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On the Interpretation of Kernels

I. Computer Simulation of Responses to Impulse Pairs[1]

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A method is presented for the use of a unit impulse response and responses to impulse pairs of variable separation in the calculation of the second-degree kernels of a quadratic system. A quadratic system may be built from simple linear terms of known dynamics and a multiplier. Computer simulation results on quadratic systems with building elements of various time constants indicate reasonably that the larger time constant term before multiplication dominates in the envelope of the off-diagonal kernel curves as these move perpendicular to and away from the main diagonal. The smaller time constant term before multiplication combines with the effect of the time constant after multiplication to dominate in the kernel curves in the direction of the second-degree impulse response, i.e., parallel to the main diagonal. Such types of insight may be helpful in recognizing essential aspects of (second-degree) kernels; they may be used in simplifying the model structure and, perhaps, add to the physical/physiological understanding of the underlying processes.

INTRODUCTION

The unit impulse response serves very well in completely describing the behavior of a linear system (Lee, 1966). It can be shown that the responses of a quadratic system to a unit impulse and to impulse pairs of variable separation can be used to calculate the second-degree[2] kernels of the quadratic system. A quadratic system can be built with simple linear (first-degree) elements with known dynamics and a multiplier as shown in Table 1. In this paper, we clarify the theoretical justification for the double impulse approach in the calculation of the second-degree kernels of a quadratic system. Also, we examine, by means of computer simulation, the effects upon the second-degree kernels due to variations in the time constants of the linear building elements. In this way we gain insight into the factors which affect the second-degree kernels of such a system.

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[2] In terms of differential equations of a system, order is equal to the highest derivative while degree is equal to the highest power of the terms. For example, $\ddot{x} = ay$ is a first-degree second-order system; $(\dot{x})^2 = ay$ is a second-degree first-order system.

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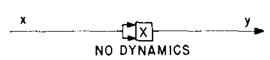
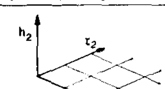
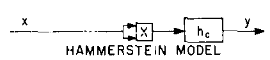
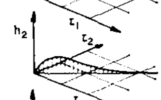
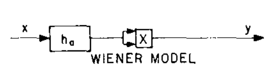
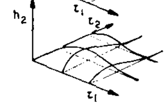
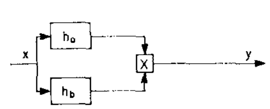
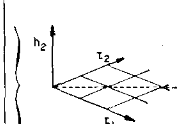
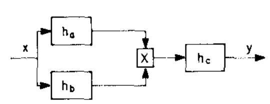
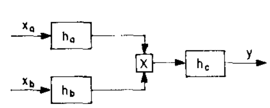
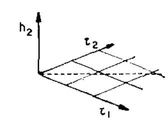
There have been very few attempts in applying physical interpretation to the kernels. Sandberg and Stark (1968) calculated the kernels of the pupillary system by means of a cross-correlation method developed by Lee and Schetzen (1961). They found that the first (h_1) and second (h_2) degree kernels are of opposite signs, and in drawing physical–physiological interpretations from their study, concluded that these represent the first terms of a saturation nonlinearity. The width of h_2 off the main diagonal is less than 0.5 sec which indicates that there is no second-degree nonlinear interaction longer than this time. Baker (1963), using double pulse monocular and binocular stimulation of the pupillary system, concluded that the important nonlinearity occurs after the binocular summation point and is attributed to amplitude saturation. Watanabe and Stark (1975) proposed a second-degree heuristic model of the pupillary system which consists of the simultaneous input through two linear elements and the multiplication of their outputs. Sandberg and Stark introduced the use of double pulses of variable separation in the calculation of the kernels of the pupillary system. The approximate kernels obtained in this manner do not correspond exactly to those found using the cross-correlation technique and they attempted interpretation of the differences. Marmarelis and Naka (1972) applied a two-input two-output cross-correlation method to a neuron chain in the catfish retina, and by inspecting the shapes of the kernels, attempted to derive some of the characteristics and topology of this system. Diskin, Boneh, and Golan (1975), using kernels to describe a civil engineering water runoff catchment system, discussed the physical constraints of the system as reflected in the kernel's structure.

TABLE 1

		DIAGRAM	$h_2(\tau_1,\tau_2)$	SYMMETRY	CHARACTERISTICS OF h_2
ONE INPUT	a.		$\delta(\tau_1)\,\delta(\tau_2)$ DIRAC FUNCTION IN ORIGIN	YES	
	b.		$h_c(\tau_1)\,\delta(\tau_1-\tau_2)$ DIAGONAL $\tau_1=\tau_2$ ONLY	YES	
	c.		$h_a(\tau_1)\,h_a(\tau_2)$ $\frac{h_2(\tau_1,c_1)}{h_2(\tau_1,c_2)}$ = CONSTANT	YES	
	d.		$h_a(\tau_1)\,h_b(\tau_2)$	1	
	e.		$\int h_a(\tau_1-\xi)\,h_b(\tau_2-\xi)\,h_c(\xi)\,d\xi$	1	
TWO INPUTS	f.		$\int h_a(\tau_1-\xi)\,h_b(\tau_2-\xi)\,h_c(\xi)\,d\xi$	IFF $h_a=h_b$	

[1] Can be made symmetrical as $h_2'(\tau_1,\tau_2) = \frac{1}{2}\{h_2(\tau_1,\tau_2) + h_2(\tau_2,\tau_1)\}$.

The Frechet (1910)–Volterra (1930) functional is a generalization of the power series expansion for an instantaneous function to that for a noninstantaneous function. The first-degree noninstantaneous term is the well-known linear convolution integral. The generalization of the convolution technique to that for a nonlinear, time-invariant, and finite memory system leads to the Frechet–Volterra representation:

$$y(t) = y_0 + y_1(t) + y_2(t) + y_3(t) + \cdots$$

with

$$y_1(t) = \int_0^t h_1(t)x(t - \tau)d\tau,$$

$$y_2(t) = \int_0^t \int_0^t h_2(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)d\tau_1 d\tau_2,$$

$$y_3(t) = \int_0^t \int_0^t \int_0^t h_3(\tau_1, \tau_2, \tau_3)x(t - \tau_1)x(t - \tau_2)x(t - \tau_3)d\tau_1 d\tau_2 d\tau_3,$$

where

$x(t)$ is the system input,

$y(t)$ is the system output, consisting of: y_0, a constant value not dependent on the input y_1; (t), the contribution due to the linear (first-degree) term; $y_2(t)$, the contribution due to the quadratic (second-degree) term, etc.;

$h_1(\tau_1)$, $h_2(\tau_1, \tau_2), \ldots$ are called the kernels of the first, second, ..., degree.

It is assumed that the system is at rest at $t = 0$.

The different degree terms in the Frechet–Volterra representation are nonorthogonal in the sense that if higher-degree terms are added in fitting the representation of the system, all the different degree kernels under consideration are modified. On the other hand, the Wiener (1958) representation, which is an orthogonalized version of the Frechet–Volterra representation, contains kernels which are not changed when higher-degree kernels are added to the representation of the system.

Table 1 shows as a preliminary survey examples of quadratic dynamical systems of increasing complexity from case a to f. Besides the block diagram there are given the expressions for $h_2(\tau_1, \tau_2)$ and illustrations of some of the characteristics of h_2. For physical realizability $h(\tau_1, \tau_2) = 0$ for all $\tau_1 < 0$ and/or $\tau_2 < 0$.

Case a shows the simplest second-degree element, i.e., a pure squarer (multiplier). In this case,

$$h_2(\tau_1, \tau_2) = \delta(\tau_1)\delta(\tau_2),$$

where the δ's are Dirac functions. This can easily be verified by substitution in the expression for $y_2(t)$. The graphical representation of h_2 has only a Dirac function in the origin $(\tau_1, \tau_2) = (0, 0)$.

Case b shows a squarer followed by linear dynamics (*Hammerstein model*). Again one can verify the expression

$$h_2(\tau_1, \tau_2) = h_c(\tau_1)\delta(\tau_1, \tau_2),$$

and one notices that for this configuration h_2 has only contributions unequal to zero for points at the "diagonal" $\tau_1 = \tau_2$.

Case c shows linear dynamics followed by a squarer (*Wiener model*). In this case,

$$h_2(\tau_1, \tau_2) = h_a(\tau_1)h_a(\tau_2).$$

This kernel is characterized by the properties

$$\frac{h_2(\tau_1, c_1)}{h_2(\tau_1, c_2)} = \text{constant} \quad \text{and} \quad \frac{h_2(c_1, \tau_2)}{h_2(c_2, \tau_2)} = \text{constant.}$$

The cases a, b, and c may serve to recognize from an experimentally determined kernel h_2 whether the system under study has predominantly a particular simplified structure. Note that there is symmetry around the diagonal $\tau_1 = \tau_2$, hence

$$h_2(\tau_1, \tau_2) = h_2(\tau_2, \tau_1).$$

In case d the outputs of two (different) linear dynamic elements are multiplied, and

$$h_2(\tau_1, \tau_2) = h_a(\tau_1)h_b(\tau_2).$$

Case e is the same as case d but with another linear dynamic element following the multiplier, and the kernel is represented in the convolution form:

$$h_2(\tau_1, \tau_2) = \int_0^\infty h_a(\tau_1 - \xi)h_b(\tau_2 - \xi)h_c(\xi)d\xi.$$

Note that, due to the symmetry in the integral for $y_2(t)$ with respect to $x(t - \tau_1)$ and $x(\tau_2 - t)$, the kernel may be made symmetrical by the procedure:

$$h_{2_{\text{symm}}}(\tau_1, \tau_2) = \tfrac{1}{2}\{h_{2_{\text{asymm}}}(\tau_1, \tau_2) + h_{2_{\text{asymm}}}(\tau_2, \tau_1)\}.$$

In the following we will always consider symmetrical kernels. Since *f* is a two-input system it does not, strictly speaking, belong to this table. This case is given, however, in order to emphasize that in this system the aysmmetry of $h_2(\tau_1, \tau_2)$ is essential if $h_a \neq h_b$:

$$y(t) = \int_0^t \int_0^t h_2(\tau_1, \tau_2)x_a(t - \tau_1)x_b(t - \tau_2)d\tau_1 d\tau_2.$$

The $h_2(\tau_1, \tau_2)$ of cases d and e are more difficult than the previous cases to determine and to recognize from experimental data. It is for this reason that in the following some simulation results are given in order to enhance the insight into

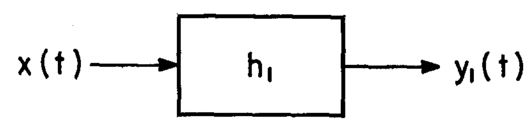


FIGURE 1

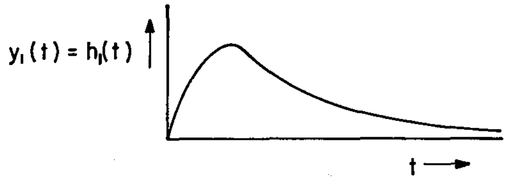


FIGURE 2

the characteristics of these kernels. The discussion will concentrate on the relation between these kernels and the impulse and biimpulse responses of the system.

I. Determination of the First-Degree Kernels of a Linear System Using a Single Impulse

Consider the response $y_1(t)$ due to input $x(t)$ through a system with a first-degree kernel $h_1(\tau)$ assuming initial conditions at $t = 0$ to be zero (Fig. 1):

$$y_1(t) = \int_0^t h_1(\tau)x(t - \tau)d\tau. \tag{1}$$

For a signal consisting of a unit impulse (Dirac function) $x(t) = \delta(t)$, the output is

$$y_1(t) = \int_0^t h_1(\tau)\delta(t - \tau)d\tau.$$

A contribution occurs only when $t = \tau$. For each t, instead of a convolution in $d\tau$, we get only one value for $y_1(t)$, namely $h_1(t)$. Hence we obtain $y_1(t) = h_1(t)$. This is a nice result because h_1 is now a function of t instead of t and τ, and the single unit impulse response of the first-degree system will be identical to $h_1(t)$ (see Fsg. 2).

II. Determination of the Second-Degree Kernels of a Purely Quadratic System by Means of Single Impulses and Impulse Pairs

Consider the response $y_2(t)$ due to input $x(t)$ through a quadratic system with second-degree kernel $h_2(\tau_1, \tau_2)$, assuming initial conditions at $t = 0$ to be zero (Fig. 3):

$$y_2(t) = \int_0^t \int_0^t h_2(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)d\tau_1 d\tau_2. \tag{2}$$

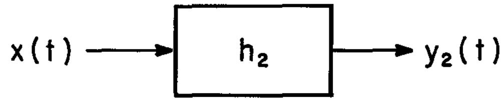


FIGURE 3

Note that the response of a purely quadratic system contains only second-degree kernel contributions. For a signal consisting of two unit impulses, $x(t) = \delta(t) + \delta(t - \theta)$, the output is

$$y_{\substack{\text{double}\\\text{impulse}}}(t) = \int_0^t \int_0^t h_2(\tau_1, \tau_2)\{\delta(t - \tau_1)\delta(t - \tau_2) + \delta(t - \theta - \tau_1)\delta(t - \theta - \tau_2) + \delta(t - \tau_1)\delta(t - \theta - \tau_2) + \delta(t - \theta - \tau_1)\delta(t - \tau_2)\}d\tau_1 d\tau_2. \quad (3)$$

Since each of the four δ products in (3) can have a value only when the arguments of both δ functions are zero, we can eliminate the terms τ_1 and τ_2 by equating these with their respective combination t and θ terms within the δ functions and then substituting into the τ_1, τ_2 terms in h_2. We obtain

$$\begin{aligned} y_{\substack{\text{double}\\\text{impulse}}}(t) &= h_2(t, t) \\ &+ h_2(t - \theta, t - \theta) \qquad (= h_2(t, t) \text{ delayed over time interval } \theta) \\ &\left.\begin{aligned} &+ h_2(t, t - \theta) \\ &+ h_2(t - \theta, t) \end{aligned}\right\} \qquad \text{(identical due to physical considerations).} \end{aligned} \quad (4)$$

The last two terms of (4) represent the nonlinear interaction within the system due to unit impulses at times t and $t - \theta$ before the observation of the output at time t. The impulses under consideration occur at times 0 and θ. Since for a quadratic system an impulse pair is indistinguishable from the same impulse pair presented in the reverse order, the values of the last two terms in (4) are identical and indistinguishable. This is a physical argument in favor of making h_2 symmetrical as indicated before.

In order to obtain a complete set of second-degree kernels up to some $\tau_1 = \tau_2 = \tau_{\text{limit}}$, we must extract from the single and double impulse experiments the values of $h_2(t, t)$, $h_2(t - \theta, t - \theta)$, $h_2(t, t - \theta)$, and $h_2(t - \theta, t)$ for t up to τ_{limit}.

A single impulse through the quadratic system will result in

$$y_{\substack{\text{single}\\\text{impulse}}}(t) = \int_0^t \int_0^t h_2(\tau_1, \tau_2)\delta(t - \tau_1)\delta(t - \tau_2)d\tau_1 d\tau_2 = h_2(t, t). \quad (5)$$

Hence the single impulse experiment gives us $h_2(t, t)$. Analogous to $h_1(t)$ for a single impulse through a first-degree system, $h_2(t, t)$ is the second-degree impulse response of the quadratic system. In the plot of $h_2(\tau_1, \tau_2)$ vs τ_1 and τ_2, $h_2(t, t)$ is the set of kernel values on the main diagonal.

We can perform a set of impulse-pair experiments using different values of impulse separation θ, along with the single impulse result, to obtain the off-diagonal kernels, where for each separation:

$$h_2(t, t - \theta) + h_2(t - \theta, t) = y_{\substack{\text{double}\\\text{impulse}}} - \underbrace{h_2(t, t) - h_2(t - \theta, t - \theta)}_{\text{from single impulse result}},$$

$$h_2(t, t - \theta) = h_2(t - \theta, t) = \tfrac{1}{2}\{y_{\substack{\text{double}\\\text{impulse}}} - h_2(t, t) - h_2(t - \theta, t - \theta)\}. \quad (6)$$

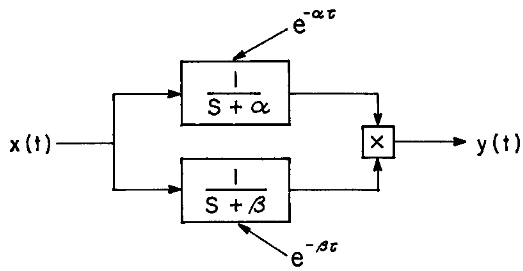


FIGURE 4

The off-diagonal kernels $h_2(t, t - \theta)$ represent the system nonlinearity due to impulses separated by an interval θ. Looking at (6), we note that $h_2(t, t - \theta)$ represents one-half of the difference between the system response to an impulse pair and the second-degree single impulse response as well as the shifted second-degree single impulse response. We call these off-diagonal kernels the separation nonlinearities.

III. Direct Calculation of Second-Degree Kernels of a Quadratic System by Means of Impulse Pair Input

Consider a quadratic system with block diagram (Fig. 4; same as Table 1, case d).

If $x(t)$ is a unit impulse, $x(t) = \delta(t)$, then

$$\begin{aligned} y_{\substack{\text{single}\\ \text{impulse}}}(t) &= \left[\int_0^t e^{-\alpha\tau}x(t-\tau)\,d\tau\right]\cdot\left[\int_0^t e^{-\beta\tau}x(t-\tau)\,d\tau\right] \\ &= \left[\int_0^t e^{-\alpha\tau}\delta(t-\tau)\,d\tau\right]\cdot\left[\int_0^t e^{-\beta\tau}\delta(t-\tau)\,d\tau\right] \\ &= e^{-\alpha t}e^{-\beta t} = e^{-(\alpha+\beta)t}. \end{aligned}$$

From (5),

$$h_2(t, t) = y_{\substack{\text{single}\\ \text{impulse}}}(t) = e^{-(\alpha+\beta)t}. \tag{7}$$

Consider an impulse pair stimulus, $x(t) = \delta(t) + \delta(t - \theta)$

$$\begin{aligned} y_{\substack{\text{double}\\ \text{impulse}}}(t) &= \left[\int_0^t e^{-\alpha\tau}(\delta(t-\tau) + \delta(t-\theta-\tau))\,d\tau\right] \\ &\qquad \cdot\left[\int_0^t e^{-\beta\tau}(\delta(t-\tau) + \delta(t-\theta-\tau))\,d\tau\right] \\ &= (e^{-\alpha t} + e^{-\alpha(t-\theta)})\cdot(e^{-\beta t} + e^{-\beta(t-\theta)}) \\ &= e^{-(\alpha+\beta)t} + e^{-\alpha t-\beta(t-\theta)} + e^{-\alpha(t-\theta)-\beta t} + e^{-(\alpha+\beta)(t-\theta)}. \end{aligned} \tag{8}$$

Substituting (7) and (8) into (6), we obtain:

$$\begin{aligned} h_2(t, t-\theta) &= \tfrac{1}{2}(e^{-\alpha t-\beta(t-\theta)} + e^{-\alpha(t-\beta)-\beta t}) \\ &= \tfrac{1}{2}(e^{-\alpha(t-\theta)-\alpha\theta-\beta(t-\theta)} + e^{-\alpha(t-\theta)-\beta(t-\theta)-\beta\theta}) \\ &= \tfrac{1}{2}(e^{-\alpha\theta} + e^{-\beta\theta})\cdot e^{-(\alpha+\beta)(t-\theta)}. \end{aligned} \tag{9}$$

Consider the term $e^{-(\alpha+\beta)(t-\theta)}$ in (9). In the direction parallel to the main diagonal and away from the origin, the off-diagonal curves have the same time constant as the curve on the main diagonal (i.e., second-degree single impulse response) in which the smaller of the two time constants, $1/\alpha$ and $1/\beta$ dominate. Consider the term $\frac{1}{2}(e^{-\alpha\theta} + e^{-\beta\theta})$ in (9). The values of α, β, and θ affect the magnitude of the off-diagonal kernels which in turn affect the time constant of the curves as these move perpendicular to and away from the main diagonal. For example, if $\alpha > \beta$ (i.e., the time constant $1/\beta > 1/\alpha$), then the larger time constant dominates in affecting the time constant of the envelope of the curves as these move away from the main diagonal.

IV. Equivalence of Second-Degree Kernels as Obtained by Impulse Pairs and by Inspection of the Two-Dimensional Convolution

Consider a quadratic system with block diagram as in Fig. 4. $y(t)$ can be obtained as the product of two convolution integrals. That is,

$$y(t) = \int_0^t e^{-\alpha\tau}x(t-\tau)d\tau \int_0^t e^{-\beta\tau}x(t-\tau)d\tau,$$

$$y(t) = \int_0^t \int_0^t \underbrace{e^{-\alpha\tau_1}e^{-\beta\tau_2}}_{h_2(\tau_1,\ \tau_2)}x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2.$$

Since the choice of τ_1 and τ_2 to be associated with α and β is arbitrary, we should average over the two combinations of τ_1 and τ_2. Hence

$$h_2(\tau_1, \tau_2) = \tfrac{1}{2}(e^{-\alpha\tau_1}e^{-\beta\tau_2} + e^{-\alpha\tau_2}e^{-\beta\tau_1}). \tag{10}$$

In order to obtain a correspondence with the result of Section III, we introduce a representation which is diagonally equivalent to the rectangular τ_1, τ_2 co-

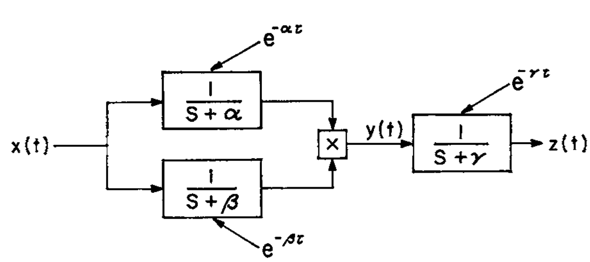


FIGURE 5

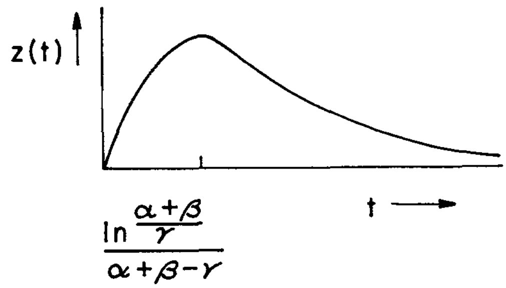


FIGURE 6

ordinates so that t and θ are made equivalent to τ_1 and $\tau_1 - \tau_2$. If the length on the graph is considered relevant, then the former are longer than the respective latter terms by a factor of $2^{\frac{1}{2}}$. Since, however, the two representations actually belong to separate domains, we can let the length scale of the diagonal system be $2^{\frac{1}{2}}$ times greater than that of the rectangular system. Substituting the equivalent terms into (10), we obtain:

$$\begin{aligned} h_2(\tau_1, \tau_2) = h_2(t, t-\theta) &= \tfrac{1}{2}(e^{-\alpha(t-\theta)}e^{-\beta t} + e^{-\alpha t}e^{-\beta(t-\theta)}) \\ &= \tfrac{1}{2}(e^{-(\alpha+\beta)(t-\theta)})(e^{-\alpha\theta} + e^{-\beta\theta}). \end{aligned} \tag{11}$$

Equation (11) is identical to (9). Note that we could have chosen $\tau_2 = t$ and $\tau_1 = t - \theta$ and the result for $h_2(t - \theta, t)$ would have been identical to (11), as would be expected, since the off-diagonal kernels are symmetric about the main diagonal.

V. Single Impulse Through a Quadratic System which is Followed by a Lag Term—in Order to Illustrate the Effect of the Lag Term

Consider the following quadratic system (Fig. 5; similar to Table 1, case e, Hammerstein model). If $x(t) = \delta(t)$, then from (7), $y(t) = e^{-(\alpha+\beta)t}$

$$\begin{aligned} z(t) &= \int_0^t e^{-\gamma\tau}e^{-(\alpha+\beta)(t-\tau)}d\tau = e^{-(\alpha+\beta)t}\int_0^t e^{(\alpha+\beta-\gamma)t}d\tau \\ &= e^{-(\alpha+\beta)t}(e^{(\alpha+\beta-\gamma)t} - 1)/(\alpha+\beta-\gamma) \\ &= (e^{-\gamma t} - e^{-(\alpha+\beta)t})/(\alpha+\beta-\gamma). \end{aligned} \tag{12}$$

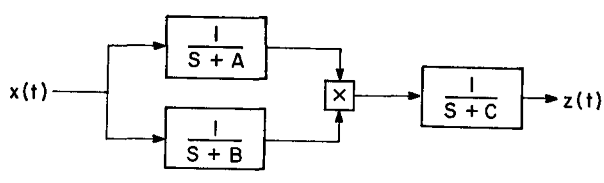


FIGURE 7

The effect of the third lag term in smoothing the peak of the $y(t)$ curve and moving the maximum $z(t)$ value down the time scale is shown in Fig. 6.

The lag term after the multiplication process is seen to increase the overall memory of the system. This is in agreement with Sandberg and Stark (1963), who showed that a linear system following a second-degree system smears out the second-degree kernels of the combined system along lines parallel to the main diagonal.

VI. Calculation of Second-Degree Kernels of Quadratic Systems by Means of Computer Simulation

A general representation of a purely quadratic system consists of two linear parallel elements whose outputs are multiplied and then act as input to a third linear element (Fig. 7).

As can be seen from Section V, the expressions for the second-degree kernels for a general quadratic system would be similar to (9), but the third element does make the expressions more cumbersome. Nine computer simulations were performed using different combinations of values of A, B, and C. A tenth simulation was concerned with a quadratic system with a biphasic component (Fig. 8). A fourth-order Runge–Kutta integration routine (PDP8/I computer, Tektronix 611 Storagescope, and Tektronix 4601 hard copy unit) was used for the simultaneous solution of the response of each element to its input. The solution of the single inpulse response of the model with selected time constants was stored in computer memory, and this was appropriately subtracted from the biimpulse response to give the second-degree kernel values for a particular separation between impulses. The second-degree curves for each model are stagger plotted with the curves shifted down by an amount equal to the biimpulse separation (see Fig. 9). The exponential filters used in each of the building elements theoretically have infinite memory. Practically, however, the memory of the filters are limited by the resolution limit of the simulation display.

The results indicate that the larger of the time constants before multiplication dominates the time constant of the envelope of the off-diagonal curves as the curves move away from the main diagonal. The tenth biphasic simulation demonstrates dramatically the dominance of the large time constant term (with negative sign for the linear lag term) on the off-diagonal curves. The smaller of the time constants before multiplication combines with that of the lag term after

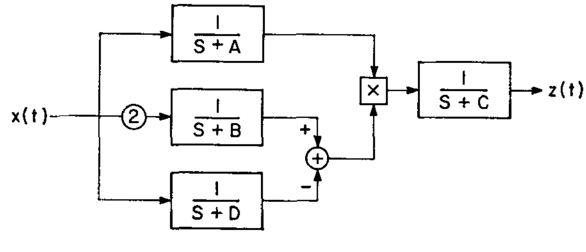
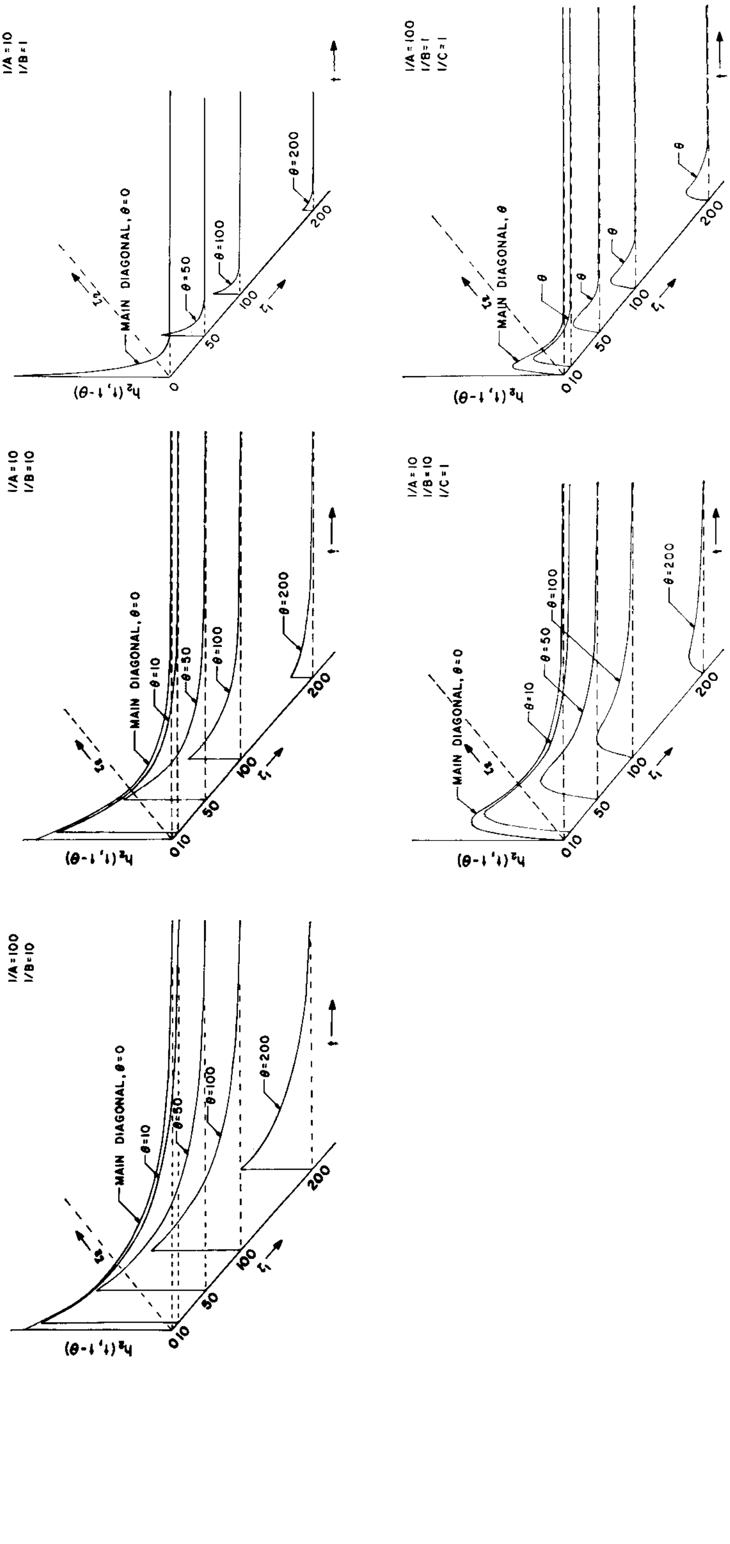


FIGURE 8


1/A = 10
1/B = 1
1/A = 10
1/B = 10
1/A = 100
1/B = 10
1/A = 100
1/B = 1
1/C = 1
1/A = 10
1/B = 10
1/C = 1
MAIN DIAGONAL, θ = 0
MAIN DIAGONAL, θ
θ = 10
θ = 50
θ = 100
θ = 200
$h_2(t, t-\theta)$
τ_1
τ_2
t
0
10
50
100
200

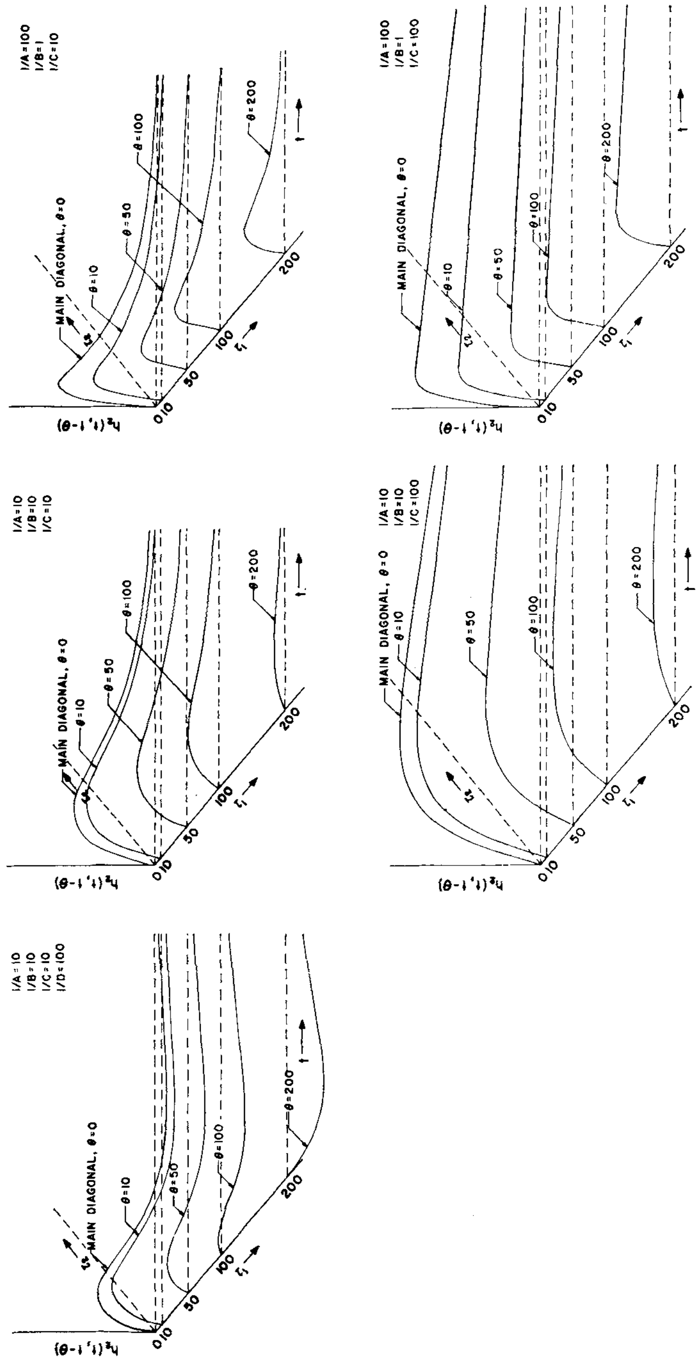


FIG. 9. Biimpulse simulation plots of quadratic kernels at time constants (TC) $1/A$, $1/B$, and $1/C$ with reference to Fig. 7. Top row (without post multiplier [X] TC) shows that smaller pre-[X] TC controls decay parallel to main diagonal and larger pre-[X] TC controls off-diagonal decay. Middle column shows post-[X] TC smears mostly parallel to main diagonal. Right column shows interaction of post-[X] TC with shortest pre-[X] TC in affecting mostly decay parallel to main diagonal. Bottom of left column refers to Fig. 8 and shows the effect of subtraction as a phase lead term. Also, large pre-[X] TC ($1/D$) acts mostly in off-diagonal direction.

the multiplication process to dominate the time constant of the envelope of kernel curves as these move parallel to the main diagonal.

DISCUSSION

The impulse-pair simulations, using various time constants for the building elements of quadratic systems, give clear and easily interpretable results. We note that the off-diagonal kernels are concerned with the nonlinear interaction of the system due to impulses separated in time. The off-diagonal curves are influenced mostly by the larger of the time constants which enter into the multiplication process. This is reasonable, since the larger time constant term determines the extent to which its effect from the first impulse extends pass the start of the second impulse and be multiplied by the smaller time constant term which concomitantly experiences the second impulse. The smaller time constant before multiplication determines the time constant of the envelope of the curves in the direction parallel to the main diagonal, i.e., the quadratic impulse response. This is reasonable since the smaller time constant term is multiplied against the larger time constant term until the smaller time constant term becomes negligible. Beyond that the second-degree impulse response is zero since the contribution due to the smaller time constant term becomes zero.

The time constant after the multiplication process affects the kernels in the directional parallel to the main diagonal. This is in agreement with Sandberg and Stark, who showed that a linear system following a second-degree system (e.g., Hammerstein model) smears out the second-degree kernels of the combined system along lines parallel to the main diagonal. Also, they showed that a linear system followed by a second-degree system (e.g., Wiener model) tends to smear the kernels both parallel and perpendicular to the main diagonal. Therefore, as Sandberg and Stark have noted, a linear system preceding a nonlinear one will in general increase the nonlinear interaction time whereas a linear system following the nonlinear one will only increase the overall memory of the system.

One of the drawbacks of the functional approach has been the computational difficulties in obtaining even up to the third-degree kernels. Watanabe and Stark introduced a method for the calculation of kernels using least squared error approximation of each degree functional constructed from a linear combination of a small number of finite base functions such as the quasi-Laguerre functions (Laguerre functions were first put forward by Wiener). These were further modified in an iterative technique to resemble the system dynamical properties. The calculation of the approximate third-degree kernels of the pupillary system, for 500 data points, required only 30 sec on the CDC 6600 computer.

The ability to construct models using simple building elements facilitates the process of generating and interpreting kernels and appears very profitable in recognizing essential aspects of the second-degree kernels (Alper and Poortvliet, 1963; Eykhoff, 1974; Hung, 1977; Stark, 1968). In this way the model structure may be simplified. Also it may add to the physical–physiological understanding of the underlying process. Future extensions of this work might involve the simulation of quadratic kernels of more general systems. The present method may also be applied to the simulation of cubic kernels, using an impulse triple.

SUMMARY

A theoretical basis has been presented for obtaining the second-degree kernels of a quadratic system by using unit impulse and impulse pairs of variable separation. The representation of the kernels obtained in this manner is seen to be equivalent to that obtained by direct inspection of the kernels in the two-dimensional convolution of the same quadratic system. Computer simulation results on quadratic systems with a third lag term show that, in agreement with the results derived for a simple quadratic system with no third lag term, the larger of the time constants before multiplication dominates the time constant of the envelope of the off-diagonal curves as the curves move away from the main diagonal. The smaller of the time constants before multiplication combines with that of the third element to dominate the time constant of the envelope of kernel curves as these move parallel with the main diagonal.

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