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VECTOR OPTIMIZATION PROBLEMS: STABILITY IN THE DECISION SPACE AND IN THE SPACE OF ALTERNATIVES

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The present article is related to [1-7], and it examines stability of vector optimization problems of the form

$$(C, X):\ \text{"max"}\ \{Cx: x \in X\}, \tag{1}$$

where $C = [c_{ij}]_{L \times n}$ is the matrix of coefficients of all particular linear efficiency criteria $c_l(x) = \sum_{j=1}^{n} c_{lj}x_j$, $c_l = \{c_{l1}, c_{l2}, \ldots, c_{ln}\} \in R^n$, $l = 1, \ldots, L$, L is the number of criteria, X is a bounded set of arbitrary structure (possibly discrete) in R^n.

Sec. 1 presents the necessary definitions and notation. Here stability and P-stability are understood in the sense of Hausdorff upper semicontinuity of certain set-valued mappings.

Sec. 2 proves that our problem is P-stable in the decision space under changes of the criterion coefficients. Necessary and sufficient conditions of stability under changes of criterion coefficients are proved, and simple sufficient conditions are given. For the problem with an unperturbed feasible region in the initial-data space, we identify the set of initial-data matrices for which the problem is stable in the decision space under changes in the criterion coefficients. We show that this set is everywhere dense in the initial-data space.

Sec. 3 examines the equivalence of the concepts of stability in the decision space and in the space of alternatives. We show that for the vector mixed-integer optimization problem these concepts are not equivalent. Some sufficient conditions of their equivalence are given. We also prove necessary and sufficient conditions of stability of problem (1) in the space of alternatives under changes in the criterion coefficients.

Sec. 4 considers yet another definition of stability (I-stability), which is based on Hausdorff lower semicontinuity. In particular we show that the set of initial-data matrices $C \in R^{L \times n}$ for which the problem (C, X^0) is not I-stable in the decision space under changes of criterion coefficients is of measure zero for $L \geq n$.

1. A solution of problem (1) is some subset of one of the following sets: the set $\Pi(C, X)$ of all Pareto-optimal (efficient) solutions, the set $P(C, X)$ of semi-efficient solutions, or the set $S(C, X)$ of strictly efficient solutions.

Recall [8] that the point $x^* \in X$ is called efficient (or Pareto-optimal) if $\nexists x \in X$: $Cx \geq Cx^*$, $Cx \neq Cx^*$; it is called weakly efficient (semi-efficient, Slater-optimal) if $\nexists x \in X$: $Cx > Cx^*$; it is called strictly efficient if $\nexists x \in X$: $x \neq x^*$, $Cx \geq Cx^*$. Clearly $S(C, X) \subseteq \Pi(C, X) \subseteq P(C, X)$.

Consider the convex cone $K = \{x \in R^n: Cx \geq 0\}$, which can be represented as the set union $K = K_0 \cup K_1 \cup K_2$, where $K_0 = \{x \in R^n: Cx = 0\}$, $K_1 = \{x \in R^n: Cx > 0\}$; $K_2 = K/(K_0 \cup K_1)$. Then $x \in \Pi(C, X) \Leftrightarrow (x + K_1 \cup K_2) \cap X = \{x\}$; $x \in P(C, X) \Leftrightarrow (x + K_1) \cap X = \{x\}$; $x \in S(C, X) \Leftrightarrow (x + K) \cap X = \{x\}$.

Let $r(C)$ be the rank of the matrix C. Recall that $K_0 = \{0\} \Leftrightarrow r(C) = n$. We have the following obvious proposition.

Proposition 1. If $r(C) = n$, then $\Pi(C, X) = S(C, X)$; if $K_1 = \varnothing$, then $P(C, X) = X$; if $K_2 = \varnothing$, then $P(C, X) = \Pi(C, X)$; if there exist nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_L$ that are not all zero at the same time and such that $\sum_{k=1}^{L} \alpha_k c_k = 0$, then $P(C, X) = X$; if $r(C) = 1$, then $K_2 = \varnothing$ and $P(C, X) = \Pi(C, X)$.

To the problem (C, X) we associate a family of perturbed problems $\{(C(u), X(u))\}$, where u is the perturbation parameter, $u \in R^k$ ($k \geq 1$), so that the initial problem corresponds to the initial value of the perturbation parameter. Here $C(u)$

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$= [c_{ij}(u)]_{L\times n}$ is the matrix of coefficients of all particular linear efficiency criteria $c_l(u, x) = \sum_{j=1}^{n} c_{lj}(u)x_j$, $c_{lj}(u) \in R^n$, $l = 1, \ldots, L$, L is the number of criteria, $X(u)$ is a bounded set in R^n. For instance, we may take $u = C \in R^L$ $(L = k)$ when $X(u) \equiv X$, and also assume that the vector $u = (C, A, b) \in R^{L\times n} \times R^{m\times n} \times R^m$ if $X(u) = X(A, b) = \{x \in R^n\colon Ax \leq b\}$, where A is an $m \times n$ matrix from $R^{m\times n}$, b is an m-vector from R^m.

The theory of stability analysis of optimization problems is based on continuity properties of the set-valued mappings $\Pi(u)$, $S(u)$, $P(u)$, where $\Pi\colon u \to \Pi(u) = \Pi(C(u), X(u))$; $S: u \to S(u) = S(C(u), X(u))$; $P: u \to P(u) = P(C(u), X(u))$, and also the mappings $\mathfrak{C}_\Pi(u) = \mathfrak{C}(C(u), \Pi(C(u), X(u))$, $\mathfrak{C}_P(u) = \mathfrak{C}(C(u), P(C(u), X(u)))$, $\mathfrak{C}_S(u) = \mathfrak{C}(C(u), S(C(u), X(u)))$, where $\mathfrak{C}: (C, Q) \to \{y = Cx \in R^L \mid x \in Q\}$.

The vector $u \in R^k$ is an element in the initial-data space. The sets $X(u)$, $\Pi(u)$, $S9u)$, $P(u)$ are subsets of the decision space R^n and elements of the set 2^{R^n} (the set of all subsets of the set R^n), while the sets $\mathfrak{C}(C(u), \Pi(C(u), X(u)))$ $\mathfrak{C}(C(u), P(C(u), X))$, $\mathfrak{C}(C(u), S(C(u), X(u)))$ are subsets of the space of alternatives R^L and elements of the set 2^{R^L}. The notion of continuity needed for stability analysis of optimization problems requires the definition on the sets 2^{R^n} and 2^{R^L} of topologies induced by the topologies of the decision space and the space of alternatives, respectively. In what follows, any real p-dimensional space R^p is treated as a normed space. We assume that the topology in the relevant spaces is induced by the metric $d(x, y) = \|x - y\|$.

Let us recall some basic definitions [6, 8-11].

Given are two normed spaces U and V. A mapping Γ which associates to every point of the set U some subset of the set V is called a set-valued mapping of U to V.

A set-value mapping Γ at the point $u^0 \in U$ is called:

a) Bergé upper semicontinuous (USC) if for every open set Ω such that $\Gamma(u^0) \subseteq \Omega$ there exists $\delta = \delta(\Omega) > 0$ such that $\Gamma(u) \subseteq \Omega$ for every $u \in O_\delta(u^0)$ (here $O_\delta(u^0) = \{u \in U : \| u - u^0 \| \leq \delta\}$);

b) Bergé lower semicontinuous (LSC) or open if for every open set Ω such that $\Gamma(u^0) \cap \Omega \neq \varnothing$ there exists $\delta = \delta(\Omega) > 0$ such that $\Gamma(u) \cap \Omega \neq \varnothing$ for every $u \in O_\delta\{u^0\}$;

c) Hausdorff USC if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\Gamma(u) \subseteq O_\varepsilon(\Gamma(u^0))$ for every $u \in O_\delta(u^0)$, where $O_\varepsilon(\Gamma(u^0)) = \{x \in V\colon \inf\{\|x - y\| \leq \varepsilon : y \in \Gamma(u^0)\}\}$;

d) Hausdorff LSC if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\Gamma(u^0) \subseteq O_\varepsilon(\Gamma(u))$ for every $u \in O_\delta(u^0)$;

e) Bergé (Hausdorff) continuous if this mapping is Bergé (Hausdorff) USC and LSC at the point $u^0 \in U$;

f) closed if for any two sequences $\{u_n\}$ and $\{x_n\}$ such that $u_n \to u^0$ and $x_n \to x^0$ the inclusion $x_n \in \Gamma(u_n)$ implies the inclusion $x^0 \in \Gamma(u^0)$.

We now give some necessary elementary properties of set-valued mappings.

1. Bergé USC $\Rightarrow$ Hausdorff USC.

The converse is not true: in general, Hausdorff USC does not imply Bergé USC.

2. Hausdorff LSC $\Rightarrow$ Bergé LSC.

In general, Bergé LSC does not imply Hausdorff LSC.

3. If the mapping $\Gamma(u)$ is closed at the point $u^0 \in U$, then the set $\Gamma(u)$ is closed at this point.

4. If the mapping $\Gamma(u)$ is Hausdorff USC at the point $u^0 \in A$ and the set $\Gamma(u^0)$ is closed, then the mapping $\Gamma(u)$ is closed at the point $u^0 \in U$.

5. If the mapping $\Gamma(u)$ is closed at the point $u^0 \in U$ and V is a compactum, then $\Gamma(u)$ is Bergé USC at the point $u^0 \in U$.

6. If the mapping $\Gamma(u)$ is Hausdorff USC at the point $u^0 \in U$ and $\Gamma(u^0)$ is a compactum, then $\Gamma(u)$ is Bergé USC at the point $u^0 \in U$.

7. If the mapping $\Gamma(u)$ is Bergé LSC at the point $u^0 \in U$ and the closure $\mathrm{cl}\,\Gamma(u^0)$ is a compactum, then $\Gamma(u)$ is Hausdorff LSC at the point $u^0 \in U$.

For the vector optimization problem, stability in the decision space is understood in the sense of Hausdorff upper semicontinuity of the set-valued mapping $\Pi(u)$ that characterizes the dependence of the set of efficient points on the initial data; stability in the space of alternatives is understood in the sense of Hausdorff upper semicontinuity of the set-valued mapping $G_\Pi(u)$ that characterizes the dependence of the components of the vector criterion on the Pareto-optimal set on the initial data.

Stability in the decision space implies that, under small perturbations of initial data, for every efficient solution of the perturbed problem there is a sufficiently close solution of the initial problem.

Stability in the space of alternatives implies that, under small perturbations in the initial data, for any combination of criterion values on the Pareto-optimal set of the perturbed problem there is an efficient solution of the initial problem with a sufficiently close vector of criterion values.

For a single-criterion problem, an analog of the notion of stability in the solution stable is stability by solution, and an analog of the notion of stability in the space of alternatives is stability by the functional.

Stability by the functional implies that sufficiently small perturbations in initial data lead to sufficiently small changes in the functional values.

Stability by solution implies that, under sufficiently small perturbations in initial data, for any optimal solution of the perturbed problem there is a sufficiently close optimal solution of the initial problem.

Other definitions of stability using various notions of continuity are also used in the literature. Following [7], we can introduce for (1) the notion of P-stability, which replaces the set of efficient points with the larger set of weakly efficient points.

Definition 1. Problem (1) is called stable (P-stable) in the decision space if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any vector $u \in R^k$ and any matrix $C(u) \in R^{L\times n}$ satisfying the inequalities

$$\| u \| < \delta, \ \| C(u) - C \| < \delta, \tag{2}$$

the set $\Pi(C(u), X(u))$ of Pareto-optimal solutions (the set $P(C(u), X(u))$ of semi-efficient solutions) of the problem $(C(u), X(u))$ is nonempty and is included in the ε-neighborhood of the set $\Pi(C, X)$ (the set $P(C, X)$),

$$\Pi(C(u), X(u)) \subseteq O_\varepsilon \Pi(C, X) \ \ (P(C(u), X(u)) \subseteq O_\varepsilon P(C, X)). \tag{3}$$

Definition 2. Problem (1) is called stable (P-stable) in the space of alternatives if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any vector $u \in R^k$ and any matrix $C(u) \in R^{L\times n}$ that satisfy inequalities (2) the set of criterion values $\mathfrak{C}_\Pi(u)$, $\mathfrak{C}_\Pi(u) = \mathfrak{C}(C(u), \Pi(C(u), \times (u)))$ ($\mathfrak{C}_P(u)$, $\mathfrak{C}_P(u) = \mathfrak{C}(C(u), P(C(u), X(u)))$ of the problem $(C(u), X(u))$ is nonempty and is included in the ε-neighborhood of the set $\mathfrak{C}_\Pi(\mathfrak{C}_P)$,

$$\mathfrak{C}_\Pi(u) \subseteq O_\varepsilon \mathfrak{C}_\Pi \ (\mathfrak{C}_P(u) \subseteq O_\varepsilon \mathfrak{C}_P). \tag{4}$$

Definition 3. Problem (1) is called stable (P-stable) in the decision space under changes in criterion coefficients if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any matrix $C(\delta) \in R^{L\times n}$, satisfying the inequalities

$$\| C(\delta) - C \| < \delta, \tag{5}$$

the set $\Pi(C(\delta), X)$ of Pareto-optimal solutions (the set $P(C(\delta), X)$ of semi-efficient solutions) of the problem $(C(\delta), X)$ is nonempty and is included in the ε-neighborhood of the set $\Pi(C, X)$ (the set $P(C, X)$),

$$\Pi(C(\delta), X) \subseteq O_\varepsilon \Pi(C, X) \ \ (P(C(\delta), X) \subseteq O_\varepsilon P(C, X)).$$

Definition 4. Problem (1) is called stable (P-stable) in the space of alternatives under changes in criterion coefficients if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any matrix $C(\delta) \in R^{L\times n}$ that satisfies inequalities (4) the set of criterion values $\mathfrak{C}_\Pi(\delta)$, $\mathfrak{C}_\Pi(\delta) = \mathfrak{C}(C(\delta), \Pi(C(\delta), X))$ ($\mathfrak{C}_P(\delta)$, $\mathfrak{C}_P(\delta) = \mathfrak{C}(C(\delta), P(C(\delta), X))$ of the problem $(C(\delta), X)$ is nonempty and is included in the ε-neighborhood of the set $\mathfrak{C}_\Pi(\mathfrak{C}_P)$.

2. It is shown in [7] that problem (1), where X is a bounded set in Z^n, is always P-stable under changes of criterion coefficients. We will prove a similar proposition for problem (1) in the general case.

Proposition 2. If the set X^0 is bounded, then the set-valued mapping $P(C) = P(C, X^0))$ is a closed mapping.

Proof. Consider any two sequences $\{C_n\}$ and $\{x_n\}$ such that $C_n \to C^0$, $x_n \to x^0$, and we have the inclusion $x_n \in P(C_n)$. We will show that $x^0 \in P(C)$. If, by contradiction, $x^0 \notin P(C^0)$, then $\exists x' \in X$ such that $C^0x' > C^0x^0$. Since $C_n \to C^0$, inequalities (5) imply the existence of $N > 0$ such that for $n > N$ we have the inequality $C_nx' > C_nx_n$. This contradicts the inclusion $x_n \in P(C_n)$.

COROLLARY 1. If X^0 is a nonempty bounded closed set in R^n, then the mapping $P(C) = P(C, X^0))$ is Hausdorff upper semicontinuous, i.e., problem (1) is P-stable under changes in criterion coefficients.

Proof follows directly from properties 5 and 1 of set-valued mappings.

COROLLARY 2. If X is a nonempty bounded closed set in $\mathbf{R}^n$ and $\Pi(C, X) = P(C, X)$, then problem (1) is stable in the decision space under changes in criterion coefficients.

Proof. By Corollary 1 problem (1) is P-stable, and thus for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any matrix $C(\delta) \in \mathbf{R}^{L\times n}$ satisfying inequalities (5) the set $P(C(\delta), X)$ of semi-efficient solutions of the problem $(C(\delta), X)$ is included in the ε-neighborhood of the set $P(C, X)$, where $P(C(\delta), X)) \subseteq O_\varepsilon P(C, X)$. Since $\Pi(C(\delta), X)) \subseteq P(C(\delta), X))$ and by assumption $\mathrm{cl}\,\Pi = P$, we have $cl\,\Pi(C(\delta), X)) \subseteq P(C(\delta), X) \subseteq O_\varepsilon P(C, X) = O_\varepsilon cl\,\Pi(C, X)) \subseteq O_{2\varepsilon}\Pi(C, X))$. Thus, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon/2) > 0$ such that for any matrix $C(\delta) \in \mathbf{R}^{L\times n}$ satisfying inequalities (5) the set $\Pi(C(\delta), X)$ of efficient solutions of the problem $(C(\delta), X)$ is included in the ε-neighborhood of the set $\Pi(C, X)$, $\Pi(C(\delta), X)) \subseteq O_\varepsilon\Pi(C, X)$.

To prove the converse, we have to consider the family of problems $\{(C_\tau, X)\}$.

The matrix C_τ is constructed by perturbing each row c_k, $k = 1, \ldots, L$, of the matrix C in the following way:

$$c_k^\tau = c_k - \tau v, \quad v = \sum_{k=1}^{L} \mu_k c_k, \quad \mu_k > 0, \quad k = 1, \ldots, L.$$

Here τ is a numerical parameter.

Consider the cone $K^* = \left\{ x \in \mathbf{R}^n : x = \sum_{k=1}^{L} \lambda_k c_k, \quad \lambda_k \geqslant 0, \ k = 1, \ldots, L \right\}$, which is the conjugate of K. Clearly [12], $v \in \mathrm{ri}\, K^*$. For definiteness, we take $\sum_{k=1}^{L} \mu_k = 1$.

Consider the cone

$$K_\tau = \{x \in \mathbf{R}^n : C_\tau x \geqslant 0\}, \qquad K_\tau = K_0^\tau \cup K_1^\tau \cup K_2^\tau;$$

$$K_0^\tau = \{x \in \mathbf{R}^n : C_\tau x = 0\}; \qquad K_1^\tau = \{x \in \mathbf{R}^n : C_\tau x > 0\}; \qquad K_2^\tau = K_\tau \setminus (K_0^\tau \cup K_1^\tau).$$

The following properties of these cones are actually proved in [5, 7]:

1) $\forall \tau \in [0, 1] : K_\tau \subseteq K$;
2) $\forall \tau > 0 : K_\tau \cap K \subseteq K_1 \cup K_0$;
3) $\forall \tau \in \mathbf{R} : K_0 \subseteq K_0^\tau$;
4) $\forall \tau \leqslant 0 : K \subseteq K_\tau$;
5) $\forall \tau < 0 : K \subseteq K_1^\tau \cup K_0^\tau$;
6) $\forall \tau < 0 : K \cap K_0^\tau \subseteq K_0$;
7) $\forall \tau_1, \tau_2 \in [-\infty, 1), \ \tau_1 < \tau_2 : K_{-\infty} = \{x \in \mathbf{R}^n : vx \geqslant 0\} \supseteq K_{\tau_1} \supseteq K_{\tau_2}$.

The following proposition holds for any nonempty set X.

Proposition 3 [7]. $\forall \tau < 0 : P(C_\tau, X) \subseteq \Pi(C, X)$.

Proposition 3 leads as a corollary to the well-known folding property [8].

COROLLARY 3. $\forall \tau < 0 \ \forall v \in ri\, K^*$: $\operatorname{argmax} \{vx : x \in X\} \subseteq P(C_\tau, X) \subseteq \Pi(C, X)$. Moreover, $\forall \varepsilon > 0 \ \exists \delta > 0$ such that for $\| C(\delta) - C \| \leqslant \delta$ $v(\delta) = \sum_{k=1}^{L} \mu_k c_k(\delta)$, $\operatorname{argmax} \{v(\delta)x : x \in X\} \subseteq O_\varepsilon \operatorname{argmax} \{vx : x \in X\} \subseteq O_\varepsilon P(C_\tau, X) \subseteq O_\varepsilon \Pi(C, X)$.

Proof. The first chain of inclusions in the corollary follows directly from Proposition 3. Now, by Proposition 1, for the problem $(C_{-\infty}, X)$: $\max \{vx : x \in X\}$ we have $\{vx : x \in X\} = P(C_{-\infty}, X) = \Pi(C_{-\infty}, X)$. By Corollary 1, the problem $(C_{-\infty}, X)$ is P-stable, and therefore the second chain of inclusions holds.

Proposition 4. $\forall \tau \in (0, 1) : P(C, X) \subseteq \Pi(C_\tau, X)$.

Proof. Let $x \in P$. Then $(x + K_1) \cap X = \{x\}$. We will show that $x \in \Pi(C_\tau, X)$, i.e., $(x + K_1^\tau \cup K_2^\tau) \cap X = \{x\}$. Indeed, $K_1^\tau \cup K_2^\tau = K_\tau \setminus K_0^\tau$. By property 3 of perturbed cones, $K_0 \subseteq K_0^\tau$. Therefore, $K_\tau \setminus K_0^\tau \subseteq K_\tau \setminus K_0$. By properties 1, 2 of perturbed cones, for $\forall \tau \in (0, 1)$ $K_\tau \subseteq K_1 \cup K_0$. Hence it follows that $K_\tau \setminus K_0^\tau \subseteq K_\tau \setminus K_0 \subseteq K_1$. Then $(x + K_1^\tau \cup K_2^\tau) \cap X \subseteq (x + K_1) \cap X = \{x\}$. We have thus shown that $x \in \Pi(C_\tau, X)$.

Proposition 5. If problem (1) is stable in the decision space, then $\mathrm{cl}\,\Pi(C, X) = P(C, X)$.

Here no additional constraints are imposed on the set X.

Proof. By contradiction assume that problem (1) is stable, but $\exists x \in P(C, X) \backslash cl\, \Pi(C, X)$. Since $x \notin cl\, \Pi(C, X)$, there exists $\varepsilon > 0$ such that $O_\varepsilon[x] \cap cl\, \Pi(C, X) = \varnothing$. By Proposition 4, $\forall\, \tau \in (0, 1)$: $P(C, X) \subseteq \Pi(C_\tau, X)$. Let $C(\delta) = C_\tau$. Recall that $C_\tau = C - \tau V$, where V is the $L \times n$ matrix with each row equal v, $\| C_\tau - C \| = \tau \, \| V \|$. If

$$\tau_0(\varepsilon) = \min\{\delta(\varepsilon) / \| V \|, 1\}, \tag{6}$$

then by stability of problem (1) $\forall \tau \in (0, \tau_0(\varepsilon))$: $\Pi(C_\tau, X)) \subseteq O_\varepsilon \Pi(C, X)$, and from the last two inclusions we obtain that $P(C, X) \subseteq O_\varepsilon \Pi(C, X)$. Therefore for $x \in P(C, X) \backslash \mathrm{ci}\, \Pi(C, X)$ there exists $x' \in O_\varepsilon\{x\} \cap \Pi(C, X)$. A contradiction.

Corollary 2 and Proposition 5 directly lead to the following assertion.

Proposition 6. If X is a nonempty bounded closed set in $\boldsymbol{R}^n$, then problem (1) is stable in the decision space under changes of criterion coefficients if and only if $cl\, \Pi(C, X) = P(C, X)$.

Let us now state some simple sufficient conditions of stability in the decision space under changes of criterion coefficients.

Proposition 7. Let X be a nonempty bounded closed set. Assume that at least one of the following conditions is satisfied:

1) $K_2 = \varnothing$,

2) there exist nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_L$ not all of which are simultaneously zero such that $\sum_{k=1}^{L} \alpha_k c_k = 0$,

3) $r(C) = 1$.

Then problem (1) is stable in the decision space under changes in the criterion coefficients.

Proof follows directly from Propositions 1 and 6.

Proposition 8. If X is a nonempty bounded closed set, then there exists $\tau_0 > 0$ such that for $\forall \tau \in (0, \tau_0]$ the problem (C_τ, X) is stable in the decision space under changes in criterion coefficients.

Proof. By Corollary 1, for $\forall\, \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ such that for $\delta < \delta(\varepsilon/2)$ we have the inclusion $P(C(\delta), X)) \subseteq O_{\varepsilon/2} P(C, X)$. By Proposition 3, $\forall \tau \in (0, 1)$: $P(C, X) \subseteq \Pi(C_\tau, X)$, therefore $O_{\varepsilon/2} P(C, X) \subseteq O_{\varepsilon/2} \Pi(C_\tau, X)$. If $\tau_0 = \tau_0(\varepsilon/2)$, where $\tau_0(\varepsilon/2)$ satisfies (6), then $\forall \tau \in (0, \tau_0(\varepsilon/2))$: $P(C_\tau, X) \subseteq O_{\varepsilon/2} P(C,X) \subseteq O_{\varepsilon/2} \Pi(C_\tau, X)$ and $O_{\varepsilon/2} P(C_\tau, X) \subseteq O_\varepsilon \Pi(C_\tau, X)$. By Corollary 1, for $\forall \varepsilon > 0\ \exists \delta_1(\varepsilon) > 0$ such that for $\delta < \delta_1(\varepsilon)$ we have the inclusion $P(C_\tau(\delta), X)) \subseteq O_{\varepsilon/2} P(C_\tau, X)$. Hence it follows that $P(C_\tau(\delta), X)) \subseteq O_\varepsilon \Pi\, (C_\tau, X)$. Since $\Pi(C_\tau(\delta), X)) \subseteq P(C_\tau(\delta), X))$, we have $\Pi(C_\tau(\delta), X)) \subseteq O_\varepsilon \Pi(C_\tau, X)$. Q.E.D.

Consider the space $R^{L \times n}$ as the space of initial data, $u = C \in R^{L \times n}$, and identify in $R^{L \times n}$ the set $G(X^0)$ of initial-data matrices C for which the problem (C, X^0) is stable in the decision space by the vector criterion.

COROLLARY 4. $\mathrm{cl}\, G(X^0) = \boldsymbol{R}^{L \times n}$.

Thus, the set $G(X^0)$ of all initial data of vector optimization problems (C, X^0) of fixed dimension with a fixed feasible region that are stable in the decision space by the vector criterion is everywhere dense.

3. Stability of linear programming problems has been studied in considerable detail [13]. Necessary and sufficient stability conditions have been derived for the linear programming problem, and it has been shown that the notions of stability by solution and stability by the functional are identical in this case. For the integer linear programming problem a similar result has been obtained in [14]. Equivalence of these stability concepts for some classes of multicriterion optimization problems, including linear and integer multicriterion problems, follows from the results of [15].

We will show that stability in the decision space implies stability in the space of alternatives for problems of the form (1).

LEMMA. Let M be a bounded set in the space $\boldsymbol{R}^n$ and $C \in R^{L \times n}$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $G \subseteq O_\delta(M)$ $C(\delta) \in O_\delta C$ we have the inclusion $\mathfrak{C}(C(\delta), G) \subseteq O_\varepsilon \mathfrak{C}(C, M)$.

Proof. Let $y \in \mathfrak{C}(C(\delta), G)$. Then the set G contains an element x such that $y = C(\delta)x$. Since $x \in G$, there exists $x^0 \in M$ such that $\|x - x^0\| \le \delta$; moreover, $\|x^0\| \le A$, where A is a constant. Then $y^0 = Cx^0 \in \mathfrak{C}(C, M)$ and $\|y - y^0\| = \|C(\delta)x - Cx^0\| = \|C(\delta)x - Cx + Cx - Cx^0\| = \|(C(\delta) - C)x + C(x - x^0\| = \|(C(\delta) - C)\,(x^0 + (x - x^0)) + C(x - x^0)\| \le \|C(\delta) - C\|\, \|x^0\| + \|C(\delta) - C\|\, \|x - x^0\| + \|C\|\, \|x - x^0\| \le \delta A + \delta^2 + \delta\, \|C\| \le \varepsilon$ if δ satisfies the inequality

$$0 < \delta \leqslant \{ -(A + \| C \|) + [(A + \| C \|)^2 + 4\varepsilon]^{1/2}\} / 2. \tag{7}$$

Proposition 9. If problem (1) is stable in the decision space, then it is stable in the space of alternatives.

Proof. Take an arbitrary $\varepsilon > 0$. Let δ_1 satisfy inequality (6). By assumption problem (1) is stable in the decision space, and so for $\varepsilon' = \delta_1$ there exists $\delta_2 > 0$ (we may assume that $\delta_2 \leq \delta_1$) such that for any vector $u(\delta_2) \in \boldsymbol{R}^k$ and any matrix $C(u(\delta_2)) \in \boldsymbol{R}^{L\times n}$, satisfying inequalities (2), for $\delta = \delta_2$ the set $\Pi(C(u(\delta_2)), X(u(\delta_2)))$ of Pareto-optimal solutions of the problem $(C(u(\delta_2)), X(u(\delta_2)))$ is nonempty and is included in the δ_1-neighborhood of the set $\Pi(C, X)$, $\Pi(C(u(\delta_2)), X(u(\delta_2))) \subseteq O_{\delta_1}\Pi(C, X)$. Since δ_2 satisfies inequality (7), by the proof of the preceding lemma we have $\mathfrak{C}(Cu(\delta_2), \Pi(C(u(\delta_2)), X(u(\delta_2))) \subseteq O_\varepsilon \mathfrak{C}(C, \Pi(C, X))$.

We have thus proved stability of problem (1) in the space of alternatives.

COROLLARY 5. If X is a nonempty bounded closed set in $\boldsymbol{R}^n$, then problem (1) is P-stable under changes of criterion coefficients in the space of alternatives.

We will now give an example of a problem of the form (1) which is stable in the space of alternatives but not stable in the decision space.

Example. Consider a vector mixed integer linear programming problem of the form (1), where $x = (x_1, x_2, x_3) \in X(A, b) = \{x \in D : Ax \leqslant b\} = \{(0, 0, 0); (0, 1, 0), (1, 1, \alpha), 1 \geqslant \alpha \geqslant 0\} \subset \boldsymbol{Z}^1 \times \boldsymbol{Z}^1 \times \boldsymbol{R}^1$,

$$C = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1,5 \\ 0,5 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to show that

$$\Pi(C, X) = \{(0, 1, 0); (1, 1, \alpha), 1 \geqslant \alpha > 0\},$$

$$\mathfrak{C}_\Pi = \{(1, 0); (-\alpha, \alpha), 1 \geqslant \alpha > 0\}.$$

The problem is stable in the space of alternatives and unstable in the decision space, because for every $\varepsilon > 0$ we have $\mathfrak{C}(C, X) \subseteq O_\varepsilon \mathfrak{C}_\Pi$ and there exists a perturbed criterion matrix

$$C(\delta) = \begin{bmatrix} -1 + \delta_1 & 1 + \delta_2 & -1 + \delta_3 \\ 0 + \delta_4 & 0 + \delta_5 & 1 + \delta_6 \end{bmatrix}$$

such that for sufficiently small δ_i, $i = 1,\ldots,6$, for instance $\delta_5 < 0$, $\delta_1 = -\delta_2$, $\delta_4 = -\delta_5$, we have $\Pi(C(\delta), X(\delta)) = X \not\subseteq O_\varepsilon \Pi(C, X)$

Proposition 10. Let $r(C) = n$. Moreover, assume that the sets $X(u)$ are jointly bounded in some neighborhood of the point $u = 0$ and the set-valued mapping $X(u)$ is closed at this point. Then stability of problem (1) in the space of alternatives implies its stability in the decision space.

Proof. Assume that problem (1) is stable in the space of alternatives. Then by definition for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any matrix $C(u) = C(\delta) \in \boldsymbol{R}^{L\times n}$, satisfying (2), (5) the set of criterion values $\mathfrak{C}_\Pi(u)$, $\mathfrak{C}_\Pi(u) = \mathfrak{C}(C(u), \Pi(C(u), X(u)))$, of the problem $(C(u), X(u))$ is nonempty and is included in the ε-neighborhood of the set $\mathfrak{C}_\Pi$. Thus, for any point $x(u) \in \Pi(C(u), X(u))$ $\exists\, \bar{x}(u) \in \Pi(C, X)$ such that

$$\| C\bar{x}(u) - C(u)x(u) \| \leqslant \varepsilon. \tag{8}$$

Since the sets $X(u)$ are nonempty, closed, and jointly bounded, we can extract convergent subsequences from the sequences $\{\bar{x}(u)\}$ and $\{x(u)\}$. Without loss of generality, assume that the sequences $\{\bar{x}(u)\}$ and $\{x(u)\}$ converge: $\bar{x}(u) \to \bar{x} \in \operatorname{cl}\Pi(C, X)$ as $u \to 0$; $x(u) \to x$. Then $C(u)x(u) \to Cx$, $C\bar{x}(u) \to C\bar{x}$, and by (8)

$$Cx = C\bar{x}. \tag{9}$$

By assumption $r(C) = n$. Recall that $K_0 = \{x \in \boldsymbol{R}^n : Cx = 0\} = \{0\} \Leftrightarrow r(C) = n$. Then from (9) we obtain that $x = \bar{x}$ and $x(u) \to \bar{x} \in \operatorname{cl}\Pi(C, X)$. By contradiction, assume that problem (1) is unstable in the decision space. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ there is a matrix $C(u) \in \boldsymbol{R}^{L\times n}$, satisfying inequalities (2) such that the set $\Pi(C(u), X(u))$ of Pareto-optimal solutions of the problem $(C(u), X(u))$ is either empty or is not included in the ε-neighborhood of the set $\Pi(C, X)$. Since problem (1) is stable in the space of alternatives, the sets $X(u)$ are nonempty and $\exists \varepsilon > 0$ such that for $u \to 0$

$$\exists\, x(u) \in \Pi(C(u), X(u))\ \forall x \in \Pi(C, X) : \| x - x(u) \| > \varepsilon. \tag{10}$$

By the above argument we may assume that the sequence $\{x(u)\}$ converges. Let $x(u) \to x$. Then by stability of problem (1) in the space of alternatives $x \in \mathrm{cl}\, \Pi(C, X)$, which contradicts (10).

The contradiction proves the proposition.

If $r(C) = n = L$, the last proposition can be proved without assuming closure of the set-valued mapping $X(u)$.

Proposition 11. Let $r(C) = n = L$ and the sets $X(u)$ are jointly bounded in some neighborhood of the point $\delta = 0$. Then stability of problem (1) in the space of alternatives implies stability of this problem in the decision space.

Proof. Assume that problem (1) is stable in the space of alternatives. Then $\exists \delta_0 > 0$ such that for $0 \le \delta < \delta_0$ the set $\Pi(C(u), X(u))$ is nonempty for any matrix $C(u) \in R^{L\times n}$ that satisfies inequality (2). We will show that $\forall \varepsilon > 0\ \exists \delta > 0 : \Pi(C(u), X(u)) \subseteq O_\varepsilon \Pi(C, X)$. By stability of problem (1) in the space of alternatives, for $\varepsilon_1 = \varepsilon \| C \| /2\ \exists\ \delta_1 > 0$ such that for $0 \le \delta < \delta_1 \le \min\{\delta_0, \varepsilon_1/\Delta\}$ (here $\Delta = \max\{\| x \| : x \in X(u),\ 0 \leqslant \delta < \delta_0\}$) for any point $x(\delta) \in \Pi(C(u), X(u))\ \exists\, \bar{x}(\delta) \in \Pi(C, X)$ such that $\| C\bar{x}(u) - C(u)x(\delta) \| \leqslant \varepsilon$. Denote $z(\delta) = C\bar{x}(\delta) - C(u)x(\delta)$. Since $r(C) = n = L$, $\exists C^{-1}$ such that $C^{-1}z(\delta) = \bar{x}(\delta) - C^{-1}C(u)x(\delta)$. Hence it follows that $\| \bar{x}(\delta) - x(\delta) \| = \| C^{-1}z(\delta) + C^{-1}C(u)x(\delta) - x(\delta) \| \leqslant \| C^{-1} \| \| z(\delta) \| + \| C^{-1}C(u) - 1 \| \| x(\delta) \| \leqslant \| C^{-1} \| \varepsilon_1 + \| C^{-1} \| \delta \| x(\delta) \| \leqslant \varepsilon$. Q.E.D.

Using Proposition 6 [6], which follows from the results of [15], we can state a sufficient condition of equivalence of the notions of stability in the space of alternatives and stability in the criterion space for problems of the form (1).

Proposition 12. If the set $\Pi(C, X)$ of problem (1) is closed, then stability of problem (1) in the decision space implies its stability in the space of alternatives, and conversely, stability in the space of alternatives implies stability in the criterion space.

An example of a vector mixed integer optimization problem of the form (1) with an open set of efficient points which is stable in the decision space is given in [6] (example 7).

Let us now prove the necessary and sufficient conditions of stability in the space of alternatives under changes in criterion coefficients.

Proposition 13. Assume that the set X is nonempty and bounded. Problem (1) is stable in the space of alternatives under changes in criterion coefficients if and only if $\mathrm{cl}\, \mathfrak{C}(C, \Pi) = \mathfrak{C}(C, P)$.

Proof. Note that $\mathrm{cl}\, \mathfrak{C}(C, \Pi) \subseteq \mathfrak{C}(C, P)$. Let $u = C$, $X(C) = X$. We will first show that the condition

$$\mathfrak{C}(C, P) \subseteq \mathrm{cl}\, \mathfrak{C}(C, \Pi) \tag{11}$$

implies stability of problem (1) in the space of alternatives under changes in criterion coefficients. By contradiction, assume that condition (11) is satisfied, but problem (1) is unstable in the space of alternatives under changes in $u \equiv C : \exists \varepsilon > 0\ \forall \delta > 0\ \exists C(\delta) : \| C(\delta) - C \| \leqslant \delta\ \ \exists\, x(\delta) \in \Pi(C(\delta), X)\ \forall y \in \mathfrak{C}_\Pi(u) = \mathfrak{C}(C, \Pi(C, X)) :$

$$\| C(\delta)x(\delta) - y \| \geqslant \varepsilon. \tag{12}$$

Without loss of generality assume that $x(\delta) \to x_0$ as $\delta \to 0$. Since $x(\delta) \in \Pi(C(\delta), X) \subseteq P(C(\delta), X)$, by Proposition 2 $x_0 \in P(C, X)$ and passing to the limit in (12) we obtain $\forall y \in \mathfrak{C}_\Pi(u) = \mathfrak{C}(C, \Pi(C, X)) : \| Cx_0 - y \| \geqslant \varepsilon$. Since $y \in \mathfrak{C}(C, \Pi(C, X))$, $Cx_0 \in \mathfrak{C}(C, P(C, X))$, we obtain a contradiction with (11).

Let us now prove that stability of problem (1) in the space of alternatives under changes of criterion coefficients implies the inclusion (11). By contradiction, assume that $\exists\, y \in \mathfrak{C}(C, P) \setminus \mathrm{cl}\, \mathfrak{C}(C, \Pi)$. Then $y = Cx_0$, where $x_0 \in P \setminus \mathrm{cl}\, \Pi$ and $\exists \varepsilon > 0$ such that

$$\forall x \in \mathrm{cl}\, \Pi : \| y - Cx \| \geqslant \varepsilon . \tag{13}$$

Let $C(\delta) = C_\tau$, where $\tau = \delta / \| v \|$. Then by Proposition 4 for $0 < \delta < \| v \|$ we have the inclusion $x_0 \in P(C, X) \subseteq \Pi(C(\delta), X) = \Pi(C_\tau, X)$ and by stability of problem (1) in the space of alternatives $\forall \varepsilon > 0 \ \exists \delta_0 > 0$, such that for $0 < \delta < \min \{\| v \|, \delta_0\} \ \exists x \in \Pi: \| y - Cx \| < \varepsilon$. This contradicts (13).

4. As we know, definitions of stability can be based on other variants of continuity. It may be useful to define stability on the basis of Hausdorff lower semicontinuity.

Definition 5. Problem (1) is called *I*-stable in the decision space under changes in criterion coefficients if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any matrix $C(\delta) \in \boldsymbol{R}^{L \times n}$ satisfying inequality (5) the set $\Pi(C, X)$ of Pareto-optimal solutions of the problem (C, X) is nonempty and is included in the ε-neighborhood of the set $\Pi(C(\delta), X)$, $\Pi(C, X)) \subseteq O_\varepsilon \Pi(C(\delta), X)$.

I-stability in the decision space implies that, under small perturbations of initial data, for any efficient solution of the initial problem there is a sufficiently close efficient solution of the perturbed problem.

The notion of *I*-stability is useful for finding solutions of families of problems of the form $(C(u), X^0)$, where $C(u) \in U \subseteq \boldsymbol{R}^{L \times n}$. Indeed, suppose that for some $\varepsilon > 0$ there exists a finite covering of the set U by δ-neighborhoods $O_\delta C(u_i)$, $i = 1, \ldots, S$, where $(C(u_i), X^0)$ is an *I*-stable problem. Then $\forall x \in \Pi(C(u_i), X^0)$, $i = 1, \ldots, S$, is an ε-approximation of some efficient solution of any problem $(C(u), X^0)$ if $\| C(u_i) - C(u) \| \leq \delta$. Solving finitely many problems from the family $(C(u), X^0)$, we obtain an idea about part of efficient solutions of any problem from this family.

The following necessary and sufficient condition of *I*-stability of the problem (C, X) with a bounded feasible region has been actually proved in [5].

Proposition 14 [5]. If the set X is nonempty and bounded, then the problem (C, X) is *I*-stable in the decision space under changes in criterion coefficients if and only if $\mathrm{cl}\, S = \mathrm{cl}\, \Pi$.

Propositions 1 and 14 directly lead to an important sufficient condition of *I*-stability under changes in criterion coefficients.

Proposition 15. If the set X is nonempty and bounded and $r(C) = n$, then the problem (C, X) is *I*-stable in the decision space under changes in criterion coefficients.

In the initial-data space $\boldsymbol{R}^{L \times n}$ identify the set $G_1(X^0)$ of initial-data matrices $C \in \boldsymbol{R}^{L \times n}$, for which the problem (C, X^0) is *I*-stable in the decision space under changes in criterion coefficients. Consider the set $C(G_1(X^0)) = \boldsymbol{R}^{L \times n} \setminus G_1(X^0)$ of initial-data matrices $C \in \boldsymbol{R}^{L \times n}$, for which the problem (C, X^0) is not *I*-stable in the decision space under changes in criterion coefficients.

Proposition 16. Assume that the set X^0 is nonempty and bounded and $L \geq n$. Then the set $C(G_1(X^0))$ is of measure zero.

Proof. Let $C \in \boldsymbol{R}^{L \times n} \setminus G_1(X^0)$. By Proposition 15, $r(C) < n$. This means that any n rows in the matrix C, e.g., $c_{i_1}, c_{i_2}, \ldots, c_{i_n}$, are linearly dependent, i.e., $\det(c_{i_1}, c_{i_2}, \ldots, c_{i_n}) = 0$. Let T be the set of all sequences of τ row indices $i_1 < i_2 < \ldots < i_n$, $\tau = \{t_1, i_2, \ldots, i_n\}$. Clearly, $|T| = C_L^n$. Thus, if $C \in \boldsymbol{R}^{L \times n} \setminus G_1(X^0)$, then it satisfies a system of C_L^n equations of the form $\det(c_{i_1}, c_{i_2}, \ldots, c_{i_n}) = 0 \ \forall \tau \in T$, Therefore $C(G_1(X^0)) \subseteq M \equiv \{C \in \boldsymbol{R}^{L \times n} \mid \det(c_{i_1}, c_{i_2}, \ldots, c_{i_n}) = 0 \ \forall \tau \in T\}$. The intersection of C_L^n surfaces defined by n-th order polynomial equations is of measure zero.

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