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CONVERGENCE OF CERTAIN INCOMPLETE BLOCK FACTORIZATION SPLITTINGS

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This note fills a logical gap in the theory of incomplete block factorizations of the generalized SSOR *type. Namely, it is shown that using the so-called factorized sparse approximate inverses it is possible to preserve the symmetry of a given Stieltjes or positive definite H-matrix A in its incomplete block factorization K and to insure simultaneously the convergence of the related splitting $A = K - R$. Bibliography: 3 titles.*

This note fills a logical gap in the existence and convergence theory for incomplete block factorizations of the generalized SSOR type for M- and H-matrices developed in [2]. In that paper it was proved, in particular, that, under certain general assumptions on the involved sparse approximate inverses to pivot blocks, the splitting $A = K - R$ of a given M- or H-matrix A, where K is an (unmodified) incomplete block factorization of A either in standard or in inverse-free form, is convergent, i.e., $\rho(K^{-1}R) < 1$. However, for inverse-free incomplete block factorizations no method for constructing sparse approximate inverses was indicated which would simultaneously ensure the convergence of the related splitting and preserve the symmetry of the original matrix in its incomplete block factorization. Since, obviously, it is desirable to be able to construct a symmetric preconditioner for a symmetric original matrix, in this note, based on the results from [2] and [3], we show that, in the case of Stieltjes or symmetric positive definite H-matrices, using the so-called factorized sparse approximate inverses provides a way for constructing symmetric inverse-free incomplete block factorizations for which the related splitting of the original matrix is convergent.

Throughout the paper the following notation is used. I_n or simply I denotes the $n \times n$ identity matrix. For two matrices X and Y of the same size the inequality $X \geq Y$ is understood componentwise and means that the matrix $X - Y$ is nonnegative. By $|X|$ we denote the matrix whose entries are absolute values of the corresponding entries of X; $\rho(X)$ means the spectral radius of a square matrix X. Finally, the notation $\operatorname{diag}(X)$ and D_X is used for the pointwise and blockwise diagonal parts of a matrix X, respectively, while $\operatorname{Diag}(X_1, \dots, X_m)$ denotes the block diagonal matrix with blocks $X_1, \dots, X_m$ on the main block diagonal.

First we recall the construction of an (unmodifed) incomplete block factorization of the generalized SSOR type in inverse-free form (see, e.g., [2]). Let $A = (A_{ij})_{i,j=1}^m$ be a symmetric block $m \times m$ matrix. We also represent it in the form

$$A = D - L - L^T,$$

where $D_A = \operatorname{Diag}(A_{11}, \dots, A_{mm})$ is the block diagonal part of A and $-L$ is its strictly lower block triangular part. An (unmodified) incomplete block factorization of A of the generalized $SSOR$ type in inverse-free form is constructed as follows:

$$K = (I - L\Delta)\Delta^{-1}(I - \Delta L^T), \tag{1}$$

where $\Delta = \operatorname{Diag}(\Delta_1, \dots, \Delta_m)$ and

$$\Delta_i = \Omega_i[(D_A - L\Delta L^T)_{ii}^{-1}], \qquad i = 1, 2, \dots, m, \tag{2}$$

whereas $\Omega_i(X^{-1})$ denotes a sparse approximate inverse to X, and in what follows Ω_i are referred to as approximation rules.

Recall next that, according to [3], a factorized sparse approximate inverse to a symmetric positive definite $n \times n$ matrix X is constructed in the following way. Fix a lower triangular sparsity pattern S, i.e., a set

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of off-diagonal positions $S \subseteq \{(i,j) : 1 \leq i \neq j \leq n\}$ such that $S \supseteq \{(i,j) : i < j\}$ and define the lower triangular matrix $G^* = (g^*_{ij})^n_{i,j=1}$ by the equations

$$\begin{aligned} g^*_{ij} &= 0, \qquad (i,j) \in S, \\ (G^* X)_{ij} &= \delta_{ij}, \qquad (i,j) \notin S, \end{aligned} \tag{3}$$

where δ_{ij} is the Kronecker symbol. Then set

$$D^* = \operatorname{diag}(G^*) \tag{4}$$

and

$$G = D^{*-1/2} G^*, \tag{5}$$

ensuring that

$$\operatorname{diag}(GXG^T) = I_n. \tag{6}$$

Now the factorized sparse approximate inverse to X, determined by the sparsity pattern S and denoted by $\mathrm{fsai}_S(X)$, is defined by the relation

$$\mathrm{fsai}_S(X) = G^T G. \tag{7}$$

The following properties of factorized sparse approximate inverses to Stieltjes matrices were established in [3].

Theorem 1. *Let X be a Stieltjes matrix and let S be a lower triangular sparsity pattern. Then the lower triangular matrix G defined by* (3)–(5) *is nonsingular and nonnegative, whereas GXG^T is a Stieltjes matrix and* $\operatorname{diag}(GXG^T) = I$.

To insure that incomplete block factorizations in inverse-free form based on factorized sparse approximate inverses are well defined for Stieltjes matrices, we need the following results.

Lemma 1. *Under the assumptions of Theorem* 1, *the following inequalities are valid:*

$$0 \leq \mathrm{fsai}_S(X) = G^T G \leq X^{-1}.$$

Proof. The left-hand-side inequality is obvious since, by Theorem 1, G is nonnegative. On the other hand, since, in view of Theorem 1, GXG^T is a Stieltjes matrix and $\operatorname{diag}(GXG^T) = I$, we have

$$GXG^T = I - R, \qquad R \geq 0, \tag{8}$$

and, therefore, $\rho(R) < 1$, because the Jacobi splitting of a Stieltjes matrix is convergent (see, e.g., [1]). It thus follows from (8) that

$$X^{-1} = G^T (I - R)^{-1} G = G^T (I + R + R^2 + \cdots) G \geq G^T G,$$

which completes the proof. □

Lemma 2 [2]. *Let $A = (A_{ij})^m_{i,j=1}$ be a Stieltjes matrix and let for $i = 1, \ldots, m$ the approximation rules Ω_i be such that, for any Stieltjes matrix X, the relations*

$$0 \leq \Omega_i(X^{-1}) \leq X^{-1}, \qquad \Omega_i(X^{-1}) = [\Omega_i(X^{-1})]^T$$

are valid. Then for $i = 1, \ldots, m$, $A_{ii} - (L\Delta L^T)_{ii}$ are Stieltjes matrices and $\Delta = \mathrm{Diag}(\Delta_1, \ldots, \Delta_m)$ is well defined by (2) *and is nonnegative.*

Using Theorem 1 and Lemmas 1 and 2 we easily derive the following existence result.

Theorem 2. *Let $A = (A_{ij})_{i,j=1}^{m}$ be a Stieltjes matrix, let S_i $(i = 1, \dots, m)$ be lower triangular sparsity patterns, and let K be an incomplete block factorization of A, constructed according to (1)–(2), with the following approximation rules Ω_i:*

$$\Omega_i\,(X^{-1}) = \mathrm{fsai}_{S_i}\,(X), \qquad i = 1, \dots, m,$$

i.e., let

$$\Delta_i = G_i^T\, G_i, \qquad i = 1, \dots, m, \tag{9}$$

be the factorized sparse approximate inverses to $A_{ii} - (L\Delta L^T)_{ii}$, determined by S_i. Then the matrix $\Delta = \mathrm{Diag}\,(\Delta_1, \dots, \Delta_m)$ is well defined, symmetric positive definite, and nonnegative.

We are now ready to establish the main result of this paper for Stieltjes matrices.

Theorem 3. *Under the hypotheses of Theorem 2, the splitting $A = K - R$ is convergent, i.e., $\rho\,(K^{-1} R) < 1$.*

Proof. Since

$$R = K - A = \Delta^{-1} + L\Delta^{-1} L^T - D_A, \tag{10}$$

for $i \neq j$, $1 \leq i,\ j \leq m$ we have

$$R_{ij} = (L\Delta L^T)_{ij} \geq 0, \tag{11}$$

because $L \geq 0$ and, by Theorem 2, $\Delta \geq 0$. Now let $i = j$. As Theorem 1 shows,

$$G_i\,(D_A - L\Delta L^T)_{ii}\, G_i^T = I - Q_{ii}, \qquad i = 1, \dots, m, \tag{12}$$

where $Q_{ii} \geq 0$. Therefore, taking into account (9), it follows from (10) and (12) that

$$G_i\, R_{ii}\, G_i^T = G_i\, \Delta_i^{-1}\, G_i^T - (I - Q_{ii}) = Q_{ii} \geq 0, \quad i = 1, \dots, m. \tag{13}$$

Relations (11) and (13) show that

$$Q = G\,R\,G^T \geq 0, \tag{14}$$

where $G = \mathrm{Diag}\,(G_1, \dots, G_m)$. On the other hand, using (10) and (14) we derive

$$G^{-T} K^{-1} A\, G^T = G^{-T} (I - K^{-1} R)\, G^T = I - G^{-T} K^{-1} G^{-1} (G\,R\,G^T) = I - (G\,K\,G^T)^{-1} Q. \tag{15}$$

We next show that the matrix GKG^T is monotone. Indeed,

$$\begin{aligned} GKG^T &= G(\Delta^{-1} - L)\,\Delta\,(\Delta^{-1} - L^T)\,G^T = G\,(G^{-1} G^{-T} - L)\,G^T\, G\,(G^{-1} G^{-T} - L^T)G^T \\ &= (I - GLG^T)(I - GLG^T)^T. \end{aligned} \tag{16}$$

Furthermore, since $G \geq 0$ is block diagonal and $L \geq 0$ is strictly lower block triangular, we can represent the inverse matrix $(I - GLG^T)^{-1}$ in the form

$$(I - GLG^T)^{-1} = \sum_{i=0}^{m-1} (GLG^T)^i \geq 0,$$

implying, in view of (16), that

$$(G\,K\,G^T)^{-1} \geq 0. \tag{17}$$

Now, (15), (14), and (17) show that $G^{-T} K^{-1} A\, G^T$ is a matrix with nonpositive off-diagonal entries and, as a matrix similar to $K^{-1} A$, it has positive eigenvalues. Therefore (see, e.g., [1]), $G^{-T} K^{-1} AG^T$ is an M-matrix and $G^{-T} K^{-1} AG^T = I - (GKG^T)^{-1} Q$ is its regular and, hence, convergent splitting, i.e.,

$$\rho\,((GKG^T)^{-1} Q) = \rho\,(G^{-T} K^{-1} RG^T) = \rho\,(K^{-1} R) < 1,$$

which completes the proof. $\square$

Now consider the case of a symmetric positive definite H-matrix A, i.e., of a symmetric H-matrix with positive diagonal entries. The comparison matrix $\mathcal{M}(A)$ for a matrix $A = (a_{ij})_{i,j=1}^n$ is defined in the usual way:

$$(\mathcal{M}(A))_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases} \qquad 1 \leq i,\, j \leq n. \tag{18}$$

We recall (see, e.g., [1]) that, by definition, A is an H-matrix if and only if $\mathcal{M}(A)$ is an M-matrix. Furthermore, if A is a symmetric positive definite H-matrix, then $\mathcal{M}(A)$ is a Stieltjes matrix and $A \geq \mathcal{M}(A)$.

As in the case of Stieltjes matrices, we first recall some known results concerning factorized sparse approximate inverses and incomplete block factorizations for H-matrices. The results summarized in the following theorem can be extracted from [3].

Theorem 4. *Let X be a symmetric H-matrix, let S be a lower triangular sparsity pattern, and let the two lower triangular matrices G^* and H^* be defined by (3) for X and $\mathcal{M}(X)$, respectively. Then G^* is nonsingular and*

$$|G^*| \leq H^*. \tag{19}$$

Furthermore,

$$|D^{*-1} G^*| \leq [\mathrm{diag}(H^*)]^{-1} H^*, \tag{20}$$

where $D^* = \mathrm{diag}(G^*)$, and

$$|I - G^* X G^{*^T} D^{*-1}| \leq I - H^* \mathcal{M}(X) H^{*^T} [\mathrm{diag}(H^*)]^{-1}. \tag{21}$$

Lemma 3. *Let $X = (x_{ij})_{i,j=1}^n$ be a symmetric H-matrix and let $\widetilde{X} = (\widetilde{x}_{ij})_{i,j=1}^n$ be a Stieltjes matrix such that $\widetilde{X} \leq \mathcal{M}(X)$. Then for a lower triangular sparsity pattern S we have*

$$|\mathrm{fsai}_S(X)| \leq \mathrm{fsai}_S(\widetilde{X}) \leq \widetilde{X}^{-1}. \tag{22}$$

Furthermore, if the two lower triangular matrices G^* and $\widetilde{G}^*$ are defined by (3) for X and $\widetilde{X}$, respectively, then

$$|G^*| \leq \widetilde{G}^*, \tag{23}$$

$$|\, D^{*-1} G^* \,| \leq \widetilde{D}^{*-1} \widetilde{G}^*, \tag{24}$$

and

$$|I - G^* X G^{*^T} D^{*-1}| \leq I - \widetilde{G}^* \widetilde{X} \widetilde{G}^{*^T} \widetilde{D}^{*-1}, \tag{25}$$

where $D^* = \mathrm{diag}(G^*)$ and $\widetilde{D}^* = \mathrm{diag}(\widetilde{G}^*)$.

Proof. Let H^* be defined by (3) for the Stieltjes matrix $\mathcal{M}(X)$. Then, by Theorem 4,

$$|G^*| \leq H^*. \tag{26}$$

On the other hand, using the assumption $\tilde{}X \leq \mathcal{M}(X)$ and the well-known fact that the inverses to M-matrices M_1 and M_2 such that $M_1 \leq M_2$ satisfy the relation $M_1^{-1} \geq M_2^{-1}$, we readily derive that

$$H^* \leq \widetilde{G}^*. \tag{27}$$

Taken together, inequalities (26) and (27) prove (23). Now we establish (24). By Theorem 4, we have

$$|D^{*-1} G^*| \leq [\mathrm{diag}(H^*)]^{-1} H^*,$$

and, thus, to prove (24) it is sufficient to make sure that

$$[\mathrm{diag}\,(H^*)]^{-1}\, H^* \le \widetilde{D}^{*-1}\, \widetilde{G}^*, \tag{28}$$

where H^* and $\widetilde{G}^*$ are defined for the two Stieltjes matrices $\mathcal{M}\,(X)$ and $\widetilde{X} \le \mathcal{M}\,(X)$, respectively. Inequality (28) is similar to inequality (20) and is established in a similar fashion (see [3]). Now the inequality

$$|\mathrm{fsai}_S\,(X)| = |G^{*^T}\, D^{*-1}\, G^*| \le \mathrm{fsai}_S\,(\widetilde{X}) = \widetilde{G}^{*^T}\, \widetilde{D}^{*-1}\, \widetilde{G}^*$$

follows from (23) and (24), whereas the inequality $\mathrm{fsai}_S\,(\widetilde{X}) \le \widetilde{X}^{-1}$ is insured by Lemma 1. It thus only remains to prove (25). Since, obviously,

$$\mathrm{diag}\,(G^*\, X\, G^{*^T}\, D^{*-1}\,) = I = \mathrm{diag}\,(\widetilde{G}^*\, \widetilde{X}\, \widetilde{G}^{*^T}\, \widetilde{D}^{*^{-1}}\,),$$

we need only to show that, for $i \neq j$,

$$\left|(G^*\, X\, G^{*^T}\, D^{*-1})_{ij}\,\right| \le \left|(\widetilde{G}^*\, \widetilde{X}\, \widetilde{G}^{*^T}\, \widetilde{D}^{*-1})_{ij}\right|,$$

while in view of (24), the last inequality will be established if we prove that

$$|\,(G^*\,X)_{ij}\,| \le |\,(\widetilde{G}^*\,\widetilde{X})_{ij}\,|, \qquad 1 \le i,\, j \le n. \tag{29}$$

For $(i,j) \notin S$ inequality (29) is obviously satisfied by the definition of the matrices G^* and $\widetilde{G}^*$. If $(i,j) \in S$, then

$$|\,(G^*\,X)_{ij}\,| = \left|\sum_{(i,k)\notin S} g^*_{ik}\, x_{kj}\right| \le \sum_{(i,k)\notin S} |\,g^*_{ik}\,| \cdot |\,x_{kj}\,| \le \sum_{(i,k)\notin S} \widetilde{g}^*_{ik}\,|\,\widetilde{x}_{kj}\,| = |\,(\widetilde{G}^*\,\widetilde{X})_{ij}\,|,$$

because the simultaneous conditions $(i,j) \in S$ and $(i,k) \notin S$ imply that $k \neq j$ and so $|\,x_{kj}\,| \le |\,\widetilde{x}_{kj}\,|$ in view of the assumption $\mathcal{M}(X) \ge \widetilde{X}$. Lemma 3 is thus completely proved. $\square$

Lemma 4 [2]. *Let $A = (A_{ij})_{i,j=1}^m$ be a symmetric positive definite H-matrix, and for $i = 1, \ldots, m$ let the approximation rules Ω_i be such that, for any symmetric positive definite H-matrix X and a Stieltjes matrix $\widetilde{X}$ satisfying the inequality $\widetilde{X} \le \mathcal{M}(X)$, the relations*

$$|\,\Omega_i\,(X^{-1})\,| \le \Omega_i\,(\widetilde{X}^{-1}) \le \widetilde{X}^{-1}, \quad \Omega_i\,(X^{-1}) = [\Omega_i\,(X^{-1})]^T$$

are valid. Furthermore, let the matrices Δ_i and $\widetilde{\Delta}_i$, $i = 1, \ldots, m$, be defined by (2) for A and a Stieltjes matrix $\widetilde{A} = \widetilde{D} - \widetilde{L} - \widetilde{L}^T$ such that $\widetilde{A} \le \mathcal{M}(A)$, respectively. Then for $i = 1, \ldots, m$, we have

$$A_{ii} - (L\Delta L^T)_{ii} \ge \mathcal{M}\,(A_{ii} - (L\Delta L^T)_{ii}) \ge \widetilde{A}_{ii} - (\widetilde{L}\widetilde{\Delta}\widetilde{L}^T)_{ii},$$

implying that $A_{ii} - (L\Delta L^T)_{ii}$ are symmetric positive definite H-matrices and

$$|\,\Delta_i\,| \le \widetilde{\Delta}_i.$$

Applying Theorem 4 and Lemmas 3 and 4, we easily establish the following result.

Theorem 5. *Let $A = (A_{ij})_{i,j=1}^m$ be a symmetric positive definite H-matrix, let $S_i\,(i = 1, \ldots, m)$ be lower triangular sparsity patterns, and let $\Delta = \mathrm{Diag}\,(\Delta_1, \ldots, \Delta_m)$ be constructed according to (2) with*

$$\Omega_i\,(X^{-1}) = \mathrm{fsai}_{S_i}\,(X), \qquad i = 1, \ldots, m,$$

i.e., let

$$\Delta_i = G_i^T\, G_i, \qquad\qquad i = 1, \ldots, m,$$

be the factorized sparse approximate inverses to $A_{ii} - (L\Delta L^T)_{ii}$ determined by S_i. Then the matrix Δ is well defined and symmetric positive definite.

The final theorem of this paper establishes the required result on the convergence of splittings related to incomplete block factorizations based on factorized sparse approximate inverses in the case of symmetric positive definite H-matrices.

Theorem 6. *Under the hypotheses of Theorem 5, the splitting $A=K-R$ is convergent, i.e., $\rho(K^{-1}R)<1$.*

Proof. Let $G=\mathrm{Diag}(G_1,\dots,G_m)$, $D^*=[\mathrm{diag}(G)]^2$, $G^*=D^{*1/2}G$, and $Q=GRG^T$. Then $\Delta=G^TG$ and

$$\begin{aligned}\rho(K^{-1}R)&=\rho(G^{-T}K^{-1}G^{-1}Q)=\rho\left(\left[D^{*1/2}GKG^TD^{*-1/2}\right]^{-1}\cdot\left[D^{*1/2}QD^{*-1/2}\right]\right)\\&\le\rho\left(\left|\left[D^{*1/2}GKG^TD^{*-1/2}\right]^{-1}\right|\cdot\left|D^{*1/2}QD^{*-1/2}\right|\right).\end{aligned}\tag{30}$$

Now let $\widetilde{A}=\mathcal{M}(A)$, so that $\widetilde{D}=D_{\widetilde{A}}=\mathcal{M}(D_A)$ and $\widetilde{L}=|L|$. Furthermore, let $\widetilde{\Delta}_i=\widetilde{G}_i^T\widetilde{G}_i=\mathrm{fsai}_{S_i}(\widetilde{A}_{ii}-(\widetilde{L}\widetilde{\Delta}\widetilde{L}^T)_{ii})$, $i=1,\dots,m$, $\widetilde{\Delta}=\mathrm{Diag}(\widetilde{\Delta}_1,\dots,\widetilde{\Delta}_m)$, $\widetilde{G}=\mathrm{Diag}(\widetilde{G}_1,\dots,\widetilde{G}_m)$, $\widetilde{D}^*=[\mathrm{diag}(\widetilde{G})]^2$, $\widetilde{G}^*=\widetilde{D}^{*1/2}\widetilde{G}$, $\widetilde{K}=(I-\widetilde{L}\widetilde{\Delta})\widetilde{\Delta}^{-1}(I-\widetilde{\Delta}\widetilde{L}^T)$, $\widetilde{R}=\widetilde{K}-\widetilde{A}$, and $\widetilde{Q}=\widetilde{G}\widetilde{R}\widetilde{G}^T$ be the corresponding matrices defined for the Stieltjes matrix $\widetilde{A}$.

First we show that

$$\left|\left[D^{*1/2}GKG^TD^{*-1/2}\right]^{-1}\right|\le\left[\widetilde{D}^{*1/2}\widetilde{G}\widetilde{K}\widetilde{G}^T\widetilde{D}^{*-1/2}\right]^{-1}.\tag{31}$$

Indeed, since (see (16))

$$\begin{aligned}&D^{*1/2}GKG^TD^{*-1/2}=D^{*1/2}(I-GLG^T)(I-GL^TG^T)D^{*-1/2}\\&=(I-D^{*1/2}GLG^TD^{*-1/2})(I-D^{*1/2}GL^TG^TD^{*-1/2})=(I-G^*LG^{*T}D^{*-1})(I-G^*L^TG^{*T}D^{*-1})\end{aligned}$$

and since, by Lemmas 4 and 3,

$$\begin{aligned}\left|(I-G^*LG^{*T}D^{*-1})^{-1}\right|&=\left|\sum_{i=0}^{m-1}(G^*LG^{*T}D^{*-1})^i\right|\le\sum_{i=0}^{m-1}\left(|G^*|\,|L|\,|D^{*-1}G^*|^T\right)^i\\&\le\sum_{i=0}^{m-1}(\widetilde{G}^*\widetilde{L}\widetilde{G}^{*T}\widetilde{D}^{*-1})^i=(I-\widetilde{G}^*\widetilde{L}\widetilde{G}^{*T}\widetilde{D}^{*-1})^{-1},\end{aligned}$$

we see that

$$\begin{aligned}&\left|\left[D^{*1/2}GKG^TD^{*-1/2}\right]^{-1}\right|\le(I-\widetilde{G}^*\widetilde{L}^T\widetilde{G}^{*T}D^{*-1})^{-1}(I-\widetilde{G}^*\widetilde{L}\widetilde{G}^{*T}\widetilde{D}^{*-1})^{-1}\\&=\left[(I-\widetilde{G}^*\widetilde{L}\widetilde{G}^{*T}\widetilde{D}^{*-1})(I-\widetilde{G}^*\widetilde{L}^T\widetilde{G}^*\widetilde{D}^{*-1})\right]^{-1}=(\widetilde{D}^{*1/2}\widetilde{G}\widetilde{K}\widetilde{G}^T\widetilde{D}^{*-1/2})^{-1}.\end{aligned}$$

This establishes inequality (31).

Next we show that

$$\left|D^{*1/2}QD^{*-1/2}\right|\le\widetilde{D}^{*1/2}\widetilde{Q}\widetilde{D}^{*-1/2}.\tag{32}$$

To this end, we first note that, for $i\ne j$,

$$(D^{*1/2}QD^{*-1/2})_{ij}=D_i^{*1/2}G_iR_{ij}G_j^TD_j^{*-1/2}=G_i^*(L\Delta L^T)_{ij}G_j^{*T}D_j^{*-1},$$

where we use the notation $D^*=\mathrm{Diag}(D_1^*,\dots,D_m^*)$, implying, in view of Lemmas 4 and 3, that for $i\ne j$

$$\left|(D^{*1/2}QD^{*-1/2})_{ij}\right|\le\widetilde{G}_i^*(\widetilde{L}\widetilde{\Delta}\widetilde{L}^T)_{ij}\widetilde{G}_j^{*T}\widetilde{D}_j^{*-1}=\widetilde{D}_i^{*1/2}\widetilde{G}_i\widetilde{R}_{ij}\widetilde{G}_j^T\widetilde{D}_j^{*-1/2}=(\widetilde{D}^{*1/2}\widetilde{Q}\widetilde{D}^{*-1/2})_{ij},$$

where $\widetilde{D}^*=\mathrm{Diag}(\widetilde{D}_1^*,\dots,\widetilde{D}_m^*)$, i.e., inequality (32) is valid for $i\ne j$. Let now $i=j$. Then, using inequality (25) of Lemma 3 and Lemma 4, we derive

$$\begin{aligned}&\left|(D^{*1/2}QD^{*-1/2})_{ii}\right|=\left|D_i^{*1/2}G_i(\Delta_i^{-1}-[A_{ii}-(L\Delta L^T)_{ii}])G_i^TD_i^{*-1/2}\right|\\&=\left|I-G_i^*[A_{ii}-(L\Delta L^T)_{ii}]G_i^{*T}D_i^{*-1}\right|\le I-\widetilde{G}_i^*[\widetilde{A}_{ii}-(\widetilde{L}\widetilde{\Delta}\widetilde{L}^T)_{ii}]\widetilde{G}_i^{*T}\widetilde{D}_i^{*-1}\\&=(\widetilde{D}^{*1/2}\widetilde{Q}\widetilde{D}^{*-1/2})_{ii}.\end{aligned}$$

Inequality (32) is thus proved. Finally, using (30)–(32) and applying Theorem 3 we obtain

$$\rho(K^{-1}R) \le \rho\,(\widetilde{D}^{*1/2}\,\widetilde{G}^{-T}\,\widetilde{K}^{-1}\,\widetilde{G}^{-1}\,\widetilde{D}^{*-1/2}\,\widetilde{D}^{*1/2}\,\widetilde{Q}\,\widetilde{D}^{*-1/2})$$
$$= \rho\,(\widetilde{G}^{-T}\,\widetilde{K}^{-1}\,\widetilde{G}^{-1}\,\widetilde{G}\,\widetilde{R}\,\widetilde{G}^{T}) = \rho\,(\widetilde{K}^{-1}\,\widetilde{R}) < 1,$$

which completes the proof of Theorem 6. □

Translated by L. Yu. Kolotilina.

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