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LAGRANGIAN FAMILIES OF JACOBIANS OF GENUS 2 CURVES

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The main result of this paper is that for any Lagrangian fibration on a projective algebraic (holomorphically) symplectic fourfold which is a compactified Jacobian of a family of curves, its base is the projective plane, and the family of curves is identified with the linear system of hyperelliptic curves of genus 2 on a $K3$ surface of degree 2. The relation between the Lagrangian structures on the Jacobians of different degrees is discussed; a nonsingularity criterion for the compactified Jacobians and for the relative Hilbert schemes of families of curves is obtained; explicit construction of the compactified Jacobians in the genus 2 case are obtained; toric techniques for computation of certain tensor holomorphic fields are given for use in the proof of the main result.

1. Introduction

This paper continues the study of Lagrangian fibrations on projective algebraic (holomorphically) symplectic varieties along the lines of [19]. Let X be a complex algebraic variety of dimension $2n$ with a holomorphic symplectic form $\alpha \in \Gamma(X, \Omega^2_X)$ ($d\alpha = 0, \alpha^{\wedge n}$ does not vanish). A Lagrangian fibration is a proper (or even projective) morphism $f : X \longrightarrow B$, whose generic fiber is Lagrangian. Applying the Stein factorization, one can suppose that the fibers of f are connected. By the Liouville theorem, the general fiber is an abelian variety of dimension n. In the classical mechanics, Lagrangian fibrations appear as a tool for integrating Hamiltonian systems. In the framework of the study of projective algebraic symplectic varieties (or more generally, of compact Kähler holomorphically symplectic varieties) started by [6, 7, 10, 15, 21], the Lagrangian fibrations were introduced in [8, 9, 12, 13, 18, 19]. The following theorem was proved in [19, Theorems 2, 5]:

Theorem. *Let C/B be a family of hyperelliptic curves of genus 2 with mild degenerations over the base $B = \mathbf{P}^2$. Then the compactified relative Jacobian $P = P^0_{C/B}$ of the family C/B is a nonsingular projective variety. Assume that it is symplectic and that the natural projection to B is a Lagrangian fibration. Then C/B is identified with the family of curves $\{\beta^{-1}(l)\}_{l \in \mathbf{P}^{2*}}$ on a hyperelliptic $K3$ surface $\beta : S \longrightarrow \mathbf{P}^2$ (the base B is identified with the dual projective plane $\mathbf{P}^{2*}$ parameterizing lines l in the last $\mathbf{P}^2$).*

This theorem describes the Lagrangian fibrations over $\mathbf{P}^2$ which are compactified families of Jacobians of genus 2 curves. All the resulting symplectic varieties P are birational to the Fujiki–Beauville symplectic fourfold $S^{[2]}$, obtained by blowing up the diagonal in the symmetric square $S^{(2)}$ of the $K3$ surface S; this birational isomorphism was described in other terms in [21]. So, in other words, the theorem states that with certain restrictions (mild degenerations), all the Jacobians which are symplectic and Lagrangian over $\mathbf{P}^2$ are Mukai transforms of certain Fujiki–Beauville symplectic fourfolds. Here, we will strengthen this result in several directions:

(1) replace P^0 by P^d; in view of the periodicity $P^d \simeq P^{d+2}$ for genus 2, this adds one variety P^1;

(2) move the condition $B = \mathbf{P}^2$ from the hypothesis to the conclusion of the theorem;

(3) replace the mild degeneration condition by a necessary and sufficient one for the nonsingularity of P.

For (1), we have to analyze the relation between a Lagrangian fibration $f : X \longrightarrow B$ and its Albanese family (having a cross section) $A(f) : A(X) \longrightarrow B$. It turns out that the Albanese family of a Lagrangian fibration is always symplectic and Lagrangian (Proposition 2.3), but for the inverse passage a certain cohomological condition should be verified (Proposition 2.6). However, this condition is automatically verified if

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$H^1(B, \Omega^2) = 0$ and all the fibers are reduced irreducible; in this case $A(f)$ is Lagrangian if and only if f is (Theorem 2.8). This implies that P^0 is Lagrangian if and only if P^1 is (Corollary 5.2).

For (2), note that, except for isotrivial fibrations, the base B of a Lagrangian fibration of relative dimension 2 is a rational surface (see [18, Sec. 3, Remark 5; 19, Sec. 2]); we eliminate the isotrivial cases by requiring that X be *irreducible symplectic* (see ***Terminology and Notations*** below for the definition, and [6] for its motivation). So, to prove that $B = \mathbf{P}^2$, it suffices to eliminate two cases: (a) B has exceptional (-1)-curves, and (b) $B = \mathbf{F}_n$ (Hirzebruch surface), $n = 0, 1, 2, \ldots$. In [19], it was shown that the families C/B giving rise to Lagrangian fibrations of Jacobians over the two-dimensional base are constructed as two-sheeted coverings of $\mathbf{P}(\mathcal{T}_B)$ ramified in the divisor of zeros of a section $\nu \in \Gamma(B, S^6\mathcal{T}_B \otimes \omega_B^2)$. We compute this group of sections on the formal neighborhood of a (-1)-curve and on Hirzebruch surfaces to see that the zero divisor is too bad: the corresponding compactified Jacobian is singular. To see this, we establish a necessary and sufficient condition on the degenerations (item (3) above) for P^d to be nonsingular (Corollary 4.4). The proof of this condition uses a more general criterion of the nonsingularity of the relative Hilbert scheme Hilb^d of a family of curves (Theorem 3.2) and a result of Altman–Iarrobino–Kleiman on the smoothness of the Abel–Jacobi map from the relative Hilbert scheme to the compactified relative Jacobian.

The structure of the paper is as follows. Section 2 gives a construction of the Albanese family of a Lagrangian fibration; having in mind applications to families with reduced irreducible fibers, we do not go into the uniqueness and compactification problems in case of multiple or reducible fibers. The relation between the fibration and its Albanese family is investigated in terms of an étale or Čech cocycle, and the conditions for moving the Lagrangian structure from one to the other are formulated.

In Sec. 3, we cite the definition of the compactified Jacobian of a family of curves and the Altman–Iarrobino–Kleiman result on its connection with the relative Hilbert scheme, and then prove the nonsingularity criterion for the relative Hilbert scheme of a family of curves, which implies also the one for its compactified Jacobian.

Section 4 deals with families of *hyperelliptic* reduced, but possibly reducible curves of genus 2. We provide an explicit construction of a nonsingular compactification $\tilde{P}$ of the relative Jacobian as the result of a blow up and a subsequent blow down which is isomorphic to the Altman–Kleiman compactification P^0 when the latter is defined, that is, when the curves are irreducible, and formulate the necessary and sufficient condition for the nonsingularity of P^0. It is also proved that for a family of hyperelliptic curves on a possibly singular $K3$ surface S of degree 2, the nonsingularity of $\tilde{P}$ is equivalent to that of S (Proposition 4.5).

In Sec. 5, we provide a construction of the compactified Jacobian P^1 of degree 1 as a compactified principal homogeneous space of J^0 associated with an element of order 2 in the Shafarevich–Tate group of B and apply the results of Sec. 2 to show that P^0 is Lagrangian if and only if P^1 is. We also make explicit in coordinates the symplectic structure on P^1 coming from a family of hyperelliptic curves on a $K3$ surface S of degree 2.

Section 6 gives a method of calculating the groups $\Gamma(B, S^6\mathcal{T}_B \otimes \omega_B^2)$ for toric varieties B, which is then applied in Sec. 7 to two toric surfaces: the blow up $\tilde{\mathbf{A}}^2$ of the origin in the affine plane $\mathbf{A}^2$ and $\mathbf{P}^2$. The computation for $\tilde{\mathbf{A}}^2$ implies that there are not enough sections on the neighborhood of the exceptional curve to generate a nonsingular family of hyperelliptic curves, and hence if P^d/B is nonsingular with a Lagrangian fibration, then B is a relatively minimal surface. The following main result is proved.

Theorem 1.1. *Let B be a nonsingular projective surface, C/B a family of hyperelliptic curves of genus* 2 *satisfying the conditions* (i), (ii) *of* (a) *of Theorem* 4.1, *and $\tilde{P}$ the nonsingular projective variety constructed in* (b) *of Theorem* 4.1, *which is a compactification of the relative Jacobian $J^0_{C/B}$ of the family C/B. Then the following assertions are true:*

(1) *$\tilde{P}$ is irreducible symplectic in such a way that the natural projection to B is a Lagrangian fibration if and only if C/B is identified with the family of curves $\{\beta^{-1}(l)\}_{l \in \mathbf{P}^{2*}}$ on a (nonsingular) hyperelliptic $K3$ surface $\beta : S \longrightarrow \mathbf{P}^2$. In particular, the only possible base surface B is the projective plane ($B = \mathbf{P}^{2*}$).*

(2) *Assume in addition that all the curves of the family C/B are irreducible. Then $\tilde{P}$ is canonically isomorphic to the Altman–Kleiman compactified Jacobian $P^0_{C/B}$, and* (1) *holds with $P^d_{C/B}$ in place of $\tilde{P}$ for all $d \in \mathbf{Z}$.*

Terminology and Notations. All the varieties are analytic or algebraic over $\mathbf{C}$. We do not distinguish varieties and associated analytic spaces. A variety is called symplectic if it is nonsingular and has a holomorphic, closed everywhere, nondegenerate 2-form. If it is also simply connected and the symplectic 2-form is unique up to a constant factor, it is called irreducible symplectic. A symplectic variety X of dimension $2n$ is called completely integrable if there exists a proper surjective holomorphic map $f : X \longrightarrow B$ with connected fibers onto a normal variety B of dimension n such that its generic fiber is Lagrangian. Such a map f is called a Lagrangian fibration. In the case where X is projective, we demand that f and B also be projective.

For a map $f : X \longrightarrow Y$, a point $x \in X$ is critical if x or $f(x)$ is singular, or if $x, f(x)$ are nonsingular and $\mathrm{rk}\, d_x f < \min\{\dim X, \dim Y\}$. A point $y \in Y$ is a critical value of f if there exists a critical $x \in f^{-1}(y)$. A fiber $f^{-1}(y)$ is called multiple if all its points are critical. A map f is nonsingular if it has no critical points. We will denote by X_{nc} the set of noncritical points of f, $Y_{\mathrm{nm}} = f(X_{\mathrm{nc}})$ the set of points $y \in Y$ such that the fiber $f^{-1}(y)$ is not multiple, Y_{nc} the set of noncritical values of f, $X_{\mathrm{ns}} = f^{-1}(Y_{\mathrm{nc}}))$ the nonsingular locus of f, $f_{\mathrm{nc}} : X_{\mathrm{nc}} \longrightarrow Y_{\mathrm{nm}}$ and $f_{\mathrm{ns}} : X_{\mathrm{ns}} \longrightarrow Y_{\mathrm{nc}}$ the restrictions of f. For another map $g : U \longrightarrow Y$, we denote by $f_U : X_U \longrightarrow U$ the base change of f; here $X_U = X \times_Y U$. For a nonsingular map $f : X \longrightarrow Y$, we denote $\Omega^p_{X/Y} = \Omega^p_X / f^*\Omega^1_Y \wedge \Omega^{p-1}_X$ the vector bundle (=locally free sheaf) of relative p-forms on X, and $\mathcal{T}_{X/Y} = (\Omega^1_{X/Y})^*$ the vertical tangent bundle (i.e., the distribution of tangent planes of the fibers of f).

2. Albanese Fibration

We fix for this section a completely integrable projective symplectic variety of dimension $2n$ and a Lagrangian fibration $f : X \longrightarrow B$. We denote by $\alpha = \alpha_X \in \Gamma(X, \Omega^2_X)$ a symplectic form on X. The following is standard.

Proposition 2.1. (i) *There is a canonical nondegenerate coupling $\hat{\alpha} : f^*_{\mathrm{ns}}\mathcal{T}_{B_{\mathrm{nm}}} \otimes \mathcal{T}_{X_{\mathrm{ns}}/B_{\mathrm{nm}}} \longrightarrow \mathcal{O}_{X_{\mathrm{ns}}}$ defined by α, which gives a canonical isomorphism $i_\alpha : f^*_{\mathrm{ns}}\Omega^1_{B_{\mathrm{nm}}} \longrightarrow \mathcal{T}_{X_{\mathrm{ns}}/B_{\mathrm{nm}}}$.*

(ii) *The connected components of fibers of f_{ns} are analytically isomorphic to the quotients $\mathbf{C}^{2n}/L$, where L is a lattice of rank $\leq 4n$ in $\mathbf{C}^{2n}$. In the algebraic situation, if f is projective, then each connected component A_i of a fiber $A = f^{-1}_{\mathrm{ns}}(b)$ $(b \in B)$ is quasi-projective and can be represented as an isotrivial fiber bundle over an abelian variety A' with $\mathbf{G}^q_a \times \mathbf{G}^p_m$ as a fiber (p, q and A' depend only on b).*

(iii) *There is a canonical way to associate with f a nonsingular surjective holomorphic map $A(f) : A(X) \longrightarrow B_{\mathrm{nm}}$ which is a family of connected complex commutative Lie groups with a cross section $e : B_{\mathrm{nm}} \longrightarrow A(X)$ of neutral elements, together with an action $\Phi : A(X) \times X_{\mathrm{ns}} \longrightarrow X_{\mathrm{ns}}$ making each connected component A_i of a fiber $A = f^{-1}_{\mathrm{ns}}(b)$ $(b \in B)$ a principal homogeneous space under $A_b = A(f)^{-1}(b)$. Moreover, if B, X are algebraic and f is projective, then there is a maximal Zariski open subset $B_{\mathrm{qp}} \subset B_{\mathrm{nm}}$ such that $\mathrm{codim}_{B_{\mathrm{nm}}}(B_{\mathrm{nm}} \setminus B_{\mathrm{qp}}) \geq 2$ over which $A(f)$ is a quasiprojective family of commutative algebraic groups and the action Φ is algebraic.*

Proof. The proofs of (i) and of the first part of (ii) are the same as in the C^∞ symplectic geometry; (ii) is a holomorphic (algebraic) analogue of the Liouville–Arnold Theorem [4]. A' is defined in the algebraic category as the Albanese variety of the irreducible components of $f^{-1}(b)$. We should check that it does not depend on the choice of the component; we will first do so in the analytic category.

Let $s_1, s_2 : U \longrightarrow X_{\mathrm{ns}}$ be two local analytic sections of f over a neighborhood U of b, meeting two connected components A_1, A_2 of $f^{-1}(b)$ at points p_1, p_2 respectively. Any holomorphic frame $\xi_1, \ldots, \xi_n$ of Ω^1_X over U can be lifted to that of $\mathcal{T}_{X_{\mathrm{ns}}/B_{\mathrm{nm}}}$ via i_α; denote the latter by $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$. Choose ξ_k closed; then the fact that α is closed implies that $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$ commute pairwise, so their flows define the fiber exponential map $\exp : \Omega^1_U \longrightarrow \mathrm{Aut}(X_{\mathrm{ns}\,U}/U)$. The exponential induces by restriction homogeneous actions of the cotangent space $T^*_b X = \Omega^1_X(b)$ on A_1, A_2. Let L_1, L_2 be the kernels of these actions. It suffices to see that $L_1 = L_2$. Let $l \in L_1$; then $\exp l(p_1) = p_1$, and by the inverse function theorem, l extends to a local section λ of Ω^1_X such that $\exp \lambda(s_1) = s_1$. But on a dense open part of U the fibers of f are compact complex tori, so $\exp \lambda$ is identity on this part, and hence, by continuity, everywhere. Hence $L_1 \subset L_2$. By symmetry, $L_1 = L_2$.

Thus, we have a holomorphic automorphism between A_1, A_2 sending p_1 to p_2. It can be extended to the holomorphic translation map $t_{s_2 - s_1} : X_{1,U} \longrightarrow X_{2,U}$, where $X_{i,U}$ denotes the open subset of $X_{\mathrm{ns}\,U}$ formed by the connected components of the fibers $f^{-1}_{\mathrm{ns}}(b)$ $(b \in B)$ containing $s_i(b)$. This map is algebraic if we are in the algebraic category and s_i are local cross sections in the étale topology. Indeed, by the GAGA principle over B_{nc}, which is the projective locus of f_{ns}, $t_{s_2 - s_1}$ is a rational map, regular over B_{nc}; the regularity on $X_{1,U}$ follows from the rationality and from the regularity in the analytic category. Hence $A_1 \simeq A_2$ in the algebraic category as well.

To prove (iii), note that the above proof of (ii) gives a local system of lattices $\mathcal{L}$ in the cotangent bundle $\Omega^1_{B_{\mathrm{nm}}}$; the fiber $\mathcal{L}_b$ is the kernel of the action of $\Omega^1_B(b)$ on $f^{-1}_{\mathrm{ns}}(b)$ by the exponential map. Define $A(X)$ to be the quotient $\Omega^1_{B_{\mathrm{nm}}}/\mathcal{L}$, and $A(f)$ the natural projection. In the algebraic category, $A(X)$ is defined locally in the étale topology over B_{nm} as a quotient by a quasi-finite algebraic equivalence relation: Let $s : U \longrightarrow X_U$ be a local section of f over an étale open $\pi : U \longrightarrow V \subset B_{\mathrm{nm}}$, and $X^\#_U$ the union of connected components of $s(u)$ in $f^{-1}_U(u)$ over all $u \in U$. Then $A(X)_V = X^\#_U/R$, where R is the equivalence relation defined by

$$R = \left\{ ((x_1, u_1), (x_2, u_2)) \in X^\#_U \times_V X^\#_U \mid \pi(u_1) = \pi(u_2),\ t_{x_1 - s(u_1)} = t_{x_2 - s(u_2)} \right\},$$

where $t_{x_i - s(u_i)}$ denotes the rational translation map on $X_{\mathrm{ns},\,\pi(u_i)}$. By [23], there exists a Zariski open $V' \subset V$, over which the quotient by R is quasi-projective. Then B_{qp} is the union of the V' taken over all V as above. □

Definition 2.2. The map $A(f) : A(X) \longrightarrow B_{\mathrm{nm}}$ is called the Albanese family of f, and $A(f)_{\mathrm{qp}} : A(X)_{\mathrm{qp}} \longrightarrow B_{\mathrm{qp}}$ its quasi-projective part.

Proposition 2.3. *The Albanese family $A(f) : A(X) \longrightarrow B_{\mathrm{nm}}$ is analytically isomorphic to the quotient of the cotangent bundle $\Omega^1_{B_{\mathrm{nm}}}$ by a local system of lattices $\mathcal{L} \subset \Omega^1_{B_{\mathrm{nm}}}$, and the natural symplectic structure on the cotangent bundle descends to $A(X)$ in such a way that all the fibers of $A(f)$ are Lagrangian.*

Proof. Let $u_1, \ldots, u_n$ be local analytic coordinates on an open $U \subset B_{\mathrm{nm}}$, and $z_1, \ldots, z_n$ the coordinates on $\Omega^1_{B_{\mathrm{nm}}}$, dual to $du_1, \ldots, du_n$. Any local analytic section s of f_{ns} over U provides the identification of $A(X)_U$ with the open subset $X^\#_U$ of X_U, defined as in the proof of (iii) of Proposition 2.1, such that s is transformed into the section of neutral elements e. With this identification, $u_1, \ldots, u_n, z_1, \ldots, z_n$ are local analytic coordinates on $X^\#_U$, and the restriction of the symplectic form α_X to $X^\#_U$ can be written in the form

$$\alpha_X = \sum_p du_p \wedge dz_p + \beta, \qquad \beta = \sum_{p,q} \beta_{pq}(u, z) du_p \wedge du_q. \tag{1}$$

There are no terms of the form $dz_p \wedge dz_q$, as the fibers of $f^\# = f|_{X^\#_U}$ are Lagrangian. The condition that α_X descends from Ω^1_U to the quotient by $\mathcal{L}$ can be written down as follows:

$$\beta(u, z + \gamma) - \beta(u, z) = d\left(\sum_p \gamma_p du_p\right), \tag{2}$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ is any local section of $\mathcal{L}$. Let $\gamma^{(1)}, \dots, \gamma^{(2n)}$ be a local basis of $\mathcal{L}$ in the neighborhood of a noncritical value of f, and $\beta_0(u, z)$ a $\mathbf{R}$-linear in z solution of (2) of the form

$$\beta_0\left(u, \sum \lambda_j \gamma^{(j)}\right) = \beta(u, 0) + d\left(\sum_p \sum_j \lambda_j \gamma_p^{(j)} du_p\right), \quad \lambda_j \in \mathbf{R}, \ \ j = 1, \dots, 2n.$$

Then the coefficients of $du_p \wedge du_q$ in $\mu(u,z) = \beta(u,z) - \beta_0(u,z)$ are fiberwise harmonic and periodic with respect to a lattice of rank $2n$; hence they are constant, hence they are zero. So $\beta = \beta_0$, and we have $\beta(u, z+\gamma) = \beta(u,z) \ \forall \ \gamma \in \mathcal{L}$. This implies that the coefficients $\beta_{pq}(z)$ do not depend on z. By continuity, the representation (1) with $\beta_{pq}(u,z) = \beta_{pq}(u)$ extends to the critical values as well. Hence the first term of α_X,

$$\alpha_0 = \sum_p du_p \wedge dz_p,$$

is also invariant under $\mathcal{L}$, and hence it descends to the quotient $A(X) = \Omega^1_{B_{\mathrm{nm}}}/\mathcal{L}$. $\square$

Remark 2.4. Our construction of $A(X)$ shows that B_{qp} contains all the points of B_{nm} over which the fibers of f are of dimension n and do not have multiple components. Indeed, for such a point b, we can find a sufficiently ample subvariety $\tilde{B}$ of dimension n in X which meets $f^{-1}(b)$ transversely; then a neighborhood of $A(f)^{-1}(b)$ can be represented as a quotient of the projective family $X_{\tilde{B}}$ by the action of $G = \mathrm{Gal}(\tilde{B}/B)$ associated with a cocycle $\sigma \in H^1(G, X_{\tilde{B}})$.

Remark 2.5. It is very plausible that $\mathrm{codim}_B B_{\mathrm{nm}} \geq 2$. In this case, we have $\mathrm{codim}_B B_{\mathrm{qp}} \geq 2$. The analogous statement for fibrations of elliptic curves whose total space has trivial canonical class can be proved by using Kodaira's description of degenerate fibers.

However, if we admit multiple fibers along a divisor C in B, we can specialize at the generic point η of C and use a finite cyclic ramified base change to get a new family $\tilde{X}_{\tilde{\eta}} \longrightarrow \tilde{B}_{\tilde{\eta}}$ with a local section; here $\tilde{\eta}$ denotes the point of the base change above η, and $\mathrm{Gal}(\tilde{B}_{\tilde{\eta}}/B_{\tilde{\eta}}) = \boldsymbol{\mu}_r$ is cyclic of order r. It has a birational model (constructed, for example, following [22]) such that the nonsingular locus of the irreducible component of the central fiber containing the image of the local section has a group structure; denote the resulting group family $Y_{\tilde{\eta}} \longrightarrow \tilde{B}_{\tilde{\eta}}$. Then the birational class of the original family is represented in the local Shafarevich–Tate group by a cocycle $\sigma \in H^1(\mathrm{Gal}(\tilde{B}_{\tilde{\eta}}/B_{\tilde{\eta}}), Y_{\tilde{\eta}})$ which specializes to a group homomorphism $\sigma_0 : \boldsymbol{\mu}_r \longrightarrow Y_0$, where Y_0 is the group central fiber of the compactification chosen. Quotienting by the trivial cocycle, we will get the Albanese family of the original map f, defined at the generic point of C. As concerns the symplectic structure, it lifts to $\tilde{X}_{\tilde{\eta}}$ and hence to $Y_{\tilde{\eta}}$, but does not descend to the Albanese family over B via the quotient by the trivial cocycle, as the quotient map is ramified.

This argument also shows that any connected component X_0 of the nonsingular locus of a generic multiple fiber f over C has a group structure; it is the quotient of A_0 by the image of σ_0. Indeed, since X is nonsingular and the canonical class of X is trivial, the map from $Y_{\tilde{\eta}}$ to X_η is nonramified and does not contract anything, so the birational equivalence $Y_0/\mathrm{im}(\sigma_0) \sim X(\eta)$ is in fact an isomorphism.

The inverse operation of constructing X from $A = A(X)$ consists of two steps: quotienting A by a cocycle $\sigma \in H^1_{\text{ét}}(B_{\mathrm{nm}}, A)$, and compactifying the quotient in a minimal way, say, with trivial canonical class. (In the analytic category, one should use the Čech cohomology $\check{H}^1(B_{\mathrm{nm}}, A)$.) For a family of commutative complex Lie or algebraic groups $h : A \longrightarrow B_{\mathrm{nm}}$, denote by $h_\sigma : A_\sigma \longrightarrow B_{\mathrm{nm}}$ (or $h(\sigma) : A(\sigma) \longrightarrow B_{\mathrm{nm}}$) the quotient by the cocycle σ. Suppose that a family h admits a symplectic structure α_0 induced from that of the cotangent bundle of B as in Proposition 2.3. Then the cohomology class $\sigma^*(\alpha_0) \in H^1_{\text{ét}}(B_{\mathrm{nm}}, \Omega^2) = H^1(B_{\mathrm{nm}}, \Omega^2) = \check{H}^1_{\mathrm{an}}(B_{\mathrm{nm}}, \Omega^2)$ is well defined. Say, in $\check{H}^1_{\mathrm{an}}$, σ is represented by a cocycle $c_{ij} \in$ Hol. Sections$(U_i \cap U_j, A)$, and $\sigma^*(\alpha_0)$ by $c^*_{ij}(\alpha_0)$, where $\{U_i\}$ is an open covering of B_{nm} in the strong topology.

Proposition 2.6. *Let B be a nonsingular analytic (resp. projective algebraic) variety of dimension n, $h : A \longrightarrow B$ a flat analytic (resp. algebraic) family of commutative complex Lie (resp. algebraic) groups whose generic fiber is a compact complex torus of dimension n, and $\sigma \in \check{H}^1_{\mathrm{an}}(B, A)$. Then the twisted family $h_\sigma : A_\sigma \longrightarrow B$ is Lagrangian with respect to some symplectic structure if and only if the following two conditions hold:*

(i) *A has a symplectic structure α_0 such that h is Lagrangian with respect to α_0, and*

(ii) *$\sigma^*(\alpha_0) \in H^1(B, \Omega^2)$ lifts to the zero cohomology class in $H^1(B, \Omega^2_{\mathrm{closed}})$.*

Proof. We give a proof for the analytic category; the translation into the language of the étale cohomology in the algebraic category is an easy exercise.

Sufficiency: Assuming (i), (ii), take an open covering $\{U_i\}$ as above and a cocycle c_{ij} representing σ. Then by (ii), there exist $\beta_i \in \Gamma(U_i, d\Omega^1)$ such that $c^*_{ij}(\alpha_0) = \beta_j - \beta_i$. The 2-forms $\alpha_i = \alpha_0 + h^*_\sigma \beta_i$ give a well-defined global symplectic form on A_σ with the required properties.

Necessity was shown in the proof of Proposition 2.3. □

Remark 2.7. (ii) of the proposition is equivalent to the following condition:

(ii)′ *$\sigma^*(\alpha_0) \in H^1(B, \Omega^2)$ lifts to an element of $H^1(B, \Omega^1)$ via the de Rham differential $d : \Omega^1 \longrightarrow \Omega^2$.*

Theorem 2.8. *Let X, B be nonsingular projective varieties of dimensions $2n, n$ respectively such that $H^1(B, \Omega^2) = 0$, and $f : X \longrightarrow B$ be a flat surjective morphism with reduced irreducible fibers whose generic fiber is an abelian variety. Then X is completely integrable symplectic with moment map f if and only if there exists a flat quasi-projective family of commutative algebraic groups $h : A \longrightarrow B$ whose generic fiber is an abelian variety of dimension n such that the following two conditions hold:*

(a) *A has a symplectic structure α_0 such that h is Lagrangian with respect to α_0, and*

(b) *$f : X \longrightarrow B$ is a compactification of the family $h_\sigma : A_\sigma \longrightarrow B$ associated with a class $\sigma \in \check{H}^1_{\mathrm{an}}(B, A)$.*

In this case $A = A(X), h = A(f)$.

Proof. Necessity follows by Proposition 2.3. By Remark 2.4, the hypotheses of the theorem imply that $B_{\mathrm{qp}} = B$. For sufficiency, note that the condition $H^1(B, \Omega^2) = 0$ implies that $\sigma^*(\alpha_0) = 0$, hence, as in the proof of Proposition 2.6, the forms $\alpha_0 + h^*_\sigma \beta_i$ give a global nondegenerate 2-form α on A_σ vanishing when restricted to the generic fiber of h_σ. By (b) and the assumption that the fibers are reduced irreducible, $\mathrm{codim}_X X \setminus A_\sigma \geq 2$, so α extends to a nondegenerate 2-form on X. It is closed by the projectivity of X. □

3. Criterion for the Nonsingularity of the Relative Hilbert Scheme

Let $\phi : X \longrightarrow S$ be a flat projective morphism whose fibers are reduced irreducible curves of arithmetic genus p. The relative Hilbert scheme $\mathrm{Hilb}^n_{X/S}$ parametrizing zero-dimensional subschemes of length n in the fibers of X over S is closely related to the Altman–Kleiman relative compactified Jacobian $P_n = P_n(X/S)$, which is a projective scheme over S representing the étale sheaf associated with the following compactified Picard functor on the S-schemes T:

$$\underline{\mathrm{Pic}}^{-}_{X/S\,(n)} = \left\{ \begin{array}{l} \text{the isomorphism classes of } T\text{-flat coherent sheafs } \mathcal{I} \\ \text{on } X \times_S T, \text{ such that for all } t \in T, \text{ the fiber} \\ \mathcal{I}(t) \text{ is torsion-free, of rank 1, and of Euler number} \\ \chi(\mathcal{I}(t)) = n \end{array} \right\}.$$

(see [14, 24, 1–3] for the construction and general properties of P^n; the idea of this compactification is due to [20]).

We will also use the notation $P^n = P^n(X/S) = P_{1-p-n}(X/S)$. The scheme P^n contains the open subscheme $\mathrm{Pic}^n_{X/S}$ parametrizing the isomorphism classes of inversible sheaves of degree n (or linear equivalence classes of divisors of degree n) on the fibers of X/S. The map sending every subscheme of a fiber $X(s)$ to the isomorphism class of its ideal sheaf is well defined as a morphism of schemes

$$\mathcal{A}^n = \mathcal{A}^n_{X/S} : \mathrm{Hilb}^n_{X/S} \longrightarrow P^n.$$

It is called the Abel–Jacobi map.

Theorem 3.1 (D'Souza–Rego–Altman–Kleiman). *The following conditions on the Abel–Jacobi map $\mathcal{A}^n$ are equivalent:*

(1) $n \geq 2p-1$ *and all the fibers* $X(s)$ *are Gorenstein;*
(2) $\mathcal{A}^n$ *is smooth of relative dimension* $n-p$*;*
(3) *all the fibers of* $\mathcal{A}^n$ *are of the same dimension.*

One concludes from this theorem that the compactified Jacobian P^{2p-1} is a nonsingular variety if and only if $\mathrm{Hilb}^{2p-1}_{X/S}$ is. As all the schemes P^n are locally isomorphic in the étale topology, this condition is also that of the nonsingularity of all the P^n. Now, we are going to formulate the criterion of the nonsingularity of $\mathrm{Hilb}^n_{X/S}$. We eliminate the case where S is singular, because then the Hilbert scheme is singular along certain fibers. We also eliminate the case where the fibers $X(s)$ have singularities of embedding codimension > 1: in this case P^n, as well as $\mathrm{Hilb}^n_{X/S}$, is not flat over S and the fibers of P^n have irreducible components which are not in the closure of invertible sheaves. Thus, we assume that S is nonsingular and that the singularities of $X(s)$ are plane (of embedding codimension 1). To formulate the nonsingularity criterion, we have to introduce some notations.

Let $H = \mathrm{Hilb}^n_{X/S}$, $Z \in H$, $Z \subset X_0$, $0 \in S$. We have a decomposition in the sum of irreducible components:

$$Z = Z_1 \coprod \dots \coprod Z_r, \quad \mathrm{Supp}\, Z_i = \{z_i\},\ i = 1, \dots, r.$$

Let $\{f_i(x_i, y_i, s_1, \dots s_m) = 0\}$ be a local (analytic or formal) equation of X at z_i, $s_1, \dots s_m$ being local parameters of S in the neighborhood of 0 and x_i, y_i those on the fiber $X(0)$ in the neighborhood of $z_i \in \mathrm{Supp}\, Z$. Let $I_i = I_Z A_i$, where $A_i = \mathbf{C}[[x_i, y_i]]$. Then there exists a resolution of I_i of the following form (see [17, Exercise 7, p. 148]):

$$0 \longrightarrow A_i^{\oplus(s-1)} \xrightarrow{u} A_i^{\oplus(s)} \xrightarrow{v} I_i \longrightarrow 0, \tag{3}$$

where $v = \Lambda^{s-1}u$, i.e., v is the vector with s components $(v_1, \dots, v_s)$ which are the minors of order $s-1$ of the matrix of size $(s-1) \times s$ defining the map u. The condition that $Z_i \subset X_0$ is equivalent to $f_{i0} := f_i(x_i, y_i, 0, \dots, 0) \in I_i$. Since I_i is generated by the minors of the matrix u, the last condition is equivalent to the existence of a representation of f_{i0} in the form of the determinant of the extended matrix:

$$f_{i0} = \det \tilde{u}, \quad \tilde{u} = \begin{pmatrix} & u & \\ \cdots & \cdots & \cdots \\ u_{s1} & \dots & u_{ss} \end{pmatrix}. \tag{4}$$

Define the ideal $\tilde{I}_i = (\Lambda^{s-1}\tilde{u})$ generated by the minors of order $s-1$ of the extended matrix $\tilde{u}$. Since the minors of u form a subset of those of $\tilde{u}$, we have the inclusion $I_i \subset \tilde{I}_i$. Define the map

$$\phi_i : T_0 S \to A_i/\tilde{I}_i,$$
$$\frac{\partial}{\partial s_k} \mapsto \frac{\partial f_i}{\partial s_k} \bmod \tilde{I}_i,$$

where $T_0 S$ denotes the tangent space of S at 0.

Theorem 3.2. *The scheme H is nonsingular at Z (that is, smooth over $\mathbf{C}$) if and only if the map*

$$\phi = \bigoplus_{i=1}^{r} \phi_i : T_0 S \longrightarrow \bigoplus_{i=1}^{r} A_i/\tilde{I}_i \tag{5}$$

is surjective.

Corollary 3.3. *The condition of the surjectivity of the map ϕ in Theorem 3.2 is a necessary and sufficient condition for the nonsingularity of the relative compactified Jacobian at all the points representing the isomorphism classes of the sheaves $I_Z \otimes \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $X(0)$, and I_Z is the ideal sheaf of the subscheme $Z \subset X(0)$.*

The following corollary gives a *sufficient* condition for the nonsingularity of H.

Corollary 3.4. *If the fiber $X(0)$ has only one singular point z, and if the deformation $\{X(s)\}_{s\in S}$ of $X(0)$ dominates the versal deformation of the singular point $z \in X(0)$, then the Hilbert scheme $\mathrm{Hilb}^n_{X/S}$ is nonsingular for all n in all the points Z with $\mathrm{Supp}\, Z = \{z\}$.*

Proof of Theorem 3.2. We will use the following criterion of the nonsingularity.

Criterion for Nonsingularity. *Let H be a scheme of finite type over $\mathbf{C}$, $h \in H$ a closed point. Let for $k \in \mathbf{Z}$, $k \geq 0$, $T_k = \mathrm{Spec}\mathbf{C}[\varepsilon]/(\varepsilon^k)$ for $k \in \mathbf{Z}$, $k \geq 1$, and $t \in T_k$ be the closed point. Then the scheme H is smooth at h if and only if for every $k \geq 1$, every morphism $\gamma_k : T_k \longrightarrow H$ such that $\gamma_k(t) = h$ extends to a morphism $\gamma_{k+1} : T_{k+1} \longrightarrow H$.*

For a proof, see [19, Sec. 2].

By the above criterion, the nonsingularity of H at Z is equivalent to the possibility of extending over T_{k+1} simultaneously for all Z_i the data (3), (4), in assuming they are already extended to T_k; this is done by induction degree by degree. The flatness of the deformation of A_i/I_i obtained in this way is guaranteed, e.g., by [5, Proposition 31]. Denote by $s_{1k}, \ldots s_{mk}, \tilde{u}_k = \tilde{u}_{ik}$ the extensions to T_k (understood as polynomials in ε of degree k), and $f_{ik} = f_i(x_i, y_i, s_{1k}, \ldots, s_{mk})$. Then, to construct the extension to T_{k+1}, we have to define $s_{1,k+1} = s_{1k} + a_1\varepsilon^{k+1}, \ldots, s_{m,k+1} = s_{mk} + a_m\varepsilon^{k+1}$, $\tilde{u}_{i,k+1} = \tilde{u}_{ik} + G_i\varepsilon^{k+1}$ with $a_1, \ldots, a_m \in \mathbf{C}$, $G_i \in \mathrm{Mat}_s(A_i)$ in such a way that $\det \tilde{u}_{i,k+1} \equiv f_i(x_i, y_i, s_{1,k+1}, \ldots, s_{m,k+1}) \bmod(\varepsilon^{k+2})$. This brings us to the equation

$$\sum_j a_j \frac{\partial f_i}{\partial s_j} = \sum_{p,q} g_{pq}\tilde{u}_i^{pq} + \text{ known terms,}$$

where $\tilde{u}_i^{pq}$ denotes the minor of order $s-1$, complementary to the element $(\tilde{u}_i)_{pq}$. This implies the assertion. □

4. Compactified Jacobian P^2

Let B be a nonsingular projective surface, and $\phi : C \longrightarrow B$ a flat surjective projective morphism whose fibers are reduced hyperelliptic curves of arithmetic genus 2. That is, for every $b \in B$, the fiber $C_b = \phi^{-1}(b)$ admits a double covering map $\mu_b : C_b \longrightarrow \mathbf{P}^1$ whose ramification divisor is of degree 6; such a curve can be defined by an affine equation $t^2 = P_6(x)$, where P_6 is a nonzero polynomial of degree ≤ 6. We also suppose that the generic fiber of ϕ is nonsingular. We will search for symplectic 4-dimensional varieties X compactifying the family of (generalized) Jacobians of degree 2 $\{J^2C_b\}_{b\in B}$ in such a way that the fibers are Lagrangian. Generically, the Jacobians J^2C_b can be obtained by the classical procedure: take the symmetric square $C_b^{(2)}$ of C_b and blow down the hyperelliptic linear series $g_1^2 = \mathbf{P}^1$, formed by pairs $((x,t),(x,-t)) \in C_b^{(2)}$. When extended to singular fibers, this construction gives a singular surface. The nonsingular part of the symmetric square $C^{(2)}$ for a reduced curve C is the symmetric square of the nonsingular part C^0 of C. The Jacobian J^dC of degree d is defined as the quotient of the group of divisors of degree d of C^0 by the linear equivalence relation on C: $D_1 \sim D_2 \Leftrightarrow D_1 - D_2 = (f)$ for a rational function f on C, regular and regularly invertible at all the singular points of C. By [25], if C is irreducible, then the generic class of $J^\pi C$, where π is the virtual genus of C, is represented in a unique way as a positive divisor $p_1 + \ldots + p_\pi$ with $p_i \in C^0$, so $C^{(\pi)}$ is birational to $J^\pi C$. This is no longer true for reducible curves C. However, it is a natural question to search for nonsingular compactifications of the family of Jacobians $\{J^2C_b\}_{b\in B}$ in resolving singularities of $C_b^{(\pi)}$. This approach gives a minimal compactification (that is, with trivial canonical class) in the case where $\pi = 2$, as the following theorem shows.

Theorem 4.1. *Let $\phi : \mathcal{C} \longrightarrow B$ be as above, $\iota : \mathcal{C} \longrightarrow \mathcal{C}$ the hyperelliptic involution given by $(x,t) \mapsto (x,-t)$ in the affine coordinates, $\mathcal{C}_B^{(2)} = \mathcal{C} \times_B \mathcal{C}/(\text{permutation})$ the symmetric square,*

$$\Delta = \{(p,p) \in \mathcal{C}_B^{(2)}\}, \qquad E = \{(p,\iota(p)) \in \mathcal{C}_B^{(2)}\}$$

the diagonal and antidiagonal respectively, and $\sigma : \mathcal{C}_B^{[2]} \longrightarrow \mathcal{C}_B^{(2)}$ the blow up of (the reduced ideal of) the diagonal in $\mathcal{C}_B^{(2)}$, $\psi = \phi_B^{[2]} : \mathcal{C}_B^{[2]} \longrightarrow B$ the natural projection. Suppose that $\mathcal{C}$ is given by the equation $y^2 = P(x, s_1, s_2)$ in the formal neighborhood of a fiber $C_{b_0} = \phi^{-1}(b_0)$ for a point $b_0 \in B$, where s_1, s_2 are local parameters of B at b_0, and P is a polynomial of degree 6 with coefficients in $\mathbf{C}[[s_1, s_2]]$. Let $Q_i = \{x = x_i, y = 0\}$, $i = 1, \dots, r$, be all the singular points of C_{b_0} (obviously, $r \le 3$). Then we have:

(a) *$\mathcal{C}_B^{[2]}$ is nonsingular in the neighborhood of the fiber $\psi^{-1}(b_0)$ if and only if the following conditions are verified:*

(i) *For any $i = 1, \dots, r$, the partial derivatives $\partial P/\partial s_1, \partial P/\partial s_2$ are linearly independent modulo the ideal $((x - x_i)^2, s_1, s_2)$.*

(ii) *For all $i \ne j$, $i, j = 1, \dots, r$, the matrix*

$$\begin{pmatrix} \partial P(x_i)/\partial s_1 & \partial P(x_i)/\partial s_2 \\ \partial P(x_j)/\partial s_1 & \partial P(x_j)/\partial s_2 \end{pmatrix}$$

is nondegenerate at $s_1 = s_2 = 0$.

(b) *If the conditions (i), (ii) of (a) are verified for all $b_0 \in B$, then the proper transform E' of the antidiagonal E in $\mathcal{C}_B^{[2]}$ represents a family of nonsingular rational (-1)-curves in the nonsingular loci of the fibers of ψ that can be contracted simultaneously by a projective map $\text{cont}_{E'/B} : \mathcal{C}_B^{[2]} \longrightarrow \tilde{P}$ to give a nonsingular projective variety $\tilde{P}$ with trivial canonical sheaf $\omega_{\tilde{P}/B}$.*

Remark 4.2. Condition (i) of (a) means that locally $\mathcal{C}$ can be given by the equation $y^2 = x^k + s_1x + s_2 + p(s_1, s_2, x)$ for an appropriate choice of local analytic parameters x, y of C_{b_0} at Q_i and s_1, s_2 of B at b_0; p is a polynomial in x with coefficients in $\mathbf{C}[[s_1, s_2]]$ of degree $\le k - 2$ such that $p(s_1, s_2, x) \equiv 0 \bmod (s_1, s_2)(x^2)$. Condition (ii) means that the reduced tangent cones at b_0 of the germs of discriminant curves $\Delta_i, \Delta_j \subset B$ of the unfoldings of the singular points Q_i, Q_j are transversal. Conditions (i) and (ii) represent a weakened form of the mild degeneration condition of [19]. The latter imposes the additional restriction that $k \le 3$.

Proof. The nonsingularity of the blow up of Δ follows from Theorem 3.2 and from the fact that whenever the singularities of fibers of a flat family of curves are plane, the relative Hilbert scheme Hilb^2 is identified with the blow up of the diagonal (see, e.g., [16]). It remains to verify that the proper transform E' of E in $\mathcal{C}_B^{[2]}$ is a family of nonsingular rational curves contained in the nonsingular locus of fibers of $\mathcal{C}_B^{[2]}$ over B.

Choosing local analytic parameters s_1, s_2 on B and x_1, y_1, x_2, y_2 on two copies of $\mathcal{C}$ as in Remark 4.2, we can consider

$$\begin{gathered} s_1, \quad s_2, \quad z_1 = x_1 + x_2, \quad z_2 = y_1 + y_2, \quad z_3 = (x_1 - x_2)^2, \\ z_4 = (x_1 - x_2)(y_1 - y_2), \quad z_5 = (y_1 - y_2)^2 \end{gathered} \tag{6}$$

as local parameters on $\mathcal{C}_B^{(2)}$. We have, in addition to (6), two equations defining the two copies of $\mathcal{C}$:

$$y_i^2 = x_i^k + s_1x_i + s_2 + p(s_1, s_2, x_i), \quad i = 1, 2. \tag{7}$$

The case $k \le 3$ has been treated in [19], so we restrict ourselves to $k = 4, 5, 6$. The elimination of x_i, y_i from (6), (7) gives the local equations for $\mathcal{C}_B^{(2)}$. We will make explicit the equations of the fiber $\mathcal{C}_B^{(2)}(b_0)$ in terms of parameters $z_1, \dots, z_5$:

$$
\begin{aligned}
k=4: \quad & z_4^2 - z_3 z_5 = 0, \ \frac{1}{4} z_1^4 - z_2^2 + \frac{3}{2} z_1^2 z_3 + \frac{1}{4} z_3^2 - z_5 = 0, \\
& \frac{1}{2} z_1^3 z_4 + \frac{1}{2} z_1 z_3 z_4 - z_2 z_5 = 0, \\
& \frac{1}{2} z_1^3 z_3 + \frac{1}{2} z_1 z_3^2 - z_2 z_4 = 0; \\
k=5: \quad & z_4^2 - z_3 z_5 = 0, \ \frac{1}{8} z_1^5 - z_2^2 + \frac{5}{4} z_1^3 z_3 + \frac{5}{8} z_1 z_3^2 - z_5 = 0, \\
& \frac{5}{16} z_1^4 z_4 + \frac{5}{8} z_1^2 z_3 z_4 + \frac{1}{16} z_3^2 z_4 - z_2 z_5 = 0, \\
& \frac{5}{16} z_1^4 z_3 + \frac{5}{8} z_1^2 z_3^2 + \frac{1}{16} z_3^3 - z_2 z_4 = 0; \\
k=6: \quad & z_4^2 - z_3 z_5 = 0, \ \frac{1}{16} z_1^6 - z_2^2 + \frac{15}{16} z_1^4 z_3 + \frac{15}{16} z_1^2 z_3^2 + \frac{1}{16} z_3^3 - z_5 = 0, \\
& \frac{3}{16} z_1^5 z_4 + \frac{5}{8} z_1^3 z_3 z_4 + \frac{3}{16} z_1 z_3^2 z_4 - z_2 z_5 = 0, \\
& \frac{3}{16} z_1^5 z_3 + \frac{5}{8} z_1^3 z_3^2 + \frac{3}{16} z_1 z_3^3 - z_2 z_4 = 0;
\end{aligned} \tag{8}
$$

The diagonal is given by $\Delta = \{z_3 = z_4 = z_5 = 0\}$ in coordinates $s_1, s_2, z_1, \dots, z_5$, and the antidiagonal by $E = \{z_2 = z_3 = z_4 = 0\}$. Let us look, for example, at the case $k = 5$. Restricting the blow up of Δ to the fiber $s_1 = s_2 = 0$, choose a chart, say $\{z_5 \neq 0\}$. Then the equations of $\psi^{-1}(b_0) = \mathcal{C}_B^{[2]}(b_0)$ will be

$$
\begin{aligned}
& z_4^2 - z_3 = 0, \ \frac{1}{8} z_1^5 - z_2^2 + \frac{5}{4} z_1^3 z_3 z_5 + \frac{5}{8} z_1 z_3^2 z_5^2 - z_5 = 0, \\
& \frac{5}{16} z_1^4 z_4 + \frac{5}{8} z_1^2 z_3 z_4 z_5 + \frac{1}{16} z_3^2 z_4 z_5^2 - z_2 = 0, \\
& \frac{5}{16} z_1^4 z_3 + \frac{5}{8} z_1^2 z_3^2 z_5 + \frac{1}{16} z_3^3 z_5^2 - z_2 z_4 = 0.
\end{aligned} \tag{9}
$$

The first and third equations allow us to eliminate z_3, z_2, and the fourth one becomes tautological. Upon elimination, (9) defines a surface given by one equation in local parameters z_1, z_4, z_5. In terms of these parameters, E' is given by $z_4 = 0$ ($z_2 = z_3 = 0$ being tautological), and $\psi^{-1}(b_0)$ is given by

$$
\frac{1}{8} z_1^5 - \left(\frac{5}{16} z_1^4 + \frac{5}{8} z_1^2 z_4^2 z_5 + \frac{1}{16} z_4 z_5^2 \right)^2 z_4^2 + \frac{5}{4} z_1^3 z_4^2 z_5 + \frac{5}{8} z_1 z_4^2 z_5^2 - z_5 = 0. \tag{10}
$$

One sees immediately that on E', that is, when $z_4 = 0$, the partial derivative of the l.h.s. of the second equation with respect to z_5 is -1, so the Jacobian of (10) is nonzero along E', and hence E' does not meet $\operatorname{Sing} \psi^{-1}(b_0)$. The intersection $E' \cap \psi^{-1}(b_0)$ is given by $s_1 = z_4 = \frac{1}{8} z_1^5 - z_5 = 0$. A similar verification in the other charts shows that $E' \cap \psi^{-1}(b_0)$ is nonsingular and irreducible.

Condition (ii) of (a) assures the nonsingularity of $\mathcal{C}_B^{(2)}$ at a point (Q_i, Q_j) with $i \neq j$; such a point is not on $E \cup \Delta$, so the blow up of Δ does not change anything in its neighborhood, and locally $\mathcal{C}_B^{(2)} \simeq \mathcal{C}_B^{[2]}$. The rest of the proof goes exactly as in loc. cit., Sec. 3: E' over B is a smooth family of nonsingular rational curves of self-intersection -1, and hence can be contracted simultaneously by the relative version of the Castelnuovo contraction theorem [26]. The differential

$$
\bar{\nu} = (x_1 - x_2) \frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2} \wedge ds_1 \wedge ds_2
$$

on $\mathcal{C}\times_B\mathcal{C}$ is invariant under the permutation $\sigma:(x_1,y_1)\mapsto(x_2,y_2)$, and hence descends to the quotient $\mathcal{C}_B^{(2)}=\mathcal{C}\times_B\mathcal{C}/\sigma$. It lifts to $\mathcal{C}_B^{[2]}$, as it is a *small* blow up (that is, without exceptional divisors), and descends to $\tilde{P}$ via the contraction map, because the image of the exceptional locus is of codimension 2.

The other types of singularities are treated in a similar way. □

Corollary 4.3. *If we assume in addition to the hypotheses of Theorem* 4.1(b) *that the curves of the family* $\mathcal{C}$ *are irreducible, then* $\tilde{P}$ *is canonically isomorphic to the Altman–Kleiman compactified Jacobian* P^0 *over* B.

Proof. It is identical to that of (v) of Theorem 4 in [19]. □

Corollary 4.4. *If the fibers of the family* $\mathcal{C}/B$ *are irreducible, then conditions* (i), (ii) *of* (a) *of Theorem* 4.1 *are equivalent to the nonsingularity of* P^d *for any* d.

Proof. As all the P^d are locally isomorphic in the étale topology, it suffices to verify the assertion for one d; take $d=3$. By Theorem 3.1, P^3 is nonsingular if and only if Hilb^3 is. By Theorem 3.2, this is equivalent to the surjectivity of the map (5) for all Z. Conditions (i), (ii) of (a) of Theorem 4.1 are an obvious reformulation of its surjectivity in the cases where $\#(\mathrm{Supp}\,Z\cap\mathrm{Sing}\,C_{b_0})=1$ or 2. This is enough for the nonsingularity of Hilb^2, but for that of Hilb^3 we should add a similar condition with 3 singular points of C_{b_0} in the support of Z. This condition, in fact, is never verified over a base of dimension 2, but fortunately, irreducible curves of arithmetic genus 2 cannot have more than 2 singular points. So, in the case where the fibers are irreducible, (i), (ii) are also equivalent to the nonsingularity of Hilb^3. □

The following proposition shows that the families of hyperelliptic curves arising from $K3$ surfaces of degree 2 in Theorem 1.1 always satisfy the conditions of (a) of Theorem 4.1.

Proposition 4.5. *Let* S *be a surface with a* $(2:1)$*-covering* $\beta:S\longrightarrow\mathbf{P}^2$ *ramified in a (possibly singular) sextic curve* D, *and* $\mathcal{C}$ *the family of hyperelliptic curves of arithmetic genus* 2 $\{\beta^{-1}(l)\}_{l\in\mathbf{P}^{2*}}$, *parametrized by the dual projective plane* $B=P^{2*}$. *Then* $\mathcal{C}$ *satisfies the conditions* (i), (ii) *of Theorem* 4.1 *if and only if* S *(or equivalently,* D*) is nonsingular.*

Proof. For (i), choose affine coordinates x,y in $\mathbf{P}^2$ and s_1,s_2 in $\mathbf{P}^{2*}$ near a singular point of a curve C_{b_0} in such a way that the equation of D is $P_6(x,y)=0$, $\mathcal{C}$ is defined by $t^2-P_6(x,s_1x+s_2)=0$, and $P_6(x,0)=x^k+c_{k+1}x^{k+1}+\ldots+c_6x^6$ ($k\geq 2$ having the same sense as in Remark 4.2). Then the nonsingularity of D in the origin is equivalent to $\partial P_6(0,0)/\partial y=c\neq 0$, i.e., $t^2-P_6(x,s_1x+s_2)=t^2-cs_1x-cs_2+\cdots$. This is equivalent to (i).

Thus, we can now suppose that D is nonsingular, and we will show that in this case (ii) is always verified. We will use the reformulation of (ii) given in Remark 4.2. Singular points on $C_{b_0}=\beta^{-1}(l_0)$ correspond to points of tangency of l_0 to D. So, the discriminant curve parametrizing the points $l\in B$ for which C_l is singular, is the dual curve $D^\vee$ of D, parametrizing all the tangent lines to D. Let l be tangent to D at $Q\in D$. Then the tangent cone to $D^\vee$ is the pencil of lines passing through Q. Hence, whenever there are two distinct points of tangency on l, the corresponding tangent cones are different. □

5. Compactified Jacobian P^1

We discuss here two constructions of a nonsingular compactification P^1 of the relative Jacobian J^1 of a family of hyperelliptic curves of arithmetic genus 2. We impose one more restriction in addition to the assumptions of Sec. 4: the curves of the family must be reduced *irreducible*.

Theorem 5.1. *Let $\phi : \mathcal{C} \longrightarrow B$ be a family of irreducible curves satisfying conditions* (i), (ii) *of Theorem* 4.1. *Then there exists an element $\sigma \in H^1_{\text{ét}}(B, J^0_{\mathcal{C}/B})$ of order 2 such that $J^1_{\mathcal{C}/B}$ is isomorphic to the twisted family $J^0_{\mathcal{C}/B}(\sigma)$. Moreover, the action of $J^0_{\mathcal{C}/B}$ on itself by translations extends to a regular action on the compactification $\tilde{P} = P^0 = P^0_{\mathcal{C}/B}$ of J^0 constructed in Theorem 4.1, so that the twisted family $P^1 = P^1_{\mathcal{C}/B}(\sigma)$ is a nonsingular compactification of $J^1 = J^1_{\mathcal{C}/B}$ with reduced irreducible fibers and* $\operatorname{codim}_{P^1} J^1 = 2$.

Proof. Identify J^2 with J^0 by translation by the hyperelliptic divisor class. Let $\mu : \mathcal{C} \longrightarrow \mathbf{P}$ be the hyperelliptic covering, where $\pi : \mathbf{P} \longrightarrow B$ is a $\mathbf{P}^1$-bundle over B isomorphic to the projectivization of the rank 2 vector bundle $\mathcal{E} := \phi_* \omega_{\mathcal{C}/B} \otimes \mathcal{L}$ with $\mathcal{L}$ invertible. Let $\mathcal{O}(1) = \mathcal{O}_{\mathbf{P}/B}(1)$ be the relative Grothendieck tautological sheaf; if $\mathcal{L}$ is sufficiently ample, $\mathcal{O}(1)$ is very ample. Each irreducible surface S from the linear system $|\mathcal{O}(1)|$ represents a rational cross section of the bundle π; it is regular over the open subset $U \subset B$. If $\Delta \subset \mathbf{P}$ is the ramification divisor of μ, then $\Delta \cap S$ is a very ample curve in Δ, hence, by the finiteness of $\pi|_\Delta$, $\pi(\Delta \cap S)$ is very ample in B. By construction, $B \setminus U \subset \pi(\Delta \cap S)$, so B can be covered by affine open subsets of the form $U_S := U \setminus \pi(\Delta \cap S)$. Let $s : U \longrightarrow \mathbf{P}$ be the cross section defined by S, and $V_S = \mu^{-1}(s(U_S))$. The map $\mu_S : \pi \circ \mu|_{V_S} : V_S \longrightarrow U_S$ is an étale double covering with the following property: the family $\phi_{V_S} : \mathcal{C}_{V_S} \longrightarrow V_S$ obtained from ϕ by the base change admits two natural regular sections $s^{\pm} : V_S \longrightarrow \mathcal{C}_{V_S}$, s^+ being the tautological one, coming from the inclusion $V_S \subset \mathcal{C}$, and $s^- = s^+ \circ \kappa$, where $\kappa : V_S \longrightarrow V_S$ is the involution transposing two points in the fibers of μ_S. Then the cocycle σ is defined on V_S by $\sigma(\kappa) = [s^- - s^+]$, where the brackets stand for the divisor class, so that the r.h.s. is a cross section of $J^0_{\mathcal{C}_{V_S}/V_S}$. The translation by $\sigma(\kappa)$ extends to $\tilde{P}$ by Corollary 4.3; it is then presented on the level of the presentation functor by tensor multiplication by the *invertible* sheaf corresponding to the divisor $s^- - s^+$. To finish the proof, it suffices to note that the Altman–Kleiman compactified Jacobians P^d are locally isomorphic in the étale topology over B. □

Now, we turn to the question whether P^1 is Lagrangian with respect to some symplectic structure. The above construction of J^1, which is an open part of P^1, and Proposition 2.3 imply that if P^1 is Lagrangian over B, then P^0 is. By Theorem 1.1, this implies that $B = \mathbf{P}^2$, hence $H^1(B, \Omega^2) = 0$, and by Theorem 2.8, we conclude the following.

Corollary 5.2. *With the hypotheses of Theorem* 5.1, *P^1 is Lagrangian over B if and only if P^0 is.*

Now we discuss another more explicit construction of P^1 for particular families of curves arising in Theorem 1.1 or Proposition 4.5 from a hyperelliptic $K3$ surface $\beta : S \longrightarrow \mathbf{P}^2$. We will also verify directly that P^1 of such a family is Lagrangian over $\mathbf{P}^{2*}$. Let W be the hypersurface of the triples $(p_1, p_2, p_3) \in S^{(3)}$ whose images $\beta(p_i)$ are collinear. It is defined in affine coordinates by one equation of the form

$$\begin{vmatrix} \xi_1 & \eta_1 & 1 \\ \xi_2 & \eta_2 & 1 \\ \xi_3 & \eta_3 & 1 \end{vmatrix}, \tag{11}$$

where (ξ_i, η_i) are coordinates on affine open charts of three factors $\mathbf{P}^2$, the three factors S being defined by $\{y_i^2 - P_6(\xi_i, \eta_i) = 0\}$, $i = 1, 2, 3$. This equation descends to $S^{(3)}$ because it is symmetric in (ξ_i, η_i), $i = 1, 2, 3$. Let $g : W \dashrightarrow \mathbf{P}^{2*}$ be the natural rational map sending each triple (p_1, p_2, p_3), such that at least two of the three points $\beta(p_1), \beta(p_2), \beta(p_3)$ are distinct, to the line $l = \overline{\beta(p_1)\beta(p_2)\beta(p_3)} \subset \mathbf{P}^2$, and $\rho : W \dashrightarrow \mathcal{C}^{(3)}_B$ the natural map $(p_1,p_2,p_3)\in W\mapsto (p_1,p_2,p_3)\in(\beta^{-1}(l))^{(3)}\subset C_B^{(3)}$, where $\phi: C\longrightarrow B=\mathbf{P}^{2*}$ is the family of curves $\beta^{-1}(l)$. Then ρ is a birational map which identifies g with the natural projection $\phi^{(3)}: C_B^{(3)}\longrightarrow B$.

To write down everything in coordinates, represent C/B in the form $\{y^2=P_6(x,s_1x+s_2)\}$, where (s_1,s_2) are affine coordinates on a chart of B. Then the cartesian cube C_B^3 (resp. S^3) has $(x_1,y_1,x_2,y_2,x_3,y_3,s_1,s_2)$ (resp. $(\xi_i,\eta_i,y_i)_{i=1,2,3}$) as affine coordinates, and ρ,ρ^{-1} are given by

$$\rho:(\xi_i,\eta_i,y_i)_{i=1,2,3}\mapsto y_i=y_i,\quad x_i=\xi_i,\quad s_1=\left[\frac{\eta_j-\eta_i}{\xi_j-\xi_i}\right],\quad s_2=\left[\eta_i-\frac{\eta_j-\eta_i}{\xi_j-\xi_i}\xi_i\right],$$

$$\rho^{-1}:(x_1,y_1,x_2,y_2,x_3,y_3,s_1,s_2)\mapsto \xi_i=x_i,\quad \eta_i=s_1x+s_2,\quad y_i=y_i\quad (i=1,2,3),$$

where $[\dots]$ denotes the symmetrization with respect to the subscripts i,j.

Let α be the symplectic form on $S^{(3)}$ induced by that on S, denoted α_S. Then there is a 1-dimensional foliation in W defined by the kernel of $\alpha|_W$. We have, up to a constant factor,

$$\alpha_S=\frac{d\xi\wedge d\eta}{2y},$$

$$\alpha=\left[\frac{d\xi_i\wedge d\eta_i}{2y_i}\right]=\left[\frac{x_idx_i}{2y_i}\right]\wedge ds_1+\left[\frac{dx_i}{2y_i}\right]\wedge ds_2,$$

so the distribution of kernels of $\alpha|_W$ is given by

$$ds_1=ds_2=\left[\frac{x_idx_i}{2y_i}\right]=\left[\frac{dx_i}{2y_i}\right]=0. \tag{12}$$

By the Abel–Jacobi theory, these are exactly the equations of the tangent distribution of the Abel–Jacobi map $\mathcal{A}: C_B^{(3)}\longrightarrow J^3_{C/B}$. In taking into account that the generic fiber of $\mathcal{A}$ is compact (it is $\mathbf{P}^1$), this implies that $\alpha|_W$ descends to $J^3_{C/B}$.

6. Computation of $\Gamma(S^6\mathcal{T}\otimes\omega^2)$

We will describe a way to compute the groups $\Lambda_{kl}=\Gamma(X,S^k\mathcal{T}_X\otimes\omega_X^l)$ for toric varieties X. Fix the following toric data: an algebraic torus $\mathbf{T}=(\mathbf{C}^*)^n$, its lattice of characters M, that of 1-parametric subgroups $N=M^\vee$, and a fan Σ in $N\otimes\mathbf{R}$ defining a toric variety $X=X_\Sigma$ of dimension n. For more details on toric varieties, see [11]. We have a weight decomposition of Λ_{kl} considered as a representation of $\mathbf{T}$:

$$\Lambda_{kl}=\bigoplus_{m\in M}\Lambda_{kl}(m), \tag{13}$$

where $\Lambda_{kl}(m)$ is the $\mathbf{T}$-semi-invariant subspace on which $\mathbf{T}$ acts by multiplication by the character x^m (we use the monomial notation for characters: if $x=(x_1,\dots,x_n)$ are coordinates on $\mathbf{T}$, then the character associated with $m\in M$ is $x^m=x_1^{m_1}\cdots x_n^{m_n}$). To describe $\Lambda_{kl}(m)$, we start with the well known description of $\Omega^1_{\mathbf{C}[X_\sigma]}=\Gamma(X_\sigma,\Omega^1_{X_\sigma})$ for an affine toric variety, i.e., in the case where the fan consists of a unique cone σ and of all its faces (see loc. cit.). The weight m subspace $\Omega^1_{\mathbf{C}[X_\sigma]}(m)$ is generated by the rational differentials of the form $x^{m-m'}dx^{m'}$ which are regular on X_σ. The regularity condition is verified at generic points of the toric divisors $F_{\sigma_i}\subset X_\sigma$, $i=1,\dots,\kappa$, where $\sigma_1,\dots,\sigma_\kappa$ are all the 1-dimensional cones (rays) of the fan (see loc. cit. for the correspondence between cones of the fan and $\mathbf{T}$-invariant subvarieties of X_Σ). Finally, we have

$$\Omega^1_{\mathbf{C}[X_\sigma]}(m)=\langle x^{m-m'}dx^{m'}\rangle_{m'\in\Pi(m)},$$

where $\Pi(m)$ is the linear span of the minimal face $\tau(m)\prec\tau$ containing m, and τ denotes the dual cone $\sigma^\vee=\{m\in M\otimes\mathbf{R}\mid\langle v,m\rangle\geq 0\ \forall\ v\in\sigma\}$. Note that in the case where X_σ is singular, $\Omega^1_{X_\sigma}$ should be understood as the sheaf of Zariski differentials, i.e., 1-forms which are regular in codimension 1.

Apply the same idea to describe the summands in the r.h.s. of (13) for an affine toric variety X_σ as groups of rational tensors $\xi \in S^k(\Omega^1_{\mathbf{C}(X_\sigma)})^* \otimes (\Omega^n_{\mathbf{C}(X_\sigma)})^{\otimes l}$ which are regular at generic points of the F_{σ_i}. The dual $(\Omega^1_{\mathbf{C}(X_\sigma)})^*$ (as a $\mathbf{C}(X_\sigma)$-vector space) is generated (as a $\mathbf{C}$-vector space) by the rational semi-invariant vector fields $x^m L_\lambda$, where $\lambda \in N, m \in M$, and L_λ denotes the invariant vector field on $\mathbf{T} \subset X_\sigma$ generating the action by translations of the 1-parametric group λ on $\mathbf{T}$. To find out when $x^m L_\lambda \in \Lambda_{1,0} = \Gamma(X_\sigma, \mathcal{T}_{X_\sigma})$, look at the open piece $\mathbf{O}_{\sigma_i} = F_{\sigma_i} \cap X_{\sigma_i}$ of F_{σ_i}, which is the $\mathbf{T}$-orbit associated with σ_i. The open subvariety $X_{\sigma_i} \simeq (\mathbf{C}^*)^{n-1} \times \mathbf{C} \subset X_\sigma$ can be endowed with coordinates $z_1 = x^{m^{(1)}}, \dots, z_n = x^{m^{(n)}}$, where $m^{(1)}, \dots, m^{(n-1)}$ form a basis of the lattice $M \cap \sigma_i^\perp$, and $m^{(n)} \in \mathrm{Int}(\sigma_i^\vee) \cap M$ completes it to a basis of M. Then $\mathbf{O}_{\sigma_i}$ is defined by the equation $z_n = 0$ in X_{σ_i}, and the condition that $x^m L_\lambda$ be regular on $\mathbf{O}_{\sigma_i}$ means that its expression in terms of the coordinates $z_1, \dots, z_n$ does not contain z_n in the denominator. Thus, if $L_\lambda \neq (\mathrm{const}) \cdot z_n \partial_n$, where ∂_p is a shorthand for $\frac{\partial}{\partial z_p}$, then the regularity on $\mathbf{O}_{\sigma_i}$ is equivalent to the condition $\mathrm{ord}_{z_n}(x^m) \geq 0$, i.e., $\langle v_i, m \rangle \geq 0$, where v_i is the primitive vector of N spanning σ_i. If $L_\lambda = (\mathrm{const}) \cdot z_n \partial_n$, then λ is $\mathbf{Z}$-proportional to v_i, and the regularity condition is $\mathrm{ord}_{z_n}(x^m) \geq -1$, i.e., $\langle v_i, m \rangle \geq -1$. Finally, we get the following formulas for $X = X_\sigma$:

$$\Gamma(X, \mathcal{T}_X) = \bigoplus_{m \in M} \Lambda_{1,0}(m), \tag{14}$$

$$\Lambda_{1,0}(m) = \begin{cases} 0, & \text{if there exist } i, j \in \{1, \dots, \kappa\} \text{ such that } \langle v_i + v_j, m \rangle \leq -2, \\ \mathbf{C} L_{v_i}, & \text{if there exists } i \in \{1, \dots, \kappa\} \text{ such that } \langle v_i, m \rangle = -1 \text{ and } \\ & \langle v_j, m \rangle \geq 0 \text{ for any } j \neq i, \\ \sum_{q=1}^{n} \mathbf{C} L_{\lambda_q}, & \text{if } \langle v_i, m \rangle \geq 0 \ \forall\, i \in \{1, \dots, \kappa\}, \text{ where } \lambda_1, \dots, \lambda_n \text{ is any lin-} \\ & \text{early independent set of elements of } N. \end{cases} \tag{15}$$

Note that the condition $\langle v_i, m \rangle \geq 0 \ \forall\, i \in \{1, \dots, \kappa\}$ is equivalent to $m \in \sigma^\vee = \tau$, and $\langle v_i, m \rangle = -1$ means that m lies in the exterior of τ, but is at the smallest possible distance from the wall $\sigma_i^\perp$ of τ. The first condition $\langle v_i + v_j, m \rangle \leq -2$ forbids that m be at a distance ≥ 2 behind $\sigma_i^\perp$, and also that it be behind two walls at once.

Now, if X_Σ is a toric variety associated with any fan Σ, one should apply the above arguments to all the cones $\sigma \in \Sigma$. Let Σ_r denote the set of all the r-dimensional cones in Σ ($l = 0, 1, \dots, n$), and N_1 the set of primitive vectors of N on the rays from Σ_1. Formulas (14), (15) remain valid for $X = X_\Sigma$ if one makes v_i run over N_1, κ being the cardinality of N_1. Similar arguments prove the analog of (14), (15) for tensors of a more general form, given by the following proposition.

Proposition 6.1. *Let $X = X_\Sigma$, Σ_1, $N_1 = \{v_1, \dots, v_\kappa\}$ be as above. Denote $B_i(m, l) = \max\{0, l - \langle v_i, m \rangle\}$, $i = 1, \dots, \kappa$, $l \in \mathbf{N}$, $m \in M$. Let T be any subset of N generating $N \otimes \mathbf{R}$ as a $\mathbf{R}$-vector space and containing N_1, $T^{(k)} = \mathrm{Symm}^k T$ its kth symmetric power, and*

$$T^{(k)}(m, l) = \left\{ (\lambda_1, \dots, \lambda_k) \in T^{(k)} \;\middle|\; \begin{array}{l} \text{for each } i = 1, \dots, \kappa, \ v_i \text{ is present in} \\ (\lambda_1, \dots, \lambda_k) \text{ at least } B_i(m, l) \text{ times} \end{array} \right\}.$$

Then we have the decomposition (13) *with*

$$\Lambda_{kl}(m) = 0 \quad \text{if} \quad \sum_{i=1}^{\kappa} B_i(m, l) > k, \tag{16}$$

and

$$\Lambda_{kl}(m) = \sum_{(\lambda_1, \dots, \lambda_k) \in T^{(k)}(m, l)} \mathbf{C} \cdot x^m L_{\lambda_1} \cdots L_{\lambda_k} \otimes \nu^l, \tag{17}$$

where $\nu = \dfrac{dx_1}{x_1} \wedge \dots \wedge \dfrac{dx_n}{x_n}$ is a $\mathbf{T}$-invariant n-differential having simple poles on all the E_{σ_i} ($\sigma_i \in \Sigma_1$).

Example 6.2 (Vector fields on $\mathbf{P}^2$). The fan Σ in $\mathbf{R}^2$ defining $\mathbf{P}^2$ consists of three 2-dimensional cones (angles) $\langle v_1, v_2\rangle$, $\langle v_2, v_3\rangle$, $\langle v_3, v_1\rangle$ and their faces: three rays $\mathbf{R}v_i$, $i = 1,2,3$, and the origin $\{0\}$, where $v_1 = e_1, v_2 = e_2, v_3 = -e_1 - e_2$ in terms of the standard basis of $\mathbf{R}^2$. Here N_1 generates $\mathbf{R}^2$, so we can choose $T = N_1 = \{v_1, v_2, v_3\}$. The corresponding vector fields are $L_{v_1} = x_1\partial_1, L_{v_2} = x_2\partial_2, L_{v_3} = -x_1\partial_1 - x_2\partial_2$. Applying (14), (15) (or (13), (16), (17) with $k = 1, l = 0$), we see that $\Lambda(m) = \Lambda_{1,0}(m)$ can be nonzero only if m is in the triangle Δ bounded by the lines $\langle v_i, m\rangle = -1$, and for $m \in \Delta$ the nonzero $\Lambda(m)$ are

$$\Lambda(-1,1) = \langle x_2\partial_1\rangle,\ \ \Lambda(0,1) = \langle x_1x_2\partial_1 + x_2^2\partial_2\rangle,$$
$$\Lambda(-1,0) = \langle\partial_1\rangle,\ \ \Lambda(0,0) = \langle x_1\partial_1, x_2\partial_2\rangle,\ \ \Lambda(1,0) = \langle x_1^2\partial_1 + x_1x_2\partial_2\rangle,$$
$$\Lambda(0,-1) = \langle\partial_2\rangle,\ \ \Lambda(1,-1) = \langle x_1\partial_2\rangle.$$

So, $\dim\Gamma(\mathbf{P}^2, \mathcal{T}_{\mathbf{P}^2}) = 8$, as was expected.

Example 6.3 (Computation of $\Gamma(S^6\mathcal{T}_{\mathbf{P}^2} \otimes \omega_{\mathbf{P}^2}^2)$). Applying (13), (16), (17) with $k = 6, l = 2$, we see that $\Lambda_{6,2}(m)$ can be nonzero only for $m \in \Delta_1$, where the triangle Δ_1 is cut out by the inequalities $\langle v_i, m\rangle \leq 2$, and for $m \in \Delta_1$, the vector spaces $\Lambda_{6,2}(m)$ are 1-dimensional, generated by $x^m L_{v_1}^{b_1} L_{v_2}^{b_2} L_{v_3}^{b_3} \otimes \nu^2$, where $b_i = B_i(m, 2)$. For example, the generator of $\Lambda_{6,2}(m)$ corresponding to the vertex $m = (-4, 2)$ of the triangle is $x_1^{-4}x_2^2(x_1\partial_1)^6 \otimes \left(\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right)^2 = \partial_1^6 \otimes (dx_1 \wedge dx_2)^2$, and so on for the other 27 lattice points of Δ_1. In particular, $\dim\Gamma(S^6\mathcal{T}_{\mathbf{P}^2} \otimes \omega_{\mathbf{P}^2}^2) = 28$. One can identify this space with that of homogeneous polynomials of degree 6 in three variables by the map

$$P_6(X, Y, Z) \mapsto P_6(\partial_1, \partial_2, -x_1\partial_1 - x_2\partial_2) \otimes (dx_1 \wedge dx_2)^2. \tag{18}$$

7. Proof of Theorem 1.1

Let $\phi : \mathcal{C} \longrightarrow B$ be a family of hyperelliptic curves of arithmetic genus 2 satisfying (i) and (ii) of Theorem 4.1. Suppose that the variety $\tilde{P}$ constructed in Theorem 4.1 is Lagrangian over B. Then, as shown in [19, formula (21)], $\mathcal{C}$ is a double covering of the projectivization $\mathbf{P} = \mathbf{P}(\mathcal{T}_B)$ of the tangent bundle of B ramified in the divisor of zeros of a section $\sigma \in \Gamma(B, S^6\mathcal{T}_B \otimes \omega_B^2)$. To make explicit this description in coordinates, choose local coordinates x, y on B; then ν can be written in the form

$$\sigma = \sum a_{ij}(x,y)\partial_x^i\partial_y^j(dx \wedge dy)^2. \tag{19}$$

To represent this as a homogeneous form on $\mathbf{P}$, one should consider ∂_x, ∂_y as homogeneous coordinates of the fibers of $\mathbf{P}$ over B; iet us denote them in this quality as $\xi = \partial_x, \eta = \partial_y$. Then $\mathcal{C}$ is defined by $t^2 - \sum a_{ij}(x,y)\xi^i\eta^j = 0$.

Lemma 7.1. *Let $E \subset B$ be an exceptional (-1)-curve with local equation $u = 0$, and η a vector field in the neighborhood of E, annihilating u. Then all the fibers of $\mathcal{C}$ over points $z \in E$ have a singularity at $\eta = 0$ (η being considered as a linear form on $\mathbf{P}$) of local analytic type $t^2 - w^p = 0$ with $p \geq 4$.*

Proof. By the rigidity of exceptional subvarieties, we can identify the formal or analytic neighborhood of E with that of the exceptional curve in the blow up $\tilde{\mathbf{A}}^2$ of the origin in the affine plane $\mathbf{A}^2$. The tensors (19) will be then represented by series in powers of x, y, which can be rewritten in the notation of Proposition 6.1 as infinite sums on m of elements of the semi-invariant spaces $\Lambda_{kl}(m)$ defined by (17) with $k = 6, l = 2$. For $\tilde{\mathbf{A}}^2$, the fan Σ has three rays, generated by $e_1, e_2, e_1 + e_2$. The corresponding vector fields are $L_{e_1} = x\partial_x, L_{e_2} = y\partial_y, L_{e_1+e_2} = x\partial_x + y\partial_y$. Then $\Lambda_{6,2}(m)$ is generated by the 'monomials'

$$\delta = x^m L_{e_1}^{k_1} L_{e_2}^{k_2} L_{e_1+e_2}^{k_3} \otimes \nu^2 = x^{m_1}y^{m_2}(x\partial_x)^{k_1}(y\partial_y)^{k_2}(x\partial_x + y\partial_y)^{k_3} \otimes \left(\frac{dx}{x} \wedge \frac{dy}{y}\right)^2 \tag{20}$$

such that

$$m_1 + k_1 - 2 \geq 0, \quad m_2 + k_2 - 2 \geq 0, \quad m_1 + m_2 + k_3 - 2 \geq 0,$$
$$k_1 + k_2 + k_3 = 6, \quad k_i \geq 0 \ (i = 1, 2, 3). \tag{21}$$

Let us rewrite (20) in a coordinate patch of the blow up, say $x = u, y = uv$:

$$\delta = u^{m_1+m_2+k_3-2} v^{m_2+k_2-2} (u\partial_u - v\partial_v)^{k_1} \partial_v^{k_2} \partial_u^{k_3} (du \wedge dv)^2.$$

Restrict it to the exceptional curve $u = 0$, in replacing ∂_u, ∂_v by the corresponding homogeneous coordinate forms ξ, η on $\mathbf{P}$:

$$\delta|_{u=0} = -v^{m_2+k_1+k_2-2} \xi^{k_3} \eta^{k_1+k_2}$$

with the additional restriction $\text{ord}_E \delta = m_1 + m_2 + k_3 - 2 = 0$. As η corresponds to the vector field ∂_v annihilating u, we have to prove that $k_1 + k_2 \geq 4$. Assume the opposite: $k_1 + k_2 < 4$. Then, by (21), $k_3 > 2$, and $m_1 + m_2 + k_3 - 2 = 0$ implies $m_1 + m_2 < 0$. Taking the sum of the first two inequalities (21), we obtain $m_1 + m_2 + k_1 + k_2 - 4 \geq 0$, which contradicts $k_1 + k_2 < 4$ and $m_1 + m_2 < 0$. □

The lemma implies that the local equation of C cannot be locally analytically equivalent to $y^2 = x^k + s_1 x + s_2 + p(s_1, s_2, x)$ with $p(s_1, s_2, x) \equiv 0 \bmod (s_1, s_2)(x^2)$, as it should be in order that (i) and (ii) of Theorem 4.1 might be satisfied (see Remark 4.2). Thus, B is relatively minimal. By Theorem 1 of loc. cit., B is rational, so it is $\mathbf{P}^2$ or a Hirzebruch surface $\mathbf{F}_n$ $(n = 0, 2, 3, \dots)$.

Lemma 7.2. *Any section $\sigma \in \Gamma(B, S^6\mathcal{T}_B \otimes \omega_B^2)$ for $B = \mathbf{F}_n$ $(n = 0, 1, 2, \dots)$ defines a family C/B of generically singular curves.*

Proof. Look at the two affine charts $\mathbf{A}^2$ in the atlas of $\mathbf{F}_n$ with transition functions $(x, y) = (x_1, 1/y_1)$. In the chart (x, y), the tensor field σ can be given by the formula (19). In coordinates (x_1, y_1), we have

$$\sigma = \sum a_{ij} \left(x_1, \frac{1}{y_1} \right) \partial_{x_1}^i (-y_1^2 \partial_{y_1})^j \frac{(dx_1 \wedge dy_1)^2}{y_1^4}.$$

It is holomorphic at $y_1 = 0$ only if $a_{ij} = 0$ for $j < 2$. This implies that if ξ, η denote the homogeneous coordinates on $\mathbf{P}$ corresponding to ∂_x, ∂_y, then $\eta = 0$ is a (at least) double point of every curve of C/B. □

Hence, the only possible case is $B = \mathbf{P}^2$. The general form of σ is represented in this case by (18). Replace, as above, the fields ∂_1, ∂_2 by corresponding homogeneous coordinates ξ, η, change the notation for coordinates in the base B from x_1, x_2 to s_1, s_2, as we have done in the preceding sections, and pass to the affine coordinate $x = \xi/\eta$ on the fibers of the $\mathbf{P}^1$-bundle $\mathbf{P}$. Then we will obtain the following family of hyperelliptic curves: $t^2 - P_6(x, 1, -s_1 x - s_2) = 0$. This is one of the affine charts of the family $\{\beta^{-1}(l)\}$ for the map $\beta : S \longrightarrow \mathbf{P}^2$ ramified in the sextic curve $P_6(X, Y, Z) = 0$, the lines $l \subset \mathbf{P}^2$ being parametrized by $Z = -s_1 X - s_2 Y$.

The regularity of the extension of the symplectic structure to the compactification $\tilde{P}$ can be proved by using its coordinate representations as in (26), loc. cit. The rest of the statements of the theorem follow from Corollaries 4.3, 4.4, 5.2, and Proposition 4.5; the regularity of the extension of the symplectic structure to P^1 follows from Theorem 5.1, together with the Riemann extension theorem, as codim $P^1 \setminus J^1 \geq 2$.

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