda)l_1 + P_\mathfrak{L}(A_2 - \lambda)^{-1}l_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}I_\mathfrak{L} & \mathbb{O} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix};$$

$$\begin{aligned}\sqrt{2}\tilde{\Gamma}_2^{\mathfrak{A}_1}\hat{f}_\lambda &= \begin{pmatrix} \Gamma_1(\hat{\gamma}(\lambda)l_1 + \hat{R}_{A_2}(\lambda)l_2) + P_\mathfrak{L}(\gamma(\lambda)l_1 + (A_2 - \lambda)^{-1}l_2) \\ -\sqrt{2}(l_1 - l_2) \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + a_{21} & a_{12} + a_{22} \\ -\sqrt{2}I_\mathfrak{L} & \sqrt{2}I_\mathfrak{L} \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.\end{aligned} \tag{9.33}$$

This implies that $\mathfrak{A}_1(\lambda)$ takes the form

$$\begin{aligned}\mathfrak{A}_1(\lambda) &= \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}I & \mathbb{O} \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix}\begin{pmatrix} a_{11}(\lambda) + a_{21}(\lambda) & a_{12}(\lambda) + a_{22}(\lambda) \\ -\sqrt{2}I_\mathfrak{L} & \sqrt{2}I_\mathfrak{L} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -2H^{-1}(\lambda) & \sqrt{2}H^{-1}(\lambda)(a_{12} + a_{22}) \\ \sqrt{2}(a_{21} + a_{22})H^{-1}(\lambda) & a_{22} - (a_{22} + a_{21})H^{-1}(\lambda)(a_{12} + a_{22}) \end{pmatrix},\end{aligned} \tag{9.34}$$

where $H(\lambda) = a_{11}(\lambda) + a_{21}(\lambda) + a_{12}(\lambda) + a_{22}(\lambda)$.

Further, the following formula for the defect subspace $\mathfrak{N}_\lambda(S)$ of the operator $S$ holds:

$$\mathfrak{N}_\lambda(S) = \{f_\lambda = \gamma(\lambda)l + (A_2 - \lambda)^{-1}l : l \in \mathfrak{L}\}. \tag{9.35}$$

Indeed, in view of (9.31), (9.32), (9.28), we have the equivalence

$$\hat{f}_\lambda \in S^* \iff \Gamma_2(\hat{\gamma}(\lambda)l_1 + \hat{R}_{A_2}(\lambda)l_2) - l_2 = 0 \iff l_1 = l_2 =: l.$$

Now for all $\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda(S)$ we find

$$\Gamma_1^{\mathfrak{A}_1}\hat{f}_\lambda = -\sqrt{2}l, \qquad \sqrt{2}\Gamma_2^{\mathfrak{A}_1}\hat{f}_\lambda = (a_{11} + a_{12} + a_{21} + a_{22})l = H(\lambda)l. \tag{9.36}$$

This implies that the Weyl function $M_1(\lambda)$ and the $\gamma$-field $\gamma_1(\lambda)$ corresponding to the BVS $\Pi^{\mathfrak{A}_1}$ take the form

$$M_1(\lambda) = -2H^{-1}(\lambda), \qquad \gamma_1(\lambda) = \sqrt{2}[\gamma(\lambda) + (A_2 - \lambda)^{-1}]H^{-1}(\lambda). \tag{9.37}$$

It is easy to check that

$$S_1 := \ker\Gamma_1^{\mathfrak{A}_1} = \ker\Gamma_2 =: A_2, \qquad S_2 := \ker\Gamma_2^{\mathfrak{A}_1} = \{\hat{f} = \{f, f' + l\} : \Gamma_2\{f, f'\} + l = \Gamma_1\{f, f'\} + P_\mathfrak{L} f = 0\}. \tag{9.38}$$

Taking account of (9.37), (9.38) and the resolvent formula (3.39), we transform the element $a'_{22}(\lambda)$ of the matrix (9.34):

$$a'_{22} := a_{22} - (a_{22} + a_{21})H^{-1}(a_{12} + a_{22}) + P_\mathfrak{L}[(A_2 - \lambda)^{-1} + \gamma_1(\lambda)M_1^{-1}(\lambda)\gamma_1^*(\bar{\lambda})] \upharpoonright \mathfrak{L}$$

$$= P_{\mathfrak{L}}[(S_1-\lambda)^{-1}+\gamma_1(\lambda)M_1^{-1}(\lambda)\gamma_1^*(\bar\lambda)] \upharpoonright \mathfrak{L} = P_{\mathfrak{L}}(S_2-\lambda)^{-1} \upharpoonright \mathfrak{L}. \tag{9.39}$$

Combining (9.34), (9.37), and (9.39) we obtain the desired coincidence of the matrix $\mathfrak{A}_1(\lambda)$ with the preresolvent matrix corresponding to the BVS $\Pi^{\mathfrak{A}_1}$

**Corollary 9.4.** *Under the assumptions of Proposition* 9.1 *the operator-valued function* $V(\lambda)$ *defined by equality* (9.23) *takes the form*

$$V(\lambda) = \begin{pmatrix} 2P_{\mathfrak{L}}(S_2-\lambda)^{-1} \upharpoonright \mathfrak{L} & I-\sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda) \\ I-\sqrt{2}\gamma_1^*(\bar\lambda) \upharpoonright \mathfrak{L} & M_1(\lambda) \end{pmatrix}, \tag{9.40}$$

*where* $S_2 = \ker \Gamma_2^{\mathfrak{A}_1}$ [*see* (9.38)], $M_1(\lambda)$ *and* $\gamma_1(\lambda)$ *are the Weyl function and the* $\gamma$*-field of the operator* $S$ *corresponding to the BVS* (9.29).

**Remark 9.3.** One can deduce formula (9.34) for $\mathfrak{A}_1(\lambda)$ from (9.25) and the following equality implied by (8.58):

$$V(\lambda) = -(X_{21}\mathfrak{A}(\lambda)+X_{22})(X_{11}\mathfrak{A}(\lambda)+X_{12})^{-1}, \tag{9.41}$$

where the matrices $X_{ij}$ are defined by (8.57).

**Theorem 9.3.** *Let* $A$ *be a Hermitian operator in* $\mathfrak{h}$ *and let* $\mathfrak{L}$ *be a subspace of* $\mathfrak{h}$ *such that* $\mathfrak{L}\cap\mathfrak{N}=\{0\}$ $(\mathfrak{N}=\mathfrak{D}(A)^{\perp})$, $\rho(A;\mathfrak{L})\neq\varnothing$, *the linear relation* $A^*$ *is* $\mathfrak{L}$*-regular and* $S_2(0)\perp\mathfrak{L}$, *where* $S_2$ *is an operator of the form in* (9.38). *Further, let* $\Pi=\{\mathfrak{L},\Gamma_1,\Gamma_2\}$ *be a BVS for* $A^*$ *and* $W_{\Pi\mathfrak{L}}(\lambda)$ *be the corresponding* $\Pi\mathfrak{L}$*-resolvent matrix. Then*

(1) $0\in\rho(w_{21}(\lambda))$ $\forall\lambda\in\rho(A;\mathfrak{L})$;

(2) $-1\in\rho(W_{\Pi\mathfrak{L}}(\lambda))$ $\forall\lambda\in\rho(A;\mathfrak{L})$;

(3) *the operator-valued function*

$$V(\lambda) = (v_{jk}(\lambda))_{j,k=1}^2 = i(I-W(\lambda))(I+W(\lambda))^{-1}J \tag{9.42}$$

*is the Weyl function of the operator* $A_0 = A \upharpoonright \mathfrak{L}^{\perp}$ *corresponding to the BVS* $\tilde\Pi^V=\{\mathfrak{L}\oplus\mathfrak{L},-\Gamma_2'',\Gamma_1''\}$ [*see* (8.52)] *and satisfies the following hypotheses:*

(a) $V(\lambda)=V^*(\bar\lambda)\in R_{\mathfrak{L}\oplus\mathfrak{L}}$;

(b) $0\in\rho(\operatorname{Im}V(i))$;

(c) $s-\lim_{y\uparrow\infty} iy \;\; v_{11}(iy) = -2I$;

(d) $s-\lim_{y\uparrow\infty} v_{21}(iy) = I$.

*Conversely, if the operator-valued function* $W(\lambda)=(w_{jk}(\lambda))_{j,k=1}^2$ *with values in* $[\mathfrak{L}\oplus\mathfrak{L}]$ *is holomorphic on the domain* $G_W$, $0\in\rho(w_{21}(\lambda))$ *for all* $\lambda\in G_W$, $-1\in\rho(W(\lambda))$ *for all* $\lambda\in G_W\setminus\mathbb{R}$, *and the operator-valued function* $V(\lambda)$ *defined by* (9.42) *satisfies hypotheses* (a)–(d), *then* $W(\lambda)$ *coincides on* $G_W$ *with a* $\Pi\mathfrak{L}$*-resolvent matrix of an* $\mathfrak{L}$*-regular Hermitian operator* $A$ *corresponding to the BVS* $\Pi=\{\mathfrak{L},\Gamma_1,\Gamma_2\}$ *such that* $\mathfrak{L}\perp S_2(0)$ *and the linear relation* $A^*$ *is* $\mathfrak{L}$*-regular. In this case* $\rho(A;\mathfrak{L})\supset G_W$.

**Proof.** *Necessity.* Condition (1) is a consequence of the definition (8.17). Condition (2) follows from the $\mathfrak{L}$-regularity of $A^*$ and was mentioned in the proof of Proposition 9.1. In view of (8.53) and (9.42), $V(\lambda)$ is connected with the Weyl function $M_2(\lambda)$ corresponding to the BVS (8.52) by the obvious equality

$$V(\lambda) = -M_2(\lambda)^{-1}, \tag{9.43}$$

which proves (3). Relations (a)–(d) follow from (9.40) and the assumption $S_2(0)\perp\mathfrak{L}$.

*Sufficiency.* Suppose that $V(\lambda)$ satisfies hypotheses (a)–(d). Then the matrix $\mathfrak{A}_1(\lambda)=KV(\lambda)K^*+C$ of the form in (9.25) satisfies the conditions of Theorem 9.1 and, therefore, it is a $\Pi\mathfrak{L}$-preresolvent matrix of a Hermitian operator $S\in C(\mathfrak{h})$ corresponding to the BVS $\Pi_1^{\mathfrak{A}_1}=\{\mathfrak{L},\Gamma_1^{\mathfrak{A}_1},\Gamma_2^{\mathfrak{A}_1}\}$ such that $S_2(0)\perp\mathfrak{L}$ $(S_i := \ker\Gamma_i^{\mathfrak{A}_1}, i=1,2)$:

$$\mathfrak{A}_1(\lambda) = \begin{pmatrix} M_1(\lambda) & \gamma_1^*(\bar\lambda)\upharpoonright\mathfrak{L} \\ P_{\mathfrak{L}}\gamma_1(\lambda) & P_{\mathfrak{L}}(S_2-\lambda)^{-1}\upharpoonright\mathfrak{L} \end{pmatrix}. \tag{9.44}$$

Here $M_1(\lambda)$ and $\gamma_1(\lambda)$ are the Weyl function and the $\gamma$-field corresponding to the BVS $\Pi_1^{\mathfrak{A}_1}$. It follows from (9.44), (9.23), and (9.25) that

$$V(\lambda) = K^{-1}[\mathfrak{A}_1(\lambda) - C](K^*)^{-1} = \begin{pmatrix} 2P_{\mathfrak{L}}(S_2-\lambda)^{-1} \upharpoonright \mathfrak{L} & I_{\mathfrak{L}} - \sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda) \\ I_{\mathfrak{L}} - \sqrt{2}\gamma_1^*(\bar\lambda) \upharpoonright \mathfrak{L} & M_1(\lambda) \end{pmatrix}. \tag{9.45}$$

Now we define a matrix $\mathfrak{A}(\lambda)$ setting [see (9.41)]

$$\mathfrak{A}(\lambda) = (a_{jk}(\lambda))_{j,k=1}^2 = -(X_{12}^*V(\lambda) + X_{22}^*)(X_{11}^*V(\lambda) + X_{21}^*)^{-1}. \tag{9.46}$$

Here the matrices $X_{ij}$ are the same as in (8.57). Making use of (9.46) and (8.57) we have

$$\mathfrak{A}(\lambda) = -\frac{1}{2}\begin{pmatrix} -v_{11} + (I+v_{12})v_{22}^{-1}(I+v_{21}) & v_{11} + (I+v_{12})v_{22}^{-1}(I-v_{21}) \\ v_{11} + (I-v_{12})v_{22}^{-1}(I+v_{21}) & -v_{11} + (I-v_{12})v_{22}^{-1}(I-v_{21}) \end{pmatrix}. \tag{9.47}$$

Hence, taking into account formula (9.45) for $V(\lambda)$ and the resolvent formula (3.39), we obtain

$$a_{22}(\lambda) = \frac{1}{2}P_{\mathfrak{L}}[2(S_2-\lambda)^{-1} - 2\gamma_1(\lambda)M_1^{-1}(\lambda)\gamma_1^*(\bar\lambda)] \upharpoonright \mathfrak{L} = P_{\mathfrak{L}}(S_1-\lambda)^{-1} \upharpoonright \mathfrak{L}; \tag{9.48}$$

$$a_{21}(\lambda) = -\frac{1}{2}P_{\mathfrak{L}}[2(S_2-\lambda)^{-1} + \sqrt{2}\gamma_1(\lambda)M_1^{-1}(\lambda)(2-\sqrt{2}\gamma_1^*(\bar\lambda))] \upharpoonright \mathfrak{L} = -P_{\mathfrak{L}}(S_1-\lambda)^{-1} \upharpoonright \mathfrak{L} - \sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda)M_1(\lambda)^{-1}; \tag{9.49}$$

$$\begin{aligned} a_{11}(\lambda) &= \frac{1}{2}[2P_{\mathfrak{L}}(S_2-\lambda)^{-1} - (2\cdot I_{\mathfrak{L}} - \sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda))M_1^{-1}(\lambda)(2\cdot I_{\mathfrak{L}} - \sqrt{2}\gamma_1^*(\bar\lambda))] \upharpoonright \mathfrak{L} \\ &= P_{\mathfrak{L}}(S_1-\lambda)^{-1} \upharpoonright \mathfrak{L} + \sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda)M_1^{-1}(\lambda) + \sqrt{2}M_1^{-1}(\lambda)\gamma_1^*(\bar\lambda) - 2M_1^{-1}(\lambda). \end{aligned} \tag{9.50}$$

We define a linear relation $A^*$ [cf. (9.28)] by the equality

$$A^* = \{\{f,f'\} = \hat f - \{0,l\} : \hat f \in S^*, \{0,l\} \in \hat{\mathfrak{L}}, \sqrt{2}l - \Gamma_1^{\mathfrak{A}_1}\hat f = 0\}. \tag{9.51}$$

It easy to check that the triple $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ with

$$\Gamma_1\{f,f'\} = \sqrt{2}\Gamma_2^{\mathfrak{A}_1}\hat f - P_{\mathfrak{L}}f, \qquad \Gamma_2\{f,f'\} = -\frac{1}{\sqrt{2}}\Gamma_1^{\mathfrak{A}_1}\hat f \tag{9.52}$$

forms a BVS for $A^*$.

We show that the $\Pi\mathfrak{L}$-resolvent matrix corresponding to the BVS $\Pi$ coincides with the matrix $\mathfrak{A}(\lambda)$ of the form in (9.47). A defect subspace $\mathfrak{N}_\lambda(A_0)$ of the operator

$$A_0 := A \cap \hat{\mathfrak{L}}^\perp = S \cap \hat{\mathfrak{L}}^\perp = S \cap \{0, \mathfrak{L}^\perp\} \tag{9.53}$$

takes the form [cf. (9.31)]

$$\mathfrak{N}_\lambda(A_0) = \mathfrak{N}_\lambda(S) \dotplus (S_2-\lambda)^{-1}\mathfrak{L} = \gamma_1(\lambda)\mathfrak{L} \dotplus (S_2-\lambda)^{-1}\mathfrak{L}, \tag{9.54}$$

that is, consists of the vectors $f_\lambda = \gamma_1(\lambda)l_1 + (S_2-\lambda)^{-1}l_2$. Put

$$\hat f_\lambda = \{f_\lambda, \lambda f_\lambda\} = \hat\gamma_1(\lambda)l_1 + \hat R_{S_2}(\lambda)l_2 - \{0,l_2\} =: \hat f - \{0,l_2\} \in \hat{\mathfrak{N}}_\lambda(A_0).$$

By the definition (9.51) $\hat f_\lambda \in A^*$ if and only if

$$\Gamma_1^{\mathfrak{A}_1}\hat f_\lambda - \sqrt{2}l_2 = \Gamma_1^{\mathfrak{A}_1}(\hat\gamma_1(\lambda)l_1 + \hat R_{S_2}(\lambda)l_2) - \sqrt{2}l_2 = M_1(\lambda)l_1 + \gamma^*(\bar\lambda)l_2 - \sqrt{2}l_2 = 0.$$

The last equality is equivalent to the following one: $l_1 = M_1(\lambda)^{-1}(\sqrt{2} - \gamma^*(\bar{\lambda}))l_2$. Thus

$$f_\lambda \in \mathfrak{N}_\lambda(A) \Longleftrightarrow f_\lambda = (S_2 - \lambda)^{-1}l + \gamma_1(\lambda)M_1^{-1}(\lambda)(\sqrt{2} - \gamma_1^*(\bar{\lambda}))l = (S_1 - \lambda)^{-1}l + \sqrt{2}\gamma_1(\lambda)M_1^{-1}(\lambda), \quad (9.55)$$

where $S_1 = \ker \Gamma_1^{\mathfrak{A}_1}$ and the defect subspace of the operator $A = \ker \Gamma_1 \cap \ker \Gamma_2$ takes the form

$$\mathfrak{N}_\lambda(A) = \{f_\lambda = [(S_1 - \lambda)^{-1} + \sqrt{2}\gamma_1(\lambda)M_1(\lambda)^{-1}]l : l \in \mathfrak{L}\}. \quad (9.56)$$

We are now in a position to find the $\gamma$-field $\gamma(\lambda)$ and the Weyl function $M(\lambda)$ corresponding to the BVS $\Pi$:

$$\Gamma_2 \hat{f}_\lambda = -\frac{1}{\sqrt{2}}\Gamma_1^{\mathfrak{A}_1}\{\hat{R}_{S_1}(\lambda)l + \sqrt{2}\hat{\gamma}_1(\lambda)M_1(\lambda)^{-1}l\} = -l. \quad (9.57)$$

Therefore

$$\gamma(\lambda)l = -[(S_1 - \lambda)^{-1} + \sqrt{2}\gamma_1(\lambda)M_1(\lambda)^{-1}]l. \quad (9.58)$$

And finally we have

$$M(\lambda)l = \Gamma_1\hat{\gamma}(\lambda)l = -\sqrt{2}\Gamma_2^{\mathfrak{A}_1}[\hat{R}_{S_1}(\lambda)l + \sqrt{2}\hat{\gamma}_1(\lambda)M_1(\lambda)^{-1}l] - P_{\mathfrak{L}}(S_1 - \lambda)^{-1}l + \sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda)M_1(\lambda)^{-1}l =$$

$$= [\sqrt{2}M_1(\lambda)^{-1}\gamma_1^*(\bar{\lambda}) - 2M_1(\lambda)^{-1} + P_{\mathfrak{L}}(S_1 - \lambda)^{-1} + \sqrt{2}P_{\mathfrak{L}}\gamma_1(\lambda)M_1(\lambda)^{-1}]l. \quad (9.59)$$

We mention also that according to (9.52)

$$A_2 := \ker \Gamma_2 = \ker \Gamma_1^{\mathfrak{A}_1} =: S_1. \quad (9.60)$$

A comparison of relations (9.47)–(9.50) and (9.58)–(9.60) leads to the equality

$$\mathfrak{A}(\lambda) = \begin{pmatrix} M(\lambda) & \gamma^*(\bar{\lambda}) \upharpoonright \mathfrak{L} \\ P_{\mathfrak{L}}\gamma(\lambda) & P_{\mathfrak{L}}(A_2 - \lambda)^{-1} \upharpoonright \mathfrak{L} \end{pmatrix}, \quad (9.61)$$

which means that $\mathfrak{A}(\lambda)$ is a $\Pi\mathfrak{L}$-preresolvent matrix corresponding to the BVS $\Pi^{\mathfrak{A}} = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ of the form in (9.52): $\mathfrak{A}(\lambda) = \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$.

We now show that $W(\lambda)$ coincides with the $\Pi\mathfrak{L}$-resolvent matrix of the operator $A$ corresponding to the BVS $\Pi^{\mathfrak{A}} = \Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$. Rewriting equalities (9.46) and (9.42) in the form

$$\mathfrak{A}(\lambda) = \begin{pmatrix} -X_{12}^* & -X_{22}^* \\ X_{11}^* & X_{21}^* \end{pmatrix} \circ V(\lambda), \qquad V(\lambda) = \begin{pmatrix} -iI & iI \\ J & J \end{pmatrix} \circ W(\lambda), \quad (9.62)$$

we calculate the composition of the linear-fractional transformations which, on account of the assumption $0 \in \rho(w_{21}(\lambda))$ for all $\lambda \in G_W$, takes the form

$$\mathfrak{A}(\lambda) = \begin{pmatrix} -X_{12}^* & -X_{22}^* \\ X_{11}^* & X_{21}^* \end{pmatrix} \begin{pmatrix} -iI & iI \\ J & J \end{pmatrix} \circ W(\lambda)$$

$$= \left[\begin{pmatrix} \mathbb{O} & \mathbb{O} \\ I & \mathbb{O} \end{pmatrix} W(\lambda) + \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}\right] \cdot \left[\begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix} W(\lambda) - \begin{pmatrix} \mathbb{O} & I \\ \mathbb{O} & \mathbb{O} \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} I & \mathbb{O} \\ w_{11} & w_{12} \end{pmatrix} \begin{pmatrix} \mathbb{O} & -I \\ w_{21} & w_{22} \end{pmatrix}^{-1} = \begin{pmatrix} w_{21}^{-1}w_{22} & w_{21}^{-1} \\ w_{11}w_{21}^{-1}w_{22} - w_{12} & w_{11}w_{21}^{-1} \end{pmatrix}. \quad (9.63)$$

Since $\mathfrak{A}(\lambda) = \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ we have by virtue of (8.17) $W(\lambda) = W_{\Pi\mathfrak{L}}(\lambda)$, that is, $W(\lambda)$ coincides with the $\Pi\mathfrak{L}$-resolvent matrix corresponding to the BVS $\Pi^{\mathfrak{A}} = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ of the form in (9.52). It remains to note that $\mathfrak{L} \perp S_2(0)$, and in view of (9.51), (9.52) we have

$$S_2 := \ker \Gamma_2^{\mathfrak{A}_1} = \{\hat{f} = \{f, f' + l\} : \Gamma_2\{f, f'\} + l = \Gamma_1\{f, f'\} + P_{\mathfrak{L}}f = 0\}. \quad (9.64)$$

This proves the theorem. □

**4. Theorem 9.4.** *Let $A$ be a Hermitian operator in $\mathfrak{h}$, $\mathfrak{L}$ be a subspace of $\mathfrak{h}$, $\rho(A;\mathfrak{L}) \neq \varnothing$, $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ be a BVS for $A^*$ such that $A_2(0) \perp \mathfrak{L}$, and $W_{\Pi\mathfrak{L}}(\lambda)$ be the corresponding $\Pi\mathfrak{L}$-resolvent matrix. Then we have:*

(1) $0 \in \rho(w_{21}(\lambda))$ *for all* $\lambda \in \rho(A;\mathfrak{L})$;

(2) *the operator-valued function*

$$\mathfrak{A}(\lambda) = \begin{pmatrix} w_{12}(\lambda)^{-1}w_{11}(\lambda) & w_{12}(\lambda)^{-1} \\ w_{22}(\lambda)w_{12}(\lambda)^{-1}w_{11}(\lambda) - w_{21}(\lambda) & w_{22}(\lambda)w_{12}(\lambda)^{-1} \end{pmatrix} \tag{9.65}$$

*may be holomorphically continued on $\mathbb{C}_+ \cap \mathbb{C}_-$ to an $R_{\mathfrak{L}\oplus\mathfrak{L}}$-function with values in $[\mathfrak{L}\oplus\mathfrak{L}]$, which satisfies hypotheses* (1)–(4) *of Theorem* 9.1.

*Conversely, if an operator-valued function $W(\lambda) = (w_{jk}(\lambda))_{j,k=1}^2$ with values in $[\mathfrak{L}\oplus\mathfrak{L}]$ is holomorphic on a domain $G_W$, $0 \in \rho(w_{21}(\lambda))$ for all $\lambda \in G_W$, and $\mathfrak{A}(\lambda)$ satisfies hypotheses* (1)–(4) *of Theorem* 9.1, *then $W(\lambda)$ is a $\Pi\mathfrak{L}$-resolvent matrix of an $\mathfrak{L}$-regular Hermitian operator $A$, corresponding to the BVS $\Pi = \{\mathfrak{L}, \Gamma_1, \Gamma_2\}$ such that $\mathfrak{L} \perp A_2(0)$.*

The proof follows from Theorem 9.1 and Proposition 8.2.

**Remark 9.4.** A result close to Theorem 9.4 is contained in [63]. Note also that the inverse problem for $\mathfrak{L}$-resolvent matrices of isometrical operators was considered in [3].

**Remark 9.5.** We omit the sharp statement of a version of Theorem 9.2 in terms of the resolvent matrix $W_{\Pi\mathfrak{L}}(\lambda)$. Note also that one can formulate the condition $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) \in S(\mathbb{R} \setminus (\alpha,\beta))$ with the help of $W_{\Pi\mathfrak{L}}(\lambda)$ in the following way:

$$W_{\Pi\mathfrak{L}}(\lambda)J_pW_{\Pi\mathfrak{L}}^*(\lambda) - J_p > 0 (\Longleftrightarrow W_{\Pi\mathfrak{L}}^*(\lambda)J_pW_{\Pi\mathfrak{L}}(\lambda) - J_p > 0), \qquad \forall\lambda \in (\alpha,\beta), \tag{9.66}$$

where $J_p = \begin{pmatrix} \mathbb{O} & I \\ I & \mathbb{O} \end{pmatrix}$. The property of $W_{\Pi\mathfrak{L}}(\lambda)$ of being holomorphic on $(\alpha,\beta)$, as well as the inequality (9.66), are implied by the elementary identity

$$W_{\Pi\mathfrak{L}}(\lambda)J_pW_{\Pi\mathfrak{L}}^*(\lambda) - J_p = Y_2^{-1}(\lambda)[\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) + \mathfrak{A}_{\Pi\mathfrak{L}}^*(\lambda)]Y_2^{-1}(\lambda)^*, \tag{9.67}$$

where the operator-valued function $Y_2(\lambda)$ is the same as in (8.20). In the case $(\alpha,\beta) = (-\infty, 0)$ it follows from (9.67) that the following equivalences hold:

$$\begin{aligned} \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) \in S(a,\infty) &\Longleftrightarrow W_{\Pi\mathfrak{L}}(\lambda)J_pW_{\Pi\mathfrak{L}}^*(\lambda) - J_p > 0 \quad \forall\lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda < a; \\ \mathfrak{A}_{\Pi\mathfrak{L}}(\lambda) \in S^-(-\infty,b) &\Longleftrightarrow W_{\Pi\mathfrak{L}}(\lambda)J_pW_{\Pi\mathfrak{L}}^*(\lambda) - J_p < 0 \quad \forall\lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda > b; \end{aligned} \tag{9.68}$$

## 10. TRUNCATED MOMENT PROBLEM

Application of Theorem 4.2 and Proposition 8.10 to the truncated Hamburger, Stieltjes, and Hausdorff moment problems enables one to describe all of its solutions as well as solutions whose supports in some intervals are finite or empty.

**1.** Let $\{s_k\}_0^{2n}$ be a strictly positive sequence, $s_0 = 1$, $\mathfrak{h} = \mathbb{C}_n[t]$ be a Euclidean space of polynomials of degree $\leq n$ endowed with the inner product

$$(f,g)_{\mathfrak{h}} = \sum_{j,k=0}^{n} s_{j+k}\alpha_j\bar{\beta}_k \quad (f = \sum_{k=0}^{n}\alpha_k t^k,\ g = \sum_{k=0}^{n}\beta_k t^k \in \mathbb{C}_n[t]). \tag{10.1}$$

Let $A$ be the operator of multiplication by $t$ in $\mathfrak{h}$ and $\{P_k(t)\}_0^n$ be the standard basis of orthogonal polynomials of the first kind. Then the following equalities hold:

$$Ae_k := AP_k(t) = tP_k(t) = b_{k-1}P_{k-1}(t) + a_kP_k(t) + b_kP_{k+1}(t) \quad (0 \leq k \leq n-1), \tag{10.2}$$

in which $b_{-1}=0$, $b_k>0$, $a_k=\bar a_k$ $(0\le k\le n-1)$, $\mathfrak{D}(A)=\mathfrak{h}_0=\mathfrak{h}\ominus\{P_n(t)\}$.

Clearly, $n_\pm(A)=1$. Consider some self-adjoint extension $A_0=A_0^*$ of $A$, whose matrix with respect to the basis $\{P_k(t)\}_0^n$ is the Jacobi matrix

$$A_0=\begin{pmatrix} a_0 & b_0 & 0 & \dots & 0 & 0\\ b_0 & a_1 & b_1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1}\\ 0 & 0 & 0 & \dots & b_{n-1} & a_n \end{pmatrix}\supset A=\begin{pmatrix} a_0 & b_0 & 0 & \dots & 0 & *\\ b_0 & a_1 & b_1 & \dots & 0 & *\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & a_{n-1} & *\\ 0 & 0 & 0 & \dots & b_{n-1} & * \end{pmatrix}. \tag{10.3}$$

The orthogonal polynomials $P_k(\lambda)$ and $Q_k(\lambda)$ of the first and the second kinds can be expressed in terms of the Jacobi matrix by the formulas

$$P_k(\lambda)=\frac{\det(\lambda-A_0^{(k)})}{b_0b_1\dots b_{k-1}},\quad Q_k(\lambda)=\frac{\det(\lambda-\tilde A_0^{(k)})}{b_0b_1\dots b_{k-1}}\quad (1\le k\le n+1), \tag{10.4}$$

in which the matrix $A_0^{(k)}$ can be obtained by removing from the Jacobi matrix its last $n+1-k$ rows and columns, and $\tilde A_0^{(k)}$ can be obtained by removing from $A_0^{(k)}$ its first row and column.

**Proposition 10.1.** *Let $A$ be a Hermitian operator of the form* (10.2) *in $\mathfrak{h}=\mathbb{C}_n[t]$, $\mathfrak{L}=\{e_0\}$. Then*

(1) *the triple $\Pi=\{\mathbb{C},\Gamma_1,\Gamma_2\}$, in which*

$$\Gamma_1\hat f=c,\quad \Gamma_2\hat f=(f,P_n(t))_{\mathfrak{h}},\quad \hat f=\{f,A_0f+ce_n\}\in A^*, \tag{10.5}$$

*forms a BVS for the linear relation $A^*$ and $A_1:=\ker\Gamma_1=A_0$;*

(2) *the corresponding Weyl function $M(\lambda)$, $\Pi\mathfrak{L}$-preresolvent and $\Pi\mathfrak{L}$-resolvent matrices $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ and $W_{\Pi\mathfrak{L}}(\lambda)$ take the form*

$$M(\lambda)=b_nP_{n+1}(\lambda)/P_n(\lambda), \tag{10.6}$$

$$\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)=\frac{1}{P_n(\lambda)}\begin{pmatrix} b_nP_{n+1}(\lambda) & 1\\ 1 & -Q_n(\lambda)\end{pmatrix},$$

$$W_{\Pi\mathfrak{L}}(\lambda)=\begin{pmatrix} -Q_n(\lambda) & -b_nQ_{n+1}(\lambda)\\ P_n(\lambda) & b_nP_{n+1}(\lambda)\end{pmatrix}. \tag{10.7}$$

**Proof.** (1) The first assertion is obvious.

2) It is easy to see that the defect subspace $\mathfrak{N}_\lambda$ is generated by the polynomial kernel $h(\lambda,t)$ at the point $\lambda$, i.e., by the vector

$$f_\lambda:=h(\lambda,t)=\sum_{k=0}^n P_k(\lambda)P_k(t)=\sum_{k=0}^n P_k(\lambda)e_k(\in\mathfrak{N}_\lambda). \tag{10.8}$$

$$\Gamma_1\hat f_\lambda=\Gamma_1\{f_\lambda,\lambda f_\lambda\}=\Gamma_1\{f_\lambda,A_1f_\lambda+b_nP_{n+1}(\lambda)e_n\}=b_nP_{n+1}(\lambda). \tag{10.9}$$

Therefore

$$\Gamma_2\hat f_\lambda=(f_\lambda,e_n)_{\mathfrak{h}}=(h(\lambda,t),P_n(t))_{\mathfrak{h}}=P_n(\lambda). \tag{10.10}$$

This implies that equality (10.6) for $M(\lambda)$ holds. Equality (10.10) yields

$$\hat\gamma(\lambda)\mathbb{I}=\frac{\hat f_\lambda}{P_n(\lambda)}\Longrightarrow\gamma(\lambda)\mathbb{I}=\frac{1}{P_n(\lambda)}\sum_{k=0}^n P_k(\lambda)e_k$$

$$\Longrightarrow\gamma^*(\bar\lambda)f=\frac{1}{P_n(\lambda)}\sum_{k=0}^n P_k(\lambda)(f,e_k).$$

Consequently,

$$a_{21}(\lambda) = P_{\mathfrak{L}}\gamma(\lambda) = P_0(\lambda)/P_n(\lambda) = P_n(\lambda)^{-1}, \qquad a_{12}(\lambda) = \gamma^*(\bar{\lambda})e_0 = P_n(\lambda)^{-1}. \tag{10.11}$$

Next we find an expression for $a_{22}(\lambda) = ((A_2 - \lambda)^{-1}\mathbb{I}, \mathbb{I})$. $(A_2 - \lambda)^{-1}\mathbb{I} = \sum_{k=0}^{n-1} c_k e_k$ since $A_2 = A \dot{+} \hat{\mathfrak{N}}$. Therefore, the vector $\vec{c} = \{c_k\}_0^{n-1}$ satisfies the equation

$$(A_0^{(n)} - \lambda)\vec{c} = \vec{\delta} = \operatorname{col}\{1, 0, \dots, 0\} \in \mathbb{C}^n.$$

Taking account of the expressions (10.4) for $P_n(\lambda)$ and $Q_n(\lambda)$ we obtain

$$c_0 = ((A_2 - \lambda)^{-1}e_0, e_0) = -Q_n(\lambda)/P_n(\lambda). \tag{10.12}$$

It is easy to see that $c_k = Q_k(\lambda) - c_0 P_k(\lambda)$ $(1 \le k \le n-1)$. Combining (10.6), (10.11), and (10.12) we obtain the expression (10.7) for $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$.

Further, the matrices $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ and $W_{\Pi\mathfrak{L}}(\lambda)$ are connected by equality (8.17). Making use of the Liouville–Ostrogradskiĭ formula [4, 6]

$$b_n[P_n(\lambda)Q_{n+1}(\lambda) - Q_n(\lambda)P_{n+1}(\lambda)] = 1,$$

we obtain from (8.17) equality (10.7) for $W_{\Pi\mathfrak{L}}(\lambda)$.

**Remark 10.1.** Formula (10.7) for $W_{\Pi\mathfrak{L}}(\lambda)$ is also implied by formula (8.22). Indeed, the subspace $\mathfrak{M}_{\lambda_0} = (A - \lambda_0)\mathfrak{D}(A)$ consists of the polynomials $f(t)$ with $f(\lambda_0) = 0$. Therefore, for all $f = \sum_0^n a_k P_k(t)$ we have

$$\mathcal{P}(\lambda)f = f(\lambda) = \sum_0^n a_k P_k(\lambda), \qquad \mathcal{Q}(\lambda)f = P_{\mathfrak{L}}(A - \lambda)^{-1}(I - \mathcal{P}(\lambda))f = \sum_0^n a_k Q_k(\lambda) \tag{10.13}$$

and hence

$$\mathcal{P}^*(\lambda) = \mathcal{P}^*(\lambda)\mathbb{I} = \sum_0^n \overline{P_k(\lambda)} e_k = h(\bar{\lambda}, t) = f_{\bar{\lambda}}(t), \tag{10.14}$$

$$\mathcal{Q}^*(\lambda) = \mathcal{Q}^*(\lambda)\mathbb{I} = \sum_1^n \overline{Q_k(\lambda)} e_k = \sum_1^n Q_k(\bar{\lambda})P_k(t). \tag{10.15}$$

Taking account of equalities (10.5), (10.8)–(10.10), we obtain from (10.14), (10.15)

$$\Gamma_2\hat{\mathcal{P}}^*(\lambda) = (h(\bar{\lambda}, t)e_n)_{\mathfrak{h}} = P_n(\bar{\lambda}), \qquad \Gamma_1\hat{\mathcal{P}}^*(\lambda) = \Gamma_1\{h(\bar{\lambda}, t), \bar{\lambda}h(\bar{\lambda}, t)\} = b_n P_{n+1}(\bar{\lambda}),$$

$$\begin{aligned}\Gamma_2\hat{\mathcal{Q}}^*(\lambda) &= \left(\sum_1^n Q_k(\bar{\lambda})e_k, e_n\right)_{\mathfrak{h}} = Q_n(\bar{\lambda}), \Gamma_1\hat{\mathcal{Q}}^*(\lambda) = \Gamma_1\{\mathcal{Q}^*(\lambda), \bar{\lambda}\mathcal{Q}^*(\lambda) + e_0\} \\ &= \Gamma_1\{\mathcal{Q}^*(\lambda), A_1\mathcal{Q}^*(\lambda) + b_n Q_{n+1}(\bar{\lambda})e_n\} = b_n Q_{n+1}(\bar{\lambda}).\end{aligned} \tag{10.16}$$

Formula (10.7) for $W_{\Pi\mathfrak{L}}(\lambda)$ is implied now by (8.22) and (10.16).

**2.** Consider the truncated Hamburger moment problem

$$s_k = \int_{-\infty}^{\infty} t^k d\sigma(t) \qquad (0 \le k \le 2n-1), \qquad s_{2n} \ge \int_{-\infty}^{\infty} t^{2n} d\sigma(t). \tag{10.17}$$

We denote by $\tilde{V}(\mathbf{s}; \mathbb{R})$ the set of all solutions $\sigma(t)$ of problem (10.17) and by $V(\mathbf{s}; \mathbb{R})$ the subclass of such $\sigma(t) \in \tilde{V}(\mathbf{s}; \mathbb{R})$ for which the inequality in (10.17) is replaced by the equality

$$s_{2n} = \int_{-\infty}^{\infty} t^{2n} d\sigma(t). \tag{10.18}$$

The next theorem was proved by Nevanlinna [38] (see also [4, 6]).

**Theorem 10.1.** *Let* $\{s_k\}_0^{2n}$ *be a strictly positive sequence,* $\{P_k(\lambda)\}_0^{n+1}$, $\{Q_k(\lambda)\}_0^{n+1}$ *be orthogonal polynomials of the first and the second kind. Then the formula*

$$\int_{-\infty}^{\infty}\frac{d\sigma(t)}{t-\lambda}=-\frac{b_nQ_{n+1}(\lambda)+\tau(\lambda)Q_n(\lambda)}{b_nP_{n+1}(\lambda)+\tau(\lambda)P_n(\lambda)} \tag{10.19}$$

*establishes a bijective correspondence between* $\sigma(t)\in\tilde V(\mathbf{s};\mathbb{R})$ *and* $\tau(\lambda)\in\tilde R$. *In this case* $\sigma(t)\in V(\mathbf{s},\mathbb{R})$ *if and only if* $\lim_{y\uparrow\infty}y^{-1}\tau(iy)=0$.

**Proof.** Let $A$ be an operator of the form in (10.2) in a Euclidean space $\mathfrak{h}=\mathbb{C}_n[t]$ with inner product (10.1). $\mathfrak{L}=\{\mathbb{I}\}$, $\Pi=\{\mathbb{C},\Gamma_1,\Gamma_2\}$ be a BVS for the linear relation $A^*$ of the form in (10.5). We show first that the set of solutions of the moment problem (10.17) coincides with the set of $\mathfrak{L}$-spectral functions of the operator $A$,

$$\sigma(t)\in\tilde V(\mathbf{s};\mathbb{R})\Longleftrightarrow\sigma(t)=(E_{\tilde A}(t)\mathbb{I},\mathbb{I}), \tag{10.20}$$

and $\sigma(t)\in V(\mathbf{s},\mathbb{R})$ [i.e., equality (10.18) holds] if and only if $\tilde A(0)=\{0\}$ ($\Longleftrightarrow\tilde A$ is an operator).

Let $\tilde A=\tilde A^*$ be a minimal extension of the operator $A$ acting in a space $\tilde{\mathfrak{h}}\supseteq\mathfrak{h}$ and $A'$ be its operator part in the case $\tilde A(0)\neq\{0\}$. It is clear that $\dim\tilde A(0)=\dim A^*(0)=1$ since $\tilde A$ is a minimal extension. Let $l_\infty\in\tilde A(0)$, $\|l_\infty\|=1$. Then the following equalities hold:

$$(A')^k\mathbb{I}=A^k\mathbb{I}(0\le k\le n-1),\qquad A^n\mathbb{I}=(A')^n\mathbb{I}+c_0l_\infty,\qquad c_0\in\mathbb{C}. \tag{10.21}$$

Indeed, assume that equalities (10.21) are proved for all $k\le p\le n-2$ and show that $(A')^{p+1}\mathbb{I}=A^{p+1}\mathbb{I}$. Since $A^k\mathbb{I}\perp A^*(0)=\{P_n(t)\}$ for all $k\le n-1$ and $\tilde A(0)\subset A^*(0)\oplus(\tilde{\mathfrak{h}}\ominus\mathfrak{h})$, we have $A^k\mathbb{I}\perp\tilde A(0)$ for all $k\le n-1$. Therefore the inclusions

$$\{A^p\mathbb{I},A^{p+1}\mathbb{I}\}\in\operatorname{gr}A\subset A',\qquad\{(A')^p\mathbb{I},(A')^{p+1}\mathbb{I}\}\in A'$$

yield $A^{p+1}\mathbb{I}=(A')^{p+1}\mathbb{I}$. The last equality in (10.21) is implied by the relations

$$\{A^{n-1}\mathbb{I},A^n\mathbb{I}\}\in\operatorname{gr}A\subset A',\{(A')^{n-1}\mathbb{I},(A')^n\mathbb{I}\}\in A'.$$

Now we obtain from (10.1) and (10.21) for $k+j\le 2n-2$ the equality

$$s_{k+j}=(t^k,t^j)_{\mathfrak{h}}=(A^k\mathbb{I},A^j\mathbb{I})=((A')^k\mathbb{I},(A')^j\mathbb{I})=\int_{\mathbb{R}}t^{k+j}d(E(t)\mathbb{I},\mathbb{I}), \tag{10.22}$$

in which $E(t)=E_{A'}(t)P_{\mathfrak{D}(A')}$ is a spectral function of the linear relation $\tilde A$. Analogously, taking account of (10.1), (10.21), we find

$$s_{2n-1}=(A^n\mathbb{I},A^{n-1}\mathbb{I})=((A')^n\mathbb{I}+c_0l_\infty,A^{n-1}\mathbb{I})=((A')^n\mathbb{I},A^{n-1}\mathbb{I})=\int_{\mathbb{R}}t^{2n-1}d(E(t)\mathbb{I},\mathbb{I}),$$

$$s_{2n}=(A^n\mathbb{I},A^n\mathbb{I})=\|(A')^n\mathbb{I}+c_0l_\infty\|^2=\|(A')^n\mathbb{I}\|^2_{L_2(d\sigma)}+|c_0|^2=\int_{\mathbb{R}}t^{2n}d(E(t)\mathbb{I},\mathbb{I})$$
$$+|c_0|^2\ge\int_{\mathbb{R}}t^{2n}d(E(t)\mathbb{I},\mathbb{I}). \tag{10.23}$$

It follows from (10.23), in particular, that

$$|c_0|^2=s_{2n}-\int_{\mathbb{R}}t^{2n}d\sigma(t),\qquad\sigma(t)=(E_{\tilde A}(t)\mathbb{I},\mathbb{I}). \tag{10.24}$$

Relations (10.22), (10.23) show that each $\mathfrak{L}$-spectral function $(E(t)\mathbb{I},\mathbb{I})$ of the operator $A$ is a solution of problem (10.17).

Conversely, assume that $\sigma(t)$ is the solution of the moment problem (10.17) and $A' = (A')^*$ is multiplication by $t$ in $L_2(d\sigma)$. Consider the self-adjoint linear relation

$$\tilde{A} = \{\{f \oplus \{0\}, A'f \oplus c\} : f' \in \mathfrak{D}(A'), c \in \mathbb{C}\} \tag{10.25}$$

in a Hilbert space $\tilde{\mathfrak{h}} = L_2(d\sigma) \oplus \mathbb{C}$ with the operator part $A'$. We define the embedding $i$ of the space $\mathfrak{h} = \mathbb{C}_n[t]$ into $\tilde{\mathfrak{h}}$, setting

$$i(t^k) = t^k \oplus \{0\} \qquad (0 \le k \le n-1), \qquad i(t^n) = t^n \oplus c_0, \tag{10.26}$$

where $c_0 = (s_{2n} - \int_{\mathbb{R}} t^{2n} d\sigma(t))^{1/2}$. By virtue of relations (10.22) and (10.23) the embedding $i : \mathfrak{h} \hookrightarrow \tilde{\mathfrak{h}}$ is isometric, which enables us to consider $\tilde{A}$ as an extension of the operator $A$ with exit in $\mathfrak{h}$. $\sigma(t)$ is an $\mathfrak{L}$-spectral function of $A$ since $\sigma(t) = (E_{A'}(t)\mathbb{I},\mathbb{I})$ and $E_{\tilde{A}}(t) = E_{A'}(t)P$, where $P$ is the orthogonal projection in $\tilde{\mathfrak{h}}$ onto $L_2(d\sigma)$:

$$\sigma(t) = (E_{\tilde{A}}(t)\mathbb{I},\mathbb{I}) = (E_{A'}(t)\mathbb{I},\mathbb{I}), \qquad \mathfrak{L} + \{\mathbb{I}\}.$$

The implications

$$\tilde{A}(0) = \{0\} \Longrightarrow c_0 = 0 \Longrightarrow \sigma(t) \in V(\mathbf{s};\mathbb{R})$$

follow from (10.21), (10.23). Conversely if $\sigma(t) \in V(\mathbf{s};\mathbb{R})$, then $c_0 = 0$ and relations (10.21) take the form $A^k\mathbb{I} = (A')^k\mathbb{I}$ $(0 \le k \le n)$. Hence, $\mathfrak{R}\,(A') \supset \mathfrak{h} \Longrightarrow \tilde{A}(0) \perp \mathfrak{h} \Longrightarrow \tilde{A}(0) = \{0\}$ since the extension $\tilde{A}$ is minimal.

Thus, in order to describe all solutions of problem (10.17) it is necessary and sufficient to describe the $\mathfrak{L}$-spectral functions $\sigma(t)$ of the operator $A$. On account of the relation $A_2 := \ker\Gamma_2 = A \dotplus \hat{\mathfrak{N}}$ we derive the desired assertion from Proposition 10.1 and Corollary 8.2.

**Remark 10.2.** As shown in Remark 10.1, formula (10.7) for $W_{\Pi\mathfrak{L}}(\lambda)$ is implied by (8.22). In view of (10.13)–(10.15), identity (8.23) takes the form

$$b_n \begin{pmatrix} Q_n(\lambda)Q_{n+1}(\mu) - Q_{n+1}(\lambda)Q_n(\mu) & Q_{n+1}(\lambda)P_n(\mu) - Q_n(\lambda)P_{n+1}(\mu) \\ P_{n+1}(\lambda)Q_n(\mu) - P_n(\lambda)Q_{n+1}(\mu) & P_n(\lambda)P_{n+1}(\mu) - P_{n+1}(\lambda)P_n(\mu) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$+ (\mu - \lambda) \begin{pmatrix} \sum_{k=1}^n Q_k(\lambda)Q_k(\mu) & -\sum_{k=1}^n Q_k(\lambda)P_k(\mu) \\ -\sum_{k=1}^n P_k(\lambda)Q_k(\mu) & \sum_{k=0}^n P_k(\lambda)P_k(\mu) \end{pmatrix} \tag{10.27}$$

and coincides with the well-known Cristoffel identity [4, 6]. Note also that the general identity (8.23), as well as the Cristoffel identity (10.27), are consequences of the Green formula.

**3.** Consider a truncated Stieltjes moment problem:

$$s_k = \int_0^\infty t^k d\sigma(t) \quad (0 \le k \le 2n-1), \quad s_{2n} \ge \int_0^\infty t^{2n} d\sigma(t). \tag{10.28}$$

As is known [36, 38], problem (10.28) is indeterminate if and only if the sequence $\{s_k\}_0^{2n}$ is strictly positive on $(0,\infty)$, i.e., the sequences $\{s_k\}_0^{2n}$, $\{s_{k+1}\}_0^{2n-2}$ are strictly positive. Following [38] we introduce the Stieltjes polynomials

$$P_{2n}^+(t) = \frac{P_n(t)}{P_n(0)}, \qquad P_{2n+1}^+(t) = b_n \begin{vmatrix} P_{n+1}(t) & P_n(t) \\ P_{n+1}(0) & P_n(0) \end{vmatrix} \tag{10.29}$$

and their conjugated polynomials $Q_{2n}^+(t)$, $Q_{2n+1}^+(t)$. Here $P_{2n}^+(t)$, $Q_{2n+1}^+(t)$ are normed by the conditions $P_{2n}^+(0) = Q_{2n+1}^+(0) = 1$.

**Proposition 10.2.** *Let the sequence $\{s_k\}_0^{2n}$ be strictly positive on $[0,\infty)$ and $A$ be a Hermitian operator in $\mathfrak{h} = \mathbb{C}_n[t]$ of the form in (10.2). Then*

(1) *$A$ is a positive operator;*

(2) *the triple $\Pi_1 = \{\mathbb{C}, \Gamma_1', \Gamma_2'\}$ with*

$$\Gamma_1'\hat{f} = \frac{1}{P_n(0)}(f, P_n)_{\mathfrak{h}}, \quad \Gamma_2'\hat{f} = b_n P_{n+1}(0)(f, P_n) - cP_n(0) \tag{10.30}$$

*forms a BVS for the linear relation $A^* = \{\hat{f} = \{f, \tilde{A}_0 f + cP_n\} : f \in \mathfrak{h}, c \in \mathbb{C}\}$ such that $A_2' := \ker \Gamma_2' = A_K$, $A_1' := \ker \Gamma_1' = A_F$;*

(3) *the corresponding Weyl function $M_1(\lambda)$ and $\Pi_1\mathfrak{L}$-resolvent matrix $W_{\Pi_1\mathfrak{L}}(\lambda)$ take the form*

$$M_1(\lambda) = -\frac{P_{2n}^+(\lambda)}{P_{2n+1}^+(\lambda)},$$

$$W_{\Pi_1\mathfrak{L}}(\lambda) == \begin{pmatrix} Q_{2n+1}^+(\lambda) & -Q_{2n}^+(\lambda) \\ -P_{2n+1}^+(\lambda) & P_{2n}^+(\lambda) \end{pmatrix}. \tag{10.31}$$

**Proof.** (1) The positiveness of the sequence $\{s_k\}_0^{2n}$ enables us to consider $\mathbb{C}_n[t]$ as a Euclidean space endowed with the scalar product (10.1). The positiveness of the operator $A$ is a consequence of the property of the sequence $\{s_{k+1}\}_0^{2n-2}$ to be strictly positive.

(2) Clearly, the triple $\Pi_1$ is a BVS for the linear relation $A^*$ and the relation $\ker \Gamma_1' = A \dot{+} \{0, e_n\} = A_F$ holds. Further it is easy to see that

$$A_2 := \ker \Gamma_2 = \{f, \tilde{A}_0 f + (f, e_n)\frac{P_{n+1}(0)}{P_n(0)}\} = \mathrm{gr}\begin{pmatrix} A_0^{(n)} & * \\ 0\ldots 0 b_{n-1} & \tilde{a}_n \end{pmatrix}, \tag{10.32}$$

where $A_0^{(n)}$ is a submatrix $A_0$ of the form in (10.3), which can be obtained by removing from the matrix $A_0$ its last row and column,

$$\tilde{a}_n = a_n + b_n P_{n+1}(0)/P_n(0) = -b_{n-1}P_{n-1}(0)/P_n(0). \tag{10.33}$$

In view of (10.4) $P_{n-1}(0)/P_n(0) = -b_{n-1}\det A_0^{(n-1)}/\det A_0^{(n)}$; hence

$$\det A_2 = \tilde{a}_n \det A_0^{(n)} - b_{n-1}^2 \det A_0^{(n-1)} = 0. \tag{10.34}$$

Equality (10.34) yields $A_2 = A_K$.

(3) Let $\hat{f}_\lambda = \{h(\lambda,t), \lambda h(\lambda,t)\}$. Then we have

$$\Gamma_1'\hat{f}_\lambda = P_n(0)^{-1}(h(\lambda,t), P_n(t))_{\mathfrak{h}} = P_n(\lambda)P_n(0)^{-1} = P_{2n}^+(\lambda),$$

$$\Gamma_2'\hat{f}_\lambda = b_n P_{n+1}(0)P_n(\lambda) - b_n P_{n+1}(\lambda)P_n(0) = -P_{2n+1}^+(\lambda). \tag{10.35}$$

Hence we obtain relation (10.31) for $M_1(\lambda)$; moreover, the above-mentioned coincidence $A_2 = A_K$ is implied by the equality $P_{2n+1}^+(0) = 0$.

Further, as in (10.16) we find

$$\Gamma_1'\hat{\mathfrak{Q}}^*(\lambda) = P_n(0)^{-1}\Big(\sum_1^n Q_k(\bar{\lambda})e_k, e_n\Big) = P_n(0)^{-1}Q_n(\bar{\lambda}) = Q_{2n}^+(\bar{\lambda}),$$

$$\Gamma_2'\hat{\mathfrak{Q}}^*(\lambda) = -b_n[Q_{2n+1}(\bar{\lambda})P_n(0) + P_{n+1}(0)Q_n(\bar{\lambda})] = -Q_{2n+1}^+(\bar{\lambda}). \tag{10.36}$$

Formula (10.31) for $W_{\Pi\mathfrak{L}}(\lambda)$ is implied by Theorem 8.1.

In the following theorem $\tilde{V}(\mathbf{s};\mathbb{R}_+)$ stands for the set of all solutions of problem (10.28) and $V(\mathbf{s};\mathbb{R}_+)$ stands for the subclass of $\sigma(t)\in\tilde{V}(\mathbf{s};\mathbb{R}_+)$ for which the inequality in (10.28) is replaced by the equality $s_{2n}=\int_0^\infty t^{2n}d\sigma(t)$.

**Theorem 10.2** [36]. *Suppose that the sequence* $\{s_k\}_0^{2n}$ *is strictly positive on* $(0,\infty)$*;* $P_{2n}^+(\lambda)$ *and* $P_{2n+1}^+(\lambda)$ *are Stieltjes polynomials,* $Q_{2n}^+(\lambda)$ *and* $Q_{2n+1}^+(\lambda)$ *are conjugated polynomials. Then the formula*

$$\int_0^\infty \frac{d\sigma(t)}{t-\lambda}=-\frac{Q_{2n}^+(\lambda)-\tau(\lambda)Q_{2n+1}^+(\lambda)}{P_{2n}^+(\lambda)-\tau(\lambda)P_{2n+1}^+(\lambda)} \tag{10.37}$$

*establishes a bijective correspondence between* $\sigma(t)\in\tilde{V}(\mathbf{s};\mathbb{R}_+)$ *and* $\tau(\lambda)\in\tilde{S}=\tilde{S}(0,\infty)$*. In this case* $\sigma(t)\in V(\mathbf{s};\mathbb{R}_+)$ *if and only if*

$$\lim_{y\uparrow\infty} y\tau(iy)=\infty. \tag{10.38}$$

**Proof.** Let $A$ be multiplication by $t$ in $\mathfrak{h}=\mathbb{C}_n[t]$, $\Pi_1=\{\mathbb{C},\Gamma_1',\Gamma_2'\}$ be a BVS of the form in (10.30). Application of Propositions 10.2 and 8.10 enable us to prove Theorem 10.2 in the same way as Theorem 10.1. We clarify only the origin of condition (10.38). Owing to the fact that $A_2=A_K$ is an operator and $\lim_{y\uparrow\infty} iyM(iy)=C<\infty$ [see (10.31)], the $M$-admissibility condition for the function $\tau(\lambda)$ takes the form

$$\lim_{y\uparrow\infty} y^{-1}(\tau(iy)+M(iy))^{-1}=0 \Longleftrightarrow \lim_{y\uparrow\infty} iy(\tau(iy)+M(iy))=\infty \Longleftrightarrow \lim_{y\uparrow\infty} y\tau(iy)=\infty.$$

**Remark 10.3.** A description of all solutions of the Stieltjes moment problem was obtained by Krein [36, 38] (canonical solutions were described earlier by T. Stieltjes).

**4.** In this section, we describe solutions of problem (10.17) that have no mass in given intervals.

Let a sequence $\{s_k\}_0^{2n}$ be strictly positive on $\mathbb{R}\setminus(\alpha,\beta)$. It is well known that this condition is equivalent to the strict positiveness of the sequences $\{s_k\}_0^{2n}$ and $\{s_k^*\}_0^{2n-2}$ with

$$s_k^*=s_{k+2}-(\alpha+\beta)s_{k+1}+\alpha\beta s_k. \tag{10.39}$$

The positiveness of the sequence (10.39) implies that an operator $A$ of the form in (10.2) acting in $\mathfrak{h}=\mathbb{C}_n[t]$ has a spectral gap $(\alpha,\beta)$ (see [79]).

Let $f(z)$ be a function holomorphic on a domain $G$, containing the segment $[\alpha,\beta]$. Denote by $\mathrm{null}_{f(z)}(\alpha,\beta)$ the number of its zeros on the interval $(\alpha,\beta)$. The notion of $\mathrm{null}_{f(z)}[\alpha,\beta]$ has the same meaning with respect to the segment $[\alpha,\beta]$.

**Lemma 10.1** [79]. *Suppose that $A$ is a Hermitian operator,* $n_\pm(A)=1$*, and one of its Weyl functions* $M(\lambda)$ *is meromorphic with noncancelable representation* $M(\lambda)=f_1(\lambda)/f_2(\lambda)$*. Then the operator $A$ has a gap* $(\alpha,\beta)$ *if and only if*

(a) $\delta:=f_1(\beta)f_2(\alpha)-f_1(\alpha)f_2(\beta)\geq 0$; (10.40)

b) *either* $\mathrm{null}_{f_2(\lambda)}(\alpha,\beta)=0$ *or* $\mathrm{null}_{f_2(\lambda)}[\alpha,\beta]=1$.

*The set* $\mathrm{Ex}_A(\alpha,\beta)$ *is infinite if and only if inequality* (10.40) *is strict.*

**Proof.** In accordance with the Calkin theorem [34], the property of an operator $A$ to have a spectral gap $(\alpha,\beta)$ is equivalent to the existence of a self-adjoint extension $\tilde{A}=\tilde{A}\in\mathrm{Ex}_A$ with the gap $(\alpha,\beta)$($\Longleftrightarrow$ $\mathrm{Ex}_A(\alpha,\beta)\neq\varnothing$). The last condition yields that some Weyl function $M_1(\lambda)$ of the operator $A$ is holomorphic on $(\alpha,\beta)$. In view of the connection $M(\lambda)=(X_{11}M_1(\lambda)+X_{12})(X_{21}M_1(\lambda)+X_{22})^{-1}$ between $M(\lambda)$ and $M_1(\lambda)$, this condition is equivalent to conditions (a) and (b) for $M(\lambda)$. □

**Corollary 10.1.** *Let a sequence* $\{s_k\}_0^{2n}$ *be strictly positive and $A$ be multiplication by $t$ in* $\mathfrak{h}=\mathbb{C}_n[t]$*. If some of its Weyl functions $M(\lambda)$ are meromorphic on* $[\alpha,\beta]$ *with a noncancelable representation* $M(\lambda)=f_1(\lambda)/f_2(\lambda)$*, then the positiveness of the sequence* $\{s_k^*\}_0^{2n-2}$ *of the form in* (10.32) *is equivalent to conditions* (a), (b) *of Lemma* 10.1*, and its strict positiveness is equivalent to conditions* (b) *and* $\delta>0$.

If the sequence $\{s_k^*\}_0^{2n-2}$ is strictly positive, then via (10.6) and Corollary 10.1 we have

$$\Delta:=P_{n+1}(\beta)P_n(\alpha)-P_{n+1}(\alpha)P_n(\beta)>0. \tag{10.41}$$

This enables us to define quasiorthogonal polynomials

$$P_\alpha(\lambda) = -\frac{1}{\Delta}\begin{vmatrix} P_{n+1}(\lambda) & P_n(\lambda) \\ P_{n+1}(\alpha) & P_n(\alpha)\end{vmatrix}, \qquad P_\beta(\lambda) = b_n\begin{vmatrix} P_{n+1}(\lambda) & P_n(\lambda) \\ P_{n+1}(\beta) & P_n(\beta)\end{vmatrix} \tag{10.42}$$

and their conjugated polynomials $Q_\alpha(\lambda)$ and $Q_\beta(\lambda)$. Note that polynomials $P_\alpha(\lambda)$ and $P_\beta(\lambda)$ are characterized by the conditions $P_\alpha(\alpha) = P_\beta(\beta) = 0$, $-P_\alpha(\beta) = Q_\beta(\beta) = 1$ in the set of quasiorthogonal polynomials.

**Proposition 10.3.** *The following assertions are equivalent:*
(a) *a sequence* $\{s_k\}_0^{2n}$ *is strictly positive on* $\mathbb{R}\setminus(\alpha,\beta)$*;*
(b) *the sequences* $\{s_k\}_0^{2n}$, $\{s_k^*\}_0^{2n-2}$ *are strictly positive;*
(c) $P_\alpha(x) < 0\ \forall x \in (\alpha,\beta]$*;*
(d) $P_\beta(x) < 0\ \forall x \in [\alpha,\beta)$.
*In this case the following statements hold:*
(1) *the triple* $\Pi_2 = \{\mathbb{C}, \Gamma_1'', \Gamma_2''\}$ *in which for all* $\hat f = \{f, \tilde A_0 f + cP_n\}$

$$\Gamma_1''\hat f = \frac{1}{\Delta b_n}\{P_{n+1}(\alpha)(f,P_n) - cP_n(\alpha)\}, \qquad \Gamma_2''\hat f = cP_n(\beta) - b_nP_{n+1}(\beta)(f,P_n), \tag{10.43}$$

*is a BVS for the relation* $A^*$, $A_1'' := \ker\Gamma_1'' = A_\alpha$*;* $A_2'' := \ker\Gamma_2'' = A_\beta$*;*
(2) *the corresponding Weyl function* $M(\lambda)$ *and* $\Pi_2\mathfrak{L}$*-resolvent* ($\mathfrak{L} = \{\mathbb{C}P_n\}$) *matrix* $W_{\Pi_2\mathfrak{L}}(\lambda)$ *take the form*

$$M_2(\lambda) = \frac{P_\alpha(\lambda)}{P_\beta(\lambda)}, \qquad W_{\Pi_2\mathfrak{L}}(\lambda) = \begin{pmatrix} -Q_\alpha(\lambda) & -Q_\beta(\lambda) \\ P_\alpha(\lambda) & P_\beta(\lambda)\end{pmatrix}. \tag{10.44}$$

**Proof.** In order to prove statement (1) it is sufficient to check the relations $\ker\Gamma_1'' = A_\alpha$, $\ker\Gamma_2'' = A_\beta$. In the case $P_n(\alpha)P_n(\beta) \neq 0$ we obtain from (10.5), (10.6), and (10.43)

$$\ker\Gamma_1'' = \ker(\Gamma_1 - M(\alpha)\Gamma_2), \qquad \ker\Gamma_2'' = \ker(\Gamma_1 - M(\beta)\Gamma_2). \tag{10.45}$$

The desired assertion follows from Proposition 4.6 [see (4.26)]. In the case $P_n(\alpha)P_n(\beta) = 0$ one should to apply relations (4.25). Note, however, that the equality $\ker\Gamma_2'' = A_\beta$ is implied by the relation

$$\det(A_2'' - \beta) = (\tilde a_n - \beta)\det(A_0^{(n)} - \beta) - b_{n-1}^2\det(A_0^{(n-1)} - \beta) = 0$$

$$(\tilde a_n = a_n + b_nP_{n+1}(\beta)P_n(\beta)^{-1}),$$

which can be proved in the same way as (10.34).

Further, relations (10.9), (10.10), and (10.43) yield

$$\Gamma_1''\hat f_\lambda = \Gamma_1''\hat P^*(\lambda) = \frac{1}{\Delta b_n}\{b_nP_{n+1}(\alpha)P_n(\lambda) - b_nP_{n+1}(\lambda)P_n(\alpha)\} = P_\alpha(\lambda), \qquad \Gamma_2''\hat f_\lambda = P_\beta(\lambda).$$

Therefore, the Weyl function $M_2(\lambda)$ takes the form in (10.44). Analogously, (10.16) and (10.43) imply

$$\Gamma_1''\hat Q^*(\bar\lambda) = \frac{1}{\Delta}[Q_n(\lambda)P_{n+1}(\alpha) - Q_{n+1}(\lambda)P_n(\lambda)] = Q_\alpha(\lambda), \qquad \Gamma_2''\hat Q^*(\bar\lambda) = Q_\beta(\lambda).$$

In accordance with Theorem 8.1, we obtain formula (10.44) for $W_{\Pi_2\mathfrak{L}}(\lambda)$.

The equivalence of conditions (a)–(d) is implied by formula (10.44) for $M_2(\lambda)$, the equalities $P_\alpha(\alpha) = P_\beta(\beta) = 0$, and Corollary 10.1. □

We denote by $\tilde V(\mathrm{s}; \mathrm{E}_m, \varkappa)$ the set of all solutions $\sigma(t)$ of the moment problem (10.17) having, in given intervals $(\alpha_j,\beta_j)$, exactly $\varkappa_j$ points of increase. Here

$$\mathrm{s} = \{s_k\}_0^{2n}, \qquad \varkappa = \{\varkappa_j\}_1^m \in \mathbb{Z}_+^m, \qquad G_m = \bigcup_{j=1}^m(\alpha_j,\beta_j), \qquad \mathrm{E}_m = \mathbb{R}\setminus G_m.$$

In particular, $\tilde{V}(\mathrm{s};\mathrm{E}_m) := \tilde{V}(\mathrm{s};\mathrm{E}_m;0)$ consists of measures $d\sigma(t)$ that have no mass in intervals $(\alpha_j,\beta_j)$ $(1 \le j \le m)$.

Let $V(\mathrm{s};\mathrm{E}_m;\varkappa)$ be the subclass of solutions $\sigma(t) \in \tilde{V}(\mathrm{s};\mathrm{E}_m;\varkappa)$ for which equality (10.18) holds.

**Theorem 10.3.** *Suppose that* $\{s_k\}_0^{2n}$ *is a sequence strictly positive on* $\mathbb{R}\setminus(\alpha,\beta)$; $P_\alpha(\lambda)$, $P_\beta(\lambda)$ *are polynomials of the form in* (10.42); $Q_\alpha(\lambda)$, $Q_\beta(\lambda)$ *are their conjugated polynomials;* $E_1 = \mathbb{R}\setminus(\alpha_1,\beta_1)$. *Then the formula*

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-\lambda} = -\frac{Q_\alpha(\lambda)\tau(\lambda)+Q_\beta(\lambda)}{P_\alpha(\lambda)\tau(\lambda)+P_\beta(\lambda)} \tag{10.46}$$

*establishes a bijective correspondence between* $\sigma(t) \in \tilde{V}(\mathrm{s};E_1;\varkappa_1)$ *and* $\tau(\lambda) \in \tilde{S}^{\varkappa_1}(\alpha,\beta)$. *In particular, the following equivalence holds:*

$$d\sigma(t) \in V(\mathrm{s};E_1) \Longleftrightarrow \tau(\lambda) \in \tilde{S}(\alpha,\beta). \tag{10.47}$$

*In this case we have:* (1) *if* $P_n(\beta) \neq 0$, *then*

$$d\sigma(t) \in V(\mathrm{s};E_1;\varkappa_1) \Longleftrightarrow \tau(\infty) + \frac{P_n(\alpha)}{b_n \Delta P_n(\beta)} \neq 0; \tag{10.48}$$

(2) *if* $P_n(\beta) = 0$ $(\Longrightarrow P_n(\alpha) \neq 0)$, *then*

$$d\sigma(t) \in V(\mathrm{s};E_1,\varkappa_1) \Longleftrightarrow \lim_{y\uparrow\infty} y^{-1}\tau(iy) = 0. \tag{10.49}$$

One can easily derive the proof from Theorems 4.2, 10.1, and Proposition 10.3.

In order to formulate the next theorem, we introduce the required notation, setting for all $j \in \mathbb{Z}_+$, $j \le n$,

$$P_n^{(j)}(\lambda) = \frac{-b_n}{\lambda-\beta_j}\begin{vmatrix} P_n(\lambda) & P_{n+1}(\lambda) \\ P_n(\beta_j) & P_{n+1}(\beta_j)\end{vmatrix}, \qquad P_{n+1}^{(j)}(\lambda) = \frac{-b_n}{\lambda-\alpha_j}\begin{vmatrix} P_n(\lambda) & P_{n+1}(\lambda) \\ P_n(\alpha_j) & P_{n+1}(\alpha_j)\end{vmatrix} \tag{10.50}$$

and, further, for $j = (j_1, j_2, \ldots, j_r) = (j', j_r) \in \mathbb{Z}_+^r$, with $m \ge j_1 > j_2 > \ldots > j_r \ge 1$

$$P_n^{(j)}(\lambda) = \frac{b_n}{\lambda-\beta_{j_r}}\begin{vmatrix} P_n^{(j')}(\lambda) & P_{n+1}^{(j')}(\lambda) \\ P_n^{(j')}(\beta_{j_r}) & P_{n+1}^{(j')}(\beta_{j_r})\end{vmatrix}, \qquad P_{n+1}^{(j)}(\lambda) = \frac{1}{\lambda-\alpha_{j_r}}\begin{vmatrix} P_n^{(j')}(\lambda) & P_{n+1}^{(j')}(\lambda) \\ P_n^{(j')}(\alpha_{j_r}) & P_{n+1}^{(j')}(\alpha_{j_r})\end{vmatrix}. \tag{10.51}$$

**Theorem 10.4.** *Suppose that the moment problem* (10.17) *is solvable and indeterminate on each of the sets* $\mathbb{R}\setminus(\alpha_j,\beta_j)$, $1 \le j \le m$. *Then it is solvable on* $E_m = \mathbb{R}\setminus\cup_{j=1}^m(\alpha_j,\beta_j)$ (*i.e.,* $\tilde{V}(\mathrm{s};E_m) \neq \varnothing$) *if and only if*

$$P_{n+1}^{(j')}(\beta_{j_r}) \ge 0 \qquad \forall j = (j_1, j_2, \ldots, j_r) \in \mathbb{Z}_+^r, \qquad m \ge j_1 > \ldots > j_r \ge 1, \qquad r \ge 2. \tag{10.52}$$

*Further, problem* (10.17) *on* $E_m$ *is indeterminate if and only if all the inequalities in* (10.52) *are strict and determinate* (*i.e., there exists a unique solution* $\sigma(t) \in \tilde{V}(\mathrm{s};\mathrm{E}_m)$) *otherwise.*

*In the former case, all solutions* $\sigma(t) \in V(\mathrm{s};\mathrm{E}_m,\varkappa)$ *are described by formula* (10.19), *in which* $\tau(\lambda) \in \tilde{\mathbb{R}}$ *and*

$$\tau_j(\lambda) = \frac{P_n(\alpha_j)\tau(\lambda) + b_n P_{n+1}(\alpha_j)}{P_n(\beta_j)\tau(\lambda) + b_n P_{n+1}(\beta_j)} \in \tilde{S}^{\varkappa_j}(\alpha_j,\beta_j).$$

*In particular*

$$\sigma(t) \in \tilde{V}(\mathrm{s};E_m) \Longleftrightarrow \tau_j(\lambda) \in \tilde{S}(\alpha_j,\beta_j) \qquad \forall j \le m.$$

*In this case the following equivalence holds:*

$$d\sigma(t) \in V(\mathrm{s};E_m) \Longleftrightarrow \lim_{y\uparrow\infty} y^{-1}\tau(iy) = 0.$$

The proof of the theorem as well as a general description of the case $\tilde{V}_A^{\mathfrak{L}}(E_m;\varkappa)$ of $\mathfrak{L}$-resolvents of an operator $A$ with gaps $(\alpha_j,\beta_j)$ $(1\le j\le m)$ will be given in another paper. Note only that the second part of this theorem (the description of all solutions $\sigma(t)\in\tilde{V}(\mathbf{s};E_m,\varkappa)$ can easily be derived from Theorem 10.1.

**Remark 10.4.** One can formulate a solvability criterion and indeterminacy conditions of problem (10.17) on each of the sets $\mathbb{R}\setminus(\alpha_j,\beta_j)$ $(1\le j\le m)$ in terms of the orthogonal polynomials, making use of the Weyl function $M(\lambda)=b_nP_{n+1}(\lambda)P_n(\lambda)^{-1}$ [cf. (10.6)]. By virtue of Corollary 10.1 these conditions take the form (for each $j\le m$):

(a) $P_{n+1}^{(j)}(\beta_{j_r})>0\Longleftrightarrow P_{n+1}(\beta_j)P_n(\alpha_j)-P_{n+1}(\alpha_j)P_n(\beta_j)>0,$ (10.53)

(b) $\mathrm{null}_{P_n(\lambda)}(\alpha_j,\beta_j)=0$ or $\mathrm{null}_{P_n(\lambda)}(\alpha_j,\beta_j)=\mathrm{null}_{P_n(\lambda)}[\alpha_j,\beta_j]=1.$ (10.54)

Thus, the solvability criterion for problem (10.17) on $E_m=\mathbb{R}\setminus\cup_1^m(\alpha_j,\beta_j)$ can be expressed by inequalities (10.52), (10.53), and conditions (10.54).

For the full (non-truncated) moment problem, Theorems 10.3, 10.4 and the corresponding abstract results on a densely defined Hermitian operator $A$ with gaps $(\alpha_j,\beta_j)$ were proved by the authors in [20, 21, 79]. Note, finally, that the problem of describing all solutions of a one-dimensional, as well as a multidimensional, moment problem, subject to prescribed localization conditions, was posed by Vladimirov (see, for example, [11]).

**5.** Consider the Hausdorff moment problem

$$s_k=\int_a^b t^k\,d\sigma(t)\qquad(0\le k\le 2n),\tag{10.55}$$

and denote by $V(\mathbf{s};[a,b])$ the set of its solutions.

Suppose that the sequence $\{s_k\}_0^{2n}$ is strictly positive on $[a,b]$, that is, the two sequences $\{s_k\}_0^{2n}$, $\{s_k'\}_0^{2n-2}$ are strictly positive, where

$$s_k'=(a+b)s_{k+1}-s_{k+2}-a\cdot bs_k.\tag{10.56}$$

In this case zeros of the orthogonal polynomials $P_k(\lambda)$ are simple and are contained in $[a,b]$. Following [36, 38] we define the polynomials

$$\underline{P}(\lambda)=(-1)^n\begin{vmatrix}P_{n+1}(\lambda)&P_n(\lambda)\\P_{n+1}(a)&P_n(a)\end{vmatrix},\qquad \overline{P}(\lambda)=\begin{vmatrix}P_{n+1}(\lambda)&P_n(\lambda)\\P_{n+1}(b)&P_n(b)\end{vmatrix},\tag{10.57}$$

and their conjugated polynomials $\underline{Q}(\lambda)$, $\overline{Q}(\lambda)$. Since $\underline{P}(a)=\overline{P}(b)=0$ the quasiorthogonal polynomials $\underline{P}(\lambda)$, $\overline{P}(\lambda)$, $\underline{Q}(\lambda)$, and $\overline{Q}(\lambda)$ have simple zeros which belong to the segment $[a,b]$. Therefore, the following inequalities hold:

$$P_n(b)\ge 0,\qquad(-1)^nP_n(a)\ge 0,\tag{10.58}$$

$$\underline{P}(b)=(-1)^n[P_{n+1}(b)P_n(a)-P_n(b)P_{n+1}(a)]=(-1)^{n+1}\overline{P}(a)>0.\tag{10.59}$$

Note also that for a sequence $\{s_k\}_0^{2n}$ nonnegative on $[a,b]$ the equality $\underline{P}(b)=\overline{P}(a)=0$ is equivalent to the property of $\{s_k'\}_0^{2n-2}$ to be singular positive.

**Proposition 10.4.** *Let a sequence $\{s_k\}_0^{2n}$ be positive on $[a,b]$, $A$ be a Hermitian operator of the form in* (10.2) *on $\mathfrak{h}=\mathbb{C}_n[t]$, $\mathfrak{L}=\{\mathbb{C}P_0\}$. Then we have:*

(1) *$a\le A\le b$;*

(2) *the set $\Pi^\mu=\{\mathbb{C},\Gamma_1^\mu,\Gamma_2^\mu\}$, in which for all $\hat{f}=\{f,A_0f+cP_n\}\in A^*$*

$$\Gamma_1^\mu\hat{f}=k[P_n(b)b_n^{-1}c-P_{n+1}(b)(f,P_n)],\quad \Gamma_2^\mu\hat{f}=(-1)^nk[P_n(a)b_n^{-1}c-P_{n+1}(a)(f,P_n)],\quad k=\sqrt{\frac{b_n}{\underline{P}(b)}},\tag{10.60}$$

*forms a BVS for the linear relation $A^*$, $\ker\Gamma_1^\mu=A_b$, $\ker\Gamma_2^\mu=A_a$;*

(3) *the corresponding Weyl function, $\Pi^\mu\mathfrak{L}$-preresolvent, and $\Pi^\mu\mathfrak{L}$-resolvent matrices take the forms*

$$
\begin{gathered}
M_\mu(\lambda) = \frac{\overline{P}(\lambda)}{\underline{P}(\lambda)}, \\
\mathfrak{A}_{\Pi^\mu\mathfrak{L}}(\lambda) = \begin{pmatrix} \overline{P}(\lambda)\underline{P}(\lambda)^{-1} & (k\underline{P}(\lambda)^{-1} \\ (k\underline{P}(\lambda))^{-1} & -\underline{Q}(\lambda)\underline{P}(\lambda)^{-1} \end{pmatrix}, \\
W_{\Pi^\mu\mathfrak{L}}(\lambda) = k\begin{pmatrix} -\underline{Q}(\lambda) & -\overline{Q}(\lambda) \\ \underline{P}(\lambda) & \overline{P}(\lambda) \end{pmatrix}.
\end{gathered}
\tag{10.61}
$$

**Proof.** (1) The inequalities $a \le A \le b$ are implied by the property of the sequence $\{s'_k\}_0^{2n-2}$ of the form in (10.56) to be positive.

(2) Since $P(b) > 0$ the constant $k = \sqrt{b_n\underline{P}(b)^{-1}}$ and, therefore, the mappings $\Gamma_j^\mu$ $(j = 1, 2)$ in (10.60) are well defined. The straightforward calculations yield that the triple $\Pi^\mu = \{\mathbb{C}, \Gamma_1^\mu, \Gamma_2^\mu\}$ is a BVS for $A^*$. On account of (10.5) and (10.6) the mappings $\Gamma_1^\mu$, $\Gamma_2^\mu$ may be rewritten (provided that $P_n(a)P_n(b) \neq 0$) in the form

$$
\Gamma_1^\mu = b_n^{-1}kP_n(b)[\Gamma_1 - M(b)\Gamma_2], \qquad \Gamma_2^\mu = (-1)^n\frac{kP_n(a)}{b_n}[\Gamma_1 - M(a)\Gamma_2]. \tag{10.62}
$$

Hence, owing to Proposition 4.6, we obtain that $\ker\Gamma_1^\mu = A_b$, $\ker\Gamma_2^\mu = A_a$. In the case $P_n(a)P_n(b) = 0$ the last equalities follow from Corollary 4.5.

(3) Relations (10.9), (10.10), and (10.60) yield

$$
\begin{aligned}
\Gamma_1^\mu\hat{f}_\lambda = \Gamma_1^\mu\hat{\mathcal{P}}^*(\bar\lambda) = k[P_n(b)P_{n+1}(\lambda) - P_{n+1}(b)P_n(\lambda)] = k\overline{P}(\lambda), \\
\Gamma_2^\mu\hat{f}_\lambda = \Gamma_2^\mu\hat{\mathcal{P}}^*(\bar\lambda) = (-1)^n k[P_n(a)P_{n+1}(\lambda) - P_{n+1}(a)P_n(\lambda)] = k\underline{P}(\lambda).
\end{aligned}
\tag{10.63}
$$

Equalities (10.63) imply that formula (10.61) for the Weyl function $M_\mu(\lambda)$ holds. In the same way we obtain from (10.15) and (10.60)

$$
\Gamma_1^\mu\hat{\mathcal{Q}}^*(\bar\lambda) = k\overline{Q}(\lambda), \qquad \Gamma_2^\mu\hat{\mathcal{Q}}^*(\bar\lambda) = k\underline{Q}(\lambda). \tag{10.64}
$$

Formula (10.61) for $W_{\Pi^\mu\mathfrak{L}}(\lambda)$ is implied now by (10.62), (10.63), and (8.22). Equality (10.61) for $\mathfrak{A}_{\Pi\mathfrak{L}}(\lambda)$ is obvious.

**Remark 10.5.** A similar statement for the BVS $\Pi^M = \{\mathbb{C}, \Gamma_1^M, \Gamma_2^M\} = \{\mathbb{C}, -\Gamma_2^\mu, \Gamma_1^\mu\}$ also holds. In this case we have $M_M(\lambda) = -\underline{P}(\lambda)/\overline{P}(\lambda)$,

$$
\mathfrak{A}_{\Pi^\mu\mathfrak{L}}(\lambda) = \begin{pmatrix} -\underline{P}(\lambda)\overline{P}(\lambda)^{-1} & (k\underline{Q}(\lambda))^{-1} \\ (k\underline{Q}(\lambda))^{-1} & -\overline{Q}(\lambda)\overline{P}(\lambda)^{-1} \end{pmatrix}, \qquad W_{\Pi^\mu\mathfrak{L}}(\lambda) = \begin{pmatrix} -\overline{Q}(\lambda) & \underline{Q}(\lambda) \\ \overline{P}(\lambda) & -\underline{P}(\lambda) \end{pmatrix}. \tag{10.65}
$$

Note also that the Weyl function $M_\mu(\lambda) = \overline{P}(\lambda)/\underline{P}(\lambda)$ $(M_M(\lambda) = -\underline{P}(\lambda)/\overline{P}(\lambda))$ coincides in the case $[a, b] = [-1, 1]$ up to the multiplier $(-1)^nP_n(b)/P_n(a)$ $((-1)^{n+1}P_n(a)/P_n(b))$ with the $Q_\mu$-function ($Q_M$-function) of a Hermitian contraction $A$ [cf. (3.73), (3.74)].

**Remark 10.6.** Relations (10.61) and (10.65) yield, in particular,

$$
((A_a - x)^{-1}e_0, e_0) = -\underline{Q}(x)\underline{P}(x)^{-1}, \qquad ((A_b - x)^{-1}e_0, e_0) = -\overline{Q}(x)\overline{P}(x)^{-1}. \tag{10.66}
$$

Therefore, from the extremal properties (3.69) of the extensions $A_a$ and $A_b$ ($A_{-1} = A_\mu, A_{+1} = A_M$ in the case $[a, b] = [-1, 1]$) follows the well-known inequalities of Markov [38]:

$$
\frac{\underline{Q}(x)}{\underline{P}(x)} = \int_a^b \frac{d\underline{\sigma}(t)}{x - t} \le \int_a^b \frac{d\sigma(t)}{x - t} \le \int_a^b \frac{d\overline{\sigma}(t)}{x - t} = \frac{\overline{Q}(x)}{\overline{P}(x)} \qquad \forall x \in \mathbb{R} \setminus [a, b], \tag{10.67}
$$

which hold for all $\sigma(t) \in V([a, b]; \mathbf{s})$, where $\mathbf{s} = \{s_k\}_0^{2n}$, $\underline{\sigma}(t) = (E_{A_a}(t)\mathbb{I}, \mathbb{I})$, $\overline{\sigma}(t) = (E_{A_b}(t)\mathbb{I}, \mathbb{I})$ are the lower and the upper main distribution of mass.

Inequalities (3.69) applied to the vectors $f_k = t^k$ yield more general inequalities ($x \in \mathbb{R} \setminus [a,b], 0 \le k \le n$):

$$\int_a^b \frac{t^{2k} d\underline{\sigma}(t)}{x-t} \le \int_a^b \frac{t^{2k} d\sigma(t)}{x-t} \le \int_a^b \frac{t^{2k} d\overline{\sigma}(t)}{x-t} \qquad \forall x \in \mathbb{R} \setminus [a,b]. \tag{10.68}$$

On the other hand, they follow from (10.67) and the obvious equalities

$$\int_a^b \frac{t^{2k} d\overline{\sigma}(t)}{t-\lambda} + \lambda^{2k} \frac{\overline{Q}(\lambda)}{\overline{P}(\lambda)} = \int_a^b \frac{t^{2k} d\sigma(t)}{t-\lambda} + \lambda^{2k} \frac{\underline{Q}(\lambda)}{\underline{P}(\lambda)} = \sum_{j=1}^{2k-1} \lambda^j s_{2k-j-1}.$$

**Remark 10.7.** It is easy to see that the resolvent matrix $W_{\Pi^\mu \mathfrak{L}}(\lambda)$ of the form in (10.59) is symplectic. Therefore,

$$\underline{P}(\lambda)\overline{Q}(\lambda) - \overline{P}(\lambda)\underline{Q}(\lambda) = \frac{1}{k^2} = \frac{\underline{P}(b)}{b_n} = \frac{(-1)^n}{b_n}[P_{n+1}(b)P_n(a) - P_n(b)P_{n+1}(a)]. \tag{10.69}$$

One can show that identity (10.69) is equivalent to identity (2.11) from [38]. Note also that the equality $\det W_{\Pi \mathfrak{L}}(\lambda) = 1$ for the symplectic matrix $W_{\Pi \mathfrak{L}}(\lambda)$ of the form in (10.7) yields the Liouville–Ostrogradskii formula.

Making use of Propositions 3.7 and 10.4, we obtain the following result of Krein [36] (see also [38]).

**Proposition 10.5** [36]. *Let a sequence $\{s_k\}_0^{2n}$ be strictly positive on $[a,b]$. Then the formula*

$$\int_a^b \frac{d\sigma(t)}{t-\lambda} = -\frac{\overline{P}(\lambda) + \tau(\lambda)\underline{P}(\lambda)}{\overline{Q}(\lambda) + \tau(\lambda)\underline{Q}(\lambda)} \tag{10.70}$$

*establishes a one-to-one correspondence between $\sigma(t) \in V(\mathbf{s}; [a,b])$ and $\tau(\lambda) \in S[a,b]$.*

A criterion of existence and a description of solutions of the Hausdorff moment problem with gaps $(\alpha_j, \beta_j)$ $(1 \le j \le m)$ will be given in another paper (cf. [24]).

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