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USING CONCEPTS OF FUZZY MATHEMATICS FOR FORMALIZATION AND SOLUTION OF COMBINATORIAL OPTIMIZATION PROBLEMS

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Combinatorial optimization methods currently play a dominant role in discrete programming. The main idea of these methods, as we know, is to utilize finiteness of the solution set and apply reduced enumeration of alternatives. The universality of combinatorial methods makes it possible to solve a wide range of optimization problems by using pure combinatorial methods without reduction to an integer problem [1-5].

However, the increasing practical demands often lead to cases when the existing models are not fully adequate for describing a real-life problem: the formal statement of the problem must allow for additional information about the object of study. In this article we focus on the use of concepts and results of fuzzy mathematics in order to improve the formal statement of combinatorial optimization problems in this respect. We also note some important properties of fuzzy formulations and highlight the possibilities of their solution by various algorithms previously developed for some classes of combinatorial optimization problems [2, 5].

A combinatorial optimization problem in general seeks to determine a combinatorial object consisting of elements of some discrete set that has certain properties and extremizes a given objective function. Formally, this can be written in the following form: find $x' \in D \subset C(A)$, that extremizes the function $F(x)$. Here $C(A)$ is the set of combinatorial objects formed from the elements of the discrete set A, D is the feasible solution set.

It is easy to see that the above definition encompasses a wide range of problems. For instance, $C(A)$ may be a nonhomogeneous (multidimensional) set of locations, samples, permutations, etc., or elements of the set A. The set D and the objective function F also allow great diversity.

In this respect, it is interesting to consider the following fragments of three interrelated problems.

Problem 1. A is a given set of objects, $|A| = n$, and B is a set of "programs," "scenarios," "events," etc., $|B| = m$, $n \leq m$. Choose $B^0 \subseteq B$ and place it in one-to-one correspondence with the set A so as to optimize the objective. The constraints of this problem are summarized in Table 1.

Problem 2. This problem differs from problem 1 only in that its constraints additionally include some characteristics of the elements of the set A (the second column of Table 2).

Problem 3. In addition to information about the elements of A, this problem also contains information about the elements of the set B (see Table 3).

Problems 1-3 have interesting practical interpretations. For instance, the set A may be a collection of production capacities, while B is the set of programs for modernizing production. The characteristics α_{ij} represent the wear-and-tear of the equipment, the energy and resource intensity, ecological factors, etc.; β_{ls} are proposals for the use of the characteristics of the objects from A, the requirements in materials, funds, scientific development, etc., proposed by the corresponding programs. Suppose that the relevant characteristics can be expressed by nonnegative numbers less than 1. In this case, we say that fuzzy sets $\tilde{A}_1, \ldots, \tilde{A}_k$ are defined on the set A and fuzzy sets $B - \tilde{B}_1, \ldots, \tilde{B}_r$ are defined on B with the respective membership functions $\alpha^{(i)} = \alpha_{ij}$, $j = 1, \ldots, k$, and $\beta^{(l)} = \beta_{ls}$, $s = 1, \ldots, r$. Problems 2 and 3 thus become combinatorial optimization problems on fuzzy sets. A combinatorial optimization problem is written in the form

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$$x^* = \underset{x \in D \subset C(A,\, \tilde{A}_1, \ldots, \tilde{A}_m;\, B,\, \tilde{B}_1, \ldots, \tilde{B}_r)}{\arg \max} f(x), \tag{1}$$

where $C(A, \tilde{A}_1, \ldots, \tilde{A}_m; B, \tilde{B}_1, \ldots, \tilde{B}_r)$ is the set of fuzzy combinatorial objects formed from the elements of the discrete sets A, B and the elements of the fuzzy sets $\tilde{A}_1, \ldots, \tilde{A}_k, \tilde{B}_1, \ldots, \tilde{B}_r$, defined on A, B by the corresponding membership functions; D is the feasible region and $f(x)$ is the objective function.

Let us now formulate problem 3 based on the general form (1).

Let $A = \{a_i\}_{i=1}^n$ be a collection of some objects (production capacities), $B = \{b_l\}_{l=1}^m$ the set of programs proposed for modernizing the objects of the set A, $n \leq m$. Now assume that fuzzy sets $\tilde{A}_1, \ldots, \tilde{A}_j$ are defined on the set A by the membership function α_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, k$, and the fuzzy sets $\tilde{B}_1, \ldots, \tilde{B}_r$ are defined on the set B by the membership function β_{ls}, $l = 1, \ldots, m$, $s = 1, \ldots, r$. The results of acting with the sets $\tilde{B}_1, \ldots, \tilde{B}_r$ on the sets $\tilde{A}_1, \ldots, \tilde{A}_j$ are represented by the values τ_{ij}^{ls} in Table 3. Find a subset of programs $B_0 \subseteq B$, $|B| = n$, and place it in a one-to-one correspondence with the elements of the set A (or, in other words, order it by the indices of the elements of the set A) so as to maximize the function

$$F(x, p, C, T) = \varphi(x, p) + \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l \in \omega(x,\, p)} \sum_{s=1}^{r} c_{ij}^{ls} \tau_{ij}^{ls}, \tag{2}$$

where x is a subset of B, $x \subset B$, and $|x| = n$; p is a permutation of the elements of the subset x placed in a one-to-one correspondence with the set A; T is the set of values τ_{ij}^{ls}, C is the "price" matrix corresponding to T, $\omega(x, p)$ is the index set defined by the set (x, p). Thus, the objective function $F(x, p, C, T)$ is the sum of two functions, one of which is independent of the matrices T and C. This representation of the objective function is useful for what follows.

Note that the values τ_{ij}^{ls} are mostly experimental data, expert estimates, results of statistical analysis, scientific forecasts, etc. Such values are of course highly approximate, and it is relevant to consider the stability of the optimization problem with the objective function (2). Let us examine this question in some detail.

Our problem is thus to find the maximum of the function (2) under the conditions of problem 3.

This problem can be solved by any of the local optimization methods [2, 4]. Suppose that a solution is some point (x^*, p^*) in a discrete metric space, whose first coordinate is a point from $M_B{}^n$ (the set of all n-element subsets of the set B) and the second coordinate is an element of the set P_n of n-element permutations. The assumptions of local optimality of the point (x^*, p^*) imply that there exists a neighborhood $\Omega_\rho(x^*, p^*)$ of radius ρ of the point (x^*, p^*) in the space $M_B^n \times P_n$ where the function $F(x, p, C, T)$ reaches a value not less than at any other point of this neighborhood [2, 4], i.e., $F(x^*, P^*, C, T) > F(x, p, C, T)$, where $(x, p) \in \Omega_p(x^*, p_n^*)$, $M_B^n \times P_n$ is the product of the two spaces.

Suppose that some $\tau_{i_0 j_0}^{l_0 s_0}$ is acted upon by the perturbation $\Delta_{i_0 j_0}^{l_0 s_0}$. We will try to answer the following question: under what conditions this perturbation does not change (x^*, p^*) as the local maximum point of the function $F(x, p, C, T)$. If $l_0 \bar{\in} \omega_\rho(x^*, p^*)$, then the value of $F(x^*, p^*, C, T)$ is independent of the perturbation $\Delta_{i_0 j_0}^{l_0 s_0}$ and thus remains unchanged. Here $\omega_\rho(x^*, p^*)$ is the index set corresponding to $\Omega_\rho(x^*, p^*)$. Let $l_0 \in \omega_\rho(x^*, p^*)$. Then the function increment is written in the form

$$\Delta F(x, p, C, P) = F(x, p, C, T) -$$

$$- \tilde{F}(x, p, C, T) = \varphi(x, p) + {\sum_{i=1}^{n}}' {\sum_{j=1}^{k}}' {\sum_{l \in \omega(x,\, p)}}' {\sum_{s=1}^{r}}' c_{ij}^{ls} \tau_{ij}^{ij} +$$

$$+ c_{i_0 j_0}^{l_0 s_0} \tau_{i_0 j_0}^{l_0 s_0} - \varphi(x, p) - {\sum_{i=1}^{n}}' {\sum_{j=1}^{k}}' {\sum_{l \in \omega(x,\, p)}}' {\sum_{s=1}^{r}}' c_{ij}^{ls} \tau_{ij}^{ls} -$$

$$- c_{i_0 j_0}^{l_0 s_0}(\tau_{i_0 j_0}^{l_0 s_0} + \Delta_{i_0 j_0}^{l_0 s_0}) = - c_{i_0 j_0}^{l_0 s_0} \Delta_{i_0 j_0}^{l_0 s_0}.$$

Here $'$ denotes summation over all indices except i_0, j_0, l_0, s_0.

TABLE 1

A	B			
	b_1	b_2	...	b_m
a_1	τ_{11}	τ_{12}	...	τ_{1m}
a_2	τ_{21}	τ_{22}	...	τ_{2m}
...	...	...	...	...
a_n	τ_{n1}	τ_{n2}	...	τ_{nm}

TABLE 2

A		B			
		b_1	b_2	...	b_m
a_1	a_{11}	τ^1_{11}	τ^2_{11}	...	τ^m_{11}
	a_{12}	τ^1_{12}	τ^2_{12}	...	τ^m_{12}
	...	...	...	...	...
	a_{1k}	τ^1_{1k}	τ^2_{1k}	...	τ^m_{1k}
a_2	a_{21}	τ^1_{21}	τ^2_{21}	...	τ^m_{21}
	a_{22}	τ^1_{22}	τ^2_{22}	...	τ^m_{22}
	...	...	...	...	...
	a_{2k}	τ^1_{2k}	τ^2_{2k}	...	τ^m_{2k}
...	...	...	...	...	...
a_n	a_{n1}	τ^1_{n1}	τ^2_{n1}	...	τ^m_{n1}
	a_{n2}	τ^1_{n2}	τ^2_{n2}	...	τ^m_{n2}
	...	...	...	...	...
	a_{nk}	τ^1_{nk}	τ^2_{nk}	...	τ^m_{nk}

Consider the various possible cases.

1. $\Delta_{i_0j_0}^{l_0s_0} > 0$. If the perturbation $\tilde{\tau}_{i_0j_0}^{l_0s_0} = \tau_{i_0j_0}^{l_0s_0} + \Delta_{i_0j_0}^{l_0s_0}$ does not enter the expression for $F(x^*, p^*)$, and the neighborhood Ω_p contains a point (x', p') such that $\tau_{i_0j_0}^{l_0s_0}$ is used to evaluate $F(x', p')$, then clearly $F(x', p', C, T) < \tilde{F}(x', p', C, P)$ and we may have a case when

$$F(x^* p^*, C, T) < \tilde{F}(x', p', C, T). \tag{3}$$

Let us estimate the perturbation $\Delta_{i_0j_0}^{l_0s_0}$ when this may happen. Let $F(x^*, p^*, C, T) - F(x', p', C, T) = \delta(x^*, x', p^*, p', C, T)$. Inequality (3) clearly holds whenever $\delta(x^*, x', p^*, p', C, T) < c_{i_0j_0}^{l_0s_0}\Delta_{i_0j_0}^{l_0s_0}$. Denote $\delta = \min\limits_{(x^*, p^*) \in \Omega_p} \delta(x^*, p^*, C, T)$. Then the inequality $c_{i_0j_0}^{l_0s_0}\Delta_{i_0j_0}^{l_0s_0} < \delta$ is a sufficient condition for the proposed model to be stable.

If the value of $F(x^*, p^*, C, T)$, itself is perturbed, it will only increase when $\Delta_{i_0j_0}^{l_0s_0} > 0$, so that the sufficient condition $c_{i_0j_0}^{l_0s_0}\,\Delta_{i_0j_0}^{l_0s_0} < \delta$ remains valid.

2. $\Delta_{i_0j_0}^{l_0s_0} < 0$. If $\Delta_{i_0j_0}^{l_0s_0}$ enters the expression for $F(x', P', C, T)$, $(x', p') \in \Omega_p$, the local maximum remains unchanged. If $\Delta_{i_0j_0}^{l_0s_0}$ is used to evaluate $F(x^*, p^*, C, T)$, we may have a case when the point (x', p') in the neighborhood Ω_p is such that $\tilde{F}(x^*, P^*, C, T) < F(x', p', C, T)$, and the problem model is unstable. Here, as in case 1, it is sufficient to have $\delta > -c_{i_0j_0}^{l_0s_0}\,\Delta_{i_0j_0}^{l_0s_0}$.

The analysis of cases 1 and 2 provides a proof of the following proposition.

THEOREM 1. The inequality $|c_{i_0j_0}^{l_0s_0}\Delta_{i_0j_0}^{l_0s_0}| < \delta$ is a sufficient condition of stability of the maximization problem of function (2).

Note that combinatorial optimization problems whose mathematical model contains fuzzy sets can be successfully solved by various local optimization methods developed for combinatorial optimization problems [2, 4]. Local optimization algorithms

TABLE 3

A		B			
		b_1	b_2	...	b_m
		$\beta_{11}\beta_{12} \cdots \beta_{1r}$	$\beta_{21}\beta_{22} \cdots \beta_{2r}$	...	$\beta_{m1}\beta_{m2} \cdots \beta_{mr}$
a_1	α_{11}	$\tau_{11}^{11}\tau_{11}^{12} \cdots \tau_{11}^{1r}$	$\tau_{11}^{21}\tau_{21}^{12} \cdots \tau_{11}^{2r}$	...	$\tau_{11}^{m1}\tau_{11}^{m2} \cdots \tau_{11}^{mr}$
	α_{12}	$\tau_{12}^{11}\tau_{12}^{12} \cdots \tau_{12}^{1r}$	$\tau_{12}^{21}\tau_{12}^{22} \cdots \tau_{12}^{2r}$	...	$\tau_{12}^{m1}\tau_{12}^{m2} \cdots \tau_{12}^{mr}$
	...	...	...	...	...
	α_{1k}	$\tau_{1k}^{11}\tau_{1k}^{12} \cdots \tau_{1k}^{1r}$	$\tau_{1k}^{21}\tau_{1k}^{22} \cdots \tau_{1k}^{2r}$	...	$\tau_{1k}^{m1}\tau_{1k}^{m2} \cdots \tau_{1k}^{mr}$
a_2	α_{21}	$\tau_{21}^{11}\tau_{21}^{12} \cdots \tau_{21}^{1r}$	$\tau_{21}^{21}\tau_{21}^{22} \cdots \tau_{21}^{2r}$	...	$\tau_{21}^{m1}\tau_{21}^{m2} \cdots \tau_{21}^{mr}$
	α_{22}	$\tau_{22}^{11}\tau_{22}^{12} \cdots \tau_{22}^{1r}$	$\tau_{22}^{21}\tau_{22}^{22} \cdots \tau_{22}^{2r}$	...	$\tau_{22}^{m1}\tau_{22}^{m2} \cdots \tau_{22}^{mr}$
	...	...	...	...	...
	α_{2k}	$\tau_{2k}^{11}\tau_{2k}^{12} \cdots \tau_{2k}^{1r}$	$\tau_{2k}^{21}\tau_{2k}^{22} \cdots \tau_{2k}^{2r}$	...	$\tau_{2k}^{m1}\tau_{2k}^{m2} \cdots \tau_{2k}^{mr}$
...	...	...	...	...	...
a_n	α_{n1}	$\tau_{n1}^{11}\tau_{n1}^{12} \cdots \tau_{n1}^{1r}$	$\tau_{n1}^{21}\tau_{n1}^{22} \cdots \tau_{n1}^{2r}$	...	$\tau_{n1}^{m1}\tau_{n1}^{m2} \cdots \tau_{n1}^{mr}$
	α_{n2}	$\tau_{n2}^{11}\tau_{n2}^{12} \cdots \tau_{n2}^{1r}$	$\tau_{n2}^{21}\tau_{n2}^{22} \cdots \tau_{n2}^{2r}$	...	$\tau_{n2}^{m1}\tau_{n2}^{m2} \cdots \tau_{n2}^{mr}$
	...	...	...	...	...
	α_{nk}	$\tau_{nk}^{11}\tau_{nk}^{12} \cdots \tau_{nk}^{1r}$	$\tau_{nk}^{21}\tau_{nk}^{22} \cdots \tau_{nk}^{2r}$	...	$\tau_{nk}^{m1}\tau_{nk}^{m2} \cdots \tau_{nk}^{mr}$

for various problems require a definition of "proximity" (a metric) in fuzzy combinatorial spaces. This can be introduced using the results of [6, 7].

In conclusion, we list some potential applications of our results:

— generating a team of operators to complete a given volume of work;

— generating a machine pool, selecting equipment;

— manufacturing mixtures from several components that meet certain requirements (diet, mixed feed, etc.);

— the fuzzy knapsack problem;

— classification of biological objects [2] (when the features are defined on the interval [0, 1]);

— a program for processing agricultural fertilizers [8], substances, materials;

— to what extent the actual technical characteristics of a system meet the advertised specifications.

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