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ON SOME APPLICATIONS OF POISSON RANDOM WALK PLANS

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In the present paper, we consider sequential Poisson random walk plans (SPRWPs). In particular, we solve such problems as the calculation of characteristics of SPRWPs with intensity parameter given, the construction of unbiased estimates of characteristics of a homogeneous Poisson process observed in the case of a sequential trial plan, the construction of the joint distribution of Smirnov statistics, and the derivation of an optimal sequential Poisson criterion of given power. Another form of representing exact formulas for calculating characteristics of SPRWPs was given by A. Dvoretsky, J. Kiefer, and J. Wolfowitz. But they note no relationship between the SPRWPs and the corresponding sequential criteria; hence those formulas became approximate but not exact for the sequential criteria.

Let $\{\tau_i\}_1^\infty$ be a sequence of independent random variables distributed by the exponential law with parameter λ, i.e., with density

$$f_\lambda(x) = \lambda e^{-\lambda x}, \qquad \lambda > 0, \qquad x \geq 0.$$

Let us define the Poisson random walk plan (PRWP) Π^G with stopping boundary G (see, e.g., [2]). The walk takes place on the set of states (t, d), $t \geq 0$, $d = 0, 1, \dots$, lying on the plane. At the time $t = 0$ the walk process $W(t) = (t, d(t))$ is at the point $(0, 0)$. Here the component $d(t)$ is the number of jumps of $W(t)$ at the time t. The walk process makes the ith jump, $i = 1, 2, \dots$, after a random time interval $\tau_1 + \tau_2 + \cdots + \tau_i$, and the coordinate d increases by 1. Between sequential jumps the coordinate d does not change. The walks stops as it attains the boundary G.

Lemma. *Let Π_1, Π_2, Π_3 be the SPRWPs with stopping boundaries $G_1 = \{(\Delta_0, 0);\ G^z\}$, $G_2 = \{(t, m),\ 0 \leq t \leq \Delta_1;\ G^z\}$, $G_3 = G_2 \cup \{(\Delta_0, 0)\}$, and intensity parameter λ, where $G^z = \{(z, j),\ j = 0, 1, \dots\}$, $0 < \Delta_0$, $0 \leq \Delta_1$, $0 < z$. Then the probability of attaining the point $\Gamma = (z, m)$, $m \in \{0, 1, \dots\}$ by the process of random walk $W(t)$ starting from the origin $O = (0, 0)$ for each of the plans is*

$$\mathbf{P}[0, \Gamma \mid G_1] \equiv b_m(\lambda, z) = \begin{cases} p_0[\lambda\Delta_0], & m = 0 \wedge z \geq \Delta_0, \\ p_m[\lambda z], & z < \Delta_0, \\ p_m[\lambda z][1 - (1 - \Delta_0/z)^m], & m \geq 1 \wedge z \geq \Delta_0, \end{cases} \tag{1}$$

$$\mathbf{P}[0, \Gamma \mid G_2] \equiv c_m(\lambda, z) = p_m[\lambda z]\begin{cases} 0, & z < \Delta_1 \vee m = 0, \\ 1 - (\Delta_1/z)^m, & z \geq \Delta_1, \end{cases} \tag{2}$$

$$\mathbf{P}[0, \Gamma \mid G_3] \equiv d_m(\lambda, z) = p_m[\lambda z]\begin{cases} 1 - (\Delta_1/z)^m, & \Delta_1 < z < \Delta_0, \\ 1 - (\Delta_1/z)^m - (1 - \Delta_0/z)^m, & \Delta_1 < \Delta_0 \leq z \wedge m \geq 1, \\ 1 - (\Delta_1/z)^m - (1 - \Delta_0/z)^m + ((\Delta_1 - \Delta_0)/z)^m, & \Delta_0 \leq \Delta_1 \leq z, \\ 0. & m = 0 \vee \Delta_1 > z, \end{cases} \tag{3}$$

respectively. Here $p_m[x] = (x^m/(m!))e^{-x}$. The distribution densities of the probabilities of attaining the state $\gamma = (y, k)$, $k = m$, for $0 < y \leq \Delta_1$ and $k = m + 1$ for $\Delta_1 < y \leq 1$, $m \in \{1, 2, \dots\}$, from the state O for the PRWP Π_2 and Π_3 with $z = 1$ are equal to

$$f_\lambda[0, \gamma \mid G_2] = \lambda\begin{cases} p_{m-1}[\lambda y], & y \leq \Delta_1, \\ c_m(\lambda, y), & y > \Delta_1, \end{cases} \tag{4}$$

$$f_\lambda[0, \gamma \mid G_3] = \lambda\begin{cases} b_{m-1}[\lambda, y], & y \leq \Delta_1, \\ d_m(\lambda, y), & y > \Delta_1, \end{cases} \tag{5}$$

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Proof. It is clear that for $m = 0$ or $\Delta_1 \geq z$, the probability sought for differs from zero only in the case of Π_1. It is equal to $\exp\{-\lambda \min(\Delta_0, z)\}$. The validity of other assertions of the lemma for $m \geq 1$ can be proved by the use of the properties of the Poisson process and the total probability formula.

Denote $F_1 = (z_1, m_1)$, $F_2 = (z_2, m_2)$, $m_1, m_2 \in \{0, 1, \dots\}$, $z_1 \geq 0$, $z_2 \geq 0$. If the stopping boundary is absent, the probability of the transition from F_1 to F_2 for the Poisson random walk is equal to

$$\mathbf{P}_\lambda[F_1, F_2] = \begin{cases} 0, & z_2 < z_1 \vee m_2 < m_1, \\ p_{m_2-m_1}[\lambda(z_2 - z_1)], & z_2 \geq z_1 \wedge m_2 \geq m_1. \end{cases} \tag{6}$$

From (6) we consecutively derive for $m \geq 1$

$$\mathbf{P}_\lambda[0, \Gamma \mid G_1] = \mathbf{P}_\lambda[0, \Gamma] = p_m[\lambda z] \quad \text{as} \quad z < \Delta_0,$$

$$\mathbf{P}_\lambda[0, \Gamma \mid G_1] = \mathbf{P}_\lambda[0, \Gamma] - \mathbf{P}_\lambda[0, (\Delta_0, 0)]\mathbf{P}_\lambda[(\Delta_0, 0), \Gamma]$$

$$= p_m[\lambda z] - p_0[\lambda\Delta_0]p_m[\lambda(z - \Delta_0)] = p_m[\lambda z][1 - ((z - \Delta_0)/z)^m] \quad \text{as} \quad z \geq \Delta_0;$$

$$\mathbf{P}_\lambda[0, \Gamma \mid G_2] = \mathbf{P}_\lambda[0, \Gamma] - \mathbf{P}_\lambda[0, (\Delta_1, m)]\mathbf{P}_\lambda[(\Delta_1, m), \Gamma]$$

$$= p_m[\lambda z] - p_m[\lambda\Delta_1]p_0[\lambda(z - \Delta_1)] = p_m[\lambda z][1 - (\Delta_1/z)^m] \quad \text{as} \quad z \geq \Delta_1;$$

$$\mathbf{P}_\lambda[0, \Gamma \mid G_3] = \mathbf{P}_\lambda[0, \Gamma \mid G_2] \quad \text{as} \quad \Delta_0 \geq z,$$

$$\mathbf{P}_\lambda[0, \Gamma \mid G_3] = \mathbf{P}_\lambda[0, \Gamma \mid G_1] - \mathbf{P}_\lambda[0, (\Delta_1, m)]\mathbf{P}_\lambda[(\Delta_1, m), \Gamma]$$

$$= p_m[\lambda z][1 - ((z - \Delta_0)/z)^m - (\Delta_1/z)^m] \quad \text{as} \quad \Delta_1 < \Delta_0 < z,$$

$$\mathbf{P}_\lambda[0, \Gamma \mid G_3] = \mathbf{P}_\lambda[0, \Gamma \mid G_2] - \mathbf{P}_\lambda[0, (\Delta_0, 0)]p_m[\lambda(z - \Delta_0)][1 - ((\Delta_1 - \Delta_0)/(z - \Delta_0))^m]$$

$$= p_m[\lambda z]\left[1 - \left(\frac{z - \Delta_0}{z}\right)^m - \left(\frac{\Delta_1}{z}\right)^m + \left(\frac{\Delta_1 - \Delta_0}{z}\right)^m\right]$$

as $\Delta_0 \leq \Delta_1 < z$. It is not difficult to obtain relations (4)–(5) by applying the formula

$$f_\lambda[0, \gamma \mid G_i] = \mathbf{P}_\lambda[0, (y, k - 1) \mid G_i]f_\lambda(0), \qquad i = 2, 3,$$

with $z = 1$. The lemma is thus proved.

To prove the main results, we introduce the following notation:

$$\langle a \rangle = \begin{cases} a - 1, & \text{if } a \text{ is an integer}, \\ \text{the integer part of } a, & \text{otherwise}; \end{cases}$$

$$Q_\lambda = \|q_{ij}(\lambda)\|_1^r = \begin{pmatrix} c_1[\lambda] & p_0[\lambda] & 0 & \dots & 0 \\ c_2[\lambda] & p_1[\lambda] & p_0[\lambda] & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{r-1}[\lambda] & p_{r-2}[\lambda] & p_{r-3}[\lambda] & \dots & p_0[\lambda] \\ d_r[\lambda] & b_{r-1}[\lambda] & b_{r-2}[\lambda] & \dots & b_1[\lambda] \end{pmatrix}$$

is the $r \times r$-matrix whose nonzero elements are

$$q_{i,i-m+1}(\lambda) = p_m[\lambda], \qquad m = 0, \dots, i - 1, \quad i = 1, \dots, r - 1;$$

$$q_{m,1}(\lambda) = c_m[\lambda] \equiv c_m(\lambda, 1) = p_m[\lambda](1 - \Delta_1^m), \qquad m = 1, \dots, r - 1;$$

$$q_{r,r-m+1}(\lambda) = b_m[\lambda] \equiv b_m(\lambda,1) = p_m[\lambda]\{1-(1-\Delta_0)^m\}, \qquad m = 1,\dots,r-1;$$

$$q_{r,1}(\lambda) = d_r[\lambda] \equiv d_r(\lambda,1) = p_r[\lambda]\begin{cases} 1-\Delta_1^r-(1-\Delta_0)^r & \text{for } \Delta_1 < \Delta_0 \le 1, \\ 1-\Delta_1^r-(1-\Delta_0)^r+(\Delta_1-\Delta_0)^r & \text{for } \Delta_0 \le \Delta_1 \le 1; \end{cases}$$

if $r = 1$, we put $Q_\lambda = d_1[\lambda]$; $\pi(h) = (\pi_1,\dots,\pi_r)$ is the row vector of size r whose only nonzero element is $\pi_h = 1$, $1 \le h \le r$; $\xi = (1,1,\dots,1)^T$ is the column vector of size r all of whose elements are equal to 1; $R_0[\lambda] = (0,0,\dots,0, e^{-\lambda\Delta_0})^T$ is a column vector of size r; $R_1[\lambda] = (\rho_1[\lambda],\dots,\rho_r[\lambda])^T$ is the column vector of size r with elements

$$\rho_m[\lambda] = 1 - \sum_{j=1}^{m+1} q_{m,i} = 1 - c_m[\lambda] - \sum_{i=0}^{m-1} p_i[\lambda], \qquad m = 1,\dots,r-1,$$

$$\rho_r[\lambda] = 1 - \sum_{i=1}^{r} q_{r,i} - e^{-\lambda\Delta_0} = 1 - d_r[\lambda] - \sum_{i=1}^{r-1} b_i[\lambda] - e^{-\lambda\Delta_0};$$

$R_1(\lambda,\Delta_1) = (g_1(\lambda,\Delta_1),\dots,g_r(\lambda,\Delta_1))^T$ is the column vector of size r with elements

$$g_m(\lambda,\Delta_1) = 1 - \sum_{i=0}^{m-1} p_i[\lambda\Delta_1], \qquad m = 1,\dots,r-1,$$

$$g_r(\lambda,\Delta_1) = 1 - \sum_{i=0}^{r-1} b_i[\lambda];$$

if $r = 1$, we put $g_1(\lambda,\Delta_1) = 1 - b_0[\lambda]$; I_r is the $r \times r$ unit matrix; $R_2(\lambda,y) = (\rho_1(\lambda,y),\dots,\rho_r(\lambda,y))^T$ is the column vector of size r with elements

$$\rho_r(\lambda,y) = \lambda\begin{cases} b_{r-1}(\lambda,y), & y \le \Delta_1, \\ d_r(\lambda,y), & y > \Delta_1, \end{cases}$$

$$\rho_m(\lambda,y) = \lambda\begin{cases} p_{m-1}[\lambda y], & y \le \Delta_1, \\ c_m(\lambda,y), & y > \Delta_1, \end{cases} \qquad m = 1,2,\dots,r-1;$$

L_0, L_1 are the lines determined by the equations $d = t - a_0$, $a_0 > 0$, $d = t + a_1$, $a_0 + a_1 > 1$, $a_1 > 0$, respectively.

Theorem. *Let Π_{01} be a SPRWP with stopping boundary composed of parallel lines L_0 and L_1; V_0 be the probability that the random walk stops due to attaining the line L_0; τ be the random stopping time; N be the number of jumps that occurred before stopping; ν be the number of continuous parts of the random walk's trajectory before stopping; $V(\lambda,t,a_0,a_1)$ be the density of distribution of τ. Then the following relations hold:*

$$V_0 = \pi(I-Q)^{-1}R_0[\lambda], \qquad \pi \equiv \pi(r_1), \qquad I \equiv I_r, \qquad Q \equiv Q_\lambda; \tag{7}$$

$$\begin{aligned} \mathbf{E}_\lambda N &= \pi(I-Q)^{-1}\left(Q^{r_0+1}\xi - R_1(\lambda,\Delta_1) + \sum_{i=0}^{r_0} Q^i R_1[\lambda]\right) + r_1(1-V_0) \\ &= \pi(I-Q)^{-1}\left(\xi - R_1(\lambda,\Delta_1) - \sum_{i=0}^{r_0} Q^i R_0[\lambda]\right) + r_1(1-V_0), \end{aligned} \tag{8}$$

$$\mathbf{E}_\lambda \nu = \mathbf{E}_\lambda N + \pi(I-Q)^{-1}Q^{r_0}R_0[\lambda], \tag{9}$$

$$\mathbf{E}_\lambda \tau = \mathbf{E}_\lambda N/\lambda, \qquad \lambda > 0, \tag{10}$$

$$V(\lambda,t,a_0,a_1) = \pi Q^h \begin{cases} R_0[\lambda], & \textit{if the random walk stops due to attaining } L_0, \\ R_2(\lambda,y), & \textit{if the random walk stops due to attaining or passing } L_1. \end{cases} \tag{11}$$

Here $r_0 = \langle a_0 \rangle$; $r_1 = \langle a_1 \rangle + 1$; $r = r_0 + r_1$; $\Delta_0 = a_0 - r_0$; $\Delta_1 = r_1 - a_1$; $h = \langle t \rangle$; $y = t - \langle t \rangle$.

Proof. We define in the domain where the random walk determined by Π_{01} is in progress the homogeneous absorbing Markov chain ζ_t, $t \geq 0$, $\zeta_0 = r_1$, with transient states $1, 2, \dots, r$ which mean that the random walk $W(t)$ is to be found at the lines

$$d(t) = t + r_1 - m, \qquad m = 1, 2, \dots, r,$$

respectively, and with two absorbing states 0 and $r+1$ meaning that $W(t)$ attains the lines $L_1^+ : d(t) = t + r$, and L_0, respectively. It is easily seen that the chain ζ_t on the set of transient states $1, 2, \dots, r$ is associated with the transition matrix Q; the vector $R_0[\lambda]$ contains the probabilities to transit to the absorbing state $r+1$ in one step, and the vector π is the initial probability distribution of ζ_t on the set of transient states. Hence, applying Theorem 3.3.7 from [3], we get (7). Formula (8) can be proved by means of the calculations

$$\mathbf{E}_\lambda N = \pi \sum_{i=1}^{\infty} i Q^{i+r_0} R_0[\lambda] + \pi \sum_{i=1}^{\infty} Q^{i-1}((r_1 + i) R_1[\lambda] - R_1(\lambda, \Delta_1))$$

$$= \pi Q^{r_0+1} \sum_{i=1}^{\infty} i Q^{i-1} R_0[\lambda] + r_1 \pi \sum_{i=1}^{\infty} Q^{i-1} R_1[\lambda] + \pi \sum_{i=1}^{\infty} i Q^{i-1} R_1[\lambda] - c$$

$$= \pi Q^{r_0+1} (I - Q)^{-2} R_0[\lambda] + r_1 \pi (I - Q)^{-1} R_1[\lambda] + (I - Q)^{-2} R_1[\lambda] - c$$

$$= \pi Q^{r_0+1} [(I - Q)^{-1} \xi - (I - Q)^{-2} R_1[\lambda]] + r_1 (1 - V_0) + \pi (I - Q)^{-2} R_1[\lambda] - c$$

$$= \pi (I - Q)^{-1} \left(Q^{r_0+1} \xi - \sum_{i=0}^{r_0} Q^i R_1[\lambda] \right) + r_1 (1 - V_0) - c = \pi (I - Q)^{-1} Q^{r_0+1} \xi + \pi \sum_{i=0}^{r_0} Q^i [\xi - (I - Q)^{-1} R_0[\lambda]] + r_1 (1 - V_0) - c$$

$$= \pi (I - Q)^{-1} \left[\xi - \sum_{i=0}^{r_0} Q^i R_0[\lambda] \right] + r_1 (1 - V_0) - c, \qquad c = \pi (I - Q)^{-1} R_1(\lambda, \Delta_1),$$

if one takes the identities

$$\sum_{i=0}^{\infty} Q^i = (I - Q)^{-1}, \qquad \sum_{i=1}^{\infty} i Q^{i-1} = (I - Q)^{-2}, \qquad (I - Q)^{-1} (R_0[\lambda] + R_1[\lambda]) = \xi$$

into account. Similarly we can obtain

$$\mathbf{E}_\lambda \nu = \pi \sum_{i=0}^{\infty} (i + 1) Q^{i+r_0} R_0[\lambda] + \pi \sum_{i=1}^{\infty} Q^{i-1}((r_1 + i) R_1[\lambda] - R_1(\lambda, \Delta_1))$$

$$= \mathbf{E}_\lambda N + \pi \sum_{i=0}^{\infty} Q^{i+r_0} R_0[\lambda] = \mathbf{E}_\lambda N + \pi (I - Q)^{-1} Q^{r_0} R_0[\lambda].$$

Formula (10) follows from the well-known Wald formula. Formula (11) can be derived by using the total probability formula and the well-known properties of Markov chains. The theorem is thus proved.

Remark. The matrix $I - Q$ is a lower almost-triangular matrix whose determinant can be recursively expanded in terms of the elements of its last row. This fact lets us use the formula

$$V_0 = e^{-\lambda a_0} \frac{D[r_1 - 1]}{D[r]},$$

where $D[m]$ is the principal minor of order m of the matrix $I - Q$. To this end, for $r > 1$, we may apply the relations

$$D[0] = 1;$$

$$D[m] = D[m-1] - \sum_{j=1}^{m-1} p_j[\lambda](p_0[\lambda])^{j-1} D[m-j] - c_m[\lambda](p_0[\lambda])^{m-1}, \qquad m = 1, \dots, r-1;$$

$$D[r] = D[r-1] - \sum_{j=1}^{r-1} b_j[\lambda](p_0[\lambda])^{j-1} D[r-j] - d_r[\lambda](p_0[\lambda])^{r-1}.$$

In the sequel we apply the SPRWPs to sequential inspection of reliability of products under the assumption that the distribution of time between failures of one item is determined by the density function $f_\lambda(x)$. The sufficient statistic for the family of distributions generated by Π_{01} is the pair of coordinates (k, τ) of the state where the random walk stops. Here k is the number of jumps of the random walk under consideration that occurred prior to the stopping time τ.

Corollary 1. *Unbiased estimates (u.e.'s) of the probability of reliable functioning $p_0[\lambda x] = e^{-\lambda x}$ for $x < a_0$ and of the intensity parameter λ for $a_1 > 1$ can be calculated by the formulas*

$$\text{u.e.}[p_0[\lambda x]] = \begin{cases} \dfrac{p_0[\lambda x] V(\lambda, \tau - x, a_0 - x, a_1 + x)}{V(\lambda, \tau, a_0, a_1)} \quad \forall \lambda > 0 & \text{if } \tau \geq x, \\ 0 & \text{if } \tau < x, \end{cases} \tag{12}$$

$$\text{u.e.}[\lambda] = \begin{cases} \dfrac{\lambda V(\lambda, \tau, a_0 + 1, a_1 - 1)}{V(\lambda, \tau, a_0, a_1)} \quad \forall \lambda > 0 & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases} \tag{13}$$

The proof of (12)–(13) is based on the fact that, in the case of a Poisson random walk, for any state γ where the random walk determined by Π_{01} stops, the relation

$$\text{u.e.}(\mu_{0,F}) = \frac{\mu_{0,F}\mu_{F,\gamma}}{\mu_{0,\gamma}}$$

holds, where $\mu_{U,Z}$ is the density of the probability of transition from U to Z with stopping boundary given.

Now, let us consider the sequential Poisson criterion (SPC) S_p minimizing the average number of trials whose durations are distributed by the exponential law with density $f_\mu(x)$. By the results $x_1, x_2, \dots, x_n$ of n sequential independent trials, based on the value

$$U_n = \sum_{i=1}^{n} s x_i, \qquad s = \frac{\mu_1 - \mu_0}{\log(\mu_1/\mu_0)}, \tag{14}$$

we either accept the null hypothesis

$$H_0: \ \mu = \mu_0, \qquad \text{if} \qquad U_n \geq a_0 + n - 1, \qquad a_0 > 0,$$

or accept the alternative

$$H_1: \ \mu = \mu_1 \qquad \text{if} \qquad U_n \leq n - a_1, \qquad a_1 + a_0 > 1,$$

or continue the trials if $a_0 + 1 < U_n < n - a_1$. Thus, the SPC just defined corresponds to the SPRWP Π_{01} and hence the validity of the assertion given below follows.

Corollary 2. *The operating characteristic $L(\mu)$, the average number of trials $M_\mu \nu$, and the average duration of trials $M_\mu \tau$ for the SPC S_p for the parameter μ can be calculated by the formulas*

$$L(\mu) = V_0, \qquad M_\mu \nu = \mathbf{E}_\lambda \nu, \qquad M_\mu \tau = \mathbf{E}_\lambda N/\mu,$$

respectively ($\lambda = \mu/s$). If the criterion is of power (α, β), then the parameter a_0 can be computed by the formula

$$a_0 = \frac{\log(1-\alpha)/\beta}{\lambda_1 - \lambda_0}, \qquad \lambda_0 = \frac{\mu_0}{s}, \qquad \lambda_1 = \frac{\mu_1}{s}. \tag{15}$$

Proof. The first part of the assertion is true by virtue of the correspondence mentioned above. To prove the second part, we use the relations

$$\lambda_0 e^{-\lambda_0} = \lambda_1 e^{-\lambda_1}, \tag{16}$$

$$L(\mu) = e^{-\lambda a_0}\varphi(\lambda e^{-\lambda}), \tag{17}$$

where $\varphi(\cdot)$ is some positive-valued function. The first relation follows from (14), and the second one, from Lemma 12.4.1 from [4]. Hence, taking the relations $1-\alpha = L(\mu_0)$, $\beta = L(\mu_1)$ into account and applying (16)–(17), we obtain

$$\frac{1-\alpha}{\beta} = \frac{L(\mu_0)}{L(\mu_1)} = \frac{\exp(-\lambda_0 a_0)\varphi(\lambda_0 \exp(-\lambda_0))}{\exp(-\lambda_1 a_0)\varphi(\lambda_1 \exp(-\lambda_1))} = \exp(a_0(\lambda_1 - \lambda_0)),$$

hence (15) follows.

It is interesting to note that the approximate Wald formula (see, e.g., [1] or [5, formula (29), p. 135]) gives the optimal value of a_0 which is only one above the true value given here.

Remark. The second parameter a_1 of the SPC S_p of given power (α, β) can be obtained as the root of one of the equations

$$L(\mu_0) = 1-\alpha, \qquad L(\mu_1) = \beta.$$

To compare the empirical distribution function F_n^+ with the known continuous distribution function $F(x)$, one can use the joint distribution of Smirnov statistics

$$D_n^+ = \sup_{|x|<\infty} [F_n^+(x) - F(x)], \qquad D_n^- = -\inf_{|x|<\infty} [F_n^+(x) - F(x)].$$

The problem on determining the distribution of (D_n^-, D_n^+) admits an exact solution, which can be obtained in the same way as in [6].

COROLLARY 3. *The relation*

$$\mathbf{P}(D_n^- < v_0, D_n^+ < v_1) = \frac{n!e^n}{n^n} Q_1^n[r_1] \tag{18}$$

is valid, where $Q_1^n[r_1]$ *is the* (r_1, r_1)*th element of the nth power of the matrix* Q *with* $a_0 = v_0 n$, $a_1 = v_1 n$, $v_0 > 0$, $v_1 > 0$. *For* $a_0 + a_1 \le 1$, *probability* (18) *is equal to zero.*

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