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INVERSE PROBLEMS OF SPECTRAL ANALYSIS FOR DIFFERENTIAL OPERATORS AND THEIR APPLICATIONS

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Introduction

Inverse problems (IP's) of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, meteorology, geophysics, and other branches of the natural sciences. IP's also play an important role in solving nonlinear evolution equations of mathematical physics. Interest in this subject has been increasing permanently because of the appearance of new important apllications, and nowadays the IP theory is intensively developed worldwide.

The greatest success in the IP theory was achieved for the Sturm–Liouville differential operator (DO)

$$-y'' + q(x)y. \tag{0.1}$$

The IP for DO (0.1) was studied by many mathematicians (see [1, 4, 6, 7, 16, 20, 24, 25, 31, 33–37, 39–41, 48, 49, 56, 57, 60–63, 65, 67, 71, 73, 75, 76, 81, 85, 89, 93] and references therein). The first result in this direction belongs to Ambarzumian [1]. He showed that if the eigenvalues of the boundary value problem

$$-y'' + q(x)y = \lambda y, \qquad q(x) \in C[0, \pi], \qquad y'(0) = y'(\pi) = 0$$

are $\lambda_k = k^2$, $k \geq 0$, then $q(x) = 0$. But this result is an exception from the rule, and the specification of the spectrum does not determine the operator (0.1) uniquely. Afterwords Borg [16] proved that the specification of two spectra of Sturm–Liouville operators uniquely determines the function $q(x)$. Levinson [56] used a different method to prove Borg's results. Tikhonov [85] obtained the uniqueness theorem for the inverse Sturm–Liouville problem on the half-line with the given Weyl function.

An important role in the spectral theory of Sturm–Liouville operators was played by the transformation operator. Marchenko ([60–61]) first applied the transformation operator to the solution of the IP. He proved that a Sturm–Liouville operator on the half-line or a finite interval is uniquely determined by specifying the spectral function. Transformation operators were also used in the fundamental paper of Gel'fand and Levitan [33], where they obtained necessary and sufficient conditions, and established a method for recovering the Sturm–Liouville operator from its spectral function.

Let us briefly formulate the main results of Borg, Marchenko, Gel'fand, and Levitan for the self-adjoint Sturm–Liouville operators on a finite interval.

Consider the boundary value problem $L = L(q(x), h, H)$ of the form

$$-y'' + q(x)y = \lambda y, \qquad 0 < x < \pi, \qquad q(x) \in \mathcal{L}(0, \pi), \tag{0.2}$$

$$U(y) \equiv y'(0) - hy(0) = 0, \qquad V(y) \equiv y'(\pi) + Hy(\pi) = 0. \tag{0.3}$$

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 54, Functional Analysis–7, 1998.

Here $q(x)$, h, and H are real, and λ is a complex parameter. Let $\varphi(x,\lambda)$ be a solution of (0.2) under the initial conditions $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=h$. It is known (see, for example, [62]) that the following representation is valid:

$$\varphi(x,\lambda)=\cos\rho x+\int_0^x K(x,t)\cos\rho t\,dt,\qquad \lambda=\rho^2,\qquad K(x,x)=h+\frac{1}{2}\int_0^x q(t)\,dt. \tag{0.4}$$

The operator $(Af)(x)+f(x)+\int_0^x K(x,t)f(t)\,dt$ is called the transformation operator.

The eigenvalues $\{\lambda_k\}_{k\geq 0}$ of the boundary value problem (0.2)–(0.3) are real and simple, and coincide with zeros of the characteristic function $\Delta(\lambda)=\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda)$. For $k\to\infty$ we have

$$\sqrt{\lambda_k}=k+\frac{\omega}{k}+o\left(\frac{1}{k}\right),\qquad \omega=\frac{1}{\pi}\left(h+H+\frac{1}{2}\int_0^\pi q(t)\,dt\right).$$

The function $\Delta(\lambda)$ is uniquely determined by its zeros:

$$\Delta(\lambda)=\pi(\lambda_0-\lambda)\prod_{k=1}^{\infty}\frac{\lambda_k-\lambda}{k^2}.$$

Denote $\alpha_k=\int_0^\pi \varphi^2(x,\lambda_k)\,dx$. It is easy to see that $\alpha_k>0$ and $\alpha_k=\frac{\pi}{2}+o\left(\frac{1}{k}\right)$ as $k\to\infty$. The set of the numbers $\{\lambda_k,\alpha_l\}_{k\geq 0}$ is called the spectral data of L.

Let the functions $\Phi(x,\lambda)$ and $S(x,\lambda)$ be solutions of (0.2) under the conditions $U(\Phi)=1$, $V(\Phi)=0$, $S(0,\lambda)=0$, $S'(0,\lambda)=1$. Clearly $\Phi(x,\lambda)=S(x,\lambda)+\mathfrak{M}(\lambda)\varphi(x,\lambda)$, where $\mathfrak{M}(\lambda)=\Phi(0,\lambda)$. The function $\mathfrak{M}(\lambda)$ is called the Weyl function. It is meromorphic with simple poles at $\lambda=\lambda_k$, and

$$\mathfrak{M}(\lambda)=\sum_{k=0}^{\infty}\frac{1}{\alpha_k(\lambda-\lambda_k)}.$$

We now formulate two uniqueness theorems of the solution of the IP.

Theorem 0.1 (Marchenko [60, 61]). *The specification of the spectral data $\{\lambda_k,\alpha_k\}_{k\geq 0}$ uniquely determines the potential $q(x)$ and the coefficients h and H.*

Theorem 0.2 (Borg [16]). *The specification of two spectra $\{\lambda_k\}$ and $\{\lambda_{k1}\}$ of the boundary value problems $L=L(q(x),h,H)$ and $L=L(q(x),h,H_1)$ $(H\neq H_1)$ uniquely determines the function $q(x)$ and the numbers h, H, and H_1.*

A method for constructing the Sturm–Liouville operator from its spectral data is based on the following theorem.

Theorem 0.3 (Gel'fand and Levitan [33]). *For each fixed x, the kernel $K(x,t)$ of the transformation operator* (0.4) *satisfies the linear integral equation*

$$K(x,t)+F(x,t)+\int_0^x K(x,s)F(s,t)\,ds=0,\qquad 0<t<x, \tag{0.5}$$

where

$$f(x,t)=\sum_{k=0}^{\infty}\left(\frac{1}{\alpha_k}\cos\sqrt{\lambda_k}x\cos\sqrt{\lambda_k}t-\frac{1}{\alpha_k^0}\cos kx\cos kt\right),\qquad \alpha_k^0=\begin{cases}\pi/2, & k>0,\\ \pi, k=0.\end{cases}$$

The potential $q(x)$ *and the numbers* h *and* H *can be constructed by the formulas* $q(x) = 2\dfrac{d}{dx}K(x,x)$, $h = K(0,0)$, $H = \pi\omega - K(\pi,\pi)$.

Equation (0.5) is called the Gel'fand–Levitan equation. Using the Gel'fand–Levitan equation one can also obtain necessary and sufficient conditions for the solvability of the IP (see [57, 62, 63]).

The IP of recovering the Sturm–Liouville equation from two spectra can be reduced to the IP from spectral data, since the numbers $\{\alpha_k\}_{k\geq 0}$ can be computed by the formula

$$\alpha_k = \frac{1}{H - H_1}\Delta_1(\Lambda_k)\dot{\Delta}(\lambda_k),$$

where $\dot{\Delta}(\lambda) = \dfrac{d}{d\lambda}\Delta(\lambda)$, and $\Delta_1(\lambda) = \varphi'(\pi,\lambda) + H_1\varphi(\pi,\lambda)$ is the characteristic function of $L = L(q(x), h, H_1)$. It is also clear that the specification of the Weyl function $\mathfrak{M}(\lambda)$ is equivalent to the specification of the spectral data $\{\lambda_k, \alpha_k\}_{k\geq 0}$.

The transformation operator method allows us to investigate also IP's for the Sturm–Liouville operator on the half-line and on the line (see [4, 62, 63, 57] and references therein).

Many works are devoted to the IP theory for partial differential equations and its applications. This direction is reflected fairly completely in [8, 12, 17, 50, 70, 74].

In recent years there appeared a new area for applications of the IP theory. In [29] G. Gardner, J. Green, M. Kruskal, and R. Miura found a remarkable method for solving some important nonlinear equations of mathematical physics connected with the use of the IP theory. This method has been described in [2, 26, 51, 59] and other works.

In contrast to the case of Sturm–Liouville operators, the IP theory for higher-order DO

$$ly \equiv y^{(n)} + \sum_{k=0}^{n-2} p_k(x)y^{(k)} \tag{0.6}$$

is nowadays far from its completeness. For $n > 2$ the IP becomes essentially more difficult, and for a long time there were only isolated fragments of the theory not constituting a general picture. However in last time there appeared new results which allow us to advance in this direction.

IP's for (0.2) were studied in [9–11, 19, 21, 22, 42–46, 52, 53, 66, 77–79, 83, 84, 87, 88, 90, 95–100, 102, 105, 107] and other works. In recent years there has been considerable interest in investigation IP's for higher-order DO's as a result of emerging of new applications in various areas of the natural sciences, in particular, in the elasticity theory, for integration of nonlinear equations of mathematical physics, and so on.

Fage [27], Leont'ev [54], and Hromov [38] determined that for $n > 2$ the transformation operators have a much more complicated structure than for the Sturm–Liouville case, which makes it more difficult to use them for solving the IP. However, in the case of analytic coefficients the transformation operators have the same "triangular" form as for Sturm–Liouville operators (see [46, 64, 77]). Sakhnovich [78–79] and Khachatryan [44–45] used a "triangular" transformation operator to investigate the IP of recovering self-adjoint DO's on the half-line from the spectral function, as well as the scattering inverse problem. The scattering inverse problem on the line has been treated in various settings in [10, 11, 19, 21, 22, 42, 84] and other works.

Leibenzon in [52–53] investigated the IP for (0.2) on a finite interval under the condition of "separation" of the spectrum. The spectra and "weight" numbers of certain specially chosen boundary value problems for the DO's (0.2) appeared as spectral data of the IP. However it was found that the "separation" condition is rather a hard restriction, since removing it leads to a violation of the uniqueness for the solution of the IP and to appearance of essential difficulties in the method. Things are more complicated for DO's on the half-line, since in the non-self-adjoint case the spectrum can have a "bad" behavior.

The present review is devoted in the main to investigations IP's for DO (0.6) on the half-line and on a finite interval and their applications. The main results in this direction obtained in last years are provided. The paper consists of 4 parts.

In Part 1, we study DO (0.6) with integrable coefficients. Sec. 1 is devoted to applying the transformation operator method to IP's for higher-order self-adjoint DO's with analytic coefficients. Results by Khachatryan constitute the base of Sec. 1.

In Sec. 2, we provide the solution of a general IP for non-self-adjoint DO (0.6) on the half-line with an arbitrary behavior of the spectrum. We introduce and study the so-called Weyl matrix $\mathfrak{M}(\lambda) = [\mathfrak{M}_{mk}(\lambda)]$ as the main spectral characteristic. The uniqueness theorem for the solution of the IP with a given Weyl matrix is proved. We give a derivation of the main equation of the IP, which is a singular linear integral equation

$$\widetilde{\varphi}(x,\lambda) = \widetilde{N}(\lambda)\varphi(x,\lambda) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\widetilde{H}(x,\lambda,\mu)}{\mu-\lambda}\varphi(x,\mu)\,d\mu, \qquad \lambda\in\gamma, \quad x \geq 0,$$

with respect to $\varphi(x,\lambda)$. Here $\varphi(x,\lambda)$ is a vector-function constructed from special solutions of the differential equation $lu = \lambda y$. The functions $\widetilde{\varphi}(x,\lambda)$, $\widetilde{N}(\lambda)$, and $\widetilde{H}(x,\lambda,\mu)$ are constructed from the given model DO

$$\widetilde{l}y = y^{(n)} + \sum_{k=0}^{n-2}\widetilde{p}_k(x)y^{(k)}$$

and from the Weyl matrix $\mathfrak{M}(\lambda)$ of DO (0.6). We give a constructive procedure, as well as necessary and sufficient conditions on the Weyl matrix when the behavior of the spectrum is arbitrary. Further, we consider a particular cases,namely, DO's with a simple spectrum and selfadjoint DO's. For second-order DO's we establish connections between the main equation of the IP and the Gel'fand–Levitan equation.

In Sec. 3, we study DO (0.6) on a finite interval. In this case there are specific difficulties connected with nontrivial structural properties of the Weyl matrix in neighbourhoods of the points of the spectrum. We provide an algorithm for the solution of the IP, as well as necessary and sufficient conditions of solvability of the IP. A counterexample shows that dropping one element of the Weyl matrix violates the uniqueness of the solution of the IP.

Section 4 is devoted to investigations of the so-called incomplete IP's, when some part of the coefficients of DO (0.6) is known a priori or there is another information about the operator. Such problems often appear in applications. As a rule, incomplete IP's are more difficult for studying. In Sec. 4, we use the so-called method of standard models, in which we construct a sequence of model DO's "approaching" the desired DO. The method allows us to obtain effective algorithms for the solution of a wide class of incomplete IP's. We also apply the method of standard models to solve an IP of the elasticity theory.

In Sec. 5, we provide the solution of the IP for DO (0.6) on the half-line with locally integrable analytic coefficients. To solve this problem, we introduce the so-called generalized Weyl functions and use connections with an IP for partial differential equations. We also use the Riemann–Fage formula [28] for the solution of the Cauchy problem for higher-order partial differential equations. Note that for $n = 2$ generalized functions for solution of IP's were applied by Marchenko [62].

Part 2 consists of two paragraphs and is devoted to investigation of higher-order DO's with nonintegrable singularities. In Sec. 6 we consider DO's with singularities on the half-line, and Sec. 7 is devoted to boundary value problems on a finite interval. The IP is studied, and completeness, expansion, and equiconvergence theorems are obtained.

In Part 3, the so-called nonlocal IP's are considered. In contrast to IP's for DO's, nonlocal IP's, because of their complicacy, have not been investigated yet. We consider two model nonlocal IP's. In Sec. 8

an IP for integro-differential operators is studied, and in Sec. 9 we consider an IP for one-dimensional perturbations of integral Volterra operators. In Part 4, we provide applications of the constructed IP theory to investigations of nonlinear integrable equations of mathematical physics.

Notations.

1. If we consider a DO l, then along with l we consider a DO $\widetilde{l}$ of the same form, but with different coefficients. We agree that if some symbol ψ denotes an object relating to l, then $\widetilde{\psi}$ denotes the analogous object relating to $\widetilde{l}$ and $\widehat{\psi} = \psi - \widetilde{\psi}$.

2. One and the same symbol C denotes various positive constants in estimates.

3. A matrix A with elements a_{ij}, $i = \overline{1,r}$, $j = \overline{1,s}$, will be written in one of the following ways:

$$A = [a_{ij}]_{i=\overline{1,r};j=\overline{1,s}} = [a_{i1}, \ldots, a_{is}]_{i=\overline{1,r}} = [a_{1j}, \ldots, a_{rj}]^T_{j=\overline{1,s}},$$

where i is the row index, j is the column index, and T is the sign for transposition. If A has the maximum rank, we shall write $A \# 0$.

4. By E we denote the identity matrix of the corresponding dimension or the identity operator on the corresponding space.

5. If for $\lambda \to \lambda_0$

$$F(\lambda) = \sum_{k=-q}^{p} \alpha_k \cdot (\lambda - \lambda_0)^k + o\big((\lambda - \lambda_0)^p\big),$$

then

$$\big[F(\lambda)\big]^{\langle k \rangle}_{|\lambda=\lambda_0} = F_{\langle k \rangle}(\lambda_0) \stackrel{\text{def}}{=} \alpha_k.$$

PART 1

DIFFERENTIAL OPERATORS WITH INTEGRABLE COEFFICIENTS

1. Transformation Operator Method

1.1. Formulation of the inverse problem. Let us consider the self-adjoint boundary value problem

$$(-1)^n y^{(2n)} + \sum_{k=0}^{n-1} (-1)^k \big(p_k(x) y^{(k)}\big)^{(k)} = \rho^{2n} y \tag{1.1}$$

on the semiaxis $(0, \infty)$ for certain boundary conditions at the point $x = 0$. If the coefficients $p_k(x)$ are summable on the semiaxis $(0, \infty)$, then (1.1) has a bounded solution $u(x, \rho)$ $(\rho > 0)$ for $x \to \infty$ which satisfies the boundary conditions and generates the Fourier expansion with respect to the eigenfunctions of the boundary value problem for (1.1). Let $y(x, \rho)$ be a solution of (1.1) that has the asymptotics

$$y(x, \rho) = \exp(i\rho x) \cdot \big(1 + o(1)\big), \qquad x \to \infty. \tag{1.2}$$

Let us assume that the function $y(x, \rho)$ is holomorphic with respect to ρ in the upper half-plane and is continuous on the real axis. Then $u(x, \rho) = (2\pi)^{-1/2} \sum\limits_{k=0}^{n} S_k(\rho) y(x, \rho\omega_k)$, $\rho > 0$, where $\omega_k = \exp(i\pi k/n)$. The solution $u(x, \rho)$ can be normalized by the condition $S_n(\rho) \equiv 1$. For brevity, let us agree to call the functions $S_k(\rho)$ phases. In Sec. 1, we consider the inverse scattering problem which consists in the reconstruction of the coefficients of (1.1) and the boundary conditions from the given phases

$$S_0(\rho),\ S_1(\rho),\ \ldots,\ S_{n-1}(\rho) \tag{1.3}$$

(for simplicity, it is assumed for the present that the point spectrum is absent).

We shall assume that the coefficients of (1.1) are holomorphic in the sector $|\arg z| < \frac{\pi}{2} - \frac{\pi}{2n}$. Under this assumption, there exists a triangular transformation which transforms the function $\exp(i\rho x)$ into the solution (1.2) of Eq. (1.1), and is holomorphic with respect to ρ for $\operatorname{Im}\rho > 0$. As shown below, the kernel $K(x,t)$ of this triangular transformation satisfies the Gel'fand–Levitan–Marchenko integral equation with a kernel $F(t,\xi)$ which is constructed in a special manner from the phases (1.3). The solution of the considered IP is the result of analysis of this integral equation, which will be called in the sequel the main equation.

A more detailed investigation of the main equation enables us to find necessary and sufficient conditions under which given functions (1.3) are the phases of a certain self-adjoint boundary value problem for (1.1).

In Sec. 1.6, we give the theorem that the boundary value problem for (1.1) is determined uniquely from its spectral matrix.

We note that the problem of reconstruction of boundary value problem for (1.1) with $n > 1$ from the spectral matrix-function, and the problem on conditions for the existence of the triangular transformation, have been considered in a series of articles by Sakhnovich. In particular, a local (in a certain sense) solution of the IP has been obtained in [78] by the Gel'fand–Levitan method.

1.2. Auxilary propositions. Let us consider the following self-adjoint differential expression, which is more general than (1.1):

$$ly \equiv (-1)^n y^{(2n)} + \sum_{k=0}^{n-1}(-1)^k\left(p_{2k}(x)y^{(k)}\right)^{(k)} - \sum_{k=0}^{n-2}(-1)^k\frac{i}{2}\left(\left(p_{2k+1}(x)y^{(k)}\right)^{(k+1)}\left(p_{2k+1}(x)y^{(k+1)}\right)^{(k)}\right). \quad (1.4)$$

Throughout this section, we will assume that the coefficients $p_k(x)$ are real for $x > 0$ and satisfy the following conditions for a certain α $(0 \le \alpha \le \infty)$:

$$\int_0^\alpha x^{2n-1-k}\left|p_k(x)\right| dx + \int_\alpha^\infty \left|p_k(x)\right| dx < \infty, \qquad k = \overline{0, 2n-2}. \quad (1.5)$$

Following [69, p. 182], let us define the quasiderivatives $y^{[k]}(x)$ $(k = \overline{0,2n})$ of the function $y(x)$ corresponding to the expression (1.4) by the equalities

$$\begin{aligned} y^{[k]} &= y^{(k)}, \qquad k = \overline{0,n};\\ y^{[n+k]} &= \frac{i}{2}p_{2n-2k-1}y^{(n-k-1)} + p_{2n-2k}y^{(n-k)}\\ &\quad - \frac{i}{2}p_{2n-2k+1}y^{(n-k+1)} - \frac{d}{dx}\left(y^{[n+k-1]}\right), \qquad k = \overline{1, n-1};\\ y^{[2n]} &= p_0 y - \frac{i}{2}p_1 y' - \frac{d}{dx}\left(y^{[2n-1]}\right). \end{aligned}$$

We will assume that ly has sense if all quasiderivatives of $y(x)$ of orders upto and including $2n-1$ exist and are absolutely continuous on each segment $[\alpha,\beta] \subset (0,\infty)$; then $ly = y^{[2n]}$.

Let $y(x)$ and $w(x)$ be functions for which (1.4) has sense. The Lagrange formula

$$\int_\alpha^\beta ly \cdot \overline{(w(x))}\, dx - \int_\alpha^\beta y(x) \cdot \overline{lw(x)}\, dx = \{y,w\}_\beta - \{y,w\}_\alpha, \qquad 0 < \alpha < \beta < \infty, \quad (1.6)$$

where

$$\{y, w\}_x = \sum_{k=0}^{n-1} \left(y^{[k]}(x)\overline{w^{[2n-1-k]}(x)} - y^{[2n-1-k]}(x)\overline{w^{[k]}(x)} \right) \tag{1.7}$$

is valid. We will also use the symbol $\{y, w\}_{x,\mu}$ for the functions y and w depending on a parameter μ.

Remark 1.1. It follows from (1.6) that the expression $\{y, w\}_{x,\mu}$ does not depend on x for the solutions $y(x, \mu)$ and $w(x, \mu)$ of the equation $ly = \mu y$ for real μ.

Remark 1.2. We can prove that each solution of the equation $ly - \mu y = f$ is continuous at the point $x = 0$ for $f \in \mathcal{L}_2(0, \infty)$.

For each nonzero ρ the equation

$$ly = \rho^{2n} y \tag{1.8}$$

has solutions $y_k(x, \rho)$ $(k = \overline{0, 2n-1})$, for which the following asymptotic formulas hold for $x \to \infty$ (see [69, p. 320]):

$$y_k(x, \rho) = \exp(i\omega_k \rho x) \cdot \big(1 + o(1)\big), \qquad k = \overline{0, 2n-1}, \tag{1.9}$$

where

$$\omega_k = \exp(i\pi k/n), \qquad k = \overline{0, 2n-1}, \tag{1.10}$$

and, in addition,

$$y_k^{[\nu]}(x, \rho) = (i\omega_k \rho)^\nu \exp(i\omega_k \rho x) \cdot \big(1 + o(1)\big), \qquad \nu = \overline{0, n-1}, \tag{1.11}$$

$$y_k^{[\nu]}(x, \rho) = (-1)^{\nu-n} (i\omega_k \rho)^\nu \exp(i\omega_k \rho x) \cdot \big(1 + o(1)\big), \qquad \nu = \overline{n, 2n-1}. \tag{1.12}$$

Let D_0^* denote the set of all functions $y \in \mathcal{L}_2(0, \infty)$ such that $ly \in \mathcal{L}_2(0, \infty)$ and let D_0 denote the set of all functions $y \in D_0^*$ such that $y^{[k]}(0) = 0$ $(k = \overline{0, 2n-1})$. Let us define operators L_0^* and L_0 $(L_0 \subset L_0^*)$ on the linear manifolds D_0^* and D_0 by setting $L_0^* y = ly$ for $y \in D_0^*$. One can prove that L_0 is a symmetric closed operator and L_0^* is adjoint to L_0 (see [69 ,p. 202]). By virtue of Remark 1.2, it follows from (1.9) that the deficiency index of L_0 is (n, n) and, consequently, L_0 admits self-adjoint extensions. The following two propositions are proved by the methods used in [69].

Lemma 1.1. *The domain D of each self-adjoint extension L of L_0 is the set of all those functions $y(x)$ from D_0^* which satisfy the boundary conditions*

$$\{y, w_k\}_0 = 0, \qquad k = \overline{1, n}, \tag{1.13}$$

where $w_k(x)$ $(k = \overline{1, n})$ are certain functions from D_0^* which are linearly independent modulo D_0 and are such that

$$\{w_j, w_k\}_0 = 0, \qquad j, k = \overline{1, n}. \tag{1.14}$$

Conversely, for arbitrary functions $w_k \in D_0^*$ $(k = \overline{1, n})$ that are linearly independent modulo D_0 and satisfy (1.14)*, the boundary conditions* (1.13) *generate the domain of a certain self-adjoint extension of the operator L_0. In particular, the Dirichlet boundary conditions $y^{(k)}(0) = 0$, $k = \overline{0, n-1}$, are of this type.*

Everywhere in the sequel, L denotes the self-adjoint extension of the operator L_0 determined by the boundary conditions (1.13).

Theorem 1.1. *The following statements are valid:*

(1) *The continuous spectrum of the operator L coincides with the semiaxis $[0, \infty)$. The point spectrum of the operator L is bounded from below and does not have any nonzero finite condensation points. The multiplicity of nonpositive eigenvalues does not exceed n, and the multiplicity of positive eigenvalues does*

not exceed $n-1$. In addition, the set of positive eigenvalues of multiplicity $n-1$ is bounded (in the case of the Dirichlet boundary conditions, the point spectrum is bounded).

(2) *If (1.5) is fulfilled for $\alpha = 0$, then the point spectrum of the operator L is bounded.*

(3) *If (1.5) is fulfilled for $\alpha = \infty$, then the point spectrum of the operator L does not contain the origin. Moreover, the number of the negative (positive) eigenvalues of multiplicity $n-1$ ($n > 1$) of the operator L are finite.*

Let $\Delta_k(\rho)$, $k = \overline{0,n}$, denote a minor of order n of the rectangular matrix $\left[\{y_k, w_j\}_{0,\rho}\right]_{k=\overline{0,n};j=\overline{1,n}}$ which does not contain the kth row of the matrix, where $y_k(x,\rho)$ is a solution (1.9) of (1.8) and the functions $w_j(x)$ are the same as in (1.13). Then the solution

$$\tilde{u}(x,\rho) = \sum_{k=0}^{n} (-1)^k \Delta_k(\rho) y_k(x,\rho) \tag{1.15}$$

of (1.8) satisfies the boundary conditions (1.13).

Note that the minors $\Delta_0(\rho)$, $-\frac{\pi}{n} \le \arg\rho \le 0$, and $\Delta_n(\rho)$, $0 \le \arg\rho \le \frac{\pi}{n}$, do not depend on the choice of the solutions $y_k(x,\rho)$ with the asymptotics indicated in (1.9)–(1.12); moreover, these minors are holomorphic in the indicated open sectors and are continuous upto the boundary ($\rho \ne 0$).

Lemma 1.2. *The zeros of the minors $\Delta_0(\rho)$ and $\Delta_n(\rho)$ coincide on the semiaxis $(0,\infty)$. Moreover, the number ρ^{2n} ($\rho \ne 0$, $-\frac{\pi}{n} \le \arg\rho \le \frac{\pi}{n}$) is an eigenvalue of the operator L if and only if $\Delta_0(\rho) = 0$ for $-\frac{\pi}{n} \le \arg\rho \le 0$ or $\Delta_n(\rho) = 0$ for $0 \le \arg\rho \le \frac{\pi}{n}$. For $\rho > 0$*

$$\left|\Delta_0(\rho)\right| = \left|\Delta_n(\rho)\right|. \tag{1.16}$$

Let T^+ (T^-) denote the set of all the numbers $\rho > 0$ ($\arg\rho = -\frac{\pi}{2n}$) such that the numbers ρ^{2n} are the eigenvalues of L, and let $T = T^+ \cup T^-$. We set

$$S_k(\rho) = (-1)^{n+k} \frac{\Delta_k(\rho)}{\Delta_n(\rho)}, \qquad \rho > 0, \quad \rho \notin T^+, \quad k = \overline{0,n}. \tag{1.17}$$

By virtue of (1.16), we have

$$\left|S_0(\rho)\right| = S_n(\rho) = 1. \tag{1.18}$$

Denote

$$u(x,\rho) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} S_k(\rho) y_k(x,\rho), \qquad \rho > 0, \quad \rho \notin T^+. \tag{1.19}$$

For each $\rho > 0$ ($\rho \notin T^+$) the function $u(x,\rho)$ is the unique (upto a constant factor) bounded solution of (1.8) that satisfies (1.13). By virtue of (1.9), we have $\sqrt{2\pi}u(x,\rho) = \exp(-i\rho x) + S_0(\rho)\exp(i\rho x) + o(1)$ for $x \to \infty$. The functions $u(x,\rho)$ and $S_0(\rho)$ do not depend on the choice of the solution (1.9). The functions $S_k(\rho)$, $0 < k < n$, depend on the choice of the solutions of (1.9) and will be defined below. For each $\rho \in T$ let $m(\rho)$ denote the multiplicity of the eigenvalue ρ^{2n}, and $\varphi_k(x,\rho)$, $k = \overline{1,m(\rho)}$, denote the corresponding orthogonal system of eigenfunctions.

Theorem 1.2. *Let $f(x), g(x) \in \mathcal{L}_2(0,\infty)$. Then the integrals*

$$F(\rho) = \int_0^\infty f(x)\overline{u}(x,\rho)\,dx, \qquad G(\rho) = \int_0^\infty g(x)\overline{u}(x,\rho)\,dx$$

are convergent in the sense of the metric of $\mathcal{L}_2(0,\infty)$ and

$$\int_0^\infty f(x)\overline{g}(x)\,dx = \int_0^\infty F(\rho)\overline{G}(\rho)\,d\rho + \sum_{\rho\in T}\sum_{k=1}^{m(\rho)} \int_0^\infty f(x)\overline{\varphi}_k(x,\rho)\,dx \cdot \int_0^\infty \overline{g}(x)\varphi_k(x,\rho)\,dx. \tag{1.20}$$

It is easily verified that the integral operator with the kernel

$$\Phi(x,t,\rho) = \sum_{k=1}^{m(\rho)} \varphi_k(x,\rho)\overline{\varphi}_k(t,\rho), \qquad \rho \in T, \quad 0 \le x,t < \infty,$$

is the orthoprojection onto the eigensubspace of the operator L corresponding to the eigenvalue ρ^{2n}. It is obvious that, by virtue of (1.9), the kernel $\Phi(x,t,\rho)$ can also be represented in the form

$$\Phi(x,t,\rho) = \sum_{k,j=1}^{n} N_{kj}(\rho) y_k(x,\rho)\overline{y}_j(t,\rho), \qquad \rho \in T, \quad \rho \neq 0, \tag{1.21}$$

where $N_{kn}(\rho) = N_{nk}(\rho) = 0$ $(k = \overline{1,n})$ for $\rho \in T^+$. Denote

$$S_{kj}(\rho) = \frac{1}{2\pi} S_k(\rho)\overline{S}_j(\rho), \qquad \rho > 0, \quad k,j = \overline{0,n}, \tag{1.22}$$

and introduce the nonnegative Hermitian matrices

$$N(\rho) = \left[N_{kj}(\rho)\right]_{k,j=\overline{1,n}}, \qquad \rho \in T, \quad \rho \neq 0, \tag{1.23}$$

$$S(\rho) = \left[S_{kj}(\rho)\right]_{k,j=\overline{0,n}}, \qquad \rho > 0. \tag{1.24}$$

The rank of the matrix $N(\rho)$ coincides with the multiplicity of the eigenvalue ρ^{2n} of the operator L, and $S(\rho)$ is a matrix of rank one.

Let E_μ $(-\infty < \mu < \infty)$ be a left-continuous spectral function (a resolution of the identity) of the operator L. Since the spectrum is bounded from below, it follows that E_μ is an integral operator whose kernel will be denoted by $E(x,t,\mu)$ It is obvious that $\Phi(x,t,\rho) = E(x,t,\rho^{2n}+0) - E(x,t,\rho^{2n})$, $\rho \in T$. It can be concluded from Theorem 1.2 that the derivative $\psi(x,t,\rho) = \dfrac{\partial}{\partial\rho} E(x,t,\rho^{2n})$, $\rho > 0$, $\rho \notin T^+$, exists, and

$$\psi(x,t,\rho) = \sum_{k,j=0}^{n} S_{kj}(\rho) y_k(x,\rho)\overline{y}_j(t,\rho), \qquad \rho > 0.$$

1.3. A triangular transformation. Let us now assume that the coefficients $p_k(x)$ in (1.4) are holomorphic in the sector

$$\Omega_a = \left\{ z : \left|\arg(z-a)\right| < \frac{\pi}{2} - \frac{\pi}{2n} \right\} \tag{1.25}$$

and satisfy the conditions

$$\int_0^a x^{2n-1-k} \left|p_k(x)\right| dx + \int_0^a x^{2n-1-k} \left(\sup_{\operatorname{Re} z = x} \left|p_k(x)\right| \right) dx < \infty, \qquad k = \overline{0, 2n-2}, \tag{1.26}$$

for a certain finite $a \ge 0$. Then for all ρ such that $\operatorname{Im}\rho \ge 0$ Eq.(1.8) has a solution $y(x,\rho)$ which can be represented on the semiaxis $[a,\infty)$ in the form

$$y(x,\rho) = \exp(i\rho x) + \int_x^\infty K(x,t)\exp(i\rho t)\, dt, \qquad a \le x, \infty, \tag{1.27}$$

where $K(x,t)$ does not depend on ρ, and for each $\xi \ge 0$ the function

$$K_0(x,\xi) = K(z, z+\xi) \tag{1.28}$$

is holomorphic with respect to z in Ω_a, and

$$\left|K_0(x,\xi)\right| \le h\left(\operatorname{Re} z + \frac{\xi}{2}\right) \tag{1.29}$$

with a certain function $h(x)$ which is nonincreasing and summable on the semiaxis (a, ∞).

The validity of the triangular representation (1.27) has been obtained in [46] and other papers. To obtain (1.27) we essentially use the analyticity of the coefficients in the sector (1.25).

By virtue of (1.29), it also follows from (1.27) that

$$\exp(i\rho x) = Y(x,\rho) + \int_x^\infty H(x,t)y(t,\rho)\,dt, \qquad a \le x < \infty, \tag{1.30}$$

where $H(x,t)$ is a solution of the integral equation

$$K(x,t) + H(x,t) + \int_x^t H(x,\xi)K(\xi,t)\,d\xi = 0, \qquad a < x \le t < \infty. \tag{1.31}$$

It follows from (1.31) that for each $\xi \ge 0$ the function

$$H_0(z,\xi) = H(z, z+\xi) \tag{1.32}$$

is holomorphic with respect to z in the sector Ω_a and, by virtue of (1.29), satisfies the inequality

$$\left|H_0(z,\xi)\right| \le C \cdot h\left(\operatorname{Re} z + \frac{\xi}{2}\right), \qquad C > 0. \tag{1.33}$$

We note that, by virtue of (1.27) and (1.29), for each $x \ge 0$ the solution $y(x,\rho)$ is holomorphic with respect to ρ in the half-plane $\operatorname{Im}\rho > 0$ and continuous on the real axis. For $x \to \infty$, uniformly with respect to ρ in the domain $\operatorname{Im}\rho \ge 0$, we have

$$y(x,\rho) = \exp(i\rho x) \cdot \bigl(1 + o(1)\bigr). \tag{1.34}$$

Remark 1.3. If, under conditions (1.5), Eq. (1.8) has two solutions $v_1(x,\rho)$ and $v_2(x,\rho)$ which, for each $x \ge 0$, are holomorphic with respect to ρ in the half-plane $\operatorname{Im}\rho > 0$ and have the asymptotics $v_k(x,\rho) = \exp(i\rho x)\cdot\bigl(1+o(1)\bigr)$ for $x \to \infty$ for each ρ, then $v_1(x,\rho) = v_2(x,\rho)$.

By the remark made above, under conditions (1.26) Eq. (1.8) has only one solution $y(x,\rho)$ which can be represented for $\operatorname{Im}\rho \ge 0$ in the form (1.27) with the kernel $K(x,t)$ satisfying the condition

$$\lim_{x\to\infty} \int_x^\infty \left|K(x,t)\right| dt = 0.$$

The kernel $K(x,t)$ is unique in representation (1.27).

In conclusion, we show that deletion of the condition of analyticity of the coefficients leads, in general, to loss of the triangular representation. Let us consider the equation

$$(-1)^n y^{(2n)} - q(x)y = \rho^{2n} y, \qquad 0 \le x < \infty, \tag{1.35}$$

in which $q(x)$ is the characteristic function of the interval $[0,1)$. Let us assume that the solution $y(x,\rho)$ of (1.35) can be represented in the form (1.27) for all $x \ge 0$ and $\operatorname{Im}\rho \ge 0$. Then $y(x,\rho)$ must be bounded in the domain $x \ge$, $\operatorname{Im}\rho \ge 0$; and, by Remark 1.3, for $x \ge 1$ we must have $y(x,\rho) = \exp(i\rho x)$. Extending this function as the solution of (1.35) to the segment $[0,1]$, we obtain the formula

$$y(x,\rho) = \frac{b(\rho)\exp(i\rho)}{2n(\rho^{2n}+1)} \sum_{k=0}^{2n-1} \frac{\omega_k \exp\bigl(i\omega_k b(\rho)(x-1)\bigr)}{\omega_k b(\rho) - \rho}, \qquad b(\rho) = \sqrt[2n]{\rho^{2n}+1}.$$

Since $\operatorname{Re}(i\omega_k) < 0$ for $k = \overline{1, n-1}$, it follows that the function $y(x,\rho)$ is unbounded for $0 \le x < 1$, $\rho \to +\infty$. We have obtained a contradiction.

1.4. Scattering data. Let the coefficients $p_k(x)$ be holomorphic in the sector (1.25) and satisfy (1.26). By Theorem 1.1, $0 \notin T$. Using (1.27), we can introduce the following solutions of (1.8):

$$y_k(x,\rho) = y(x,\rho\omega_k), \qquad \mathrm{Im}(\rho\omega_k) \geq 0, \quad k = \overline{0,n}. \tag{1.36}$$

It is obvious that (1.9) is valid for solutions (1.36) by virtue of (1.34). It is also obvious that for $\rho > 0$ solutions (1.36) form a basis of the subspace of bounded solutions of (1.8).

Everywhere in the sequel, we will assume that the functions $S_k(\rho)$ introduced via (1.17), and also the matrices $N(\rho)$ and $S(\rho)$, introduced via (1.21)–(1.24), correspond to solutions (1.36).

Let us consider the data set

$$\big(T,\ N(\rho)(\rho \in T),\ S(\rho)(\rho > 0)\big), \tag{1.37}$$

which we agree to call in the sequel the scattring data.

1.5. Inverse scattering problem. Let us consider the IP of recovering L from the data set (1.37). To solve this problem, let us consider the function

$$\widetilde{F}(x,t,r,R) = \int_r^R \int_a^x \int_a^t \sum_{k,j=0}^n S_{kj}(\rho)\exp(i\omega_k\rho\xi)\exp(-i\overline{\omega}_j\rho\eta)\,d\eta d\xi d\rho$$

$$+ \sum_{\substack{\rho\in T\\ r<|\rho|<R}} \int_a^x \int_a^t \sum_{k,j=1}^n N_{kj}(\rho)\exp(i\omega_k\rho\xi)\exp(-i\overline{\omega}_j\overline{\rho}\eta)\,d\eta d\xi, \qquad a \leq x,t < \infty, \quad 0 < r < R.$$

Theorem 1.3. *For arbitrary $x,t \in [a,\infty)$ the limit $\widetilde{F}(x,t) = \lim\limits_{\substack{r\to 0\\ R\to\infty}} \widetilde{F}(x,t,r,R)$ exists and is finite. Moreover, the function $\widetilde{F}(x,t)-\min(x,t)$ has continuous partial derivatives of second order in the domain $a < x,t < \infty$. The derivative $F(x,t) = \dfrac{\partial^2}{\partial x\partial t}\big(\widetilde{F}(x,t) - \min(x,t)\big)$, $a < x,t < \infty$, satisfies the relation $F(x,t) = \overline{F}(x,t)$ and, in addition,*

$$F(x,t) = H(x,t) + \int_t^\infty H(x,\eta)\overline{H}(t,\eta)\,d\eta, \qquad a < x \leq t, \tag{1.38}$$

$$F(x,t) + K(x,t) + \int_x^\infty K(x,\eta)F(\eta,t)\,d\eta = 0, \qquad a < x \leq t. \tag{1.39}$$

The proof of this theorem can be carried out with the help of the Parseval equality (1.20) and the formulas (1.27) and (1.30) in the same way as in the case $n = 1$ (see [63, pp. 185–188]).

Equation (1.38) shows that the function $F(x,t)$ is the same for all self-adjoint extensions of the operator It follows from (1.38) that for each $\xi \geq 0$ the function $F_0(z,\xi) = F(z,z+\xi)$ can be analytically continued with respect to z from the semi-axis (a,∞) into the sector Ω_a and satisfies the inequality

$$|F_0(z,\xi)| \leq Ch\left(\mathrm{Re}\,z + \frac{\pi}{2}\right), \tag{1.40}$$

where $h(x)$ is the same as in (1.33).

For each $z \in \Omega_a$ we define the integral operators H_z, H_z^*, and G_z:

$$H_z f(\xi) = \int_{\xi}^{\infty} H_0(z+\xi, \eta-\xi) f(\eta)\, d\eta, \qquad H_z^* f(\xi) = \int_0^{\xi} \overline{H}_0(z+\eta, \xi-\eta) f(\eta)\, d\eta, \qquad 0 < \xi < \infty,$$

$$G_z f(\xi) = \int_0^{\infty} G(z,\xi,\eta) f(\eta)\, d\eta, \qquad G(z,\xi,\eta) = \begin{cases} F_0(z+\xi, \eta-\xi), & \xi \le \eta, \\ \overline{F}_0(\overline{z}+\eta, \xi-\eta), & \xi \ge \eta. \end{cases}$$

By virtue of (1.33) and (1.40), the operators H_z, H_z^*, and G_z are completely continuous in each space $\mathcal{L}_p(0,\infty)$, $1 \le p \le \infty$, and by virtue of (1.38) we have

$$E + G_z = (E + H_z)(E + H_z^*). \tag{1.41}$$

It follows from (1.41) that for each $z \in \Omega_a$ the operator $E + G_z$ has an inverse in $\mathcal{L}_p(0,\infty)$, $1 \le p \le \infty$.

Theorem 1.4. *For each $z \in \Omega_a$ the kernel $K_0(z,\xi)$, as a function of $\xi \ge 0$, satisfies the integral equation*

$$F_0(z,\xi) + K_0(z,\xi) + \int_0^{\infty} K_0(z,\eta) G(z,\eta,\xi)\, d\eta = 0 \tag{1.42}$$

and is the unique solution of this equation in $\mathcal{L}_1(0,\infty)$.

Integral equation (1.42) is obtained from (1.39) by a change of variables, and its unique solvability follows from the invertibility of the operator $E + G_z$ and from the relation $G(z,\xi,\eta) = \overline{G}(\overline{z},\xi,\eta)$.

Theorem 1.5. *The data set* (1.37) *uniquely determines the corresponding self-adjoint operator L with the coefficients $p_k(x)$ which are holomorphic in a certain domain, containing sector* (1.25) *and the interval* $(0,a]$ *and satisfy* (1.26).

Proof. By Theorem 1.4, the data set (1.37) uniquely determines the kernel $K_0(z,\xi)$, $z \in \Omega_a$, $\xi \ge 0$, and, consequently, the kernel $K(x,t)$ also $(a < x \le t < \infty)$. By (1.27), for $\operatorname{Im}\rho \ge 0$ the solution $y(x,\rho)$ of (1.8) is uniquely determined on the semiaxis (a,∞) which, in its turn, uniquely determines the coefficients $p_k(x)$ of (1.8) on the semiaxis (a,∞). We note that the functions $p_k(x)$ can be determined immediately with the help of the kernel $K(x,t)$. By virtue of analyticity, the coefficients $p_k(x)$ are uniquely determined for all $x > 0$. But then the solution $u(x,\rho)$ of (1.8) which satisfies the boundary conditions (1.13) for all $\rho > 0$ is uniquely determined by (1.36) and (1.19). It remains to consider the fact that in the boundary conditions (1.13), generating the desired operator L, we can take as $w_k(x)$ an arbitrary function from D_0^* that coincides with the function $u(x,\rho_k)$, where $\rho_1, \rho_2, \dots, \rho_n$ are certain positive numbers, in a neighborhood of the origin. This fact can be proved easily.

1.6. Recovery of the operator from its spectral matrix. Let the operator L with the domain D and the spectral kernel $E(x,t,\mu)$ be the same as in Sec. 1.2. Let us define the matrix $\sigma(\mu) = \left[\sigma_{kj}(\mu)\right]_{k,j=\overline{1,n}}$, $-\infty < \mu < \infty$, by the formula

$$\sigma_{kj}(\mu) = \Big\{v_k(t), \big\{v_j(x), E(x,t,\mu)\big\}\Big\}_{|x=t=0}, \qquad k,j = \overline{1,n},$$

where $v_k(x)$ $(k = \overline{1,n})$ are certain functions from D_0^* that are lineary independent modulo D. Then $\sigma(\mu)$ is the spectral matrix of the operator L (see [69, pp. 255, 273]), corresponding to the system of the solutions $u_k(x,\mu)$ $(k = \overline{1,n})$ of the equation $lu = \mu u$, and satisfy the conditions

$$\{u_k, w_j\}_{0,\mu} = 0, \qquad \{u_k, v_j\}_{0,\mu} = \delta_{kj}, \qquad k,j = \overline{1,n},$$

where $w_j(x)$ are the same as in the boundary conditions (1.13).

Theorem 1.6. *A self-adjoint differential operator L with the coefficients which satisfy* (1.26) *for $a = 0$ is uniquely determined from its spectral matrix $\sigma(\mu)$.*

The proof of this theorem is based on the fact that the data set (1.37) is determined uniquely from the matrix $\sigma(\mu)$.

2. Recovery of Non-self-adjoint Differential Operators from the Weyl Matrix

2.1. The uniqueness theorem.

2.1.1. We consider a differential equation (DE) and linear forms (LF) $L = (l, U)$ of the form

$$ly \equiv y^{(n)} + \sum_{\nu=0}^{n-2} p_\nu(x) y^{(\nu)} = \lambda y, \qquad 0 \le x \le T \le \infty, \tag{2.1}$$

$$U_{\xi a}(y) = y^{(\sigma_{\xi a})}(a) + \sum_{\nu=0}^{\sigma_{\xi a}-1} u_{\xi\nu a} y^{(\nu)}(a), \qquad \xi = \overline{1,n} \tag{2.2}$$

on the half-line ($T = \infty$) or on the finite interval ($T < \infty$). Here $p_\nu(x) \in \mathcal{L}(0,T)$ are complex-valued integrable functions; $a = 0$ for $T = \infty$, and $a = 0, T$ for $T < \infty$; $0 \le \sigma_{\xi a} \le n-1$, $\sigma_{\xi a} \ne \sigma_{\eta a}$ $(\xi \ne \eta)$.

Let $\lambda = \rho^n$. It is known (see [69, p. 53]) that the ρ-plane can be partitioned into sectors S of angle $\frac{\pi}{n}$ ($\arg\rho \in \left(\frac{\nu\pi}{n}, \frac{(\nu+1)\pi}{n}\right)$, $\nu = \overline{0, 2n-1}$) in which the roots $R_1, \dots, R_n$ of the equation $R^n - 1 = 0$ can be numbered in such a way that

$$\operatorname{Re}(\rho R_1) < \operatorname{Re}(\rho R_2) < \dots < \operatorname{Re}(\rho R_n), \qquad \rho \in S. \tag{2.3}$$

Let the functions $\Phi(x,\lambda) = \left[\Phi_m(x,\lambda)\right]_{m=\overline{1,n}}$ be solutions of (2.1)satisfying the conditions $U_{\xi 0}(\Phi_m) = \delta_{\xi m}$, $\xi = \overline{1,m}$, and $U_{\eta T}(\Phi_m) = 0$, $\eta = \overline{1, n-m}$ (for $T < \infty$), $\Phi_m(x,\lambda) = \left(\exp(\rho R_m x)\right)$, $x \to \infty$, $\rho \in S$ (for $T = \infty$). Here and in the sequel, $\delta_{\xi,m}$ is the Kronecker symbol. Denote $\mathfrak{M}_{mk}(\lambda) = U_{k0}(\Phi_m)$, $k = \overline{m+1, n}$. The functions $\Phi_m(x,\lambda)$ and $\mathfrak{M}_{mk}(\lambda)$ are called the Weyl solutions (WS's) and the Weyl functions (WF's), respectively. The matrix $\mathfrak{M}(\lambda) = \left[\mathfrak{M}_{mk}(\lambda)\right]_{m,k=\overline{1,n}}$, $\mathfrak{M}_{mk}(\lambda) = \delta_{mk}$, $k = \overline{1,m}$, is called the Weyl matrix (WM) or the spectrum of L. Thus, $\mathfrak{M}(\lambda) = U_0(\Phi(x,\lambda))$, where $U_a = [U_{\xi a}]^T_{\xi=\overline{1,n}}$. We note that

$$\Phi(x,\lambda) = \mathfrak{M}(\lambda) C(x,\lambda), \tag{2.4}$$

where $C(x,\lambda) = \left[C_m(x,\lambda)\right]_{m=\overline{1,n}}$ are the solutions of (2.1) under the conditions $U_{\xi 0}(C_m) = \delta_{\xi m}$, $\xi = \overline{1,n}$.

Formulation of the inverse problem. Given the WM $\mathfrak{M}(\lambda)$, construct the DE and LF $L = (l, U)$.

In 2.1, we study the properties of the WF's and prove the uniqueness theorem of recovering the DE and LF (2.1)–(2.2) on the half-line and on the finite interval from the given WM $\mathfrak{M}(\lambda)$ when the behavior of the spectrum is arbitrary. Below, in Sec. 3, we provide a counterexample showing that dropping one element of the WM violates the uniqueness of the solution of the IP.

2.1.2. Let $\alpha \in (0,T)$, $\rho_\alpha = 2n \cdot \max\limits_\nu \|p_\nu\| \mathcal{L}(\alpha, T)$. It is known (see, for example, [69, p. 58]) that in each sector S with the property (2.3) there exists a fundamental system of solutions (FSS) $B_\alpha = \left\{y_k(x,\rho)\right\}_{k=\overline{1,n}}$ of the DE (2.1) of the form

$$y_k^{(\nu)}(x,\rho) = (\rho R_k)^\nu \exp(\rho R_k x) \cdot \left(1 + O(\rho^{-1})\right), \qquad |\rho| \to \infty, \quad x \ge \alpha, \tag{2.5}$$

where for $x \geq \alpha$ and $r_k = k$ the functions $y_k(x,\rho)$ satisfy the equations

$$y_k(x,\rho) = \exp(\rho R_k x) - \int_\alpha^x \sum_{j=1}^{r_k} R_j \exp\big(\rho R_j(x-t)\big) M_t(y_k)\,dt + \int_x^T \sum_{j=r_k+1}^{n} R_j \exp\big(\rho R_j(x-t)\big) M_t(y_k)\,dt,$$

$$M_t(y_k) = \frac{1}{n}\rho^{1-n} \sum_{\mu=0}^{n-2} p_\mu(t) y_k^{(\mu)}(t,\rho). \tag{2.6}$$

The functions $y_k^{(\nu)}(x,\rho)$, $\nu = \overline{0,n-1}$, are regular for each $x \geq 0$ with respect to $\varepsilon \in S_\alpha = \{\rho : \rho \in S, |\rho| > \rho_\alpha\}$, are continuous for $x \geq 0$, $\rho \in \overline{S}_\alpha$ and have the estimate

$$\left| y_k^{(\nu)}(x,\rho)\cdot(\rho R_k)^{-\nu}\exp(-\rho R_k x) - 1\right| \leq \rho_\alpha \cdot |\rho|^{-1}, \qquad x \geq \alpha, \quad \rho \in \overline{S}_\alpha.$$

As $|\rho| \to \infty$, $\rho \in S$,

$$\det\big[y_k^{(\nu-1)}(x,\rho)\big]_{k,\nu=\overline{1,n}} = \rho^{\frac{n(n-1)}{2}} \cdot \det\big[R_k^{\nu-1}\big]_{k,\nu=\overline{1,n}} \cdot \big(1 + O(r^{-1})\big).$$

Moreover, we require the FSS $B_{\alpha m} = \{y_1^0(x,\rho), \dots, y_m^0(x,\rho), y_{m+1}(x,\rho), \dots, y_n(x,\rho)\}$ of the DE (2.1), where $y_k(x,\rho) \in B_\alpha$, $k = \overline{m+1,n}$ and the functions $y_k^0(x,\rho)$, $k = \overline{1,m}$ are solutions of (2.6) for $x \geq \alpha$ and $r_k = m$. Furthermore, the functions $y_k^{0(\nu)}(x,\rho)$, $\nu = \overline{0,n-1}$ are continuous for $x \geq 0$, $\rho \in \overline{S}_\alpha$, are regular with respect to $\rho \in S_\alpha$ for each $x \geq 0$, and $y_k^{0(\nu)}(x,\rho) = O\big(\rho^\nu \exp(\rho R_m x)\big)$, $x \geq \alpha$, $|\rho| \to \infty$, $\rho \in S$.

2.1.3. Let $\omega_\xi(R) = R^{\sigma_{\xi 0}}$,

$$\Omega(j_1,\dots,j_p) = \det\big[\omega_{j_\nu}(R_k)\big]_{\nu,k=\overline{1,p}},$$
$$\Omega_\mu(j_1,\dots,j_p) = \det\big[\omega_{j_\nu}(R_k)\big]_{\nu=\overline{1,p};k=\overline{1,p+1}\setminus\mu},$$

$$\mu_{mk}^0 = \big(\Omega(\overline{1,m})\big)^{-1} \cdot \Omega\big(\overline{1,m-1},k\big),$$
$$a_{mk}^0 = (-1)^{m+k}\big(\Omega(\overline{1,m})\big)^{-1} \cdot \Omega_k\big(\overline{1,m-1}\big),$$

and also $\Gamma = \{\lambda : \operatorname{Im}\lambda = 0\}$, $\Phi_{\pm 1} = \{\lambda : \pm\lambda > 0\}$. Let Π and $\Pi_{\pm 1}$ are the λ-plane with the cuts Γ and $\Gamma_{\pm 1}$ respectively.

Theorem 2.1. (1) *Let $T < \infty$. Then the WF's $\mathfrak{M}_{mk}(\lambda)$ are meromorphic in λ and*

$$\begin{cases} \mathfrak{M}_{mk}(\lambda) = \big(\Delta_{mm}(\lambda)\big)^{-1} \cdot \Delta_{mk}(\lambda), \\ \Delta_{mk}(\lambda) \overset{\text{def}}{=} (-1)^{m+k} \det\big[U_{\xi T}(C_\nu)\big]_{\xi=\overline{a,n-m};\nu=\overline{m,n}\setminus k}. \end{cases} \tag{2.7}$$

(2) *Let $T = \infty$. Then the WF's $\mathfrak{M}_{mk}(\lambda)$ are regular in $\Pi_{(-1)^{n-m}}$ with the exception of an at most countable bounded set Λ'_{mk} of poles. For $(-1)^{n-m}\lambda \geq 0$ the following limits exist and are finite off the bounded sets $\Lambda_{mk}^\pm$:*

$$\mathfrak{M}_{mk}^\pm(\lambda) = \lim_{\substack{z\to 0\\ \operatorname{Re} z>0}} \mathfrak{M}_{mk}(\lambda \pm iz).$$

Proof. Let $T=\infty$, $\{y_k(x,\rho)\}_{k=\overline{1,n}}$ be the FSS B_0 of (2.1). Using the boundary conditions on $\Phi_m(x,\lambda)$ we obtain

$$\begin{cases} \Phi_m(x,\lambda)=\sum\limits_{k=1}^{m} a_{mk}(\rho)y_k(x,\rho), \\ a_{mk}(\rho)=(-1)^{m+k}\big(\Delta^0_{mm}(\rho)\big)^{-1}\det\left[U_{\xi 0}(y_\nu)\right]_{\xi=\overline{1,m-1};\nu=\overline{1,m}\setminus k}, \\ \Delta^0_{mk}(\rho)\overset{\text{def}}{=}\det\left[U_{\xi 0}(y_\nu)\right]_{\nu=\overline{1,m};\xi=\overline{1,m},k}, \end{cases} \tag{2.8}$$

Since $\mathfrak{M}_{mk}(\lambda)=U_{k0}(\Phi_m(x,\lambda))$, it follows from (2.8) that

$$\mathfrak{M}_{mk}(\lambda)=\big(\Delta^0_{mm}(\rho)\big)^{-1}\cdot\Delta^0_{mk}(\rho). \tag{2.9}$$

Using the asymptotic properties (2.5) of the functions $y_k^{(\nu)}(x,\rho)$, we have for $|\rho|\to\infty$, $\rho\in\overline{S}$:

$$\begin{cases} a_{mk}(\rho)=\rho^{-\sigma_{m0}}\big(a^0_{mk}+O(\rho^{-1})\big), \\ \Phi_m(x,\lambda)=\rho^{-\sigma_{m0}}\sum\limits_{k=1}^{m}\exp(\rho R_k x)\cdot\big(a^0_{mk}+O(r^{-1})\big), \end{cases} \tag{2.10}$$

$$\begin{cases} \Delta^0_{mk}(\rho)=\rho^{\sigma_{10}+\cdots+\sigma_{m-1,0}+\sigma_{k0}}\cdot\Omega(\overline{1,m-1},k)\cdot\big(1+O(\rho^{-1})\big), \\ \mathfrak{M}_{mk}(\lambda)=\rho^{\sigma_{k0}-\sigma_{m0}}\mu^0_{mk}\cdot\big(1+O(\rho^{-1})\big). \end{cases} \tag{2.11}$$

Repeating the preceding arguments for the FSS $B_{\alpha m}$ we get that

$$\begin{cases} \mathfrak{M}_{mk}(\lambda)=\big(\Delta^1_{mm}(\rho)\big)^{-1}\Delta^1_{mk}(\rho), \\ \Delta^1_{mk}(\rho)\overset{\text{def}}{=}\det\left[U_{\xi 0}(y^0_\nu)\right]_{\nu=\overline{1,m};\xi=\overline{1,m-1},k}. \end{cases} \tag{2.12}$$

Let

$$G=\left\{\rho:\arg\rho\in\left(\big((-1)^{n-m}-1\big)\frac{\pi}{2n},\big((-1)^{n-m}+3\big)\frac{\pi}{2n}\right)\right\}.$$

The domain G consists of two sectors S with the same collection $\{R_\xi\}_{\xi=\overline{1m}}$. Consequently, the functions $\Delta^1_{mk}(\rho)$ are regular for $\rho\in G$, $|\rho|>\rho_\alpha$ and continuous for $\rho\in\overline{G}$, $|\rho|\geq\rho_\alpha$. The theorem is obtained from this, in view of (2.9), (2.11), (2.12), and the arbitrariness of α.

Let $\Lambda_{mk}=\Lambda'_{mk}\cup\Lambda^+_{mk}\cup\Lambda^-_{mk}$ and $\Lambda=\bigcup\limits_{m,k}\Lambda_{mk}$. We say that the spectrum of L has finite multiplicity, if for some $p\geq 1$ we have that $\mathfrak{M}(\lambda)=O\big((\lambda-\lambda_0)^{-p}\big)$, $\lambda\to\lambda_0$, $\lambda_0\in\Lambda$. For example, if $p_\nu(x)\exp(\varepsilon x)\in\mathcal{L}(0,\infty)$, $\varepsilon>0$, then the spectrum of L has finite multiplicity. It is known that, in general, the spectrum can have infinite multiplicity.

Let $T<\infty$. Using the boundary conditions on $\Phi_m(x,\lambda)$, we obtain

$$\Phi_m(x,\lambda)=\big(\Delta_{mm}(\lambda)\big)^{-1}\cdot\det\left[C_\nu(x,\lambda),U_{1T}(C_\nu),\ldots,U_{n-m,T}(C_\nu)\right]_{\nu=\overline{n,m}}, \tag{2.13}$$

and consequently, the relations (2.7) are valid. Theorem 2.1 is proved.

For $T<\infty$ we denote by $\Lambda_m=\{\lambda_{lm}\}_{l\geq 1}$ the set of zeros (with multiplicities) of the entire function $\Delta_{mm}(\lambda)$ and $\Lambda=\bigcup\limits_{m=1}^{n-1}\Lambda_m$. The numbers $\{\lambda_{lm}\}$ coincide with eigenvalues of the boundary value problems S_m for the DE (2.1) under the conditions $U_{\eta T}(y)=0$, $\xi=\overline{1,m}$, $\eta=\overline{1,n-m}$. For $l\to\infty$

$$\lambda_{lm}=(-1)^{n-m}\cdot\left(\frac{\pi}{T}\left(\sin\frac{\pi m}{n}\right)^{-1}\left(l+\chi_{m0}+O\left(\frac{1}{l}\right)\right)\right)^n. \tag{2.14}$$

Denote by $G_{\delta,m}$ the λ-plane without circles $|\lambda-\lambda_0|<\delta$, $\lambda_0\in\Lambda_m$, $G_\delta=\bigcap_{m=1}^{n-1} G_{\delta,m}$. Let

$$\begin{gathered} s_{mk}=\sigma_{k0}-\frac{n(n-1)}{2}+\sum_{\xi=1}^{m-1}\sigma_{\xi0}+\sum_{\eta=1}^{n-m}\sigma_{\eta T}, \\ \Delta^1_{mk}(\rho)=\det\left[U_{10}(y_\nu),\dots,U_{m-1,0}(y_\nu),U_{k0}(y_\nu),U_{1T}(y_\nu),\dots,U_{n-m,T}(y_\nu)\right]_{\nu=\overline{1,n}}, \end{gathered} \tag{2.15}$$

where $\{y_\nu(x,\rho)\}_{\nu=\overline{1,n}}$ is the FSS B_0 in a sector S with the property (2.3). Then

$$\begin{gathered} \Phi_m(x,\lambda)=\sum_{k=1}^{n} a_{mk}(\rho)y_k(x,\rho), \\ a_{mk}(\rho)=\frac{(-1)^{m+k}}{\Delta^1_{mm}(\rho)}\det\left[U_{10}(y_\nu),\dots,U_{m-1,0}(y_\nu),U_{1T}(y_\nu),\dots,U_{n-m,T}(y_\nu)\right]_{\nu=\overline{1,n}\setminus k}. \end{gathered} \tag{2.16}$$

Since $\Delta_{mm}(\lambda)=\Delta^1_{mk}(\rho)\cdot\left(\det\left[U_{\xi0}(y_\nu)\right]_{\xi,\nu=\overline{1,n}}\right)^{-1}$ then, using (2.15), (2.16) and asymptotic properties (2.5) of the functions $y_k^{(\nu)}(x,\rho)$, we obtain for $|\lambda|\to\infty$, $\arg((-1)^{n-m}\lambda)=\beta\neq0$, $\rho\in S$:

$$\begin{cases} a_{mk}(\rho)=\rho^{-\sigma_{m0}}\left(a^0_{mk}+O(\rho^{-1})\right), & k=\overline{1,m}, \\ a_{mk}(\rho)=O\left(\rho^{-\sigma_{m0}}\exp\left(\rho(R_m-R_k)T\right)\right), & k=\overline{m+1,n}, \end{cases} \tag{2.17}$$

$$\Delta_{mk}(\lambda)=\rho^{s_{mk}}\frac{\Omega(\overline{1,m-1},k)}{\Omega(\overline{1,n})}\det\left[R_\nu^{\sigma_{jT}}\right]_{\substack{\nu=\overline{m+1,n}\\ j=\overline{1,n-m}}}\exp\left(T\rho\sum_{j=m+1}^{n}R_j\right)\cdot\left(1+O(\rho^{-1})\right), \tag{2.18}$$

$$\begin{cases} \mathfrak{M}_{mk}(\lambda)=\rho^{\sigma_{k0}-\sigma_{m0}}\mu^0_{mk}\cdot\left(1+O(\rho^{-1})\right), \\ \Phi_m(x,\lambda)=\rho^{-\sigma_{m0}}\sum\limits_{k=1}^{m}\exp(\rho R_k x)\left(a^0_{mk}+O(\rho^{-1})\right), & x\in[0,T), \end{cases} \tag{2.19}$$

and also

$$\begin{cases} |\Delta_{mm}(\lambda)|>C\left|\rho^{s_{mm}}\exp\left(T\rho\sum_{j=m+1}^{n}R_j\right)\right|, & \lambda\in G_{\delta,m}, \\ \left|\Phi_m^{(\nu)}\right|<C\left|\rho^{\nu-\sigma_{m0}}\exp(\rho R_m x)\right|, & \lambda\in G_{\delta,m} \end{cases} \tag{2.20}$$

$$\Delta_{mk}(\lambda)=O\left(\rho^{s_{mk}}\exp\left(T\rho\sum_{j=m+1}^{n}R_j\right)\right), \qquad |\lambda|\to\infty. \tag{2.21}$$

2.1.4. Denote by W_ν the set of functions $f(x)$, $0<x<T$, such that $f(x)$, $f'(x)$, $\dots$, $f^{(\nu-1)}(x)$ are absolutely continuous and $f^{(k)}(x)\in\mathcal{L}(0,T)$, $k=\overline{0,\nu}$. Let $N\geq0$ be a fixed integer. We say that $L\in V_N$ if $p_\nu(x)\in W_{\nu+N}$, $\nu=\overline{0,n-2}$. We shall assume below that $L\in V_N$. We define $p_n(x)=1$, $p_{n-1}(x)=0$, and $u_{\xi\nu a}=\delta_{\nu,\sigma_{\xi a}}$, $\nu\geq\sigma_{\xi a}$. Let

$$\langle y(x),z(x)\rangle_l=\sum_{\nu,j=0}^{n-1}\mathcal{L}_{\nu j}(x)y^{(\nu)}(x)z^{(j)}(x), \tag{2.22}$$

$$\begin{cases} \mathcal{L}_{\nu j}(x)=\sum\limits_{s=j}^{n-\nu-1}(-1)^s C_s^j p_{s+\nu+1}^{(s-j)}(x), & \nu+j\leq n-1, \\ \mathcal{L}_{\nu j}(x)=0, & \nu+j>n-1. \end{cases} \tag{2.23}$$

We consider the DE and the LF $L^* = (l^*, U^*)$:

$$l^* z \equiv (-1)^n z^{(n)} + \sum_{\nu=0}^{n-2} (-1)^\nu \big(p_\nu(x) z\big)^\nu = \lambda z, \tag{2.24}$$

$$U^*_{\xi a}(z) = z^{(\sigma^*_{\xi a})}(a) + \sum_{\nu=0}^{\sigma^*_{\xi a}-1} u^*_{\xi\nu a} z^{(\nu)}(a), \qquad \sigma^*_{\xi a} = n - 1 - \sigma_{n+1-\xi,a}, \tag{2.25}$$

where the LF $U^*_a = \big[(-1)^{n-1-\sigma_{ka}} U^*_{n-k+1,a}\big]_{k=\overline{1,n}}$ are determined by the relation

$$\langle y, z\rangle_{l|x=a} = U_a(y) U^*_a(z) = \sum_{k=1}^n (-1)^{n-1-\sigma_{ka}} U_{ka}(y) U^*_{n-k+1,a}(z).$$

It is clear that $L^* \in V_N$. Thus, for any sufficiently smooth functions $y(x)$ and $z(x)$

$$ly \cdot z - y \cdot l^* z = \frac{d}{dx} \langle y, z\rangle_l. \tag{2.26}$$

In particular, if the functions $y(x, \lambda)$ and $z(x, \mu)$ are solutions of the DE's $ly = \lambda y$ and $l^* z = \mu z$, then

$$\frac{d}{dx} \langle y, z\rangle_l = (\lambda - \mu) y z. \tag{2.27}$$

For definiteness, it will be assumed below that $\sigma_{\xi a} = n - \xi$.

Assume that the functions $\Phi^*_m(x, \lambda)$, $m = \overline{1,n}$ are solutions of DE (2.24) under the conditions

$$\begin{aligned} &U^*_{\xi 0}(\Phi^*_m) = \delta_{\xi m}, && \xi = \overline{1,m} && (T \le \infty), \\ &U^*_{\eta T}(\Phi^*_m) = 0, && \eta = \overline{1, n-m} && (T < \infty), \\ &\varphi^*_m(x, \lambda) = O\big(\exp(\rho R^*_m x)\big), && x \to \infty, \quad \rho \in S && (T = \infty), \quad R^*_m = -R_{n-m+1}. \end{aligned}$$

Let $\mathfrak{M}^*_{mk}(\lambda) = U^*_{k0}(\Phi^*_m)$, $\Phi^*(x, \lambda) = \big[(-1)^{k-1} \Phi^*_{n-k+1}(x, \lambda)\big]^T_{k=\overline{1,n}}$, $\mathfrak{M}^*(\lambda) = U^*_0(\Phi^*)$. We introduce the FSS $C^*(x, \lambda) = \big[(-1)^{k-1} C^*_{n-k+1}(x, \lambda)\big]^T_{k=\overline{1,n}}$ of DE (2.24) under the conditions $U^*_{\xi 0}(C^*_m) = \delta_{\xi m}$, $\xi = \overline{1,n}$. Then

$$\Phi^*(x, \lambda) = C^*(x, \lambda) \mathfrak{M}^*(\lambda). \tag{2.28}$$

The properties of the WF's $\mathfrak{M}^*_{mk}(\lambda)$ are completely analogous to those of the WF's $\mathfrak{M}_{mk}(\lambda)$. For $T < \infty$

$$\begin{aligned} \mathfrak{M}^*_{mk}(\lambda) &= \big(\Delta^*_{mm}(\lambda)\big)^{-1} \Delta^*_{mk}(\lambda), \\ \Delta^*_{mk}(\lambda) &= (-1)^{m+k} \det \big[U^*_{\xi T}(C^*_\nu)\big]_{\substack{\xi=\overline{1,n-m} \\ \nu=\overline{m,n}\setminus k.}} \end{aligned}$$

For $T = \infty$ the WF's $\mathfrak{M}^*_{mk}(\lambda)$ are regular in $\Pi_{(-1)^m}$ except for an at most countable set $\Lambda^{*\prime}_{mk}$ of poles, and for $(-1)^m \lambda \ge 0$ the following limits exist and are finite off the bounded sets $\Lambda^{*,\pm}_{mk}$:

$$\mathfrak{M}^{*,\pm}_{mk}(\lambda) = \lim_{\substack{z\to 0 \\ \operatorname{Re} z > 0}} \mathfrak{M}^*_{mk}(\lambda \pm iz).$$

Lemma 2.1. $\mathfrak{M}^*(\lambda) = \big(\mathfrak{M}(\lambda)\big)^{-1}$.

Indeed, it follows from (2.27) that $\frac{d}{dx} \langle \Phi_k(x, \lambda), \Phi^*_j(x, \lambda)\rangle = 0$. Hence, for $k + j \le n$

$$\sum_{\nu=1}^n (-1)^{\nu-1} \mathfrak{M}_{k\nu}(\lambda) \mathfrak{M}^*_{j,n-\nu+1}(\lambda) = 0, \tag{2.29}$$

i.e., $\mathfrak{M}(\lambda) \mathfrak{M}^*(\lambda) = E$.

Let $y(x)$ be a sufficiently smooth function. We write

$$\mathbf{y}(x) = [y^{(\nu)}(x)]_{\nu=\overline{0,n-1}}.$$

Lemma 2.2. *Assume that the functions $y_k(x)$, $k = \overline{1, n-1}$, are solutions of DE (2.1), and $z_j(x) = \det[y_k^{(\nu)}(x)]_{k=\overline{1,n-1};\nu=\overline{0,n-1}\backslash n-j-1}$. Then*

$$z_j(x) = \sum_{s=0}^{j}(-1)^s \big(p_{n-s}(x)z_0(x)\big)^{(j-s)}, \qquad j = \overline{0, n-1}, \tag{2.30}$$

$$l^* z_0(x) = \lambda z_0(x), \qquad \det\big[\mathbf{y}_1(x), \dots, \mathbf{y}_{n-1}(x), \mathbf{y}(x)\big] = \langle y(x), z_0(x)\rangle_l. \tag{2.31}$$

Lemma 2.3.

$$\Phi_m^*(x,\lambda) = \det\Big[\Phi_n^{(s)}(x,\lambda), \dots, \Phi_{n-m+2}^{(s)}(x,\lambda), \Phi_{n-m}^{(s)}(x,\lambda), \dots, \Phi_1^{(s)}(x,\lambda)\Big]_{s=\overline{0,n-2}}. \tag{2.32}$$

Proof. Denote the right-hand side of (2.32) by $y_m^*(x,\lambda)$. It follows from (2.31) that $l^* y_m^*(x,\lambda) = \lambda y_m^*(x,\lambda)$ and

$$\begin{aligned}\det\Big[\Phi_n(x,\lambda), \dots, \Phi_{n-m+2}(x,\lambda), \Phi_{n-m}(x,\lambda), \dots, \Phi_1(x,\lambda), \mathbf{y}(x)\Big]_{|x=a}\\ = \langle y(x), y_m^*(x,\lambda)\rangle_{l|x=a} = \sum_{k=1}^{n}(-1)^{k-1}U_{ka}(y)U_{n-k+1,a}^*(y_m^*).\end{aligned} \tag{2.33}$$

In (2.33), we take $y(x) = \Phi_n(x,\lambda)$, ..., $y(x) = \Phi_{n-m+1}(x,\lambda)$ successively, and obtain that $U_{\xi 0}^*(y_m^*) = \delta_{\xi m}$, $\xi = \overline{1, m}$. For $T < \infty$, $a = T$ we take $y(x) = \Phi_1(x,\lambda)$, ..., $y(x) = \Phi_{n-m}(x,\lambda)$ successively, and obtain $u_{\eta T}^*(y_m^*) = 0$, $\eta = \overline{1, n-m}$. For $T = \infty$, from the definition of the functions $y_m^*(x,\lambda)$ and the asymptotic properties of the WS's $\Phi_m^{(s)}(x,\lambda)$, we obtain that

$$y_m^*(x,\lambda) = O\big(\exp(\rho R_m^* x)\big), \qquad x \to \infty, \quad \rho \in S.$$

Consequently, $y_m^*(x,\lambda) = \Phi_m^*(x,\lambda)$. Lemma 2.3 is proved.

2.1.5. In this subsection, we obtain the uniqueness theorem for the solution of the IP. Let $C_M(x,\lambda) = [\mathbf{C}_m(x,\lambda)]_{m=\overline{1,n}}^T$, $\Phi_M(x,\lambda) = [\Phi_m(x,\lambda)]_{m=\overline{1,n}}^T$. Then (2.4) takes the form

$$\Phi_M(x,\lambda) = C_M(x,\lambda)\mathfrak{M}^T(\lambda). \tag{2.34}$$

Since $\det \mathfrak{M}(\lambda) = 1$, (2.34) and the Ostrogradskii–Liouville theorem imply

$$\det \Phi_M(x,\lambda) = \det C_M(x,\lambda) = (-1)^{\frac{n(n-1)}{2}}. \tag{2.35}$$

Let $L, \widetilde{L} \in V_N$. We define the matrix $\mathrm{P}(x,\lambda) = [\mathrm{P}_{jk}(x,\lambda)]_{j,k=\overline{1,n}}$ by the formula $\mathrm{P}(x,\lambda) = \Phi_M(x,\lambda) \cdot \big(\widetilde{\Phi}_M(x,\lambda)\big)^{-1}$ or

$$\begin{aligned}\mathrm{P}_{jk} &= \det\Big[\widetilde{\Phi}_\nu^{(n-1)}(x,\lambda), \dots, \widetilde{\Phi}_\nu^{(k)}(x,\lambda), \widetilde{\Phi}_\nu^{(j-1)}(x,\lambda), \widetilde{\Phi}_\nu^{(k-2)}(x,\lambda), \dots, \widetilde{\Phi}_\nu(x,\lambda)\Big]_{\nu=\overline{1,n}}\\ &= \sum_{\nu=1}^{n}(-1)^{\nu+k-n-1}\Phi_\nu^{(j-1)}(x,\lambda)\\ &\qquad \times \det\Big[\widetilde{\Phi}_n^{(s)}(x,\lambda), \dots, \widetilde{\Phi}_{\nu-1}^{(s)}(x,\lambda), \widetilde{\Phi}_{\nu+1}^{(s)}(x,\lambda), \dots, \widetilde{\Phi}_1^{(s)}(x,\lambda)\Big]_{s=\overline{0,n-1}\backslash k-1}.\end{aligned} \tag{2.36}$$

We remark that the idea of using mappings of the solution spaces of DE's for solving the IP is due to Leibenzon [52–53].

From (2.36) and the asymptotic properties of the WS's $\Phi_m(x,\lambda)$ and $\widetilde{\Phi}_m(x,\lambda)$ we obtain the estimates

$$\left|\mathrm{P}_{jk}(x,\lambda)\right| < C|\rho|^{j-k}, \qquad \left|\mathrm{P}_{1k}(x,\lambda) - \delta_{1k}\right| < \frac{C}{|\rho|}, \qquad j,k=\overline{1,n} \tag{2.37}$$

($\lambda \in G_\delta$ for $T<\infty$). Let

$$\left\langle [y_\nu]_{\nu=\overline{0,n-1}},\ [z_j]_{j=\overline{0,n-1}} \right\rangle_l \stackrel{\text{def}}{=} \sum_{\nu,j=0}^{n-1} \mathcal{L}_{\nu j}(x) y_\nu z_j.$$

Lemma 2.4. *Let $\widetilde{y}(x)$ be a sufficiently smooth function. Then*

$$\mathrm{P}(x,\lambda)\widetilde{\mathbf{y}}(x) = \sum_{k=1}^{n} (-1)^{k-1} \left\langle \widetilde{y}(x), \widetilde{\Phi}^*_{n-k+1}(x,\lambda) \right\rangle_{\widetilde{l}} \Phi_k(x,\lambda), \tag{2.38}$$

$$\left\langle \left(\mathrm{P}(x,\lambda) - \mathrm{P}(x,\mu)\right)\widetilde{\Phi}_k(x,\lambda),\ \Phi^*_j(x,\mu) \right\rangle_l = \left\langle \Phi_k(x,\lambda), \Phi^*_j(x,\mu) \right\rangle_l - \left\langle \widetilde{\Phi}_k(x,\lambda), \widetilde{\Phi}^*_j(x,\mu) \right\rangle_{\widetilde{l}}. \tag{2.39}$$

Proof. Let us use (2.36). We have

$$\mathrm{P}(x,\lambda)\widetilde{\mathbf{y}}(x) = \sum_{k=1}^{n} (-1)^{k-1} \Phi_k(x,\lambda) \det\left[\widetilde{\Phi}_n(x,\lambda), \dots, \widetilde{\Phi}_{k+1}(x,\lambda), \widetilde{\Phi}_{k-1}(x,\lambda), \dots, \widetilde{\Phi}_1(x,\lambda), \widetilde{\mathbf{y}}(x)\right].$$

From this, using Lemmas 2.2 and 2.3, we obtain (2.38). Further,since $\mathrm{P}(x,\lambda)\widetilde{\Phi}_k(x,\lambda) = \Phi_k(x,\lambda)$, it follows that

$$\left\langle \mathrm{P}(x,\lambda)\widetilde{\Phi}_k(x,\lambda), \Phi^*_j(x,\mu) \right\rangle_l = \left\langle \Phi_k(x,\lambda), \Phi^*_j(x,\mu) \right\rangle_l. \tag{2.40}$$

By (2.38),

$$\left\langle \mathrm{P}(x,\mu), \widetilde{\Phi}_k(x,\lambda), \Phi^*_j(x,\mu) \right\rangle = \sum_{s=1}^{n} (-1)^{s-1} \left\langle \widetilde{\Phi}_k(x,\lambda), \widetilde{\Phi}^*_{n-s+1}(x,\mu) \right\rangle_{\widetilde{l}} \cdot \left\langle \Phi_s(x,\mu), \Phi^*_j(x,\mu) \right\rangle_l.$$

According to (2.27), $\left\langle \Phi_s(x,\mu), \Phi^*_j(x,\mu) \right\rangle_l$ does not depend on x. Using the conditions on the WS's for $x=0$ and $x=T$, we find that

$$\left\langle \Phi_s(x,\mu), \Phi^*_j(x,\mu) \right\rangle_l = (-1)^{s-1} \delta_{s,n-j+1}.$$

Thus,

$$\left\langle \mathrm{P}(x,\mu)\widetilde{\Phi}_k(x,\lambda), \Phi^*_j(x,\mu) \right\rangle_l = \left\langle \widetilde{\Phi}_k(x,\lambda), \widetilde{\Phi}^*_j(x,\mu) \right\rangle_{\widetilde{l}},$$

which together with (2.40) yields (2.39). Lemma 2.4 is proved.

Theorem 2.2. *If $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$, then $L = \widetilde{L}$.*

Thus, the specification of the WM $\mathfrak{M}(\lambda)$ uniquely determines DE and LF (2.1), (2.2). We remark that the deletion of a single element from the WM leads to nonuniqueness of the solution of the IP.

Proof. We transform the matrix $\mathrm{P}(x,\lambda)$. For this we use (2.34). Under the conditions of the theorem,

$$\mathrm{P}(x,\lambda) = \Phi_M(x,\lambda)\left(\widetilde{\Phi}_M(x,\lambda)\right)^{-1} = C_M(x,\lambda)\mathfrak{M}^T(\lambda)\left(\widetilde{\mathfrak{M}}^T(\lambda)\right)^{-1}\left(\widetilde{C}_M(x,\lambda)\right)^{-1} = C_M(x,\lambda)\left(\widetilde{C}_M(x,\lambda)\right)^{-1}.$$

In view of (2.35) this leads us to conclude that for each fixed x the matrix-valued function $\mathrm{P}(x,\lambda)$ is entire analytic function in λ. Using (2.37) and Liouville's theorem ([72, p. 209]), we get that $\mathrm{P}_{11}(x,\lambda) \equiv 1$, $\mathrm{P}_{1k} \equiv 0$ for $k=\overline{2,n}$. But then $\Phi_m(x,\lambda) \equiv \widetilde{\Phi}_m(x,\lambda)$ for all x,λ, and m, and hence $l=\widetilde{L}$. Theorem 2.2 is proved.

2.2. Solution of the inverse problem on the half-line.

We consider the DE and LF (2.1)–(2.2) on the half-line ($T=\infty$). In 2.2, we present a solution of the IP of recovering L from the WM $\mathfrak{M}(\lambda)$ when the behavior of the spectrum is arbitrary. We give a derivation of the main equation of the IP, which is a singular linear integral equation. We obtain necessary and sufficient conditions on the WM and algorithm for the solution of the IP. The main results of 2.2 are contained in Theorems 2.3 and 2.5.

2.2.1. We formulate some auxiliary assertions.

Lemma 2.5. *The functions*

$$\mathfrak{M}_{mk}(\lambda) - \mathfrak{M}_{m,m+1}(\lambda)\mathfrak{M}_{m+1,k}(\lambda), \qquad \mathfrak{M}^*_{n-m,k}(\lambda) - \mathfrak{M}^*_{n-m,n-m+1}(\lambda)\mathfrak{M}^*_{n-m+1,k}(\lambda),$$
$$\Phi_m(x,\lambda) - \mathfrak{M}_{m,m+1}(\lambda)\Phi_{m+1}(x,\lambda), \qquad \Phi^*_{n-m}(x,\lambda) - \mathfrak{M}_{m,m+1}(\lambda)\Phi^*_{n-m+1}(x,\lambda)$$

are regular for $\lambda \in \Gamma_{(-1)^{n-m}} \setminus \Lambda$.

Let $L, \widetilde{L} \in V_N$. In the λ-plane we consider the contour $\gamma = \gamma_{-1} \cup \gamma_0 \cup \gamma_1$ (with counterclockwise circuit), where γ_0 is a bounded closed contour encircling the set $\Lambda \cup \widetilde{\Lambda} \cup \{0\}$ (i.e., $\Lambda \cup \widetilde{\Lambda} \cup \{0\} \subset \operatorname{int}\gamma_0$), and $\gamma_{\pm 1}$ is the two-sided cut along the arc $\{\lambda : \pm\lambda > 0,\ \lambda \notin \operatorname{int}\gamma_0\}$. Let $J_\gamma = \{\lambda : \lambda \notin \gamma \cup \operatorname{int}\gamma_0\}$.

Lemma 2.6. *The following relations hold:*

$$\widetilde{\Phi}(x,\lambda) = \Phi(x,\lambda) - \frac{1}{2\pi i}\int_\gamma \frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\Phi(x,\mu)\,d\mu, \qquad \lambda \in J_\gamma, \tag{2.41}$$

$$\frac{\langle \Phi(x,\lambda), \Phi^*(x,\mu)\rangle_l}{\lambda-\mu} - \frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu} = \frac{1}{2\pi i}\int_\gamma \frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*(x,\xi)\rangle_{\widetilde{l}}}{\lambda-\xi}\cdot\frac{\langle \Phi(x,\xi), \Phi^*(x,\mu)\rangle_l}{\xi-\mu}\,d\xi,$$
$$\lambda,\mu \in J_\gamma. \tag{2.42}$$

In (2.41) (and everywhere below, where necessary) the integral is understood in the principal value sense ([30, p. 27]).

Proof. Using Cauchy's theorem ([72, p. 166) and (2.37), we obtain

$$\begin{cases} \mathrm{P}_{1k}(x,\lambda) = \delta_{1k} + \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{\mathrm{P}_{1k}(x,\xi)}{\lambda-\xi}\,d\xi, & \lambda \in J_\gamma, \\ \dfrac{\mathrm{P}_{jk}(x,\lambda)-\mathrm{P}_{jk}(x,\mu)}{\lambda-\mu} = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{\mathrm{P}_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi, & \lambda,\mu \in J_\gamma. \end{cases} \tag{2.43}$$

By (2.38) and (2.43),

$$\sum_{k=1}^n \mathrm{P}_{1k}(x,\lambda)\widetilde{y}^{(k-1)}(x) = \widetilde{y}(x) + \frac{1}{2\pi i}\int_\gamma \frac{\langle \widetilde{y}(x), \widetilde{\Phi}^*(x,\xi)\rangle_{\widetilde{l}}}{\lambda-\xi}\Phi(x,\xi)\,d\xi.$$

Setting here $\widetilde{y}(x) = \widetilde{\Phi}(x,\lambda)$, we obtain (2.41). Similarly, by (2.38) and (2.43)

$$\frac{\mathrm{P}(x,\lambda)-\mathrm{P}(x,\mu)}{\lambda-\mu}\widetilde{\mathrm{y}}(x) = \frac{1}{2\pi i}\int_\gamma \sum_{s=1}^n (-1)^{s-1}\frac{\langle \widetilde{y}(x), \widetilde{\Phi}^*_{n-s+1}(x,\xi)\rangle_{\widetilde{l}}}{(\lambda-\xi)(\xi-\mu)}\Phi_s(x,\xi)\,d\xi.$$

From this, by (2.39), we obtain (2.42). Lemma 2.6 is proved.

Let

$$Y = [\delta_{j,k-1}]_{j=\overline{1,n-1};k=\overline{1,n}}, \qquad A_0(\lambda) = \widehat{\mathfrak{M}}(\lambda)\widetilde{\mathfrak{M}}^{-1}(\lambda),$$
$$\widetilde{A}_0(\lambda) = \widehat{\mathfrak{M}}(\lambda)\mathfrak{M}^{-1}(\lambda), \qquad \mathfrak{M}_\partial(\lambda) = \operatorname{diag}[\mathfrak{M}_{m,m+1}(\lambda)]_{m=\overline{1,n-1}}.$$

For real λ we define the matrices

$$f(x,\lambda)=\big[f_k(x,\lambda)\big]_{k=\overline{2,n}},\qquad f^*(x,\lambda)=\big[(-1)^{k-1}f^*_{n-k+1}(x,\lambda)\big]^T_{k=\overline{1,n-1}}$$

according to the formulas

$$f_k(x,\lambda)=\chi\big((-1)^{n-k+1}\lambda\big)\Phi_k(x,\lambda),\qquad f^*_k(x,\lambda)=\chi\big((-1)^{k-1}\lambda\big)\Phi^*_k(x,\lambda),$$

where $\chi(\lambda)$ is the Heaviside function. For $\lambda\in\gamma$ let

$$a(\lambda)=\chi_{+1}(\lambda)\chi_{-1}(\lambda)YA_0(\lambda)Y^T,\qquad N(\lambda)=E+\frac{1}{2}a(\lambda),$$

$$\widetilde{a}(\lambda)=\chi_{+1}(\lambda)\chi_{-1}(\lambda)Y\widetilde{A}_0(\lambda)Y^T,\qquad \widetilde{N}(\lambda)=E-\frac{1}{2}\widetilde{a}(\lambda),$$

where $\chi_{\pm1}(\lambda)=1$ for $\lambda\in\gamma_0\cup\gamma_{\pm1}$ and $\chi_{\pm1}(\lambda)=0$ for $\lambda\in\gamma_{\mp1}$. For $\lambda,\mu\in\gamma$ we define the matrices

$$\varphi(x,\lambda)=\big[\varphi_k(x,\lambda)\big]_{k=\overline{2,n}},\qquad g^*(x,\lambda)=\big[g^*_k(x,\lambda)\big]^T_{k=\overline{2,n}},$$

$$G^*(x,\lambda)=\big[G_k(x,\lambda)\big]^T_{k=\overline{1,n}},\qquad r(x,\lambda,\mu)=\big[r_{kj}(x,\lambda,\mu)\big]_{k,j=\overline{2,n}}$$

according to the formulas

$$\varphi(x,\lambda)=\begin{cases}Y\Phi(x,\lambda), & \lambda\in\gamma_0,\\ f(x,\lambda), & \lambda\in\gamma_1\cup\gamma_{-1},\end{cases}\qquad g^*(x,\lambda)=\begin{cases}-\Phi^*(x,\lambda)A_0(\lambda)Y^T, & \lambda\in\gamma_0,\\ -f^*(x,\lambda)\widehat{\mathfrak{M}}_{\partial}(\lambda), & \lambda\in\gamma_1\cup\gamma_{-1},\end{cases}$$

$$G^*(x,\lambda)=g^*(x,\lambda)Y,\qquad r(x,\lambda,\mu)=\frac{\langle\varphi(x,\lambda),g^*(x,\mu)\rangle_l}{\lambda-\mu}.$$

Similarly, we define the matrices $\widetilde{\varphi}(x,\lambda)$, $\widetilde{g}^*(x,\lambda)$, $\widetilde{G}^*)x,\lambda)$, and $\widetilde{r}(x,\lambda,\mu)$ with $\widetilde{\Phi}$, $\widetilde{f}$, $\widetilde{\Phi}^*$, $\widetilde{f}^*$, and $\widetilde{A}_0$ instead of Φ, f, Φ^*, f^*, and A_0. Finally, the matrices $\widetilde{\Gamma}(\lambda,\mu)=\big[\widetilde{\Gamma}_{j\nu}(\lambda,\mu)\big]_{j,\nu=\overline{1,n}}$ and $\widetilde{A}(\mu)=\big[\widetilde{A}_{j\nu}(\mu)\big]_{j,\nu=\overline{1,n}}$, $\mu\in\gamma$, are defined according to the formulas

$$\widetilde{\Gamma}(\lambda,\mu)=-\langle\widetilde{\Phi}(x,\lambda),\widetilde{G}^*(x,\mu)\rangle_{\widetilde{l}|x=0},$$

$$\widetilde{A}_{j\nu}(\mu)=\delta_{j,\nu-1}\chi_{(-1)^{n-j}}(\mu)\widehat{\mathfrak{M}}_{j,j+1}(\mu),\qquad \mu\in\gamma_1\cup\gamma_{-1},$$

$$\widetilde{A}(\mu)=\widetilde{A}_0(\mu),\qquad \mu\in\gamma_0.$$

Since $A_0(\lambda)-\widetilde{A}_0(\lambda)=\widetilde{A}_0(\lambda)A_0(\lambda)$, then $a(\lambda)-\widetilde{a}(\lambda)=\widetilde{a}(\lambda)a(\lambda)$. From this we obtain

$$\widetilde{N}(\lambda)N(\lambda)-\frac{1}{4}\widetilde{a}(\lambda)a(\lambda)=E,\qquad \widetilde{N}(\lambda)a(\lambda)-\widetilde{a}(\lambda)N(\lambda)=0. \tag{2.44}$$

Theorem 2.3.

$$\widetilde{\varphi}(x,\lambda)=\widetilde{N}(\lambda)\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \widetilde{r}(x,\lambda,\mu)\varphi(x,\mu)\,d\mu,\qquad \lambda\in\gamma, \tag{2.45}$$

$$\widetilde{N}(\lambda)r(x,\lambda,\mu)-\widetilde{r}(x,\lambda,\mu)N(\mu)+\frac{1}{2\pi i}\int_\gamma \widetilde{r}(x,\lambda,\xi)r(x,\xi,\mu)\,d\xi=0. \tag{2.46}$$

Equation (2.45) is the desired main equation of the IP.

Proof. By (2.4), (2.28), Lemmas 2.1 and 2.5, relations (2.41) and (2.42) give us

$$\widetilde{\Phi}(x,\lambda)=\Phi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \frac{\langle\widetilde{\Phi}(x,\lambda),\widetilde{g}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\varphi(x,\mu)\,d\mu,\qquad \lambda\in J_\gamma. \tag{2.47}$$

By continuity, it follows from (2.47) that

$$\widetilde{f}(x,\lambda) = f(x,\lambda) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\langle \widetilde{f}(x,\lambda), \widetilde{g}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\varphi(x,\mu)\,d\mu, \qquad \lambda\in\gamma_1\cup\gamma_{-1}. \tag{2.48}$$

For $\lambda\in\gamma_0$, we have from (2.47) by the Sokhotskii formulas [30] that

$$Y\widetilde{\Phi}(x,\lambda) = Y\Phi(x,\lambda) - \frac{1}{2}\widetilde{a}(\lambda)\varphi(x,\lambda) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{g}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\varphi(x,\mu)\,d\mu,$$

which together with (2.48) yields (2.45). The relation (2.46) is proved analogously.

We shall assume below for simplicity that $L, \widetilde{L}\in V_N$ are chosen so that

$$\widehat{\mathfrak{M}}_{m,m+1}(\lambda) = O(\rho^{-n-2}), \qquad |\lambda|\to\infty. \tag{2.49}$$

Then

$$\begin{cases} \left|\widetilde{g}^{*(\nu)}(x,\mu)\right| < C\left|\theta^{\nu-j-n}\exp(-\theta R_j x)\right|, & \mu=\theta^n, \\ \left|\widetilde{g}^{*(\nu)}(x,\mu)\varphi^{(s)}(x,\mu)\right| < C|\theta|^{\nu+s-2n}, & \mu\in\gamma_1\cup\gamma_{-1}, \\ \left|\varphi_j^{(s)}(x,\mu)\right| < C\left|\theta^{j+s-n}\exp(\theta R_j x)\right|. \end{cases} \tag{2.50}$$

Let

$$\varkappa_{\nu s}(x) = \frac{1}{2\pi i}\int\limits_{\gamma} \widetilde{g}^{*(\nu)}(x,\mu)\varphi^{(s)}(x,\mu)\,d\mu, \qquad \nu+s\le n-1; \tag{2.51}$$

$$\begin{cases} t_{j\nu}(x) = -\sum\limits_{\beta=\nu+1}^{j} C_j^{\beta}C_{\beta-1}^{\nu}\varkappa_{\beta-\nu-1,j-\beta}(x), & j>\nu, \\ t_{j\nu}(x) = \delta_{j\nu}, & j\le\nu, \end{cases} \qquad j,\nu=\overline{0,n}; \tag{2.52}$$

$$\xi_\nu(x) = \sum_{s=0}^{n-\nu-1}\sum_{j=\nu+1}^{n-s}\Big(C_{j+s}^{j}C_{j-1}^{\nu}\widetilde{p}_{j+s}(x)\varkappa_{j-\nu-1,s}(x)$$
$$+\delta_{s0}(-1)^{j-\nu}\sum_{r=0}^{j-\nu-1}C_{j-\nu-1}^{r}\widetilde{p}_j^{(j-\nu-1-r)}(x)\varkappa_{r0}(x)\Big), \qquad \nu=\overline{0,n-2}; \tag{2.53}$$

$$\varepsilon_\nu(x) = \xi_\nu 9x) - \sum_{j=\nu+1}^{n-2}\varepsilon_j(x)t_{j\nu}(x), \tag{2.54.}$$

The following lemma establishes a connection between the coefficients of the DE's and LF's L and $\widetilde{L}$.

Lemma 2.7.

$$p_\nu(x) = \widetilde{p}_\nu(x) + \varepsilon_\nu(x), \qquad \widetilde{u}_{\xi\nu 0} = \sum_{j=0}^{n-1}u_{\xi j0}t_{j\nu}(0). \tag{2.55}$$

Proof. Differentiating (2.47) with respect to x and using (2.27), (2.51), and (2.52), we get

$$\sum_{\nu=0}^{n} t_{j\nu}(x)\widetilde{\Phi}^{(\nu)}(x,\lambda) = \Phi^{(j)}(x,\lambda) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{g}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\varphi^{(j)}(x,\mu)\,d\mu, \qquad \lambda\in J_\gamma. \tag{2.56}$$

It follows from (2.47) that

$$\widetilde{l}\widetilde{\Phi}(x,\lambda) = l\Phi(x,\lambda) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{g}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\, l\varphi(x,\mu)\,d\mu + \frac{1}{2\pi i}\int\limits_{\gamma} \langle \widetilde{\Phi}(x,\mu), \widetilde{g}^*(x,\mu)\rangle_{\widetilde{l}} \cdot \varphi(x,\mu)\,d\mu, \quad \lambda \in J_\gamma. \tag{2.57}$$

By (2.57), in view of (2.56) and (2.22), we have

$$\widetilde{l}\widetilde{\Phi}(x,\lambda) = \sum_{j=0}^{n} p_j(x) \sum_{\nu=0}^{n} t_{j\nu}(x)\widetilde{\Phi}^{(\nu)}(x,\lambda) + \sum_{\nu,j=0}^{n-1} \widetilde{\mathcal{L}}_{\nu j}(x)\widetilde{\Phi}^{(\nu)}(x,\lambda)\varkappa_{j0}(x)$$

and hence

$$p_\nu(x) = \widetilde{p}_\nu(x) - \sum_{j=\nu+1}^{n} p_j(x)t_{j\nu}(x) - \sum_{j=0}^{n-1} \widetilde{\mathcal{L}}_{\nu j}(x)\varkappa_{j0}(x).$$

Using (2.23), (2.52), and (2.53) we get $\widehat{p}_\nu(x) = \varepsilon_\nu(x)$, and the first relation (2.55) is proved. The second assersion of the lemma is proved analogously.

Let

$$\gamma'' = \Big\{\lambda : \lambda \in \gamma_1 \cup \gamma_{-1},\ \inf|\lambda-\mu| \geq \delta_0,\ \mu \in \gamma_0\Big\}, \quad \delta_0 > 0; \qquad \gamma' = \gamma \setminus \gamma''.$$

Thus, $\gamma = \gamma' \cup \gamma''$.

Lemma 2.8.

$$\big|\widetilde{r}_{kj}(x,\lambda,\mu)\big| < \frac{C_x\big|\exp((\rho R_k - \theta R_j)x)\big|}{|\rho|^{n-k}|th|^{n+j}(|\rho-\theta|+1)}$$

for $\mu \in \gamma''$, $\lambda \in \gamma$ *or for* $\mu \in \gamma$, $\lambda \in \gamma''$, *and*

$$\big|\widetilde{r}_{kj}^{(\nu+1)}(x,\lambda,\mu)\big| < \frac{C_x\big|\exp((\rho R_k - \theta R_j)x)\big|}{|\rho|^{n-k}|th|^{n+j}}\big(|\rho|+|\theta|\big)^\nu$$

for $\lambda,\mu \in \gamma$, $\nu = \overline{0,n-1}$.

Assume for definiteness that $\arg\rho \in (0, 2\pi/n)$. Denote

$$\Omega(x,\lambda) = \operatorname{diag}\big[\rho^{k-n}\exp(\rho R_k x)\big]_{k=\overline{2,n}},$$

$$\varphi^+(x,\lambda) = \Omega^{-1}(x,\lambda)\varphi(x,\lambda), \qquad r^+(x,\lambda,\mu) = \Omega^{-1}(x,\lambda)r(x,\lambda,\mu)\Omega(x,\mu),$$

$$a^+(x,\lambda) = \Omega^{-1}(x,\lambda)a(\lambda)\Omega(x,\lambda), \qquad N^+(x,\lambda) = \Omega^{-1}(x,\lambda)N(\lambda)\Omega(x,\lambda).$$

We define the matrices $\widetilde{\varphi}^+(x,\lambda)$, $\widetilde{r}^+(x,\lambda,\mu)$, $\widetilde{a}^+(x,\lambda)$, and $\widetilde{N}^+(x,\lambda)$ similarly. Then

$$\big|\widetilde{\varphi}_j^{+(\nu)}(x,\lambda)\big| < C|\rho|^\nu, \qquad \lambda \in \gamma, \quad \nu = \overline{0,n-1},$$

$$\big|\widetilde{R}_{kj}^{+}(x,\lambda,\mu)\big| < \frac{C_x}{|\theta|^{2n}(|\rho-\theta|+1)}, \qquad \lambda \in \gamma, \quad \mu \in \gamma'' \ \text{ or } \ \lambda \in \gamma'', \quad \mu \in \gamma,$$

$$\big|\widetilde{r}_{kj}^{+(\nu+1)}(x,\lambda,\mu)\big| < C_x|th|^{-2n}\big(|\rho|+|\theta|\big)^\nu, \qquad \lambda,\mu \in \gamma, \quad \nu = \overline{0,n-1},$$

and the functions $\widetilde{r}_{kj}^{+}(x,\lambda,\mu)$ are continuous for $\lambda,\mu \in \gamma_1$ and $\lambda,\mu \in \gamma_{-1}$, while for $\lambda,\mu \in \gamma_0$

$$\widetilde{r}^+(x,\lambda,\mu) = \frac{\widetilde{a}^+(x,\lambda)}{\mu-\lambda} + \widetilde{H}^+(x,\lambda,\mu),$$

where $\widetilde{H}^+(x,\lambda,\mu)$ is a continuous function. The functions $r^+(x,\lambda,\mu)$ and $\varphi^+(x,\lambda)$ have analogous properties. It follows from (2.44) and Theorem 2.3 that the following theorem is valid.

Theorem 2.4.

$$\widetilde{\varphi}^+(x,\lambda) = \widetilde{N}^+(x,\lambda)\varphi^+(x,\lambda) + \frac{1}{2\pi i}\int\limits_\gamma \widetilde{r}^+(x,\lambda,\mu)\varphi^+(x,\mu)\,d\mu, \quad \lambda\in\gamma; \tag{2.58}$$

$$\widetilde{N}^+(x,\lambda)r^+(x,\lambda,\mu) - \widetilde{r}^+(x,\lambda,\mu)N^+(x,\mu) + \frac{1}{2\pi i}\int\limits_\gamma \widetilde{r}^+(x,\lambda,\xi)r^+(x,\xi,\mu)\,d\xi = 0, \quad \lambda,\mu\in\gamma; \tag{2.59}$$

$$\begin{cases} \widetilde{N}^+(x,\lambda)N^+(x,\lambda) - \frac{1}{4}\widetilde{a}^+(x,\lambda)a^+(x,\lambda) = E, \\ \widetilde{N}^+(x,\lambda)a^+(x,\lambda) - \widetilde{a}^+(x,\lambda)N^+(x,\lambda) = 0. \end{cases} \tag{2.60}$$

We introduce the Banach space $B = \mathcal{L}_2^{n-1}(\gamma') \oplus \mathcal{L}_\infty^{n-1}(\gamma'')$ of vector-valued functions $z(\lambda) = [z_j(\lambda)]_{j=\overline{1,n-1}}$, $\lambda\in\gamma$, with the norm

$$\|z\|_B = \sum_{j=1}^{n-1}\Big(\|z_j\|_{\mathcal{L}_2\gamma'} + \|z_j\|_{\mathcal{L}_\infty\gamma''}\Big).$$

For fixed $x\geq 0$ we consider on B the linear operators

$$\begin{cases} \widetilde{A}z(\lambda) = \widetilde{N}^+(x,\lambda)z(\lambda) + \dfrac{1}{2\pi i}\displaystyle\int\limits_\gamma \widetilde{r}^+(x,\lambda,\mu)z(\mu)\,d\mu, & \lambda\in\gamma, \\ Az(\lambda) = N^+(x,\lambda)z(\lambda) - \dfrac{1}{2\pi i}\displaystyle\int\limits_\gamma r^+(x,\lambda,\mu)z(\mu)\,d\mu, & \lambda\in\gamma. \end{cases} \tag{2.61}$$

Lemma 2.9. *For a fixed x, the operators A and $\widetilde{A}$ are bounded linear operators on B, and $\widetilde{A}A = A\widetilde{A} = E$.*

Proof. The boundedness of A and $\widetilde{A}$ is obvious. Using the formula for interchanging the order of integration in a singular integral ([30, p. 60]), we obtain that

$$\frac{1}{2\pi i}\int\limits_\gamma \widetilde{r}^+(x,\lambda,\xi)\,d\xi\,\frac{1}{2\pi i}\int\limits_\gamma r^+(x,\xi,\mu)z(\mu)\,d\mu$$

$$= \frac{1}{4}\widetilde{a}^+(x,\lambda)a^+(x,\lambda) + \frac{1}{2\pi i}\int\limits_\gamma\left(\frac{1}{2\pi i}\int\limits_\gamma \widetilde{r}^+(x,\lambda,\xi)r^+(x,\xi,\mu)\,d\xi\right)z(\mu)\,d\mu.$$

Then it follows from (2.59)–(2.61) that

$$\widetilde{A}Az(\lambda) = \left(\widetilde{N}^+(x,\lambda)N^+(x,\lambda) - \frac{1}{4}\widetilde{a}^+(x,\lambda)a^+(x,\lambda)\right)z(\lambda)$$

$$-\frac{1}{2\pi i}\int\limits_\gamma\left(\widetilde{N}^+(x,\lambda)r^+(x,\lambda,\mu) - \widetilde{r}^+(x,\lambda,\mu)N^+(x,\mu) + \frac{1}{2\pi i}\int\limits_\gamma \widetilde{r}^+(x,\lambda,\xi)r^+(x,\xi,\mu)\,d\xi\right)z(\mu)\,d\mu = z(\lambda),$$

i.e., $\widetilde{A}A = E$. Similarly, $A\widetilde{A} = E$.

Corollary 2.1. *For $x\geq 0$ the main equation* (2.45) *of the IP has a unique solution $\Omega^{-1}(x,\lambda)\varphi(x,\lambda)$ in the class B and $\sup\limits_x\|\Omega^{-1}(x,\lambda)\varphi(x,\lambda)\|_B < \infty$.*

2.2.2. Denote by M the set of matrices $\mathfrak{M}(\lambda) = [\mathfrak{M}_{mk}(\lambda)]_{m,k=\overline{1,n}}$ such that (1) $\mathfrak{M}_{mk}(\lambda) = \delta_{mk}$, $m\geq k$, and $\mathfrak{M}_{mk}(\lambda) = O(\rho^{m-k})$, $|\lambda|\to\infty$, $m<k$; (2) the functions $\mathfrak{M}_{mk}(\lambda)$ are regular in $\Pi_{(-1)^{n-m}}$ with the exception of an at most countable bounded set Λ'_{mk} of poles, and are continuous in $\overline{\Pi}_{(-1)^{n-m}}$

with the exception of bounded sets Λ_{mk}; (3) the functions $\mathfrak{M}_{mk}(\lambda) - \mathfrak{M}_{m,m+1}(\lambda)\mathfrak{M}_{m+1,k}(\lambda)$ are regular for $\lambda \in \Gamma_{(-1)^{n-m}} \setminus \Lambda$, $\Lambda = \bigcup_{m,k} \Lambda_{mk}$ (in general, the set Λ is different for each matrix $\mathfrak{M}(\lambda)$).

Theorem 2.5. *A matrix $\mathfrak{M}(\lambda) \in M$ is the WM for $L \in V_N$ if and only if the following conditions hold:*

(1) (asymptotic) *there exist $\widetilde{L} \in V_N$ such that $\widehat{\mathfrak{M}}_{m,m+1}(\Lambda) = O(\rho^{-n-2})$, $|\lambda| \to \infty$;*
(2) (condition P) *for $x \geq 0$ Eq.* (2.45) *has a unique solution in the class $\Omega^{-1}(x,\lambda)\varphi(x,\lambda) \in B$ and $\sup_x \|\Omega^{-1}(x,\lambda)\varphi(x,\lambda)\|_B < \infty$;*
(3) $\varepsilon_\nu(x) \in W_{\nu+N}$, $\nu = \overline{0, n-2}$, *where the functions $\varepsilon_\nu(x)$ are defined by* (2.51)–(2.54).

Under these conditions the DE and LF are constructed according to (2.55).

It can be shown by a counterexample that conditions (2) and (3) in Theorem 2.5 are essential.

The necessity part of Theorem 2.5 was proved above in 2.2.1. The proof of the sufficiency is in [100].

2.3. Differential operators with a simple spectrum.

We consider DE and LF (2.1)–(2.2) on the half-line ($T = \infty$). If the spectrum of L has finite multiplicity, then the main equation obtained in Sec. 2.2 cab be contracted to the set $\Gamma \cup \Lambda$. For convenience we confine ourselves here to the case of a simple spectrum. For DO's with a simple spectrum the main equation can be transformed to the form (2.68)–(2.70), and the WM is uniquely determined from the so-called spectral data (see Definition 2.2). In particular, if only the discrete spectrum is perturbed, then the main equation of the IP is a linear algebraic system (2.73). For $n = 2$, from the main equation, using Fourier transform, we obtain the Gel'fand–Levitan equation.

2.3.1. Definition 2.1. We shall say that L has a simple spectrum if for each $\lambda_0 \in \Lambda' \stackrel{\text{def}}{=} \Lambda \setminus \{0\} \subset \Pi$ there exist finite limits

$$\mathfrak{M}_{\langle -1\rangle}(\lambda_0) = \lim_{\lambda\to\lambda_0} (\lambda - \lambda_0)\mathfrak{M}(\lambda), \qquad \mathfrak{M}^*_{\langle -1\rangle}(\lambda_0) = \lim_{\lambda\to\lambda_0} (\lambda - \lambda_0)\mathfrak{M}^*(\lambda)$$

and

$$\mathfrak{M}_{mk}(\lambda) = O(\rho^{m-k}), \qquad \lambda \to 0, \tag{2.62}$$

Lemma 2.10. *If L has a simple spectrum, then Λ is a finite set, and*

$$\mathfrak{M}(\lambda) = E + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{M^0(\mu)}{\mu - \lambda}d\mu + \sum_{\lambda_0 \in \Lambda'} \frac{Q(\lambda_0)}{\lambda - \lambda_0}, \qquad \operatorname{Im}\lambda \neq 0, \tag{2.63}$$

where

$$M_0(\lambda) = \mathfrak{M}^+(\lambda) - \mathfrak{M}^-(\lambda),$$

$$Q(\lambda_0) = \begin{cases} \frac{1}{2}\mathfrak{M}_{\langle -1\rangle}(\lambda_0), & \lambda_0 \in \Lambda'_{\text{Re}} \stackrel{\text{def}}{=} \Lambda' \cap \Gamma, \\ \mathfrak{M}_{\langle -1\rangle}(\lambda_0), & \lambda_0 \in \Lambda'_{\text{Im}} \stackrel{\text{def}}{=} \Lambda' \setminus \Lambda'_{\text{Re}}, \end{cases}$$

$$M^0(\lambda) = [M^0_{mk}(\lambda)]_{m,k=\overline{1,n}}, \quad Q(\lambda_0) = [Q_{mk}(\lambda_0)]_{m,k=\overline{1,n}}.$$

Thus the WM is uniquely determined from $\{M^0(\lambda)\}_{\lambda\in\Gamma}$, $\{\lambda_0, \mathfrak{M}_{\langle -1\rangle}(\lambda_0)\}_{\lambda_0\in\Lambda'}$. But really to construct $\mathfrak{M}(\lambda)$ we need less.

Definition 2.2. Assume that L has a simple spectrum. The set

$$\mathfrak{M}' = \Big(\{M(\lambda)\}_{\lambda\in\Gamma}, \quad \{\lambda_0, \mathfrak{M}_{\langle -1\rangle}(\lambda_0)\}_{\lambda_0\in\Lambda'}\Big),$$

where $M(\lambda) = \operatorname{diag}[M_m(\lambda)]_{m=\overline{1,n-1}}$, $M_m(\lambda) = \mathfrak{M}^+_{m,m+1}(\lambda) - \mathfrak{M}^-_{m,m+1}(\lambda)$, is called the spectral data of L.

The specification of the spectral data uniquely determines the WM $\mathfrak{M}(\lambda)$. Indeed, by virtue of Lemma 2.5, the functions $\mathfrak{M}_{mk}(\lambda) - \mathfrak{M}_{m,m+1}(\lambda)\mathfrak{M}_{m+1,k}(\lambda)$ are regular for $\lambda \in \Gamma_{(-1)^{n-m}} \setminus \Lambda$ and hence $M^0_{mk}(\lambda) = M_m(\lambda)\mathfrak{M}_{m+1,k}(\lambda)$. Then it follows from (2.63) that

$$\mathfrak{M}_{mk}(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} M_m(\mu)\mathfrak{M}_{m+1,k}(\mu)\frac{d\mu}{\lambda-\mu} + \sum_{\lambda_0\in\Lambda'}\frac{Q_{mk}(\lambda_0)}{\lambda-\lambda_0}, \qquad k > m. \tag{2.64}$$

Thus, the specification of the spectral data uniquely determines the WM $\mathfrak{M}(\lambda)$. To construct L from $\mathfrak{M}'$ we can construct $\mathfrak{M}(\lambda)$ by the recurrent formulas (2.64), and then use the method described in Sec. 2.2. But if there is more precise asymptotics of $\mathfrak{M}(\lambda)$ in the neighborhoods of the points of Λ', we can contract the main equation to the set $\Gamma \cup \Lambda$ and solve the IP directly from the spectral data $\mathfrak{M}'$.

Assume that for each $\lambda_0 \in \Lambda'$ the following asymptotics are valid for $\lambda \to \lambda_0$:

$$\begin{cases} \mathfrak{M}(\lambda) = \dfrac{\mathfrak{M}_{\langle -1\rangle}(\lambda_0)}{\lambda-\lambda_0} + \mathfrak{M}_{\langle 0\rangle}(\lambda_0) + (\lambda-\lambda_0\mathfrak{M}_{\langle 1\rangle}(\lambda_0) + o(\lambda-\lambda_0), \\ \mathfrak{M}^*(\lambda) = \dfrac{\mathfrak{M}^*_{\langle -1\rangle}(\lambda_0)}{\lambda-\lambda_0} + \mathfrak{M}^*_{\langle 0\rangle}(\lambda_0) + (\lambda-\lambda_0\mathfrak{M}^*_{\langle 1\rangle}(\lambda_0) + o(\lambda-\lambda_0). \end{cases} \tag{2.65}$$

We shall say that $L \in V'_N$ if $L \in V_N$ and (2.62) and (2.65) hold.

Let $L, \tilde{L} \in V'_N$, $\lambda_0 \in \Lambda'$. For simplicity, we assume that $\widehat{M}(\lambda) = O(\rho^{-n-2})$, $|\lambda| \to \infty$. Denote $\mathfrak{N}(\lambda_0) = \mathfrak{M}_{\langle -1\rangle}(\lambda_0)\big(\mathfrak{M}_{\langle 0\rangle}(\lambda_0)\big)^{-1}$, $J = \Lambda' \cup \tilde{\Lambda}'$, $J_0 = J \cap \Gamma$, $\varphi_{\langle 0\rangle}(x,\lambda_0) = Y\Phi_{\langle 0\rangle}(x,\lambda_0)$, $\lambda_0 \in J$; $Q_1(\lambda_0) = \mathfrak{N}(\lambda_0)$ $(\lambda_0 \in J \setminus J_0)$, $Q_1(\lambda_0) = \frac{1}{2}\mathfrak{N}(\lambda_0)$ $(\lambda_0 \in J_0)$. For $\lambda_0 \in J$, $s = -1, 0$, we define the matrices

$$\tilde{D}_s(x,\lambda,\lambda_0) = -\left[\frac{\langle\tilde{\Phi}(x,\lambda),\tilde{\Phi}^*(x,\mu)\rangle_{\tilde{l}}}{\lambda-\mu}\right]^{\langle s\rangle}_{|\mu=\lambda_0}, \qquad \tilde{d}_s(x,\lambda,\lambda_0) = -\left[\frac{\langle\tilde{f}(x,\lambda),\tilde{\Phi}^*(x,\mu)\rangle_{\tilde{l}}}{\lambda-\mu}\right]^{\langle s\rangle}_{|\mu=\lambda_0},$$

Lemma 2.11. *The following relations hold:*

$$\varkappa_{\nu s}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\tilde{f}^{*(\nu)}(x,\mu)\widehat{M}(\mu)f^{(s)}(x,\mu) + \sum_{\lambda_0\in J_0}\frac{\big(\pm\tilde{\Phi}^{*(\nu)}(x,\lambda_0)\tilde{\mathfrak{N}}(\lambda_0)\hat{\mathfrak{N}}(\lambda_0)Y^T\big)}{(\mu-\lambda_0)^2}\varphi^{(s)}_{\langle 0\rangle}(x,\lambda_0)\right\}d\mu$$

$$- \sum_{\lambda_0\in J}\big(\tilde{\Phi}^{*(\nu)}_{\langle 0\rangle}(x,\lambda_0)Q_1(\lambda_0)Y^T\big)\varphi^{(s)}_{\langle 0\rangle}(x,\lambda_0), \qquad \nu+s \le n-1, \tag{2.66}$$

$$\tilde{\Phi}(x,\lambda) = \Phi(x,\lambda) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\frac{\langle\tilde{\Phi}(x,\lambda),\tilde{f}^*(x,\mu)\rangle_{\tilde{l}}}{\lambda-\mu}\widehat{M}(\mu)f(x,\mu)\right.$$

$$\left. + \sum_{\lambda_0\in J_0}\big(\pm\tilde{D}_{-1}(x,\lambda,\lambda_0)\hat{\mathfrak{N}}(\lambda_0)Y^T\big)\cdot\frac{\varphi_{\langle 0\rangle}(x,\lambda_0)}{(\mu-\lambda_0)^2}\right\}d\mu + \sum_{\lambda_0\in J}\big(\tilde{D}_0(x,\lambda,\lambda_0)\hat{Q}_1(\lambda_0)Y^T\big)\varphi_{\langle 0\rangle}(x,\lambda_0), \tag{2.67}$$

where we write + (−) when λ_0 lies on the upper (lower) side of the cut.

Denote

$$Y' = [\delta_{j,k-1}]_{j=\overline{1,n-2};k=\overline{1,n-1}}, \qquad Y_0 = [\delta_{j,k-1}]_{j,k=\overline{1,n-1}}, \qquad \chi(x,\lambda) = Y'f(x,\lambda),$$

$$V_1(z_0) = Y_0\left(\widehat{M}_{\langle -1\rangle}(z_0) + \widehat{M}_{\langle 0\rangle}(z_0)Y\mathfrak{N}(z_0)Y^T\right), \qquad z_0 \in J_0,$$

$$V_2(z_0) = Y_0\left(\widehat{M}_{\langle 0\rangle}(z_0) + \widehat{M}_{\langle 1\rangle}(z_0)Y\mathfrak{N}(z_0)Y^T\right), \qquad z_0 \in J_0,$$

$$V_3(z_0) = Y_0\widehat{M}_{\langle -1\rangle}(z_0)Y'^T, \quad z_0 \in J_0; \qquad V_k(z_0) = 0, \quad z_0 \in J\setminus J_0, \quad k = \overline{1,3}.$$

Theorem 2.6.

$$\widetilde{f}(x,\lambda) = f(x,\lambda) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg\{\frac{\langle \widetilde{f}(x,\lambda), \widetilde{f}^*(x,\lambda)\rangle_{\widetilde{l}}}{\lambda-\mu}\widehat{M}(\mu)f(x,\mu)$$

$$+\sum_{\lambda_0\in J_0}\big(\pm\widetilde{d}_{-1}(x,\lambda,\lambda_0)\widehat{\mathfrak{N}}(l_0)Y^T\big)\cdot\frac{\varphi_{\langle 0\rangle}(x,\lambda_0)}{(\mu-\lambda_0)^2}\Bigg\}\,d\mu$$

$$+\sum_{\lambda_0\in J}\big(\widetilde{d}_0(x,\lambda,\lambda_0)\widehat{Q}_1(\lambda_0)Y^T\big)\varphi_{\langle 0\rangle}(x,\lambda_0),\qquad \lambda\in\Gamma, \tag{2.68}$$

$$\widetilde{\varphi}_{\langle 0\rangle}(x,z_0) = \varphi_{\langle 0\rangle}(x,z_0) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg\{\left[\frac{\langle Y\widetilde{\Phi}(x,\lambda), \widetilde{f}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\right]^{\langle 0\rangle}_{|\lambda=z_0}\widehat{M}(\mu)f(x,\mu)$$

$$+\frac{V_1(z_0)\varphi_{\langle 0\rangle}(x,z_0)}{(\mu-z_0)^2} + \sum_{\lambda_0\in J_0}\big(\pm Y\widetilde{D}_{-1,\langle 0\rangle}(x,z_0,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T\big)\frac{\varphi_{\langle 0\rangle}(x,\lambda_0)}{(\mu-\lambda_0)^2}\Bigg\}\,d\mu$$

$$\mp\frac{1}{2}\big(V_2(z_0)\varphi_{\langle 0\rangle}(x,z_0) + V_3(z_0)\chi_{\langle 1\rangle}(x,z_0)\big)$$

$$+\sum_{\lambda_0\in J}\big(Y\widetilde{D}_{0,\langle 0\rangle}(x,z_0,\lambda_0)\widehat{Q}_1(\lambda_0)Y^T\big)\varphi_{\langle 0\rangle}(x,\lambda_0),\qquad z_0\in J, \tag{2.69}$$

$$\widetilde{\chi}_{\langle 1\rangle}(x,z_0) = \chi_{\langle 1\rangle}(x,z_0) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg\{\left[\frac{\langle \widetilde{\chi}(x,\lambda), \widetilde{f}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\right]^{\langle 1\rangle}_{|\lambda=z_0}\widehat{M}(\mu)f(x,\mu)$$

$$+\sum_{\lambda_0\in J_0}\big(\pm Y'\widetilde{d}_{-1,\langle 1\rangle}(x,z_0,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T\big)\frac{\varphi_{\langle 0\rangle}(x,\lambda_0)}{(\mu-\lambda_0)^2}\Bigg\}\,d\mu$$

$$+\sum_{\lambda_0\in J}\big(Y'\widetilde{d}_{0,\langle 1\rangle}(x,z_0,\lambda_0)\widehat{Q}_1(\lambda_0)Y^T\big)\varphi_{\langle 0\rangle}(x,\lambda_0),\qquad z_0\in J_0, \tag{2.70}$$

Relations (2.68)–(2.70) are the main equations of the IP with respect to $\{f(x,\lambda)\}_{\lambda\in\Gamma}$, $\{\varphi_{\langle 0\rangle}(x,z_0)\}_{z_0\in J}$, $\{\chi_{\langle 1\rangle}(x,z_0)\}_{z_0\in J_0}$. They allow us to solve the IP of recovering the DE and LF $L\in V_N'$ from the given spectral data $\mathfrak{M}'$. For constructing L we need solve the main equations (2.68)–(2.70) for each fixed $x\geq 0$ and then find the DE and LF via (2.55), where the functions $\varepsilon_\nu(x)$ and $t_{j\nu}(x)$ are defined by (2.66) and (2.52)–(2.54).

2.3.2. Consider a perturbation of the discrete spectrum. Let $L,\widetilde{L}\in V_N'$ and

$$\widehat{\mathfrak{M}}_{m,m+1}(\lambda) = \sum_{\lambda_0\in J}\frac{\widehat{Q}_{m,m+1}(\lambda_0)}{\lambda-\lambda_0},$$

i.e., $\widehat{M}(\lambda)\equiv 0$. Denote

$$\widetilde{P}(x,\lambda,\lambda_0) = \widetilde{D}_0(x,\lambda,\lambda_0)\widehat{Q}_1(\lambda_0)Y^T,\qquad \widetilde{G}(x,z_0,\lambda_0) = \big[Y\widetilde{P}(x,\lambda,\lambda_0)\big]^{\langle 0\rangle}_{|\lambda=z_0},$$

$$\widetilde{g}^*(x,\lambda_0) = -\widetilde{\Phi}^*_{\langle 0\rangle}(x,\lambda_0)\widehat{Q}_1(\lambda_0)Y^T,\qquad \varphi(x) = \big[\varphi_{\langle 0\rangle}(x,\lambda_0)\big]_{\lambda_0\in J},$$

$$\widetilde{g}^*(x) = \big[\widetilde{g}^*(x,\lambda_0)\big]^T_{\lambda_0\in J},\qquad \widetilde{G}(x) = \big[\widetilde{G}(x,z_0,\lambda_0)\big]_{z_0,\lambda_0\in J}.$$

Theorem 2.7.

$$\varkappa_{\nu s}(x) = \widetilde{g}^{*(\nu)}(x)\varphi^{(s)}(x), \tag{2.71}$$

$$\widetilde{\Phi}(x,\lambda) = \Phi(x,\lambda) + \sum_{\lambda_0 \in J} \widetilde{P}(x,\lambda,\lambda_0)\varphi_{\langle 0\rangle}(x,\lambda_0), \tag{2.72}$$

$$\widetilde{\varphi}(x) = \big(E + \widetilde{G}(x)\big)\varphi(x). \tag{2.73}$$

We note that (2.71)–(2.72) is the particular case of (2.66)–(2.67) when the integrals in (2.66) and (2.67) are equal to zero. Further, multiplying (2.72) by Y at the left, we obtain

$$\widetilde{\varphi}_{\langle 0\rangle}(x,z_0) = \varphi_{\langle 0\rangle}(x,z_0) + \sum_{\lambda_0 \in J} \widetilde{G}(x,z_0,\lambda_0)\varphi_{\langle 0\rangle}(x,\lambda_0),$$

i.e., (2.73) is valid.

Equation (2.73) is the main equation of the IP. For each fixed $x \geq 0$ (2.73) is a linear algebraic system with respect to $\varphi(x)$ and $\det[E + \widetilde{G}(x)] \neq 0$. To solve the IP we must find $\varphi(x)$ from (2.73) and then construct the DE and LF L via (2.55), where the functions $\varepsilon_\nu(x)$ and $t_{j\nu}(x)$ are defined by (2.71) and (2.52)–(2.54).

2.3.3. Connection with the Gel'fand–Levitan equation.

We consider the DO

$$l_1 y = -y'' + q(x)y, \quad x > 0; \qquad y'(0) - hy(0) = 0, \tag{2.74}$$

where $q(x)$ and h are real. Let $\varphi(x,\lambda)$ be a solution of the DE $l_1 y = \lambda y$ under the conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$, and let $\sigma(\lambda)$ be the spectral function of the DO (2.74), which can be uniquely expressed in terms of the WF [57]. Then the main equation (2.45) of the IP becomes

$$\widetilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \int_{\infty}^{\infty} \left(\int_0^x \widetilde{\varphi}(t,\lambda)\widetilde{\varphi}(t,\mu)\,dt \right) \varphi(x,\mu)\,d\widehat{\sigma}(\mu)$$

after contraction of the contour to the real axis. Assume for definiteness that $\widetilde{q}(x) = \widetilde{h} = 0$. Then $\widetilde{\varphi}(x,\lambda) = \cos\sqrt{\lambda}x$. Using the transformation operator (0.4) we get with the help of the Fourier cos-transformation the Gel'fand–Levitan equation

$$K(x,t) + F(x,t) + \int_0^x K(x,\tau)F(t,\tau)\,d\tau = 0, \qquad F(x,t) = \int_{\infty}^{\infty} \cos\sqrt{\mu}x \cos\sqrt{\mu}t\,d\widehat{\sigma}(\mu).$$

2.4. The self-adjoint case.

We consider DE and LF $L = (l, U)$ of the form (2.1)–(2.2) on the half-line $x > 0$ ($T = \infty$). In Secs. 2.1, 2.2, we obtained a solution of the IP for the general non-self-adjoint case. The central role was played there by the main equation of the IP. One of the conditions under which an arbitrary matrix $\mathfrak{M}(\lambda)$ is the WM for a non-self-adjoint DO is the requirement that the main equation must have a unique solution. It is difficult to verify this condition in the general case. In connection with this, an important problem is that of obtaining sufficient conditions for solvability of the main equation, and extraction of classes of operators for which unique solvability can be proved. One of such classes is the class of self-adjoint operators. Here we investigate the IP for the self-adjoint case. We prove unique solvability of the main equation, and obtain necessary and sufficient conditions, along with a procedure for construction of an operator from its WM. Some difference in the notations is pointed below.

2.4.1. For definiteness, let $n = 2m$ and $\sigma_{\xi 0} = n - \xi$. We assume that $L = L^*$, where the adjoint pair $L^* = (l^*, U^*)$ is defined by the relations

$$l^* z = z^{(n)} + \sum_{\nu=0}^{n-2} (-1)^\nu \left(\overline{p_\nu(x)} z\right)^{(\nu)},$$

$$\langle y(x), z(x) \rangle_{l|x=0}^{0} = \sum_{\xi=1}^{n} (-1)^{\xi-1} U_{\xi 0}(y) \overline{U^*_{n-\xi+1}(z)},$$

where

$$\langle y(x), z(x) \rangle_l^0 = \sum_{\nu+j \le n-1} (-1)^s C_s^j p_{s+\nu+1}^{(s-j)}(x) y^{(\nu)}(x) \overline{z^{(j)}(x)}.$$

It was proved in Sec. 2.1 that the WF's $\mathfrak{M}_{k\xi}(\lambda)$ are regular in $\Pi_{(-1)^k}$, and are continuous in $\overline{\Pi}_{(-1)^k} \setminus \{0\}$ with the exception of the bounded sets $\Lambda_{k\xi}$. We have $\mathfrak{M}_{k\xi}(\lambda)\rho^{\xi-k} = O(1)$ as $|\lambda| \to \infty$. Let

$$\Lambda = \bigcup_{k,\xi} \Lambda_{k\xi}, \qquad \mathfrak{N}_k(\lambda) = \frac{1}{2\pi i}\left(\mathfrak{M}^-_{k,k+1}(\lambda) - \mathfrak{M}^+_{k,k+1}(\lambda)\right),$$

$$\mathfrak{M}^\pm_{k\xi}(\lambda) = \lim \mathfrak{M}_{k\xi}(\lambda + iz), \quad z \to 0, \quad \operatorname{Re} z > 0, \quad -\infty < \lambda < \infty.$$

To simplify the computations we confine ourselves to the case where there is no discrete spectrum. For definiteness, let $\widetilde{p}_\nu(x) = \widetilde{u}_{\xi\nu 0} = 0$.

Definition 2.3. L is said to be in V_N^+ if $p_\nu(x) \in W_{\nu+N}$, $L = L^*$, $\Lambda = \emptyset$, $\mathfrak{M}_{k\xi}(\lambda)\rho^{\xi-k} = O(1)$ as $|\lambda| \to 0$, and $\mathfrak{N}_k(\lambda) = O(\rho^{-n-2})$ as $|\lambda| \to \infty$. We solve the IP in the classes V_N^+.

Theorem 2.8. *Assume that $L \in V_N^+$. Then the WM has the following properties:*

(1) $\mathfrak{M}_{k\xi}(\lambda) = \delta_{k\xi}$, $k \ge \xi$;
(2) *the functions $\mathfrak{M}_{k\xi}(\lambda)$ are regular in $\Pi_{(-1)^k}$ and continuous in $\overline{\Pi}_{(-1)^k} \setminus \{0\}$;*
(3) *the functions $\mathfrak{M}_{k\xi}(\lambda)\rho^{\xi-k}$ are bounded;*
(4) *the functions $\mathfrak{M}_{k\xi}(\lambda) - \mathfrak{M}_{k,k+1}(\lambda)\mathfrak{M}_{k+1,\xi}(\lambda)$ are regular for $\lambda \in \Gamma_{(-1)^k}$;*
(5) $\widehat{\mathfrak{N}}(\lambda) = O(\rho^{-n-2})$ *as* $|\lambda| \to \infty$;
(6) $\mathfrak{M}_{n-k,n-k+1}(\lambda) = \overline{\mathfrak{M}_{k,k+1}(\overline{\lambda})}$, $k = \overline{1,m}$;
(7) $(-1)^m \mathfrak{N}_m(\lambda) > 0$, $\lambda \in \Gamma_{(-1)^m}$.

We remark that $\mathfrak{N}_k(\lambda) \equiv 0$ for $\lambda \in \Gamma_{(-1)^{k-1}}$, and the functions $\rho\mathfrak{N}_k(\lambda)$ are continuous and bounded for $\lambda \in \Gamma_{(-1)^k}$.

Theorem 2.9. *Assume that $L \in V_N^+$. Then the WM is uniquely determined by the specification of the functions $\mathfrak{N}_1(\lambda), \dots, \mathfrak{N}_m(\lambda)$ according to the formulas*

$$\mathfrak{N}_{n-j}(\lambda) - \overline{\mathfrak{N}_j(\lambda)}, \quad j = \overline{1, m-1}; \qquad \mathfrak{M}_{kj}(\lambda) = \int_{\Gamma_{(-1)^k}} \frac{\mathfrak{N}_k(\mu)\mathfrak{M}_{k+1,\xi}(\mu)}{\lambda - \mu} d\mu, \quad \xi > k.$$

We set

$$\varphi(x, \lambda) = \left[\chi((-1)^{k-1}\lambda)\Phi_k(x, \lambda)\right]_{k=\overline{2,n}}$$

(a column vector),

$$\widetilde{q}(x, \lambda) = \left[(-1)^{k-1}\chi((-1)^{k-1}\lambda)\widetilde{\Phi}_{n-k+2}(x, \lambda)\widehat{\mathfrak{N}}_k(\lambda)\right]^T_{k=\overline{2,n}}$$

(a row vector) (here T is a sign for transposition, and $\chi(\lambda)$ is the Heaviside function),

$$\begin{cases} \gamma_\nu(x) = \int\limits_{-\infty}^{\infty} \big|\overline{q^{(\nu)}(x,\lambda)}\varphi(x,\lambda)\big|\,d\lambda, \qquad \gamma_{\nu s}(x) = \int\limits_{-\infty}^{\infty} \overline{q^{(\nu)}(x,\lambda)}\varphi^{(s)}(x,\lambda)\,d\lambda, \\ t_{j\nu}(x) = -\sum\limits_{\beta=\nu+1}^{j} C_j^\beta C_{\beta-1}^\nu \gamma_{\beta-\nu-1,j-\beta}(x), \quad j>\nu, \qquad t_{j\nu}(x) = \delta_{j\nu}, \quad j \le \nu, \\ \xi_\nu(x) = (-1)^\nu \gamma_{n-\nu-1,0}(x) + \sum\limits_{s=0}^{n-\nu-1} C_n^s C_{n-s-1}^\nu \gamma_{n-s-\nu-1,s}(x), \end{cases} \tag{2.75}$$

$$\psi_\nu(x) = \xi_\nu(x) - \sum_{j=\nu+1}^{n-2} \psi_j(x) t_{j\nu}(x), \qquad \nu = \overline{0, n-2}. \tag{2.76}$$

Theorem 2.10. *For a fixed $x \ge 0$ the vector-valued function $\varphi(x,\lambda)$ is a solution of the linear integral equation*

$$\widetilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \int\limits_{\infty}^{\infty} \frac{\langle \widetilde{\varphi}(x,\lambda), q(x,\mu)\rangle_{\widetilde{l}}^{0}}{\lambda-\mu}\varphi(x,\mu)\,d\mu. \tag{2.77}$$

Equation (2.77) is called the main equation of the IP.

Theorem 2.11. *The following relations hold:*

$$p_\nu(x) = \psi_\nu(x), \qquad u_{\xi\nu 0} + \sum_{j=\nu+1}^{n-1} u_{\xi j 0} t_{j\nu}(0) = 0. \tag{2.78}$$

2.4.2. In this section, we give a solvability theorem for the main equation, along with a solution of the IP. Notation: M is the set of matrices $\mathfrak{M}(\lambda) = [\mathfrak{M}_{k\xi}(\lambda)]_{k,\xi=\overline{1,n}}$ with the properties (1)–(7) in Theorem 2.8. Assume for definiteness that $\arg\rho \in (0, 2\pi/n)$. We let $\Omega(x,\lambda) = \operatorname{diag}\left[\rho^{n-k}\exp(-\rho R_k x)\right]_{k=\overline{2,n}}$ and introduce the Banach space $B = \mathcal{L}_\infty^{n-1}(-\infty,\infty)$ of vector-valued functions $z(\lambda) = [z_j(\lambda)]_{j=\overline{1,n-1}}$, $z_j(\lambda) \in \mathcal{L}_\infty(-\infty,\infty)$ with the norm

$$\|z\|_B = \sum_{j=1}^{n-1} \|z_j\|_{\mathcal{L}_\infty(-\infty,\infty)}.$$

Theorem 2.12. *Let $\mathfrak{M}(\lambda) \in M$. Then for each fixed $x \ge 0$ equation* (2.77) *has a unique solution in the class $\Omega(x,\lambda)\varphi(x,\lambda) \in B$.*

We indicate briefly the scheme of proof of Theorem 2.12. It suffices to prove that the homogeneous equation

$$h(x,\lambda) + \int\limits_{-\infty}^{\infty} \frac{\langle \widetilde{\varphi}(x,\lambda), q(x,\mu)\rangle_{\widetilde{l}}^{0}}{\lambda-\mu} h(x,\mu)\,d\mu = 0, \qquad \Omega(x,\lambda)h(x,\lambda) \in B, \tag{2.79}$$

has only the zero solution. We consider the function

$$B(x,\lambda) = \sum_{j=1}^{n} (-1)^{j-1} H_j(x,\lambda)\overline{H_{n-j+1}(x,\overline{\lambda})},$$

where the vector-valued function $H(x,\lambda) = [H_j(x,\lambda)]_{j=\overline{1,n}}$ is defined by the relation

$$H(x,\lambda) = -\int_{-\infty}^{\infty} \frac{\langle \tilde{\Phi}(x,\lambda), q(x,\mu)\rangle_l^0}{\lambda-\mu} h(x,\mu)\, d\mu.$$

For each fixed $x \geq$ the functions $H(x,\lambda)$ and $B(x,\lambda)$ have the following properties:

(1) the functions $H_k(x,\lambda)$ are regular in $\Pi_{(-1)^k}$; $H_k(x,\lambda) = h_k(x,\lambda)$ for $k = \overline{2,n}$ and $\lambda \in \Gamma_{(-1)^{k-1}}$, and the function $B(x,\lambda)$ is regular in $\Pi_{(-1)^m}$;

(2) the functions $H_k(x,\lambda) - \mathfrak{M}_{k,k+1}(\lambda)H_{k+1}(\lambda)$ are regular for $\lambda \in \Gamma_{(-1)^k}$, and the function $B(x,\lambda) + (-1)^m \mathfrak{M}_{m,m+1}(\lambda) H_{m+1}(x,\lambda)\overline{H_{m+1}(x,\overline{\lambda})}$ is regular for $\lambda \in \Gamma_{(-1)^m}$;

(3) the following equalities hold:

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|\lambda|=R} B(x,\lambda)\, d\lambda = 0, \qquad \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} B(x,\lambda)\, d\lambda = 0.$$

From these properties it follows that

$$\int_{-\infty}^{\infty} |h_{m+1}(x,\lambda)|^2 \mathfrak{N}_m(\lambda)\, d\lambda = 0,$$

hence $h_{m+1}(x,\lambda) \equiv 0$. From this, using (2.79) and the properties of the functions $H_k(x,\lambda)$, we obtain $h(x,\lambda) \equiv 0$.

Theorem 2.13. *A matrix $\mathfrak{M}(\lambda) \in M$ is the WM for an $L \in V_N^+$ if and only if*

$$\sup_{x\geq 0} \gamma_\nu(x) < \infty, \qquad \psi_\nu(x) \in W_{\nu+N}, \quad \nu = \overline{0, n-2},$$

where $\gamma_\nu(x)$ and $\psi_\nu(x)$ are constructed according to the formulas (2.75) *and* (2.76), *and $\varphi(x,\lambda)$ is the solution of the main equation* (2.77). *Under these conditions the DE and LF $L = (l, U)$ are constructed according to the formulas* (2.78).

3. Inverse Problems on a Finite Interval

We consider DE and LF $L \in V_N$ of the form (2.1)–(2.2) on a finite interval ($T < \infty$). In Sec. 3, we provide a solution of the IP of recovering L from the given WM $\mathfrak{M}(\lambda)$. We use the notations and the results of Sec. 2.1. For IP's on a finite interval there are specific difficulties connected with the properties S_1 and S_2 of the WM $\mathfrak{M}(\lambda)$ (see Lemmas 3.1 and 3.2). We obtain necessary and sufficient conditions on the WM, a procedure of constructing coefficients of the DE and LF from the given WM $\mathfrak{M}(\lambda)$, study the stability problem. A counterexample in Sec. 3.3 shows that dropping one element of the WM leads to nonuniqueness of the solution of the IP.

3.1. Properties of the Weyl solutions and the Weyl matrix.

We shall say that $L \in V_N'$ if $L \in V_N$ and the functions $\Delta_{mm}(\lambda)$, $m = \overline{1, n-1}$, have only simple zeros. If $L \in V_N'$, then the WM $\mathfrak{M}(\lambda)$ and $\mathfrak{M}^*(\lambda)$ have only simple poles. For simplicity, in the sequel we shall assume that $L \in V_N'$.

Denote $\beta_{lmk} = \operatorname{res}_{\lambda_{lm}} \mathfrak{M}_{mk}(\lambda)$. It follows from (2.7), (2.20), and (2.21) that

$$\big|\mathfrak{M}_{mk}(\lambda)\big| < C|\rho|^{m-k}, \qquad \lambda \in G_\delta, \tag{3.1}$$

(G_δ is the λ-plane without the circles $|\lambda-\lambda_0|<\delta,\ \lambda_0\in\Lambda$), and hence

$$\mathfrak{M}_{mk}(\lambda)=\sum_{l=1}^{\infty}\frac{\beta_{lmk}}{\lambda-\lambda_{lm}},\qquad \beta_{lmk}=\frac{\Delta_{mk}(\lambda_{lm})}{\dot{\Delta}_{mm}(\lambda_{lm})}$$

Thus, the WF $\mathfrak{M}_{mk}(\lambda)$ is uniquely determined by its zeros and residues $\{\lambda_{lm},\ \beta_{lmk}\}_{l\geq 1}$.

For $\lambda_0\in\Lambda$ we define the matrix $\mathfrak{N}(\lambda_0)=\left[\mathfrak{N}_{jk}(\lambda_0)\right]_{j,k=\overline{1,n}}$ via $\mathfrak{N}(\lambda_0)=\mathfrak{M}_{\langle -1\rangle}(\lambda_0)\left(\mathfrak{M}_{\langle 0\rangle}(\lambda_0)\right)^{-1}$. Since $\mathfrak{M}_{mk}(\lambda)=\delta_{mk},\ m\geq k$, it follows that $\mathfrak{N}_{jk}(\lambda_0)=0$ for $j\geq k$. The following relations hold:

$$\begin{cases}\mathfrak{N}(\lambda_0)\mathfrak{N}(\lambda_0)=0,\\ \Phi_{\langle -1\rangle}(x,\lambda_0)=\mathfrak{N}(\lambda_0)\Phi_{\langle 0\rangle}(x,\lambda_0),\\ \Phi^*_{\langle -1\rangle}(x,\lambda_0)=-\Phi^*_{\langle 0\rangle}(x,\lambda_0)\mathfrak{N}(\lambda_0).\end{cases}\tag{3.2}$$

It follows from $l\Phi_m(x,\lambda)=\lambda\Phi_m(x,\lambda)$ and (2.13) that

$$U_{n-m+1,T}\left(\Phi_m(x,\lambda)\right)=(-1)^{n-m}\left(\Delta_{mm}(\lambda)\right)^{-1}\Delta_{m-1,m-1}(\lambda),\tag{3.3}$$

$$\begin{cases}l\Phi_{m,\langle -1\rangle}(x,\lambda_0)=\lambda_0\Phi_{m,\langle -1\rangle}(x,\lambda_0),\\ l\Phi_{m,\langle 0\rangle}(x,\lambda_0)=\lambda_0\Phi_{m,\langle 0\rangle}(x,\lambda_0)+\Phi_{m,\langle -1\rangle}(x,\lambda_0).\end{cases}\tag{3.4}$$

We prove two important properties of the WM. Define $\Lambda_0=\Lambda_m=\emptyset$.

Lemma 3.1 (property S_1). *If $\lambda_0\notin\Lambda_m$, then $\mathfrak{N}_{j,m+1}(\lambda_0)=\cdots=\mathfrak{N}_{jn}(\lambda_0)=0,\ j=\overline{1,m}$. If, moreover, $\lambda_0\in\Lambda_{\nu+1}\cap\cdots\cap\Lambda_{m-1},\ \lambda_0\notin\Lambda_\nu,\ 1\leq\nu+1<m\leq n$, then $\mathfrak{N}_{\nu+1,m}(\lambda_0)\neq 0$.*

Proof. The first assertion of the lemma will be proved by induction. Since $\lambda_0\notin\Lambda_m$, it follows from (2.13) that $\Phi_{m,\langle -1\rangle}(x,\lambda_0)=0$. On the other hand, in view of (3.2), we get

$$\Phi_{m,\langle -1\rangle}(x,\lambda_0)=\mathfrak{N}_{m,m+1}(\lambda_0)\Phi_{m+1,\langle 0\rangle}(x,\lambda_0)+\cdots+\mathfrak{N}_{mn}(\lambda_0)\Phi_{n,\langle 0\rangle}(x,\lambda-0).$$

Applying here the LF $U_{m+1,0},\ldots,U_{n0}$, we find successively $\mathfrak{N}_{m,m+1}(\lambda_0)=\cdots=\mathfrak{N}_{mn}(\lambda_0)=0$.

Assume that $\mathfrak{N}_{j,m+1}(\lambda_0)=\cdots=\mathfrak{N}_{jn}(\lambda_0)=0,\ j=\overline{m-s+1,m},\ s\geq 1$. According to (3.2), we have

$$\Phi_{m-s,\langle -1\rangle}(x,\lambda_0)=\mathfrak{N}_{m-s,m-s+1}(\lambda_0)\Phi_{m-s+1,\langle 0\rangle}(x,\lambda_0)+\cdots+\mathfrak{N}_{m-s,n}(\lambda_0)\Phi_{n,\langle 0\rangle}(x,\lambda_0)$$

or

$$\begin{aligned}\Phi_{m-s,\langle -1\rangle}(x,\lambda_0)&-\sum_{i=1}^{s}\mathfrak{N}_{m-s,m-s+1}(\lambda_0)\Phi_{m-s+i,\langle 0\rangle}(x,\lambda_0)\\ &=\sum_{i=s+1}^{n-m+s}\mathfrak{N}_{m-s,m-s+i}(\lambda_0)\Phi_{m-s+i,\langle 0\rangle}(x,\lambda_0)\overset{\text{def}}{=}\psi(x).\end{aligned}\tag{3.5}$$

Since $\Phi_{m,\langle -1\rangle}(x,\lambda_0)=0$, (3.4) implies that the functions $\Phi_{m-s,\langle -1\rangle}(x,\lambda_0)$ and $\Phi_{m,\langle 0\rangle}(x,\lambda_0)$ are solutions of the DE $ly=\lambda_0 y$. Further, using (3.2) and the assumption of the induction, we obtain

$$\begin{aligned}&\sum_{i=1}^{s-1}\mathfrak{N}_{m-s,m-s+i}(\lambda_0)\Phi_{m-s+i,\langle -1\rangle}(x,\lambda_0)\\ &=\sum_{i=1}^{s-1}\mathfrak{N}_{m-s,m-s+i}(\lambda_0)\sum_{\nu=m-s+i+1}^{m}\mathfrak{N}_{m-s+i,\nu}(\lambda_0)\Phi_{\nu,\langle 0\rangle}(x,\lambda_0)\\ &=\sum_{\nu=m-s+2}^{m}\Phi_{\nu,\langle 0\rangle}(x,\lambda_0)\sum_{i=1}^{\nu-m+s-1}\mathfrak{N}_{m-s,m-s+i}(\lambda_0)\mathfrak{N}_{m-s+i,\nu}(\lambda_0)=0,\end{aligned}$$

and consequently, the function $\sum_{i=1}^{s-1} \mathfrak{N}_{m-s,m-s+i}(\lambda_0)\Phi_{m-s+i,\langle 0\rangle}(x,\lambda_0)$ is a solution of the DE $ly = \lambda_0 y$. This and (3.5) imply that $l\psi(x) = \lambda_0\psi(x)$. Using (3.5) again, we compute $U_{\xi 0}(\psi) = U_{\eta T}(\psi) = 0$, $\xi = \overline{1,m}$, $\eta = \overline{1,n-m}$. Since λ_0 is not an eigenvalue of S_m, we conclude that $\psi(x) \equiv 0$. Applying the LF $U_{m+1,0}, \ldots, U_{n0}$ to (3.5), we find successively $\mathfrak{N}_{m-s,k}(\lambda_0) = 0$, $k = \overline{m+1,n}$.

Let us go on to the second assertion of the lemma. Since $\Delta_{\nu\nu}(\lambda_0) \neq 0$, $\Delta_{ss}(\lambda_0) = 0$, $s = \overline{\nu+1, m-1}$, it follows from (3.3) that $U_{n-s+1,T}(\Phi_{s,\langle 0\rangle}(x,\lambda_0)) \neq 0$, $s = \overline{\nu+2, m-1}$, $\Phi_{\nu+1,\langle -1\rangle}(x,\lambda_0) \not\equiv 0$. Assume that $\mathfrak{N}_{\nu+1,m}(\lambda_0) = 0$. Then

$$\Phi_{\nu+1,\langle -1\rangle}(x,\lambda_0) = \mathfrak{N}_{\nu+1,\nu+2}(\lambda_0)\Phi_{\nu+2,\langle 0\rangle}(x,\lambda_0) + \cdots + \mathfrak{N}_{\nu+1,m-1}(\lambda_0)\Phi_{m-1,\langle 0\rangle}(x,\lambda_0).$$

Applying the LF $U_{n-m+2,T}, \ldots, U_{n-\nu-1,T}$ successively, we obtain $\mathfrak{N}_{\nu+1,m-1}(\lambda_0) = \cdots = \mathfrak{N}_{\nu+1,\nu+2}(\lambda_0) = 0$, i.e., $\Phi_{\nu+1,\langle -1\rangle}(x,\lambda_0) \equiv 0$. Lemma 3.1 is proved.

Denote $A_s(\lambda_0) = [\mathfrak{N}_{j\nu}(\lambda_0)]_{j=\overline{1,n-s};\nu=\overline{n-s,n}}$, $s = \overline{1,n-1}$.

Lemma 3.2 (property S_2).

$$\operatorname{rank} A_s(\lambda_0) \leq 1, \qquad s = \overline{1,n-1}.$$

Proof. We will prove the lemma by induction. Let us show that $\operatorname{rank} A_1(\lambda_0) \leq 1$. Indeed, if $\Delta_{n-2,n-2}(\lambda_0) = \Delta_{n-1,n-1}(\lambda_0) = 0$, then from (3.3) we have $U_{2T}(\Phi_{n-1,\langle -1\rangle}(x,\lambda_0)) = 0$, $U_{2T}(\Phi_{n-1,\langle 0\rangle}(x,\lambda_0)) \neq 0$. Applying the LF U_{2T} to the equality $\Phi_{\langle -1\rangle}(x,\lambda_0) = \mathfrak{N}(\lambda_0)\Phi_{\langle 0\rangle}(x,\lambda_0)$, we obtain

$$\mathfrak{N}_{j,n-1}(\lambda_0)U_{2T}(\Phi_{n-1,\langle 0\rangle}(x,\lambda_0)) + \mathfrak{N}_{j,n}(\lambda_0)U_{2T}(\Phi_{n,\langle 0\rangle}(x,\lambda_0)) = 0, \qquad j = \overline{1,n-1},$$

and hence $\operatorname{rank} A_1(\lambda_0) \leq 1$. If $\Delta_{n-1,n-1}(\lambda_0) \neq 0$ or $\Delta_{n-2,n-2}(\lambda_0) \neq 0$, then, by Lemma 3.1, $\mathfrak{N}_{jn}(\lambda_0) = 0$, $j = \overline{1,n-1}$, i.e., $\operatorname{rank} A_1(\lambda_0) \leq 1$.

Assume that the relations $\operatorname{rank} A_k(\lambda_0) \leq 1$, $k = \overline{1,s-1}$, have been proved. If $\Delta_{n-s-1,n-s-1}(\lambda_0) = \Delta_{n-s,n-s}(\lambda_0) = 0$, it follows from (3.3) that

$$U_{s+1,T}(\Phi_{n-s,\langle -1\rangle}(x,\lambda_0)) = 0, \qquad U_{s+1,T}(\Phi_{n-s,\langle 0\rangle}(x,\lambda_0)) \neq 0,$$

hence

$$\sum_{k=n-s}^{n} \mathfrak{N}_{jk}(\lambda_0)U_{s+1,T}(\Phi_{k,\langle 0\rangle}(x,\lambda_0)) = 0, \qquad j = \overline{1,n-s}. \tag{3.6}$$

We take a fixed nonzero row of the matrix $A_s(\lambda_0)$

$$[\mathfrak{N}_{\nu,n-s}(\lambda_0), \ldots, \mathfrak{N}_{\nu n}(\lambda_0)] \neq [0, \ldots, 0].$$

Since $\operatorname{rank} A_{s-1}(\lambda_0) \leq 1$, it follows that $\mathfrak{N}_{jk}(\lambda_0) = \alpha_j\mathfrak{N}_{\nu k}(\lambda_0)$, $k = \overline{n-s+1,n}$. Then from (3.6) we derive

$$(\mathfrak{N}_{j,n-s}(\lambda_0) - \alpha_j\mathfrak{N}_{\nu,n-s}(\lambda_0))U_{s+1,T}(\Phi_{n-s,\langle 0\rangle}(x,\lambda_0)) = 0,$$

or $\mathfrak{N}_{j,n-s}(\lambda_0) = \alpha_j\mathfrak{N}_{\nu,n-s}(\lambda_0)$. Hence $\operatorname{rank} A_s(\lambda_0) \leq 1$. If $\Delta_{n-s-1,n-s-1}(\lambda_0) \neq 0$ or $\Delta_{n-s,n-s}(\lambda_0) \neq 0$, we obtain from Lemma 3.1 that

$$\mathfrak{N}_{j,n-s+1}(\lambda_0) = \cdots = \mathfrak{N}_{jn}(\lambda_0) = 0, \qquad j = \overline{1,n-s},$$

i.e., $\operatorname{rank} A_s(\lambda_0) \leq 1$. Lemma 3.2 is proved.

Denote by M the set of meromorphic matrices $\mathfrak{M}(\lambda) = [\mathfrak{M}_{mk}(\lambda)]_{m,k=\overline{1,n}}$, $\mathfrak{M}_{mk}(\lambda) = \delta_{mk}$ $(m \geq k)$, having only simple poles Λ (in general, the set Λ is different for each matrix $\mathfrak{M}(\lambda)$) and such that (3.1) is valid, and for each $\lambda_0 \in \Lambda$ the matrix $\mathfrak{M}(\lambda)$ has the properties S_1 and S_2, where the sets $\Lambda_m = \{\lambda_{lm}\}_{l\geq 1}$, $\lambda_{lm} \neq \lambda_{l_0,m}$ $(l =\neq l_0)$ are defined as follows: if $\lambda_0 \in \Lambda$, $\mathfrak{N}_{kj}(\lambda_0) \neq 0$, then $\lambda_0 \in \Lambda_k \cap \cdots \cap \Lambda_{j-1}$.

It is clear that if $\mathfrak{M}(\lambda) \in M$, then $\mathfrak{N}(\lambda_0)\mathfrak{N}(\lambda_0) = 0$ for $\lambda_0 \in \Lambda$. If $L \in V_N'$ and $\mathfrak{M}(\lambda)$ is the WM for L, then $\mathfrak{M}(\lambda) \in M$.

Lemma 3.3. *Assume that the matrix* $\mathfrak{M}(\lambda) = [\mathfrak{M}_{mk}(\lambda)]_{m,k=\overline{1,n}}$, $\mathfrak{M}_{mk}(\lambda) = \delta_{mk}$ $(m \geq k)$, *has a simple pole at a point* λ_0. *For the matrix* $\mathfrak{M}^*(\lambda) \overset{\text{def}}{=} (\mathfrak{M}(\lambda))^{-1}$ *to have a simple pole at* λ_0 *it is necessary and sufficient that* $\mathfrak{N}(\lambda_0)\mathfrak{N}(\lambda_0) = 0$.

Proof. The necessity part of the lemma is obvious. We prove the sufficiency. Let $\mathfrak{N}(\lambda_0)\mathfrak{N}(\lambda_0) = 0$. Denote by X_p the set of matrices $A = [A_{\nu j}]_{\nu,j=\overline{1,n}}$ such that $A_{\nu j} = 0$ for $j - \nu < n - p$. It is clear that if $A \in X_p$, $B \in X_q$, then $AB \in X_{p+q-n}$. Since $\mathfrak{M}(\lambda)\mathfrak{M}^*(\lambda) = E$, it follows that

$$\mathfrak{M}^*(\lambda) = \sum_{k=1-n}^{\infty} (\lambda - \lambda_0)^k \mathfrak{M}^*_{\langle k\rangle}(\lambda_0), \qquad \sum_{j=-1}^{n-k-1} \mathfrak{M}^*_{\langle -j-k\rangle}(\lambda_0)\mathfrak{M}_{\langle j\rangle}(\lambda_0) = 0, \quad k = \overline{1, n-1}.$$

From this, in view of the relation $\mathfrak{N}(\lambda_0)\mathfrak{N}(\lambda_0) = 0$, we obtain

$$\mathfrak{M}^*_{\langle -k\rangle}(\lambda_0) = -\mathfrak{M}^*_{\langle 1-k\rangle}(\lambda_0)\mathfrak{N}(\lambda_0) - \left(\sum_{j=1}^{n-k-1} \mathfrak{M}^*_{\langle -j-k\rangle}(\lambda_0)\mathfrak{M}_{\langle j\rangle}(\lambda_0)\right)\left(\mathfrak{M}_{\langle 0\rangle}(\lambda_0)\right)^{-1},$$

$$\mathfrak{M}^*_{\langle 1-k\rangle}(\lambda_0)\mathfrak{N}(\lambda_0) = -\left(\sum_{j=1}^{n-k} \mathfrak{M}^*_{\langle -j-k+1\rangle}(\lambda_0)\mathfrak{M}_{\langle j\rangle}(\lambda_0)\right)\left(\mathfrak{M}_{\langle 0\rangle}(\lambda_0)\right)^{-1}\mathfrak{N}(\lambda_0), \qquad k = \overline{2, n-1}.$$

Since $\mathfrak{N}(\lambda_0) \in X_{n-1}$, $\mathfrak{M}^*_{\langle -1\rangle}(\lambda_0) \in X_{n-k}$, we can find $\mathfrak{M}^*_{\langle 1-k\rangle}(\lambda_0)\mathfrak{N}(\lambda_0)$, $\mathfrak{M}^*_{\langle -k\rangle}(\lambda_0) \in X_{n-k-2}$, $k = \overline{2, n-1}$. Repeating this procedure several times we obtain $\mathfrak{M}^*_{\langle -k\rangle}(\lambda_0) = 0$, $k = \overline{2, n-1}$. Lemma 3.3 is proved.

Corollary 3.1. *If* $\mathfrak{M}(\lambda) \in M$, *then the matrix* $\mathfrak{M}^*(\lambda) \overset{\text{def}}{=} (\mathfrak{M}(\lambda))^{-1}$ *has only simple poles.*

Let $\widetilde{L} \in V'_N$, $\mathfrak{M}(\lambda) \in M$. Denote

$$\widetilde{D}(x, \lambda, \lambda_0) = \left[-\frac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda - \mu}\right]^{\langle 0\rangle}_{|\mu=\lambda_0}, \qquad \widetilde{D}_{\langle k\rangle}(x, z_0, \lambda_0) = \left[\widetilde{D}(x,\lambda,\lambda_0)\right]^{\langle k\rangle}_{|\lambda=z_0}, \qquad k = 0, -1.$$

Using (2.27), (3.2), and Lemma 2.1, we obtain the following fact.

Lemma 3.4.

$$\begin{cases} \widetilde{D}(x,\lambda,\lambda_0)\mathfrak{N}(\lambda_0) = \dfrac{\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*_{\langle -1\rangle}(x,\lambda_0)\rangle_{\widetilde{l}}}{\lambda - \lambda_0}, \\ \widetilde{\mathfrak{N}}(z_0)\widetilde{D}_{\langle 0\rangle}(x, z_0, \lambda_0) = \left[-\dfrac{\langle \widetilde{\Phi}_{\langle -1\rangle}(x, z_0), \widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}}}{z_0 - \mu}\right]^{\langle 0\rangle}_{|\mu=\lambda_0}, \end{cases} \tag{3.7}$$

$$\widetilde{D}_{\langle -1\rangle}(x, z_0, \lambda_0) = \widetilde{\mathfrak{N}}(z_0)\widetilde{D}_{\langle 0\rangle}(x, z_0, \lambda_0) - \delta(z_0, \lambda_0)E, \tag{3.8}$$

$$\begin{cases} \widetilde{\mathfrak{N}}(z_0)\widetilde{D}_{\langle 0\rangle}(x, z_0, \lambda_0)\widetilde{\mathfrak{N}}(\lambda_0) = \dfrac{\langle \widetilde{\Phi}_{\langle -1\rangle}(x, z_0), \widetilde{\Phi}^*_{\langle -1\rangle}(x, \lambda_0)\rangle_{\widetilde{l}}}{z_0 - \lambda_0} & (z_0 \neq \lambda_0), \\ \widetilde{\mathfrak{N}}(z_0)\widetilde{D}_{\langle 0\rangle}(x, z_0, \lambda_0)\widetilde{\mathfrak{N}}(\lambda_0) = \widetilde{\mathfrak{N}}(\lambda_0) - \langle \widetilde{\Phi}_{\langle -1\rangle}(x, \lambda_0), \widetilde{\Phi}^*_{\langle 0\rangle}(x, \lambda_0)\rangle_{\widetilde{l}} & (z_0 = \lambda_0), \end{cases} \tag{3.9}$$

where $\delta(z_0, \lambda_0) = 0$ $(z_0 \neq \lambda_0)$, $\delta(z_0, \lambda_0) = 1$ $(z_0 = \lambda_0)$.

From Lemma 3.4, in virtue of the equalities

$$\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}|x=0} = \widetilde{\mathfrak{M}}(\lambda)\widetilde{\mathfrak{M}}^*(\mu),$$
$$\langle \widetilde{\Phi}(x,\lambda), \widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}|x=T} = \widetilde{U}_T(\widetilde{\Phi}(x,\lambda))\widetilde{U}^*_T(\widetilde{\Phi}^*(x,\mu)),$$

we obtain

Corollary 3.2.

$$\widetilde{D}_{k\nu}(0,\lambda,\lambda_0) = -\frac{\delta_{k\nu}}{\lambda-\lambda_0}, \qquad k \geq \nu,$$
$$\widetilde{D}_{k\nu}(0,\lambda,\lambda_0) = \mathcal{F}_{k\nu}\big(\widetilde{\mathfrak{M}}_{j,j+s}; s=\overline{1,\nu-k}; j=\overline{1,n-1}\big), \qquad k<\nu,$$
$$\widetilde{D}_{k\nu}(T,\lambda,\lambda_0) = \big(\widetilde{D}(T,\lambda,\lambda_0)\widetilde{\mathfrak{N}}(\lambda_0)\big)_{k\nu} = \big(\widetilde{\mathfrak{N}}(z_0)\widetilde{D}_{\langle 0\rangle}(T,z_0,\lambda_0)\big)_{k\nu} = 0, \qquad k<\nu,$$
$$\big(\widetilde{\mathfrak{N}}(z_0)\widetilde{D}_{\langle 0\rangle}(T,z_0,\lambda_0)\widetilde{\mathfrak{N}}(\lambda_0)\big)_{k\nu} = \delta(z_0,\lambda_0)\widetilde{\mathfrak{N}}_{k\nu}(z_0), \qquad k<\nu.$$

Denote $Y = [\delta_{j,k-1}]_{j=\overline{1,n-1};k=\overline{1,n}}$, $\mathfrak{N}_0(\lambda_0) = \mathfrak{N}(\lambda_0)$, $\mathfrak{N}_1(\lambda_0) = \widetilde{\mathfrak{N}}(\lambda_0)$; then

$$\widetilde{P}_\varepsilon(x,\lambda,\lambda_0) = \widetilde{D}(x,\lambda,\lambda_0)\mathfrak{N}_\varepsilon(\lambda_0)Y^T, \quad \varepsilon=0,1, \qquad P(x,\lambda,\lambda_0) = \widetilde{D}(x,\lambda,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T,$$
$$\widetilde{G}_\varepsilon(x,z_0,\lambda_0) = Y\widetilde{D}_{\langle 0\rangle}(x,z_0,\lambda_0)\mathfrak{N}_\varepsilon(\lambda_0)Y^T, \quad \varepsilon=0,1, \qquad \widetilde{G}(x,z_0,\lambda_0) = Y\widetilde{D}_{\langle 0\rangle}(x,z_0,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T,$$
$$\widetilde{g}^*_\varepsilon(x,\lambda_0) = -\widetilde{\Phi}^*_{\langle 0\rangle}(x,\lambda_0)\mathfrak{N}_\varepsilon(\lambda_0)Y^T, \quad \varepsilon=0,1, \qquad \widetilde{g}^*(x,\lambda_0) = -\widetilde{\Phi}^*_{\langle 0\rangle}(x,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T,$$
$$\widetilde{\Lambda}(\lambda_0) = \lambda_0 E + Y\widetilde{\mathfrak{N}}(\lambda_0)Y^T, \qquad \Lambda(\lambda_0) = \lambda_0 E + Y\mathfrak{N}(\lambda_0)Y^T,$$
$$\widetilde{\varphi}(x,\lambda_0) = Y\widetilde{\Phi}_{\langle 0\rangle}(x,\lambda_0).$$

Lemma 3.5.

$$\widetilde{l}\widetilde{\varphi}(x,\lambda_0) = \widetilde{\Lambda}(\lambda_0)\widetilde{\varphi}(x,\lambda_0), \tag{3.10}$$
$$\widetilde{P}'_\varepsilon(x,\lambda,\lambda_0) = \widetilde{\Phi}(x,\lambda)\widetilde{g}^*_\varepsilon(x,\lambda_0), \qquad \widetilde{G}'_\varepsilon(x,z_0,\lambda_0) = \widetilde{\varphi}(x,z_0)\widetilde{g}^*_\varepsilon(x,\lambda_0), \qquad \varepsilon=0,1, \tag{3.11}$$
$$\widetilde{P}(x,\lambda,\lambda_0)\big(\lambda E - \Lambda(\lambda_0)\big) = \langle\widetilde{\Phi}(x,\lambda),\widetilde{g}^*(x,\lambda_0)\rangle_{\widetilde{l}}, \tag{3.12}$$
$$\widetilde{\Lambda}(z_0)\widetilde{G}(x,z_0,\lambda_0) - \widetilde{G}(x,z_0,\lambda_0)\Lambda(\lambda_0) - \delta(z_0,\lambda_0)Y\widehat{\mathfrak{N}}(\lambda_0)Y^T = \langle\widetilde{\varphi}(x,\zeta_0),\widetilde{g}^*(x,\lambda_0)\rangle_{\widetilde{l}}. \tag{3.13}$$

If $\widetilde{\mathfrak{N}}(\lambda_0)\mathfrak{N}(\lambda_0) = 0$, *then*

$$\widetilde{P}_\varepsilon(x,\lambda,\lambda_0)\cdot\big(\lambda E - \Lambda(\lambda_0)\big) = \langle\widetilde{\Phi}(x,\lambda),\widetilde{g}^*_\varepsilon(x,\lambda_0)\rangle_{\widetilde{l}}, \qquad \varepsilon=0,1, \tag{3.14}$$
$$\widetilde{\Lambda}(z_0)\widetilde{G}_\varepsilon(x,z_0,\lambda_0) - \widetilde{G}_\varepsilon(x,z_0,\lambda_0)\Lambda(\lambda_0) - \delta(z_0,\lambda_0)Y\mathfrak{N}_\varepsilon(\lambda_0)Y^T = \langle\widetilde{\varphi}(x,z_0),\widetilde{g}^*_\varepsilon(x,\lambda_0)\rangle_{\widetilde{l}}, \qquad \varepsilon=0,1. \tag{3.15}$$

Proof. In virtue of (3.2) and (3.4) we compute

$$\widetilde{l}\widetilde{\Phi}_{\langle 0\rangle}(x,\lambda_0) = \big(\lambda_0 E + \widetilde{\mathfrak{N}}(\lambda_0)\big)\widetilde{\Phi}_{\langle 0\rangle}(x,\lambda_0).$$

This implies (3.10). Using (2.27) we derive $\widetilde{D}'(x,\lambda,\lambda_0) = -\widetilde{\Phi}(x,\lambda)\widetilde{\Phi}^*_{\langle 0\rangle}(x,\lambda_0)$, and consequently, (3.11) is proved. Further, it follows from (3.7) that $(\lambda-\lambda_0)\widetilde{D}(x,\lambda,\lambda_0) + \widetilde{D}(x,\lambda,\lambda_0)\widetilde{\mathfrak{N}}(\lambda_0) = -\langle\widetilde{\Phi}(x,\lambda),\widetilde{\Phi}^*_{\langle 0\rangle}(x,\lambda_0)\rangle_{\widetilde{l}}$. Multiplying this equality by $\mathfrak{N}_\varepsilon(\lambda_0)Y^T$, we obtain

$$\widetilde{P}_\varepsilon(x,\lambda,\lambda_0)\big(\lambda E - \Lambda(\lambda_0)\big) + \widetilde{P}_1(x,\lambda,\lambda_0)Y\mathfrak{N}(\lambda_0)Y^T = \langle\widetilde{\Phi}(x,\lambda),\widetilde{g}^*_\varepsilon(x,\lambda_0)\rangle_{\widetilde{l}}, \qquad \varepsilon=0,1. \tag{3.16}$$

This yields (3.12) and (3.14). It follows from (3.8) that

$$\big[Y\widetilde{P}_\varepsilon(x,\lambda,\lambda_0)\big]^{\langle -1\rangle}_{|\lambda=z_0} = Y\widetilde{\mathfrak{N}}(z_0)\big[\widetilde{P}_\varepsilon(x,\lambda,\lambda_0)\big]^{\langle 0\rangle}_{|\lambda=z_0} - \delta(z_0,\lambda_0)Y\mathfrak{N}_\varepsilon(\lambda_0)Y^T.$$

Using (3.16) we arrive at

$$\widetilde{G}_\varepsilon(x,z_0,\lambda_0)\big(z_0E - \Lambda(\lambda_0)\big) + Y\big[\widetilde{P}_\varepsilon(x,\lambda,\lambda_0)\big]^{\langle -1\rangle}_{|\lambda=\lambda_0} + \widetilde{G}_1(x,z_0,\lambda_0)Y\mathfrak{N}(\lambda_0)Y^T = \langle\widetilde{\varphi}(x,z_0),\widetilde{g}^*_\varepsilon(x,\lambda_0)\rangle_{\widetilde{l}}.$$

From this, in view of the equality

$$Y\widetilde{\mathfrak{N}}(z_0)\big[\widetilde{P}_\varepsilon(x,\lambda,\lambda_0)\big]^{\langle 0\rangle}_{|\lambda=z_0} = Y\widetilde{\mathfrak{N}}(z_0)Y^T\widetilde{G}_\varepsilon(x,z_0,\lambda_0),$$

we have

$$\widetilde{\Lambda}(z_0)\widetilde{G}_\varepsilon(x,z_0,\lambda_0) - \widetilde{G}_\varepsilon(x,z_0,\lambda_0)\Lambda(\lambda_0) + \widetilde{G}_1(x,z_0,\lambda_0)Y\mathfrak{N}(\lambda_0)Y^T \\ - \delta(z_0,\lambda_0)Y\mathfrak{N}_\varepsilon(\lambda_0)Y^T = \langle\widetilde{\varphi}(x,z_0),\widetilde{g}_\varepsilon^*(x,\lambda_0)\rangle_{\widetilde{l}}.$$

This yields (3.13) and (3.15). Lemma 3.5 is proved.

3.2. Solution of the IP. Necessary and sufficient conditions.

We consider $L,\widetilde{L}\in V_N'$. Denote

$$\xi_l = \sum_{m=1}^{n-1}\left(|\lambda_{lm}-\widetilde{\lambda}_{lm}| + \sum_{k=m+1}^{n}\left|\mathfrak{N}_{mk}(\lambda_{lm}) - \widetilde{\mathfrak{N}}_{mk}(\widetilde{\lambda}_{lm})\right|\cdot l\right)l^{1-n}.$$

In the sequel we shall assume that the numbers λ_{lm} and $\widetilde{\lambda}_{lm}$ are numbered in such a way that $\lambda_{lm}\neq\lambda_{l_0,m_0}$, $\widetilde{\lambda}_{lm}\neq\widetilde{\lambda}_{l_0,m_0}$, $\lambda_{lm}\neq\widetilde{\lambda}_{l_0,m_0}$ for $l\neq l_0$, $|m-m_0|=1$. This is possible and it means that "common" poles have the same number l.

Lemma 3.6.

$$\widetilde{\Phi}(x,\lambda) = \Phi(x,\lambda) + \sum_{\lambda_0\in I}\widetilde{P}(x,\lambda,\lambda_0)\varphi(x,\lambda_0), \tag{3.17}$$

$$\widetilde{\varphi}(x,\zeta_0) = \varphi(x,z_0) + \sum_{\lambda_0\in I}\widetilde{G}(x,z_0,\lambda_0)\varphi(x,\lambda_0), \qquad z_0\in I, \tag{3.18}$$

$$\widetilde{G}(x,z_0,\varkappa_0) - G(x,z_0,\varkappa_0) = \sum_{\lambda_0\in I}\widetilde{G}(x,z_0,\lambda_0)G(x,\lambda_0,\varkappa_0), \qquad z_0,\varkappa_0\in I, \tag{3.19}$$

where $I=\Lambda\cup\widetilde{\Lambda}$, $\varphi(x,\lambda_0)=Y\Phi_{\langle 0\rangle}(x,\lambda_0)$, $G(x,z_0,\lambda_0)=YD_{\langle 0\rangle}(x,z_0,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T$, *the series converge "with brackets,"*

$$\sum_{\lambda_0\in I} = \lim_{k\to\infty}\sum_{\lambda_0\in I_k}, \qquad I_k = I\cap\{\lambda : |\lambda|\le R_k\},$$

and the circumferences $|\lambda|=R_k$ *are at a positive distance from the set* I.

Proof. Using (3.2) and (3.7) we obtain

$$\operatorname*{res}_{\mu=\lambda_0}\left[-\frac{\langle\widetilde{\Phi}(x,\lambda),\widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\Phi(x,\mu)\right] = \widetilde{D}(x,\lambda,\lambda_0)\widehat{\mathfrak{N}}(\lambda_0)Y^T\varphi(x,\lambda_0).$$

Hence

$$\operatorname*{res}_{\mu=\lambda_0}\left[-\frac{\langle\widetilde{\Phi}(x,\lambda),\widetilde{\Phi}^*(x,\mu)\rangle_{\widetilde{l}}}{\lambda-\mu}\Phi(x,\mu)\right] = \widetilde{P}(x,\lambda,\lambda_0)\varphi(x,\lambda_0). \tag{3.20}$$

In the λ-plane we consider a contour $\gamma=\gamma^+\cup\gamma^-$, $\gamma^\pm=\{\lambda : \pm\operatorname{Im}\lambda = C_0,\ -\infty<\mp\operatorname{Re}\lambda<\infty\}$ such that $I,\widetilde{I}\subset\{\lambda : |\operatorname{Im}\lambda|<C_0\}$. Put $J_\gamma=\mathbb{C}\setminus\operatorname{int}\gamma$. Then the relation (2.41) is valid (the proof is the same as for the half-line). Using (3.20) and the residue theorem [72, p. 239] we obtain (3.17). Equality (3.18) follows from (3.17). Equality (3.19) is proved analogously. Lemma 3.6 is proved.

Denote

$$Y_k = \operatorname{diag}[\delta_{\nu k}]_{\nu=\overline{1,n-1}}, \qquad \lambda_{l0k} = \lambda_{lk}, \qquad \lambda_{l1k} = \widetilde{\lambda}_{lk},$$
$$\widetilde{\Lambda}_{l\varepsilon} = \sum_{k=1}^{n-1} Y_k \widetilde{\Lambda}(\lambda_{l\varepsilon k}), \qquad \widetilde{\varphi}_{l\varepsilon} = \sum_{k=1}^{n-1} Y_k \widetilde{\varphi}(x, \lambda_{l\varepsilon k}),$$
$$\widetilde{g}^*_{l\varepsilon}(x) = \sum_{k=1}^{n-1} \widetilde{g}^*_{\varepsilon}(x, \lambda_{l\varepsilon k}) Y^k, \qquad \widetilde{P}_{l\varepsilon}(x,\lambda) = \sum_{k=1}^{n-1} \widetilde{P}_{\varepsilon}(x, \lambda, \lambda_{l\varepsilon k}) Y^k,$$
$$\widetilde{G}_{(l_0,\varepsilon_0),(l,\varepsilon)}(x) = \sum_{k,k_0=1}^{n-1} Y_{k_0} G_{\varepsilon}(x, \lambda_{l_0,\varepsilon_0,k_0}, \lambda_{l\varepsilon k}) Y_k, \qquad \varepsilon, \varepsilon_0 = 0,1.$$

Analogously we define $\Lambda_{l\varepsilon}$, $\varphi_{l\varepsilon}(x)$, $G_{(l_0,\varepsilon_0),(l,\varepsilon)}(x)$. Let V' be a set of indices $v = (l, \varepsilon)$, $l \geq 1$, $\varepsilon = 0,1$ (ε changes quicker), and V be a set of indices $j = (v, k) = (l, \varepsilon, k)$, $x \in V'$, $k = \overline{1, n-1}$ (k changes quicker). We introduce the matrices

$$\widetilde{\varphi}(x) = [\widetilde{\varphi}_v(x)]_{v\in V'} = [\widetilde{\varphi}_j(x)]_{j\in V}, \qquad \widetilde{g}^*(x) = [\widetilde{g}^*_v(x)]^T_{v\in V'} = [\widetilde{g}^*_j(x)]^T_{j\in V},$$
$$\widetilde{G}(x) = [\widetilde{G}_{v_0,v}(x)]_{v_0,v\in V'} = [\widetilde{G}_{j_0,j}(x)]_{j_0,j\in V}, \qquad v_0 = (l_0, \varepsilon_0), \quad j_0 = (v_0, k_0) = (l_0, \varepsilon_0, k_0),$$
$$\widetilde{\Lambda} = \operatorname{diag}[\widetilde{\Lambda}_v]_{v\in V'}, \qquad J = \operatorname{diag}[(-1)^{\varepsilon} E]_{v\in V'}, \qquad J_1 = [\delta_{l_0,l}\theta_{v_0,v}]_{v_0,v\in V'},$$
$$\theta_{vv} = E, \qquad \theta_{(l,0),(l,1)} = -E, \qquad \theta_{(l,1),(l,0)} = 0, \qquad E = [\delta_{\mu k}]_{\mu,k=\overline{1,n-1}}.$$

Analogously we define the matrices φ, G, Λ. Denote

$$w^*_{lk}(x) = l^{k-n+1} \exp\left(-xl \cot \tfrac{k\pi}{n}\right), \quad w_{l0k}(x) = \xi_l w^*_{lk}(x), \quad w_{l1k}(x) = w^*_{lk}(x), \quad W(x) = \operatorname{diag}[w_j(x)]_{j\in V}.$$

The following estimates are valid:

$$\left|\widetilde{\varphi}^{(\nu)}_j(x)\right| < Cl^{\nu} w^*_{lk}(x), \qquad j \in V, \qquad \left|\widetilde{\varphi}^{(\nu)}_{l0k}(x) - \widetilde{\varphi}^{(\nu)}_{l1k}(x)\right| < C\xi_l l^{\nu} w^*_{lk}(x),$$
$$\left|\widetilde{g}^{*(\nu)}_j(x)\right| < Cl^{\nu}\left(w^*_{lk}(x)\right)^{-1}, \qquad \left|\widetilde{g}^{*(\nu)}_{l0k}(x) - \widetilde{g}^{*(\nu)}_{l1k}(x)\right| < C\xi_l l^{\nu}\left(w^*_{lk}(x)\right)^{-1},$$
$$\left|\widetilde{G}_{j_0,j}(x)\right| < \frac{C}{|l-l_0|+1} \cdot \frac{w^*_{l_0,k_0}(x)}{w^*_{lk}(x)}, \qquad \left|\widetilde{G}^{(\nu+1)}_{j_0,j}(x)\right| < C(l+l_0)^{\nu} \cdot \frac{w^*_{l_0,k_0}(x)}{w^*_{lk}(x)},$$
$$\left|\widetilde{G}_{j_0,(l0k)}(x) - \widetilde{G}_{j_0,(l1k)}(x)\right| < \frac{C\xi_l}{|l-l_0|+1} \cdot \frac{w^*_{l_0,k_0}(x)}{w^*_{lk}(x)},$$
$$\left|\widetilde{G}_{(l_0,0,k_0),j}(x) - \widetilde{G}_{(l_0,1,k_0),j}(x)\right| < \frac{C\xi_{l_0}}{|l-l_0|+1} \cdot \frac{w^*_{l_0,k_0}(x)}{w^*_{lk}(x)},$$

The same estimates are valid for $\varphi(x)$ and $G(x)$.

In view of our notations, (3.17)–(3.19) become

$$\widetilde{\Phi}(x,\lambda) = \Phi(x,\lambda) + \sum_{l=1}^{\infty} \left(\widetilde{P}_{l0}(x,\lambda)\varphi_{l0}(x) - \widetilde{P}_{l1}(x,\lambda)\varphi_{l1}(x)\right), \tag{3.21}$$
$$\widetilde{\varphi}(x) = \left(E + \widetilde{G}(x)J\right)\varphi(x), \tag{3.22}$$
$$\left(E + \widetilde{G}(x)J\right)\left(E - G(x)J\right) = E, \tag{3.23}$$

and as above, the series in (3.22) and (3.23) converge "with brackets." Further, according to Lemma 3.5, we have

$$\widetilde{l}\widetilde{\varphi}(x) = \widetilde{\Lambda}\widetilde{\varphi}(x), \tag{3.24}$$

$$\widetilde{P}'_{l\varepsilon}(x,\lambda) = \widetilde{\Phi}(x,\lambda)\widetilde{g}^*_{l\varepsilon}(x), \qquad \widetilde{G}'(x) = \widetilde{\varphi}(x)\widetilde{g}^*(x), \tag{3.25}$$

$$\sum_{\varepsilon=0}^{1}(-1)^{\varepsilon}\Big(\widetilde{P}_{l\varepsilon}(x,\lambda)(\lambda E - \Lambda_{l\varepsilon}) - \big\langle\widetilde{\Phi}(x,\lambda), \widetilde{g}^*_{l\varepsilon}(x)\big\rangle_{\widetilde{l}}\Big)\varphi_{l\varepsilon}(x) = 0, \tag{3.26}$$

$$\big(\widetilde{\Lambda}(E + \widetilde{G}(x)J) - (E + \widetilde{G}(x)J)\Lambda\big)\varphi(x) = \big\langle\widetilde{\varphi}(x), \widetilde{g}^*(x)\big\rangle_{\widetilde{l}} J\varphi(x). \tag{3.27}$$

Let

$$\sum_{l=1}^{\infty}\xi_l \cdot l^{n-1} < \infty.$$

Denote

$$\varkappa_{\nu s}(x) = \widetilde{g}^{*(\nu)}(x)J\varphi^{(s)}(x) = \sum_{l=1}^{\infty}\Big(\widetilde{g}^{*(\nu)}_{l0}(x)\varphi^{(s)}_{l0}(x) - \widetilde{g}^{*(\nu)}_{l1}(x)\varphi^{(s)}_{l1}(x)\Big), \qquad \nu + s \le n-1. \tag{3.28}$$

The functions $t_{j\nu}(x)$, $\xi_\nu(x)$, and $\varepsilon_\nu(x)$ are defined by (2.52)–(2.54).

Lemma 3.7.

$$p_\nu(x) = \widetilde{p}_\nu(x) + \varepsilon_\nu(x), \qquad u_{\xi\nu a} = \sum_{j=0}^{n-1} u_{\xi j a}t_{j\nu}(a), \quad a = 0, T. \tag{3.29}$$

Proof. Differentiating (3.21) with respect to x and using (3.25), (3.28), and (2.52), we obtain

$$\sum_{\nu=0}^{n} t_{j\nu}(x)\widetilde{\Phi}^{(\nu)}(x,\lambda) = \Phi^{(j)}(x,\lambda) + \sum_{l=1}^{\infty}\Big(\widetilde{P}_{l0}(x,\lambda)\varphi^{(j)}_{l0}(x) - \widetilde{P}_{l1}(x,\lambda)\varphi^{(j)}_{l1}(x)\Big). \tag{3.30}$$

Further, in view of (3.21), (3.24), and (3.26), we have

$$\widetilde{l}\widetilde{\Phi}(x,\lambda) = l\Phi(x,\lambda) + \sum_{l=1}^{\infty}\Big(\widetilde{P}_{l0}(x,\lambda)l\varphi_{l0}(x) - \widetilde{P}_{l1}(x,\lambda)l\varphi_{l1}(x)\Big) + \big\langle\widetilde{\Phi}(x,\lambda), \widetilde{g}^*(x)\big\rangle_{\widetilde{l}} \cdot J\varphi(x). \tag{3.31}$$

From (3.31), in virtue of (3.30) and (2.22), as in the proof of Lemma 2.7, we obtain $p_\nu(x) = \widetilde{p}_\nu(x) + \varepsilon_\nu(x)$, $\nu = \overline{0, n-2}$.

Denote

$$\widetilde{\widetilde{U}}_{\xi a}(y) = \sum_{\nu=0}^{n-1}\left(\sum_{j=0}^{n-1} u_{\xi j a}t_{j\nu}(a)\right) y^{(\nu)}(a).$$

It follows from (3.30) that

$$\widetilde{\widetilde{U}}\big(\widetilde{\Phi}_k(x,\lambda)\big) = U_{\xi a}\big(\Phi_k(x,\lambda)\big) + \sum_{l=1}^{\infty}\sum_{\nu=2}^{n}\left(\left(\sum_{j=1}^{\nu-1}\widetilde{D}_{kj}(a,\lambda,\lambda_{l,\nu-1})\mathfrak{N}_{j\nu}(\lambda_{l,\nu-1})\right)U_{\xi a}\big(\Phi_{\nu,\langle 0\rangle}(x,\lambda_{l,\nu-1})\big)\right.$$
$$\left. - \left(\sum_{j=1}^{\nu-1}\widetilde{D}_{kj}(a,\lambda,\widetilde{\lambda}_{l,\nu-1})\widetilde{\mathfrak{N}}_{j\nu}(\widetilde{\lambda}_{l,\nu-1})\right)U_{\xi a}\big(\Phi_{\nu,\langle 0\rangle}(x,\widetilde{\lambda}_{l,\nu-1})\big)\right).$$

For $a = 0$, using Corollary 3.2, we compute $\widetilde{\widetilde{U}}_{\xi 0}(\widetilde{\Phi}_k) = \delta_{\xi k}$, $\xi \le k$, and consequently, $\widetilde{\widetilde{U}}_{\xi 0} = \widetilde{U}_{\xi 0}$. Analogously we find that $\widetilde{\widetilde{U}}_{\xi T} = \widetilde{U}_{\xi T}$. Lemma 3.7 is proved.

Denote

$$\widetilde{\psi}(x) = W^{-1}(x)J_1\widetilde{\varphi}(x), \qquad \widetilde{H}(x) = W^{-1}(x)J_1\widetilde{G}(x)JJ_1^{-1}W(x),$$
$$\psi(x) = W^{-1}(x)J_1\varphi(x), \qquad H(x) = W^{-1}(x)J_1G(x)JJ_1^{-1}W(x).$$

It is obvious that

$$\begin{cases} \left|\widetilde{\psi}_j^{(\nu)}(x)\right| < Cl^\nu, \qquad \left|\widetilde{H}_{j_0,j}(x) < \dfrac{C\xi_l}{|l-l_0|+1}, \\ \left|\widetilde{H}_{j_0,j}^{(\nu+1)}(x)\right| < C\cdot(l+l_0)^\nu\xi_l, \qquad j_0, j \in V. \end{cases} \tag{3.32}$$

Analogous estimates are valid for $\psi(x)$ and $H(x)$.

Then (3.22) and (3.23) become

$$\widetilde{\psi}(x) = \big(E + \widetilde{H}(x)\big)\psi(x), \tag{3.33}$$

$$\big(E + \widetilde{H}(x)\big)\big(E - H(x)\big) = E, \tag{3.34}$$

and the series in (3.33) and (3.34) converge absolutely and uniformly for $x \in [0,T]$. Interchanging places for L and $\widetilde{L}$ we obtain analogously

$$\psi(x) = \big(E - H(x)\big)\widetilde{\psi}(x), \qquad \big(E - H(x)\big)\big(E + \widetilde{H}(x)\big) = E. \tag{3.35}$$

We consider the Banach space m of bounded sequences $\alpha = [\alpha_j]_{j\in V}$ with the norm $\|\alpha\|_m = \sup_j |\alpha_j|$. It follows from (3.32), (3.34), and (3.35) that for each fixed $x \in [0,T]$ the operator $E + \widetilde{H}(x)$, acting from m to m, is a linear bounded operator,

$$\left\|\widetilde{H}(x)\right\|_{m\to m} = \sup_{j_0} \sum_j \left|\widetilde{H}_{j_0,j}(x)\right| < C\sum_l \xi_l,$$

and $E + \widetilde{H}(x)$ has a bounded inverse operator.

Theorem 3.1. *For a matrix $\mathfrak{M}(\lambda) \in M$ to be the WM for $L \in V_N'$ it is necessary and sufficient that the following conditions hold:*

(1) (asymptotics) *there exists $\widetilde{L} \in V_N'$ such that*

$$\sum_{l=1}^\infty \xi_l l^{n-1} < \infty;$$

(2) (condition P) *for each fixed $x \in [0,T]$, the linear bounded operator $E + \widetilde{H}(x)$ acting from m to m has a bounded inverse operator;*

(3) $\varepsilon_\nu(x) \in W_{\nu+N}$, $\nu = \overline{0, n-2}$, *where the functions $\varepsilon_\nu(x)$ are given by* (3.28), (2.52)-(2.54), *and* $\varphi(x) = J_1^{-1}W(x)\big(E + \widetilde{H}(x)\big)^{-1}\widetilde{\psi}(x)$.

Under these conditions the DE and LF $L = (l, U)$ are constructed according to (3.29).

The necessity part of Theorem 3.1 was proved above. The proof of sufficiency is in [96].

The method described above allows also to study stability of the solution of the IP from the WM. Let $\widetilde{L} \in V_N'$ and choose $L \in V_N'$ such that

$$\Lambda^0 \overset{\text{def}}{=} \sum_{l=1}^\infty \xi_l l^{n-1} < \infty.$$

The quantity Λ^0 will describe nearness of the WM $\mathfrak{M}(\lambda)$ and $\widetilde{\mathfrak{M}}(\lambda)$.

Theorem 3.2. *There exists $\delta > 0$ (which depends on $\widetilde{L}$) such that if $\Lambda^0 < \delta$, then*

$$\max_{0\le x\le T} \left|p_\nu^{(j)}(x) - \widetilde{p}_\nu^{(j)}(x)\right| < C\Lambda^0, \qquad 0 \le j \le \nu \le n-2; \qquad |u_{\xi\nu a} - \widetilde{u}_{\xi\nu a}| < C\Lambda^0,$$

where C depends only on $\widetilde{L}$.

The proof of Theorem 3.2 is in [96].

Sometimes it is more convenient to work in $\mathcal{L}_2(0,T)$. We shall say that $L \in V'_{N2}$ if $L \in V'_N$ and $p_\nu^{(\nu+N)}(x) \in \mathcal{L}_2(0,T)$. Similarly to Theorem 3.1, we prove the following theorem.

Theorem 3.3. *For a matrix $\mathfrak{M}(\lambda) \in M$ to be the WM for $L \in V'_{N2}$ it is necessary and sufficient that the following conditions hold:*

(1) (asymptotics) *there exists $\widetilde{L} \in V'_{N2}$ such that*

$$\left(\sum_{l=1}^{\infty} \xi_l l^{n+N-1}\right)^2 < \infty;$$

(2) *condition P is fulfilled.*

We note that for "small" perturbations condition P is fulfilled automatically, i.e., the following theorem holds.

Theorem 3.4. *Let $\widetilde{L} \in V'_{N2}$ be given. Then there exists $\delta > 0$ (which depends on $\widetilde{L}$) such that if the matrix $\mathfrak{M}(\lambda) \in M$ satisfies the condition*

$$\Lambda^+ \stackrel{\text{def}}{=} \left(\sum_{l=1}^{\infty} \left(\xi_l l^{n+N-1}\right)^2\right)^{1/2} < \delta,$$

then there exists a unique $L \in V'_{N2}$ for which the matrix $\mathfrak{M}(\lambda)$ is the WM. Then

$$\left\|p_\nu^{(j)}(x) - \widetilde{p}_\nu^{(j)}(x)\right\|_{\mathcal{L}_2(0,T)} < C\Lambda^+, \qquad 0 \le j \le \nu + N,$$

$$\left|u_{\xi\nu a} - \widetilde{u}_{\xi\nu a}\right| < C\Lambda^+,$$

where the constant C depends only on $\widetilde{L}$.

3.3. Counterexample.

For definiteness, let $n = 3$. Let us show that dropping $\mathfrak{M}_{13}(\lambda)$ from WM $\mathfrak{M}(\lambda)$ leads to nonuniqueness of the solution of the IP. In the other words, WF $\mathfrak{M}_{12}(\lambda)$ and $\mathfrak{M}_{23}(\lambda)$ do not uniquely determine DE and LF L.

We consider $\widetilde{L} = (\widetilde{l}, \widetilde{U})$ of the form

$$\widetilde{l}y = y''', \qquad \widetilde{U}_{1a}(y) = y''(a) + \widetilde{\alpha}_a y'(a), \qquad \widetilde{U}_{2a}(y) = y'(a), \qquad \widetilde{U}_{3a}(y) = y(a), \qquad a = 0, T.$$

Let the functions $\widetilde{X}_k(x,\lambda)$ be solutions of the equation $y''' = \lambda y = \rho^3 y$ under the conditions $\widetilde{X}_k^{(\nu-1)}(0,\lambda) = \delta_{\nu k}$, $\nu, k = \overline{1,3}$. Then

$$\widetilde{X}_k(x,\lambda) = \frac{1}{3}\sum_{j=1}^{3} (\rho R_j)^{1-k} \exp(\rho R_j x). \tag{3.36}$$

In particular, for $\lambda = 0$ $\widetilde{X}_k(x,0) = \dfrac{x^{k-1}}{(k-1)!}$.

It is clear that for $\lambda = 0$

$$\widetilde{\Delta}_{11}(0) = \widetilde{\Delta}_{22}(0) = \widetilde{\Delta}_{12}(0) = 0 \tag{3.37}$$

for any $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_T$. We choose the coefficients $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_T$ such that the functions $\widetilde{\Delta}_{11}(\lambda)$ and $\widetilde{\Delta}_{22}(\lambda)$ have only simple zeros. Let us show that such choice is possible. By symmetry, it is sufficient to consider the function $\widetilde{\Delta}_{22}(\lambda) = \widetilde{X}_1''(T,\lambda) + \widetilde{\alpha}_T \widetilde{X}_1'(T,\lambda)$. Using (3.36) we obtain

$$\begin{cases} \widetilde{\Delta}_{22}(\lambda) = \lambda \widetilde{X}_2(T,\lambda) + \widetilde{\alpha}_T \lambda \widetilde{X}_3(T,\lambda), \\ 3\dot{\widetilde{\Delta}}_{22}(\lambda) = \big(2\widetilde{X}_2(T,\lambda) + T\widetilde{X}_1(T,\lambda)\big) + \widetilde{\alpha}_T\big(\widetilde{X}_3(T,\lambda) + T\widetilde{X}_2(T,\lambda)\big). \end{cases} \tag{3.38}$$

Denote by $\mathcal{B}$ the set of zeros of the function

$$\widetilde{\Delta}(\lambda) \stackrel{\text{def}}{=} \lambda \widetilde{X}_2(T,\lambda)\big(\widetilde{X}_3(T,\lambda) + T\widetilde{X}_2(T,\lambda)\big) - \lambda \widetilde{X}_3(T,\lambda)\big(2\widetilde{X}_2(T,\lambda) + T\widetilde{X}_1(T,\lambda)\big),$$

and by $\mathcal{B}(\widetilde{\alpha}_T) = \big\{\lambda_0 : \widetilde{\Delta}_{22}(\lambda_0) = \dot{\widetilde{\Delta}}_{22}(\lambda_0) = 0\big\}$ the set of nonsimple zeros of $\widetilde{\Delta}_{22}(\lambda)$. It is obvious that $\mathcal{B}(\widetilde{\alpha}_T)$ is a finite set. If $\lambda_0 \in \mathcal{B}(\widetilde{\alpha}_T)$, then, in virtue of (3.38), $\widetilde{\Delta}(\lambda_0) = 0$, i.e., $\mathcal{B}(\widetilde{\alpha}_T) \subset \mathcal{B}$. Further, if $\widetilde{\alpha}_T^0 \neq \widetilde{\alpha}_T$ and $\lambda_0 \in \mathcal{B}(\widetilde{\alpha}_T) \cap \mathcal{B}(\widetilde{\alpha}_T^0)$, then (3.38) implies that

$$\lambda_0 \widetilde{X}_2(T,\lambda_0) = \lambda_0 \widetilde{X}_3(T,\lambda_0) = 2\widetilde{X}_2(T,\lambda_0) + T\widetilde{X}_1(T,\lambda_0) = \widetilde{X}_3(T,\lambda_0) + T\widetilde{X}_2(T,\lambda_0) = 0.$$

Since $2\widetilde{X}_2(T,0) + T\widetilde{X}_1(T,0) = 3T \neq 0$ it follows that $\lambda_0 \neq 0$, and hence $\widetilde{X}_1(T,\lambda_0) = \widetilde{X}_2(T,\lambda_0) = \widetilde{X}_3(T,\lambda_0) = 0$. But this is impossible. Thus, if $\widetilde{\alpha}_T^0 \neq \widetilde{\alpha}_T$, then $\mathcal{B}(\widetilde{\alpha}_T^0) \cap \mathcal{B}(\widetilde{\alpha}_T) = \emptyset$. From this and from the relation $\mathcal{B}(\widetilde{\alpha}_T) \subset \mathcal{B}$ and continuity of $\mathcal{B}(\widetilde{\alpha}_T)$ we conclude that there exists $\widetilde{\alpha}_T$ such that $\mathcal{B}(\widetilde{\alpha}_T) = \emptyset$.

We define the matrix $\mathfrak{M}(\lambda) = \big[\mathfrak{M}_{mk}(\lambda)\big]_{m,k=\overline{1,3}}$, $\mathfrak{M}_{mk}(\lambda) = \delta_{mk}$, $m \geq k$, by

$$\mathfrak{M}_{12}(\lambda) = \widetilde{\mathfrak{M}}_{12}(\lambda), \qquad \mathfrak{M}_{23}(\lambda) = \widetilde{\mathfrak{M}}_{23}(\lambda), \qquad \mathfrak{M}_{13}(\lambda) = \widetilde{\mathfrak{M}}_{13}(\lambda) + \frac{\theta}{\lambda}, \tag{3.39}$$

where θ is a complex number. It follows from (3.37) and (3.39) that for sufficiently small θ $\mathfrak{M}(\lambda) \in M$ and it satisfies the conditions of Theorem 3.4. Then, according to Theorem 3.4, there exists $L \in V_{N2}'$ for which $\mathfrak{M}(\lambda)$ is the WM.

3.4. Differential operators with a "separate" spectrum.

We consider the IP of recovering DO's of the form (2.1) under the condition of "separation" of the spectrum. In this case, to construct the DO, we need not all the WM but only its part. More exactly, the DO is uniquely determined from given $n-1$ WF's. We provide a rule how to choose sets of the WF's which guarantee uniqueness of the solution of the IP. We give the solution of the IP from chosen WF's. It is shown that obtained theorems contain results of Leibenzon [52, 53]. Further, we give a solution of the IP of recovering the DO from a system of $2n-2$ spectra. It is shown that this problem can be reduced to the IP from the WF's.

3.4.1. We consider DE and LF (2.1)–(2.2). For definiteness we confine ourselves to the case where $T < \infty$. Let Λ_{mk}, $1 \leq m \leq k \leq n$, be the set of zeros (with multiplicity) of the entire function

$$\Delta_{mk}(\lambda) = (-1)^{m+k} \det \big[U_{\eta T}(C_\nu)\big]_{\eta=\overline{1,n-m};\,\nu=\overline{m,n}\setminus k}.$$

The set Λ_{mk} coincides with the set of eigenvalues of the boundary value problem S_{mk} for the DE (2.1) under the conditions $U_{\xi 0}(y) = U_{\eta T}(y) = 0$, $\eta = \overline{1,n-m}$, $\xi = \overline{1,m-1},k$. In particular, $\Lambda_{mm} = \Lambda_m$.

Let r $(1 \leq r \leq n)$ be a fixed natural, $\theta_m = \max(r, m-1)$. Assume that

$$\Lambda_{mm} \cap \Lambda_{m+1,\theta_m} = \emptyset, \qquad m = \overline{1,n-2}. \tag{3.40}$$

Everywhere in Sec. 3.4 we assume that the condition (3.40) of "separation" of the spectrum is fulfilled. In this case for recovering the DE and LF we need $n-1$ WF's. The IP is formulated as follows.

Problem 3.1. Given the WF's $\big\{\mathfrak{M}_{m,\theta_m}(\lambda)\big\}_{m=\overline{1,n-1}}$ construct the DE and LF $L = (l, U)$.

First of all we study the uniqueness of the solution of the IP.

Theorem 3.5. *If $\mathfrak{M}_{m,\theta_m}(\lambda) = \widetilde{\mathfrak{M}}_{m,\theta_m}(\lambda)$, $m = \overline{1,n-1}$, then $L = \widetilde{L}$.*

For definiteness, we prove Theorem 3.5 for $r = 2$. In this case the "separation" condition (3.40) means that

$$\Lambda_{mm} \cap \Lambda_{m+1,m+1} = \emptyset, \qquad m = \overline{1, n-2}, \tag{3.41}$$

and the conditions of Theorem 3.5 become

$$\mathfrak{M}_{m,m+1}(\lambda) = \widetilde{\mathfrak{M}}_{m,m+1}(\lambda), \qquad m = \overline{1, n-1}. \tag{3.42}$$

To prove Theorem 3.5 we use the following auxiliary statement.

Lemma 3.8. *Assume that for a certain m, $1 \le m \le n-1$, a number λ_0 is a zero of $\Delta_{mm}(\lambda)$ of multiplicity $\varkappa_m \ge 1$, and $\Delta_{m+1,m+1}(\lambda_0) \ne 0$. Then in a neighbourhood of the point $\lambda = \lambda_0$ we have the representation*

$$\Phi_m(x,\lambda) = \xi_m(x,\lambda) + \sum_{\nu=1}^{\varkappa_m} \frac{c_{\nu m}}{(\lambda - \lambda_0)^\nu} \Phi_{m+1}(x,\lambda), \tag{3.43}$$

where the function $\xi_m(x,\lambda)$ is regular at $\lambda = \lambda_0$.

Proof of Theorem 3.5. Assume that for a certain m, $1 \le m \le n-1$, a number λ_0 is a zero of $\Delta_{mm}(\lambda)$ of multiplicity $\varkappa_m$, i.e., $\lambda_0 \in \Lambda_{mm}$. Then it follows from (3.41) that $\lambda_0 \notin \Lambda_{m+1,m+1}$, i.e., $\Delta_{m+1,m+1}(\lambda_0) \ne 0$, and, by Lemma 3.8, we have the representation (3.43) in a neighbourhood of $\lambda = \lambda_0$. Applying the LF $U_{m+1,0}$ to both sides of (3.43) and taking into account the relations $U_{m+1,0}\big(\Phi_m(x,\lambda)\big) = \mathfrak{M}_{m,m+1}(\lambda)$, $U_{m+1,0}\big(\Phi_{m+1}(x,\lambda)\big) = 1$, we obtain

$$\mathfrak{M}_{m,m+1}(\lambda) = U_{m+1,0}\big(\xi_m(x,\lambda)\big) + \sum_{\nu=1}^{\varkappa_m} \frac{c_{\nu m}}{(\lambda - \lambda_0)^\nu}.$$

Hence $c_{\nu m} = \big[\mathfrak{M}_{m,m+1}(\lambda)\big]^{\langle -\nu \rangle}_{\lambda=\lambda_0}$. By virtue of (3.42), we get $c_{\nu m} = \tilde{c}_{\nu m}$. It follows from Lemma 3.8 and (2.35) that for each fixed $\lambda_0 \in \Lambda$ we have the following representation in a neighbourhood of the point $\lambda = \lambda_0$:

$$\big[\Phi_m^{(\nu-1)}(x,\lambda)\big]_{\nu,m=\overline{1,n}} = \big[\xi_m^{(\nu-1)}(x,\lambda)\big]_{\nu,m=\overline{1,n}} \cdot \big[\theta_{\nu,m}(\lambda)\big]_{\nu,m=\overline{1,n}},$$

where the functions $\xi_m(x,\lambda)$ are regular at $\lambda = \lambda_0$,

$$\det\big[\xi_m^{(\nu-1)}(x,\lambda)\big]_{\nu,m=\overline{1,n}} = (-1)^{n(n-1)/2},$$

and $\theta_{\nu m}(\lambda) = \tilde{\theta}_{\nu m}(\lambda)$. Hence, for each fixed $x \in [0,T]$ the matrix $\mathrm{P}(x,\lambda)$, defined in Sec. 2.1, is entire in λ. Further, as in the proof of Theorem 2.2, we obtain that $L = \tilde{L}$. Theorem 3.5 is proved.

The counterexample from Sec. 3.3 shows that omitting the requirement of "separation" of the spectrum leads to a violation of the uniqueness for the solution of the IP.

3.4.2. Here we provide necessary and sufficient conditions and an algorithm of solution of the IP. For simplicity, we confine ourselves to the case of $L \in V'_{N2}$.

Lemma 3.9. *If $\lambda_0 \in \Lambda_m \cap \dots \cap \Lambda_{\mu-1}$, $\lambda_0 \notin \Lambda_\mu$, then $\mathfrak{N}_{\xi\mu}(\lambda_0) \ne 0$, $\xi = \overline{m, \mu-1}$; $\mathfrak{N}_{\xi j}(\lambda_0) = 0$ for $j = \overline{\xi+1, n} \setminus \mu$.*

Indeed, by the condition of the lemma $\Delta_{\xi\xi}(\lambda_0) = 0$ for $\xi = \overline{m, \mu-1}$. Denote $h_\xi(x,\lambda) = \Delta_{\xi\xi}(\lambda)\Phi_\xi(x,\lambda)$. It follows from (2.13) that the functions $h_\xi(x,\lambda)$ are entire in λ. If $h_\xi(x,\lambda_0) \equiv 0$, then from (2.13) follows that $\Delta_{\xi-1,\xi-1}(\lambda_0) = \Delta_{\xi s}(\lambda_0) = 0$, $s = \overline{\xi, n}$. But it is impossible, by virtue of (3.40). Thus $h_\xi(x,\lambda_0) \not\equiv 0$ and hence, $\Phi_{\xi,\langle -1 \rangle}(x,\lambda_0) \not\equiv 0$. According to (3.2) we obtain

$$\Phi_{\xi,\langle -1 \rangle}(x,\lambda_0) = \sum_{j=\xi+1}^{n} \mathfrak{N}_{\xi j}(\lambda_0)\Phi_{j,\langle 0 \rangle}(x,\lambda_0).$$

Therefore

$$\sum_{j=\xi+1}^{n} \left|\mathfrak{N}_{\xi j}(\lambda_0)\right| \neq 0.$$

Further, since $\Delta_{\mu-1,\mu-1}(\lambda_0) = 0$, $\Delta_{\mu\mu}(\lambda_0) \neq 0$, it follows from Lemma 3.1 that $\mathfrak{N}_{\mu-1,j}(\lambda_0) = 0$, $j = \overline{\mu+1,n}$; $\mathfrak{N}_{\mu-1,\mu}(\lambda_0) \neq 0$. From this and from Lemma 3.2 we obtain the assertion of Lemma 3.9.

By (3.40) and Lemma 3.9, for each $l \geq 1$, $m = \overline{1,n-1}$, there exist natural μ_{lm} $(m+1 \leq \mu_{lm} \leq \theta_m)$ such that

$$\Delta_{kk}(\lambda_{lm}) = 0, \quad k = \overline{m+1, \mu_{lm}-1}, \qquad \Delta_{\mu_{lm},\mu_{lm}}(\lambda_{lm}) \neq 0.$$

Furthermore, $\Delta_{\mu_{lm},\theta_m}(\lambda_{lm}) \neq 0$, and consequently $\mathfrak{M}_{\mu_{lm},\theta_m,\langle 0\rangle}(\lambda_{lm}) \neq 0$. It follows from Lemma 3.9 that

$$\mathfrak{N}_{m,\mu_{lm}}(\lambda_{lm}) \neq 0, \qquad \mathfrak{N}_{mj}(\lambda_{lm}) = 0, \quad j = \overline{m+1,n} \setminus \mu_{lm}. \tag{3.44}$$

Further, from the equality $\mathfrak{N}\mathfrak{M}_{\langle 0\rangle} = \mathfrak{M}_{\langle -1\rangle}$ we obtain

$$\mathfrak{N}_{mk}(\lambda_{lm}) = \mathfrak{M}_{mk,\langle -1\rangle}(\lambda_{lm}) - \sum_{j=m+1}^{k-1} \mathfrak{N}_{mj}(\lambda_{lm})\mathfrak{M}_{jk,\langle 0\rangle}(\lambda_{lm}). \tag{3.45}$$

Hence $\mathfrak{M}_{m,\theta_m,\langle -1\rangle}(\lambda_{lm}) = \mathfrak{N}_{m,\mu_{lm}}(\lambda_{lm})\mathfrak{M}_{\mu_{lm},\theta_m,\langle 0\rangle}(\lambda_{lm})$ or

$$\mathfrak{N}_{m,\mu_{lm}}(\lambda_{lm}) = \left(\mathfrak{M}_{\mu_{lm},\theta_m,\langle 0\rangle}(\lambda_{lm})\right)^{-1} \cdot \beta_{l,m,\theta_m}. \tag{3.46}$$

Relations (3.44) and (3.45) give us the connections allowing to find the WM $\mathfrak{M}(\lambda)$ from the given WF's $\left\{\mathfrak{M}_{m,\theta_m}(\lambda)\right\}_{m=\overline{1,n-1}}$ (or, what is the same, from their poles and residues $\{\lambda_{lm}, \beta_{l,m,\theta_m}\}_{l\geq 1}$). Thus, our IP can be reduced to the IP of recovering L from the given WM $\mathfrak{M}(\lambda)$.

For simplicity, we formulate necessary and sufficient conditions for $r = 2$. In this case we have

$$\theta_m = m+1, \qquad \mathfrak{N}_{m,m+1}(\lambda_{lm}) = \mathfrak{M}_{m,m+1,\langle -1\rangle}(\lambda_{lm}) \neq 0,$$
$$\mathfrak{N}_{mj}(\lambda_{lm}) = 0, \qquad j = \overline{m+2,n}.$$

So, for $r = 2$ the numbers ξ_l, defined in Sec. 3.2, have the form

$$\xi_l = \sum_{m=1}^{n-1} \left(\left|\lambda_{lm} - \widetilde{\lambda}_{lm}\right| + \left|\beta_{l,m,m+1} - \widetilde{\beta}_{l,m,m+1}\right| \cdot l\right) \cdot l^{1-n},$$

where $\beta_{lmk} = \mathfrak{M}_{mk,\langle -1\rangle}(\lambda_{lm})$. From (3.45) we obtain the equality

$$\beta_{lmk} = \beta_{l,m,m+1}\mathfrak{M}_{m+1,k,\langle 0\rangle}(\lambda_{lm}). \tag{3.47}$$

From the given $\left\{\mathfrak{M}_{m,m+1}(\lambda)\right\}_{m=\overline{1,n-1}}$ (or, what is the same, from $\left\{\lambda_{lm}, \beta_{l,m,m+1}\right\}_{l\geq 1}$), using (3.47), one can construct recurrently the WF's $\mathfrak{M}_{mk}(\lambda)$ for $k > m+1$. Thus, the WM $\mathfrak{M}(\lambda)$ is constructed. Furthermore, the properties S_1 and S_2 for $\mathfrak{M}(\lambda)$ are clearly fulfilled. Thus, the following Theorems 3.6 and 3.7 succeed from Theorems 3.3 and 3.4.

Theorem 3.6. *For meromorphic functions $\left\{\mathfrak{M}_{m,m+1}(\lambda)\right\}_{m=\overline{1,n-1}}$ with simple poles $\{\lambda_{lm}\}_{l\geq 1}$, $\lambda_{lm} \neq \lambda_{l_0,m+1}$ $(l, l_0 \geq 1)$ and residues $\beta_{l,m,m+1} \neq 0$ to be the WF's for $L \in V'_{N2}$ it is necessary and sufficient that the following conditions hold:*

(1) $\left|\mathfrak{M}_{m,m+1}(\lambda)\right| < C|\rho|^{-1}$, $\lambda \in G_\delta$;

(2) *there exists $\widetilde{L} \in V'_{N2}$ such that*

$$\sum_l \left(\xi_l l^{n+N-1}\right)^2 < \infty;$$

(3) *condition P of Theorem 3.1 is fulfilled.*

Theorem 3.7. *Let $\widetilde{L} \in V'_{N2}$ be given. Then there exists $\delta > 0$ (which depends on $\widetilde{L}$) such that if the numbers $\{\lambda_{lm},\ \beta_{l,m,m+1}\}_{l\ge 1, m=\overline{1,n-1}}$, $\lambda_{lm} \neq \lambda_{l_0,m}$ $(l \neq l_0)$, $\lambda_{lm} \neq \lambda_{l_0,m+1}$ $(l, l_0 \ge 1)$, $\beta_{l,m,m+1} \neq 0$ satisfy the condition*

$$\Lambda^+ \stackrel{\text{def}}{=} \left(\sum_{l=1}^{\infty} \left(\xi_l l^{n+N-1}\right)^2 \right)^{1/2} < \delta,$$

then there exists a unique $L \in V'_{N2}$ for which $\{\lambda_{lm},\ \beta_{l,m,m+1}\}_{l\ge 1}$ are poles and residues of the WF's $\mathfrak{M}_{m,m+1}(\lambda)$. In addition,

$$\left\|p_j^{(\nu)}(x) - \widetilde{p}_j^{(\nu)}(x)\right\|_{\mathcal{L}_2(0,T)} < C\Lambda^+, \quad 0 \le \nu \le j+N; \qquad \left|u_{\xi\nu a} - \widetilde{u}_{\xi\nu a}\right| < C\Lambda^+,$$

where the constant C depends only on $\widetilde{L}$.

We note that to solve the IP it is not necessary to find the WM $\mathfrak{M}(\lambda)$ since the main equation of the IP for $r = 2$ can be constructed directly from λ_{lm} and $\beta_{l,m,m+1}$.

Remark. From Theorems 3.6 and 3.7 results of Leibenzon [52, 53] follow. Indeed, in [52, 53] the IP of recovering the DE and LF is studied from the given $\{\lambda_{lm},\ \alpha_{lm}\}_{l\ge 1, m=\overline{1,n-1}}$ under the "separation" condition (3.41), where α_{lm} are "weight" numbers connected with the residues $\beta_{l,m,m+1}$ of the WF's $\mathfrak{M}_{m,m+1}(\lambda)$ by the formula

$$\beta_{l,m,m+1} = \left(\dot{\Delta}_{mm}(\lambda_{lm})\right)^{-1} \Delta_{m,m+1}(\lambda_{lm}) = (-1)^{n-m} (\alpha_{lm})^{-1}.$$

Thus, the specification of the numbers $\{\lambda_{lm}, \alpha_{lm}\}$ is equivalent to the specification of the WF's $\{\mathfrak{M}_{m,m+1}(\lambda)\}_{m=\overline{1,n-1}}$, and the problem of Leibenzon is a particular case of Problem 3.1.

3.4.3. We consider the IP of recovering the DE and LF (2.1)–(2.2) from a system of $2n-2$ spectra. Denote by $\{\lambda^1_{lm}\}_{l\ge 1}$ the eigenvalues of S_{m,θ_m}. The IP is formulated as follows.

Problem 3.2. Given the spectra $\{\lambda_{lm},\ \lambda^1_{lm}\}_{l\ge 1, m=\overline{1,n-1}}$ construct the DE and LF $L = (l, U)$.

Let us show that this IP can be reduced to Problem 3.1 of recovering L from the WF's $\{\mathfrak{M}_{m,\theta_m}(\lambda)\}_{m=\overline{1,n-1}}$.

Let $\Lambda_{mk} = \{\lambda_{lmk}\}_{l\ge 1}$, i.e., the numbers $\{\lambda_{lmk}\}_{l\ge 1}$ are eigenvalues of S_{mk}. It follows from (2.21) that the function $\Delta_{mk}(\lambda)$ is entire in λ of the order $1/n$. Since Λ_{mk} is the set of zeroes of $\Delta_{mk}(\lambda)$, we have by Borel's theorem [55, p. 31]

$$\Delta_{mk}(\lambda) = B_{mk} \cdot \prod_{l=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{lmk}}\right), \qquad B_m k = \text{const}$$

(the case where $\lambda = 0$ is the eigenvalue of S_{mk}, requires minor modifications). Then

$$\frac{\Delta_{mk}(\lambda)}{\widetilde{\Delta}_{mk}(\lambda)} = \frac{B_{mk}}{\widetilde{B}_{mk}} \cdot \prod_{l=1}^{\infty} \frac{\widetilde{\lambda}_{lmk}}{\lambda_{lmk}} \cdot \prod_{l=1}^{\infty} \left(1 - \frac{\widetilde{\lambda}_{lmk} - \lambda_{lmk}}{\widetilde{\lambda}_{lmk} - \lambda}\right).$$

For $(-1)^{n-m+1}\Lambda \to \infty$, we have

$$\frac{B_{mk}}{\widetilde{B}_{mk}} \cdot \prod_{l=1}^{\infty} \frac{\widetilde{\lambda}_{lmk}}{\lambda_{lmk}} = 1,$$

and hence

$$\Delta_{mk}(\lambda) = \widetilde{\Delta}_{mk}(\lambda) \prod_{l=1}^{\infty} \left(1 - \frac{\widetilde{\lambda}_{lmk} - \lambda_{lmk}}{\widetilde{\lambda}_{lmk} - \lambda}\right). \tag{3.48}$$

In particular, from this we obtain that the characteristic function $\Delta_{mk}(\lambda)$ of the boundary value problem S_{mk} is uniquely determined by its zeros. Furthermore, the function $\Delta_{mk}(\lambda)$ can be constructed by (3.48), where $\widetilde{L} = (\widetilde{l}, \widetilde{U})$ are known DE and LF (for example, with zero coefficients). Then, in view of (2.7) we obtain the following statement.

Lemma 3.10. *If* $\Lambda_m = \widetilde{\Lambda}_m$, $\Lambda_{mk} = \widetilde{\Lambda}_{mk}$, *then* $\mathfrak{M}_{mk}(\lambda) \equiv \widetilde{\mathfrak{M}}_{mk}(\lambda)$.

Thus, the specification of two spectra of S_{mm} and S_{mk} uniquely determines the WF $\mathfrak{M}_{mk}(\lambda)$.

From Theorem 3.5 and Lemma 3.10 we obtain the following uniqueness theorem of solution of the IP from a system of $2n-2$ spectra.

Theorem 3.8. *If* $\lambda_{lm} = \widetilde{\lambda}_{lm}$, $\lambda^1_{lm} = \widetilde{\lambda}^1_{lm}$, $l \geq 1$, $m = \overline{1, n-1}$, *then* $L = \widetilde{L}$.

Thus, the DE and LF are uniquely determined from the given $2n-2$ spectra of the boundary value problems S_{mm}, S_{m,θ_m}, $m = \overline{1, n-1}$.

To solve the IP of recovering L from $2n-2$ spectra $\{\lambda_{lm}, \lambda^1_{lm}\}$ we can construct the characteristic functions $\Delta_{mm}(\lambda)$, $\Delta_{m,\theta_m}(\lambda)$, and then the WF's $\mathfrak{M}_{m,\theta_m}(\lambda)$ by the formula

$$\mathfrak{M}_{m,\theta_m}(\lambda) = \big(\Delta_{mm}(\lambda)\big)^{-1}\Delta_{m,\theta_m}(\lambda),$$

and residues β_{l,m,θ_m} of the WF's $\mathfrak{M}_{m,\theta_m}(\lambda)$ by the formula

$$\beta_{l,m,\theta_m} = \big(\dot{\Delta}_{mm}(\lambda_{lm})\big)^{-1}\Delta_{m,\theta_m}(\lambda_{lm})$$

and use the results of Sec. 3.4.2. Thus, the IP from $2n-2$ spectra can be reduced to the IP of recovering the DE and LF $L = (l, U)$ from the WF's $\{\mathfrak{M}_{m,\theta_m}(\lambda)\}_{m=\overline{1,n-1}}$ or, what is the same, from the poles and residues of the WF's $\{\lambda_{lm}, \beta_{l,m,\theta_m}\}_{l\geq 1, m=\overline{1,n-1}}$.

3.5. Stability of the solution of the IP from spectra.

Stability of the solution of the IP from the WM was studied in Sec. 3.2. Things are more complicated for the IP from a system of spectra. Here we study stability of the solution of the IP in the uniform norm from spectra. It is shown that small perturbations of the spectra lead to small perturbations of the operator. Here we use a method which leads to a development of ideas of Levinson [56]. For brevity, we confine ourselves to formulations of results for fourth-order self-adjoint DO's with symmetric coefficients. Analogous results are valid for DO's of an arbitrary order.

Let $\{\lambda_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be eigenvalues of the boundary value problems Q_i, $i = 1, 2$, for the DE

$$ly \equiv y^{(4)} - \big(q_2(x)y'\big)' + q_0(x)y = \lambda y = \rho^4 y, \qquad q_j(\pi - x) = q_j(x) \tag{3.49}$$

under the boundary conditions

$$\begin{aligned} &y(0) = y''(0) = y(\pi) = y''(\pi) = 0 \qquad &&(\text{for } Q_1),\\ &y(0) = y'(0) = y''(0) = y(\pi) = 0 \qquad &&(\text{for } Q_2), \end{aligned}$$

respectively. Here $q_i(x)$ are real, and $q_i^{(\nu)}(x)$ are continuous on $[0, \pi]$ for $0 \leq i - \nu \leq 2$. We shall assume that the spectra of Q_1 and Q_2 are simple, and $\lambda_n \neq \gamma_m$ for all $n, m \geq 1$.

Theorem 3.9. *If* $\lambda_n = \widetilde{\lambda}_n$, $\gamma_n = \widetilde{\gamma}_n$, $n \geq 1$, *then* $q_i(x) \equiv \widetilde{q}_i(x)$, $i = 0, 2$.

Theorem 3.10. *There exists* $\delta > 0$ (*which depends on* Q_i) *such that if*

$$\Lambda \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \Big(\big|\lambda_n - \widetilde{\lambda}_n\big| + \big|\gamma_n - \widetilde{\gamma}_n\big|\Big) < \delta,$$

then

$$\max_{0\leq x\leq \pi} \left| \frac{d^\nu}{dx^\nu} \int_0^x \big(q_i(t) - \widetilde{q}_i(t)\big)dt \right| < C\Lambda, \qquad 0 \leq i - \nu \leq 2,$$

where C depends only on Q_i.

We give an outline of the proof. Let the functions $a_i(x,\lambda)$, $b_i(x,\lambda)$, $i=1,2$, be solutions of (3.49) under the initial conditions $a_i^{(\nu-1)}(0,\lambda)=b_i^{(\nu-1)}(\pi,\lambda)=\delta_{2i,\nu}$, $\nu=\overline{1,4}$, and let $a_3(x,\lambda)=a_1(x,\lambda)a_2(\pi,\lambda)-a_2(x,\lambda)a_1(\pi,\lambda)$, $b_3(x,\lambda)=a_3(\pi-x,\lambda)$, $\Delta(\lambda)=a_3''(\pi,\lambda)$, $\delta(\lambda)=a_2(\pi,\lambda)$. Clearly, $b_3(x,\lambda_n)=k_n a_3(x,\lambda_n)$, where $k_n=(-1)^{n+1}\operatorname{sign}\delta(\lambda_n)$. The eigenvalues $\{\lambda_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ of Q_1 and Q_2 coincide with the zeros of $\Delta(\lambda)$ and $\delta(\lambda)$ respectively.

Let $f(x)\in C^2[0,\pi]$, $f(0)=f(\pi)=0$, and let $\Gamma_N=\{\lambda:|\lambda|=R_N\}$ be circumferences in the λ-plane with radii $R_N\to\infty$ such that Γ_N are $\varepsilon>0$ distant from the spectra $\{\lambda_n\}$ and $\{\gamma_n\}$. Then

$$\lim_{N\to\infty}\max_{0\leq x\leq\pi}\left|f(x)-\frac{1}{2\pi i}\int_{\Gamma_N} y(x,\lambda)\,d\lambda\right|=0,$$

$$y(x,\lambda)=\big(\tilde\delta(\lambda)\big)^{-1}\left(b_2(x,\lambda)\int_0^x\tilde a_2(t,\lambda)f(t)\,dt+a_2(x,\lambda)\int_x^\pi\tilde b_2(t,\lambda)f(t)\,dt\right)$$
$$-\big(\tilde\delta(\lambda)\tilde\Delta(\lambda)\big)^{-1}\left(b_3(x,\lambda)\int_0^x\tilde a_3(t,\lambda)f(t)\,dt+a_3(x,\lambda)\int_x^\pi\tilde b_3(t,\lambda)f(t)\,dt\right),$$

and consequently

$$\sum_{n=1}^\infty\tilde A_n\tilde a_3(x,\tilde\lambda_n)=\sum_{n=1}^\infty\left(\tilde A_n a_3(x,\tilde\lambda_n)+y_n(x)\int_0^x\tilde a_3(t,\tilde\lambda_n)f(t)\,dt\right.$$
$$\left.-r_n(x)\int_0^x\tilde a_2(t,\tilde\gamma_n)f(t)\,dt-w_n(x)\int_x^\pi\tilde b_2(t,\tilde\gamma_n)f(t)\,dt\right),\tag{3.50}$$

where

$$\tilde A_n=\tilde k_n\big(\dot{\tilde\Delta}(\tilde\lambda_n)\tilde\delta(\tilde\lambda_n)\big)^{-1}\int_0^\pi f(t)\tilde a_3(t,\tilde\lambda_n)\,t,$$

$$r_n(x)=\Phi_{n1}b_2(x,\tilde\gamma_n)+\Phi_{n2}b_1(x,\tilde\gamma_n),\qquad w_n(x)=\Phi_{n1}a_2(x,\tilde\gamma_n)+\Phi_{n2}a_1(x,\tilde\gamma_n),$$

$$y_n(x)=\big(\dot{\tilde\Delta}(\lambda_n)\tilde\delta(\tilde\lambda_n)\big)^{-1}\Big(\big(b_3(x,\tilde\lambda_n)-b_3(x,\lambda_n)\big)-k_n\big(a_3(x,\tilde\lambda_n)-a_3(x,\lambda_n)\big)\Big),$$

$$\Phi_{n1}=\frac{\tilde a_1(\pi,\tilde\gamma_n)}{\dot{\tilde\delta}(\tilde\gamma_n)\tilde\Delta(\tilde\gamma_n)}\big(a_1(\pi,\tilde\gamma_n)-\tilde a_1(\pi,\tilde\gamma_n)\big),\qquad \Phi_{n2}=\frac{\tilde a_1(\pi,\tilde\gamma_n)\delta(\tilde\gamma_n)}{\dot{\tilde\delta}(\tilde\gamma_n)\tilde\Delta(\tilde\gamma_n)},$$

and the series converge absolutely and uniformly for $[0,\pi]$.

Let now $f(x)$ satisfy the conditions $f(x)\in C^4[0,\pi]$, $lf(x)\in C^2[0,\pi]$, $f^{(\nu)}(0)=f^{(\nu)}(\pi)=0$, $\nu=0,2$; $(lf)(0)=(lf)(\pi)=0$. We aplly the operator $\tilde l$ to (3.50). On the other hand, we set lf instead of f in (3.50). After the corresponding subtraction of the obtained relations and comparison the coefficient for

f, f', and f'', we obtain formulas

$$\int_0^x \widehat{q}_{2-j}(t)\,dt = \sum_{n=1}^{\infty} \int_0^x \Big(F_j\big[y_n(t), \widetilde{a}_3(t,\widetilde{\lambda}_n)\big] - F_j\big[r_n(t), \widetilde{a}_2(t,\widetilde{\gamma}_n)\big] + F_j\big[w_n(t), \widetilde{b}_2(t,\widetilde{\gamma}_n)\big]\Big)dt, \qquad j=0,2, \tag{3.51}$$

$$\frac{d^{\mu+1}}{dx^{\mu+1}} \int_0^x \widehat{q}_2(t)\,dt = \sum_{n=1}^{\infty} \Big(F_\mu\big[y_n(x), \widetilde{a}_3(x,\widetilde{\lambda}_n)\big] - F_\mu\big[r_n(x), \widetilde{a}_2(x,\widetilde{\gamma}_n)\big] + F_\mu\big[w_n(x), \widetilde{b}_2(x,\widetilde{\gamma}_n)\big]\Big), \qquad \mu=0,1, \tag{3.52}$$

$$F_0[y,z] = -4(yz)', \qquad F_1[y,z] = -6y''z - 8y'z' - 2yz'' + \widehat{q}_2 yz,$$
$$F_2[y,z] = 4y'''z + 6y''z' + 4y'z'' + 2yz''' - 2q_2y'z - q_2'yz - (q_2 + \widehat{q}_2)yz'.$$

From here we obtain the assertions of the theorems. We omit a rather complicated part of the proof connected with getting estimates (with constants not depending on $\widetilde{l}$) for the functions from the right-hand sides of (3.50)–(3.52).

4. Method of Standard Models

4.1. Formulation of the inverse problem. Information condition. We consider the DE and LF $L = (l, U)$ of the form (2.1)–(2.2) on the half-line or on the finite interval ($T \le \infty$) and study the IP of recovering N ($1 \le N \le n-1$) coefficients of the DE from given N Weyl functions provided that these N functions are piecewise analytic (the rest of $n-N-1$ coefficients of the DE are a priori known arbitrary integrable functions).

Let the sets of positive integers $\varkappa = \{\varkappa_j\}_{j=\overline{1,N}}$, $I = \{(k_i, \gamma_i),\ i = \overline{1,N}\}$, $2 \le \varkappa_1 < \cdots < \varkappa_N \le n$, $1 \le k_i < \gamma_i \le n$, be given. The IP is formulated as follows.

Problem 4.1. Given the WF's $\{\mathfrak{M}_{k_i,\gamma_i}(\lambda)\}_{i=\overline{1,N}}$ and the coefficients $p_\nu(x)$, $n-\nu \notin \varkappa$, construct the functions $\{p_{n-\varkappa_j}(x)\}_{j=\overline{1,N}}$.

For convenience, we number the given WF's in a different way. Let $I = \{(m_\tau, \gamma_{\tau\eta}),\ \tau = \overline{1,\theta},\ \eta = \overline{1,N_\tau}\}$, $1 \le m_\tau < \gamma_{\tau 1} < \cdots < \gamma_{t,N_\tau} \le n$, $N_1 + \cdots + n_\theta = N$, $m_\tau \ne m_{\tau'}$ ($\tau \ne \tau'$). Denote $\mathfrak{M}_s(\lambda) = \mathfrak{M}_{m_\tau,\gamma_{\tau\eta}}(\lambda)$, $s = \overline{1,N}$. Here and below, the positive integer s has a unique representation $s = N_1 + \cdots + N_{\tau-1} + \eta$, $1 \le \eta \le N_\tau$. Then our IP can be written as follows.

Problem 4.2. Given the WF's $\{\mathfrak{M}_s(\lambda)\}_{s=\overline{1,N}}$ and the coefficients $p_\nu(x)$, $n-\nu \notin \varkappa$, construct the functions $\{p_{n-\varkappa_j}(x)\}_{j=\overline{1,n}}$.

Note that using the given WF's we can find not only the DE, but also the coefficients of the LF. However we assume for brevity that the LF are known. We also assume that the enumeration of R_k in (2.3) is chosen for the sector $S_0 = \{\rho : \arg\rho \in (0, \frac{\pi}{n})\}$. Denote

$$\omega_\xi^*(R) = (-R)^{\sigma_{\xi_0}^*},$$
$$\Omega^*(j_1, \ldots, j_p) = \det\big[\omega_{j_\nu}^*(R_k^*)\big]_{\nu,k=\overline{1,p}},$$
$$\Omega_\mu^*(j_1, \ldots, j_p) = \det\big[\omega_{j_\nu}^*(R_k^*)\big]_{\nu=\overline{1,p};k=\overline{1,p+1}\setminus\mu}.$$

Let us give a classification of the IP. For this we consider the matrices $A_l = [A_{lsj}]_{s=\overline{1,N};j=\overline{1,p}}$, where p is such that $l \in [\varkappa_p, \varkappa_{p+1})$, $\varkappa_0 = 1$, $\varkappa_{M+1} = \infty$ (if $l \ge \varkappa_N$, then A_l is a square matrix); here

$$A_{lsj} = \frac{(-1)^{n+m_\tau+\gamma_{\tau\eta}+1}}{\Omega(\overline{1,m_\tau})\Omega^*(\overline{1,n-m_\tau})} \sum_{\mu=1}^{m_\tau} \sum_{\nu=1}^{n-m_\tau} \frac{(-1)^{\mu+\nu}\Omega_m(\overline{1,m_\tau-1})\Omega_\nu^*(\overline{1,n-m_\tau} \setminus n-\gamma_{\tau\eta}+1)}{R_\mu^{\varkappa_j}(-R_\mu - R_\nu^*)^{l-\varkappa_j+1}}.$$

Definition 4.1. If $A_l \neq 0$ for all $l \geq 1$, then the set $\{\mathfrak{M}_s(\lambda)\}_{s=\overline{1,N}}$ of the WF's is said to be a $P_\varkappa$-system.

This definition distinguishes the classes of WF's which have sufficient information for solution of the IP. We therefore call $A_l \neq 0$, $l \geq 1$, an information condition. It is easily shown that this condition is independent of the choice of a sector.

4.2. Solution of the incomplete inverse problems. Let us study the IP for $P_\varkappa$-systems. To solve the IP we use the so-called method of standard models in which we construct a sequence of model DO's of the form (2.1) "approaching" the unknown operator. The method allows to obtain an algorithm of the solution of the IP. We first formulate some auxiliary propositions.

Lemma 4.1. *Let $r(x) = (\alpha!)^{-1}x^\alpha(h + p(x))$, $\alpha \geq 0$, $p(x) \in C[0,b]$, $p(0) = 0$, $h(x,z) = \exp(-zx)(1 + z^{-1}\xi(x,z))$, where the function $\xi(x,z)$ is continuous and bounded for $x \in [0,b]$, $|z| \geq z_0$, $z \in Q \overset{\text{def}}{=} \{z : \arg z \in [-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0], \delta_0 > 0\}$. Then for $|z| \to \infty$, $z \in Q$, we have*

$$\int_0^b z(x)H(x,z)\,dx = z^{-\alpha-1}\big(h + o(1)\big).$$

Denote

$$\psi_\tau(x,\lambda) = \Phi_{m_\tau}(x,\lambda), \qquad \beta_\tau = m_\tau + N_\tau, \qquad \gamma_{\tau 0} = m_\tau, \qquad M_\tau = \{k : k = \overline{1,m_\tau};\ \gamma_{\tau 1}, \dots, \varphi_{\tau,N_\tau}\},$$
$$M_{\tau\eta} = \{k : k \in M_\tau,\ k \neq \gamma_{\tau\eta}\}, \qquad q_\tau = \det[R_k^\mu]_{k=\overline{\beta_\tau+1,n};\mu=\overline{\beta_\tau,n-1}}, \qquad Q = [q_{\xi\nu}^\tau]_{\xi=\overline{1,n};\nu=\overline{0,n-1}},$$

where

$$q_{\xi\nu}^\tau = (-1)^{\nu-\beta_\tau+1}(q_\tau)^{-1}\det[R_k^\mu]_{k=\overline{\beta_\tau+1,n};\mu=\overline{\xi-1,\beta_\tau,n-1}\setminus\nu}$$

for $\xi = \overline{1,\beta_\tau}$, $\nu = \overline{\beta_\tau, n-1}$ and $q_{\xi\nu}^\tau = \delta_{\nu,\xi-1}$ otherwise.

The following lemma allows to solve the IP in steps.

Lemma 4.2. *For a fixed $a \in (0,T)$ the WS's $\psi_\tau(x,\lambda)$, $\tau = \overline{1,\theta}$, satisfy the boundary conditions*

$$U_{\xi a}^\tau(\psi_\tau) = \mathfrak{N}_\xi^\tau(\lambda,a), \qquad \xi = \overline{1,\beta_\tau}, \tag{4.1}$$

where

$$U_{\xi a}^\tau(y) = \sum_{\nu=0}^{n-1} Q_{\xi\nu}^\tau(\lambda,a)y^{(\nu)}(a), \qquad \mathfrak{N}_\xi^\tau(\lambda,a) = J_{\xi 0}^\tau(\lambda,a) + \sum_{\eta=1}^{N_\tau} J_{\xi\eta}^\tau(\lambda,a)\mathfrak{M}_s(\lambda),$$

$$J_{\xi\eta}^\tau(\lambda,a) = \frac{(-1)^{n-N_\tau+\eta-1}}{\Delta_\tau(\lambda,a)}\det\left[z_k^{(\mu)}(a,\lambda), U_{j0}(z_k)\right]_{k=\overline{1,n};\mu=\overline{\xi-1,\beta_\tau,n-1};j\in M_{\tau\eta}},$$

$$Q_{\xi\nu}^\tau(\lambda,a) = \frac{(-1)^{\nu-\beta_\tau+1}}{\Delta_\tau(\lambda,a)}\det\left[z_x^{(\mu)}(a,\lambda), U_{j0}(z_k)\right]_{k=\overline{1,n};\mu=\overline{\xi-1,\beta_\tau,n-1}\setminus\nu;j\in M_{\tau\eta}}, \qquad \nu = \overline{\beta_\tau, n-1},$$

$$Q_{\xi\nu}^\tau(\lambda,a) = \delta_{\xi,\nu-1}, \qquad \nu = \overline{0,\beta_\tau - 1},$$

$$\Delta_\tau(\lambda,a) = \det\left[z_k^{(\mu)}(a,\lambda), U_{j0}(z_k)\right]_{k=\overline{1,n};\mu=\overline{\beta_\tau,n-1};j\in M_{\tau\eta}}.$$

Here $\{z_k(x,\lambda)\}_{k=\overline{1,n}}$, $x \in [0,a]$, is a certain FSS of the DE (2.1). For $|\lambda| \to \infty$, $\arg\lambda = \varphi \in (0,\pi)$ we have $Q_{\xi\nu}^\tau(\lambda,a) = \rho^{\xi-1-\nu}q_{\xi\nu}^\tau(1 + O(\rho^{-1}))$.

Indeed, from the relation

$$\psi_\tau(x,\lambda) = \sum_{\mu=1}^n b_{\tau\mu}(\lambda)z_\mu(x,\lambda)$$

for $x = 0$ and $x = a$ we obtain

$$\sum_{\mu=1}^{n} b_{\tau\mu}(\lambda) z_{\mu}^{(\nu)}(a,\lambda) = \psi_{\tau}^{(\nu)}(a,\lambda), \qquad \sum_{\mu=1}^{n} b_{\tau\mu}(\lambda) U_{j0}\big(z_{\mu}(x,\lambda)\big) = \delta_{j,m_\tau},$$

$$\sum_{\mu=1}^{n} b_{\tau\mu}(\lambda) U_{\gamma_{\tau\eta},0}\big(z_{\mu}(x,\lambda)\big) = \mathfrak{M}_s(\lambda), \qquad \nu = \overline{0, n-1}, \quad j = \overline{1, m_\tau}, \quad \eta = \overline{1, N_\tau}.$$

Solving this algebraic system for each $\tau = \overline{1,\theta}$ with respect to $\{b_{\tau\mu}(\lambda)\}_{\mu=\overline{1,N}}$, $\{\psi_{\tau}^{(\nu)}(a,\lambda)\}_{\nu=\overline{0,\beta_\tau-1}}$, we obtain (4.1). Choosing $z_k(x,\lambda) = y_k(x,\rho)$, where $\{y_k(x,\rho)\}_{k=\overline{1,N}}$ is the FSS B_0 and using the asymptotic properties (2.5) of the functions $y_k(x,\rho)$, we obtain the asymptotic formula for $Q_{\xi\nu}^{\tau}(\lambda,a)$. We observe that the functions $\mathfrak{N}_{\xi}^{\tau}(\lambda,a)$ and $Q_{\xi\nu}^{\tau}(\lambda,a)$ are defined from L for $x \in (0,a)$ and from the WF's $\{\mathfrak{M}_s(\lambda)\}_{s=\overline{1,N}}$.

We define $U_{\xi a}^{\tau}(y) = y^{(\xi-1)}(a)$ for $\xi = \overline{\beta_\tau+1, n}$ and denote

$$\omega_{\xi\tau}^{1}(R) = \sum_{\nu=0}^{n-1} q_{\xi\nu}^{\tau} R^{\nu}, \qquad \xi = \overline{1,n}.$$

The functions $\omega_{\xi\tau}^{1}(R)$ are the characteristic polynomials for the LF $U_{\xi a}^{\tau}$ (they do not depend on $a \in (0,T)$). Define LF $U_{\xi a}^{\tau,*}$, $\xi = \overline{1,n}$, from the relation

$$\langle y, z\rangle_{|x=a} = \sum_{\xi=1}^{n} U_{n-\xi+1,a}^{\tau}(y) U_{\xi a}^{\tau,*}(z).$$

Denote

$$\omega_{\xi\tau}^{\tau,*}(R) = \sum_{\nu=0}^{n-1} q_{\xi\nu}^{\tau,*} R^{\nu}$$

the characteristic polynomials for the LF $U_{\xi a}^{\tau,*}$. It is clear that $q_{\xi\nu}^{\tau,*} = (-1)^{\nu} D_{n+1-\xi,n-1-\nu}^{\tau}$, where $D_\tau = [D_{\mu\nu}^{\tau}]_{\mu=\overline{1,n};\nu=\overline{0,n-1}}$ is the matrix of algebraic minors of Q_τ. For Ω_τ^1, $\Omega_{\tau\mu}^1$, $\Omega_\tau^{1,*}$, and $\Omega_{\tau\mu}^{1,*}$ the same formulas are used as for Ω, Ω_μ, Ω^*, and Ω_μ^* with $\omega_{\xi\tau}^{1}(R)$ and $\omega_{\xi\tau}^{1,*}(R)$ replacing $\omega_\xi(R)$ and $\omega_\xi^*(R)$.

Let us show that

$$\operatorname{rank}\left[\omega_{\xi\tau}^{1}(R_k)\right]_{\xi=\overline{1,\beta_\tau};k=\overline{1,m_\tau}} = m_\tau. \tag{4.2}$$

Indeed, $\left[\omega_{\xi\tau}^{1}(R)\right]_{\xi=\overline{1,n}} = Q_\tau [R^{\xi-1}]_{\xi=\overline{1,n}}$. Since $\det Q_\tau = 1$, $\det[R_k^{\xi-1}]_{k,\xi=\overline{1,n}} \neq 0$, it follows that $\Omega_\tau(R_1,\ldots, R_n) \neq 0$. However $\omega_{\xi\tau}^{1}(R) = (q_\tau)^{-1} \det\left[R^{\nu}, R_{\beta_\tau+1}^{\nu}, \ldots, R_n^{\nu}\right]_{\nu=\xi-1,\overline{\beta_\tau+1,n}}$ $(\xi = \overline{1,\beta_\tau})$. Then $\omega_{\xi\tau}^{1}(R_k) = 0$, $\xi = \overline{1,\beta_\tau}$, $k = \overline{\beta_\tau+1, n}$. Hence $\Omega_\tau^1(R_1,\ldots, R_{\beta_\tau}) \neq 0$, i.e., (4.2) is valid.

Let $\{\varepsilon_{ji}\}_{j=\overline{1,\beta_\tau}}$ denote a permutation of the numbers $\overline{1,\beta_\tau}$ such that $\Omega_\tau^1(\varepsilon_{1\tau},\ldots,\varepsilon_{m_\tau,\tau}) \neq 0$. By (4.2), such a permutation exists. Let the functions $\psi_s^*(x,\lambda,a)$, $a \geq 0$, $x \in (a,T)$ be solutions of the DE (2.24) with the following conditions: $U_{\xi 0}^*(\psi_s^*) = \delta_{\xi,n-\gamma_{\tau\eta}+1}$ $(\xi = \overline{1, n-m_\tau})$ for $a = 0$; $U_{\xi a}^{\tau,*}(\psi_s^*) = \delta_{\xi,n-\varepsilon_{m_\tau}+\eta,\tau+1}$ $(\xi = \overline{1, n-\beta_\tau}$, $n-\varepsilon_{m_\tau+1,\tau}+1, \ldots, n-\varepsilon_{\beta_\tau,\tau}+1)$ for $a > 0$, and also $U_{\xi T}^*(\psi_s^*) = 0$, $\xi = \overline{1, m_\tau}$ for $T < \infty$, and $\psi_s^*(x,\lambda,a) = O\big(\exp(\rho R_{n-m_\tau}^* x)\big)$, $x \to \infty$ for $T = \infty$ $(a \geq 0)$. Denote $\Lambda_{nj}^{\tau}(\lambda,a) = -U_{n-\varepsilon_{j\tau}+1,a}^{\tau,*}(\psi_s^*)$,

$a>0$. For $l\ge 1$, $k=1,2$ we consider the matrices $A_l^k=[A_{lsj}^k]$, $s=\overline{1,N}$, $j=\overline{1,p}$, where

$$A_{lsj}^1=\frac{\Omega(\overline{1,m_\tau-1})}{R_{m_\tau}^{\varkappa_j}\Omega(\overline{1,m_\tau})}\sum_{\nu=1}^{n-m_\tau}\frac{(-1)^{n-\beta_\tau+\eta+\nu}}{(-R_{m_\tau}-R_\nu^*)^{l+1-\varkappa_j}}$$
$$\times\frac{\Omega_{\tau\nu}^{1,*}\big(\overline{1,n-\beta_\tau};n-\varepsilon_{m_\tau+1,\tau}+1,\dots,n-\varepsilon_{\beta_\tau,\tau}+1\setminus n-\varepsilon_{m_\tau+\eta,\tau}+1\big)}{\Omega_\tau^{1,*}\big(\overline{1,n-\beta_\tau};n-\varepsilon_{m_\tau+1,\tau}+1,\dots,n-\varepsilon_{\beta_\tau,\tau}+1\big)},$$
$$A_{lsj}^2=\frac{\Omega(\overline{1,m_\tau-1})(-1)^{m_\tau+\gamma_{\tau\eta}+1}\Omega^*\big(\overline{1,n-m_\tau}\setminus n-\gamma_{\tau\eta}+1\big)}{R_{m_\tau}^{\varkappa_j}\Omega(\overline{1,m_\tau})(R_{m_\tau+1}-R_{m_\tau})^{l-\varkappa_j+1}\Omega^*(\overline{1,n-m_\tau})}.$$

We shall write $A[a,b]$ ($PA[a,b]$) for the set of functions analytic (piecewise-analytic) on $[a,b]$. Let $p_{n-\varkappa_j}(x)\in PA[0,T)$, $r_{lj}^a=p_{n-\varkappa_j}^{l-\varkappa_j}(a+0)$, $r_l^a=[r_{lj}^a]_{j=\overline{1,p}}$. Denote by $\mathrm{P}_\varkappa^a$ the set of $\widetilde L$ such that $p_k(x)=\widetilde p_k(x)$, $x>0$, $k\ne n-\varkappa_j$, $j=\overline{1,N}$, and $L=\widetilde L$ for $x\in(0,a)$.

Lemma 4.3. *Let $p_{n-\varkappa_j}(x)$, $\widetilde p_{n-\varkappa_j}(x)\in PA[0,T)$, $\widetilde L\in\mathrm{P}_\varkappa^a$, $r_\mu^a=\widetilde r_\mu^a$, $\mu=\overline{1,l-1}$. Then for $|\lambda|\to\infty$, $\arg\lambda=\varphi\in(0,\pi)$, $k=1,2$, there exist finite limits*

$$X_{ls}^k(a)=\lim \mathrm{P}_{sk}(\rho,a)\cdot\rho^l,$$

where

$$\mathrm{P}_{s1}(\rho,a)=B_s(\lambda,a)\rho^{v_{s1}(a)}\exp(-\rho R_{m_\tau}a),$$
$$\mathrm{P}_{s2}(\rho,a)=B_s(\lambda,0)\rho^{v_{s2}(a)}\exp(-\rho(R_{m_\tau+1}-R_{m_\tau})a),$$
$$B_s(\lambda,0)=\widehat{\mathfrak{M}}_s(\lambda),\qquad B_s(\lambda,a)=\mathfrak{N}_{\varepsilon_{m_\tau+\eta,\tau}}^\tau(\lambda,a)+\sum_{j=1}^{m_\tau}\widetilde\Lambda_{\eta j}^\tau(\lambda,a)\mathfrak{N}_{\varepsilon_{j\tau}}^\tau(\lambda,a)\quad(a>0),$$
$$v_{s1}(0)=v_{s2}=\sigma_{m_\tau,0}-\sigma_{\gamma_{\tau\eta},0},\qquad v_{s1}(a)=\sigma_{m_\tau,0}-\varepsilon_{m_\tau+\eta,\tau}+1\quad(a>0).$$

Moreover,

$$\begin{cases}A_l(R_l^0-\widetilde r_l^0)=[X_{ls}]_{s=\overline{1,N}}, & X_{ls}=X_{ls}^k(0),\\ A_l^k(r_l^a-\widetilde r_l^a)=\big[X_{ls}^k(a)\big]_{s=\overline{1,N}}, & a>0.\end{cases}\tag{4.3}$$

Proof. Let us show that if $\widetilde L\in\mathrm{P}_\varkappa^a$, then

$$\sum_{j=1}^N\int_a^T\widehat p_{n-\varkappa_j}(x)\psi_\tau^{(n-\varkappa_j)}(x,\lambda)\widetilde\psi_s^*(x,\lambda,a)\,dx=B_s(\lambda,a).\tag{4.4}$$

Indeed, let $\widetilde L\in\mathrm{P}_\varkappa^0$. Then

$$\sum_{j=1}^N\int_0^T\widehat p_{n-\varkappa_j}(x)\psi_\tau^{(n-\varkappa_j)}(x,\lambda)\widetilde\psi_s^*(x,\lambda,0)\,dx=\int_0^T\big(l\psi_\tau(x,\lambda)-\widetilde l\psi(x,\lambda)\big)\widetilde\psi_s^*(x,\lambda,0)\,dx.$$

It follows from (2.26) that

$$\int_0^T\widetilde l\psi_\tau(x,\lambda)\widetilde\psi_s^*(x,\lambda,0)\,dx=\Big|_0^T\big\langle\psi_\tau(x,\lambda),\widetilde\psi_s^*(x,\lambda,0)\big\rangle_{\widetilde l}+\int_0^T\psi_\tau(x,\lambda)\widetilde l^*\widetilde\psi_s^*(x,\lambda,0)\,dx.$$

From this we obtain

$$\sum_{j=1}^N\int_0^T\widehat p_{n-\varkappa_j}(x)\psi_\tau^{(n-\varkappa_j)}(x,\lambda)\widetilde\psi_s^*(x,\lambda,0)\,dx=U_{n-m_\tau+1,0}^*\big(\widetilde\psi_s^*(x,\lambda,0)\big)+\mathfrak{M}_s(\lambda).$$

For $l=\widetilde{l}$ we have $\widetilde{\mathfrak{M}}_s(\lambda)=-U^*_{n-m_\tau+1,0}\big(\widetilde{\psi}^*_s(x,\lambda,0)\big)$, and hence

$$\sum_{j=1}^{N}\int_0^T \widehat{p}_{n-\varkappa_j}(x)\psi_\tau^{(n-\varkappa_j)}(x,\lambda)\widetilde{\psi}^*_s(x,\lambda,0)\,dx=\widehat{\mathfrak{M}}_s(\lambda).$$

Thus, for $a=0$ (4.4) is proved. For $a>0$ the proof is completely analogous.

For definiteness, let $T=\infty$. Let $\{y_\mu(x,\rho)\}_{\mu=\overline{1,n}}$ be the FSS B_0. It follows from (2.8) and (2.10) that

$$\psi_\tau(x,\lambda)=\sum_{\mu=1}^{m_\tau}c_{\tau\mu}(\rho)y_\mu(x,\rho),\tag{4.5}$$

$$c_{\tau\mu}(\rho)=\rho^{-\sigma_{m_\tau,0}}\left(\frac{(-1)^{m_\tau+\mu}}{\Omega(\overline{1,m_\tau})}\Omega_\mu(\overline{1,m_\tau-1})+O(\rho^{-1})\right),\qquad |\rho|\to\infty.\tag{4.6}$$

Similarly,

$$\widetilde{\psi}^*_s(x,\lambda,a)=\sum_{\nu=1}^{n-m_\tau}\widetilde{c}^*_{s\nu}(\rho,a)\widetilde{y}^*_\nu(x,\rho),\qquad a\ge 0,\tag{4.7}$$

$$\widetilde{c}^*_{s\nu}(\rho,0)=\rho^{t_{\tau\eta}}\left(\frac{(-1)^{n-\gamma_{\tau\eta}+\nu+1}}{\Omega^*(\overline{1,n-m_\tau})}\Omega^*_\nu\big(\overline{1,n-m_\tau}\setminus n-\gamma_{\tau\eta}+1\big)+O(\rho^{-1})\right),$$

$$\begin{aligned}\widetilde{c}^*_{s\nu}(\rho,a)=\rho^{t^1_{\tau\eta}}\exp(-\rho R^*_\nu a)\Bigg((-1)^{n-\beta_\tau+\eta+\nu}\\ \times\frac{\Omega^{1;*}_{\tau\nu}\big(\overline{1,n-\beta_\tau},n-\varepsilon_{m_\tau+1,\tau}+1,\dots,n-\varepsilon_{\beta_\tau,\tau}+1\setminus n-\varepsilon_{m_\tau+\eta,\tau}+1\big)}{\Omega^{1;*}_\tau\big(\overline{1,n-\beta_\tau},n-\varepsilon_{m_\tau+1,\tau}+1,\dots,n-\varepsilon_{\beta_\tau,\tau}+1\big)}+O(\rho^{-1})\Bigg),\qquad a>0,\end{aligned}\tag{4.8}$$

where $t_{\tau\eta}=\sigma_{\gamma_{\tau\eta}}+1-n$, $t^1_{\tau\eta}=\varepsilon_{m_\tau+\eta,\tau}-n$, and $\widetilde{y}^*_\nu(x,\rho)=\exp(\rho R^*_\nu x)(1+O(\rho^{-1}))$ is the FSS B_0 for $\widetilde{l}^*$. Since $r^a_\mu=\widetilde{r}^a_\mu$, $\mu=\overline{1,l-1}$, it follows that

$$\widehat{p}_{n-\varkappa_j}(x)=\frac{(x-a)^{l-\varkappa_j}}{(l-\varkappa_j)!}\big(r^a_{lj}-\widetilde{r}^a_{lj}+o(1)\big),\qquad x\to a+0.$$

By Lemma 4.1 we get

$$\int_0^T\widehat{p}_{n-\varkappa_j}(x)y_\mu^{(n-\varkappa_j)}(x,\rho)\widetilde{y}^*_\nu(x,\rho)\,dx=\frac{(r^a_{lj}-\widetilde{r}^a_{lj})\exp\big(\rho(R_\mu+R^*_\nu)a\big)}{\rho^{l+1-n}R_\mu^{\varkappa_j}(-R_\mu-R^*_\nu)^{l+1-\varkappa_j}}\cdot\big(1+o(1)\big),\qquad |\rho|\to\infty.\tag{4.9}$$

Substituting (4.5) and (4.7) in (4.4) we obtain

$$\sum_{\mu=1}^{m_\tau}\sum_{\nu=1}^{n-m_\tau}c_{\tau\mu}(\rho)\widetilde{c}^*_{s\nu}(\rho,a)\sum_{j=1}^{N}\int_a^T\widehat{p}_{n-\varkappa_j}(x)y_\mu^{(n-\varkappa_j)}(x,\rho)\widetilde{y}^*_\nu(x,\rho)\,dx=B_s(\lambda,a),\qquad a\ge 0,\quad \widetilde{L}\in\mathrm{P}^a_\varkappa.\tag{4.10}$$

In particular,

$$\sum_{\mu=1}^{m_\tau}\sum_{\nu=1}^{n-m_\tau}c_{\tau\mu}(\rho)\widetilde{c}^*_{s\nu}(\rho,0)\sum_{j=1}^{N}\int_a^T\widehat{p}_{n-\varkappa_j}(x)y_\mu^{(n-\varkappa_j)}(x,\rho)\widetilde{y}^*_\nu(x,\rho)\,dx=B_s(\lambda,0),\qquad a\ge 0,\quad \widetilde{L}\in\mathrm{P}^a_\varkappa.\tag{4.11}$$

From (4.10) and (4.11), in view of the asymptotic formulas (4.6) and (4.8), we obtain the assertion of Lemma 4.3.

Lemma 4.4. *Let $\widetilde{L}\in\mathrm{P}^0_\varkappa$, $k=1\vee 2$, $\alpha\in(0,T)$, and let $A_l\;\#\;0$, $A^k_l\;\#\;0$, $l\ge 1$. If for $|\lambda|\to\infty$, $\arg\lambda=\varphi\in(0,\pi)$ we have $\lim \mathrm{P}_{sk}(\rho,a)\rho^l=0$ for all $l\ge 1$, $a\in[0,\alpha)$, $s=\overline{1,N}$, then $\widetilde{L}\in\mathrm{P}^\alpha_\varkappa$.*

Proof. Indeed, if there exist $a < \alpha$ and $l \geq 1$ such that $\widetilde{L} \in \mathrm{P}^a_\varkappa$, $r^a_\mu = \widetilde{r}^a_\mu$, $\mu = \overline{1, l-1}$, $r^a_l \neq \widetilde{r}^a_l$, then it follows from (4.3) and the conditions of the lemma that $X^k_{l,s_0}(a) \neq 0$ for a certain s_0. This contradiction proves the lemma.

From the propositions proved above we obtain the following theorem.

Theorem 4.1. *Let $A_l \# 0$, $l \geq 1$. Then Problem 4.2 has a unique solution in the class $p_{n-\varkappa_j}(x) \in A[0, T)$, $j = \overline{1, N}$. If, further, $A^1_l \# 0$, $l \geq 1$, then Problem 4.2 has a unique solution in the class $p_{n-\varkappa_j}(x) \in PA[0, T)$, $j = \overline{1, N}$.*

The solution of Problem 4.2 can be found by applying the following algorithm.

Algorithm 4.1. (1) Take $a = 0$.

(2) Compute $\{r^a_l\}_{l \geq 1}$. For this we do successively for $l = 1, 2, \ldots$ the following operations: construct $\widetilde{L} \in \mathrm{P}^a_\varkappa$ such that $\widetilde{r}^a_\mu = r^a_\mu$, $\mu = \overline{1, l-1}$, and find r^a_l from (4.3) for $k = 1$.

(3) Construct L for $x \in (a, \alpha)$ by the formula

$$p_{n-\varkappa_j}(x) = \sum_{l=0}^{\infty} r^a_{l+\varkappa_j, j} \frac{(x-a)^l}{l!}, \qquad j = \overline{1, N}. \tag{4.12}$$

(4) If $\alpha < T$, then we put $a := \alpha$ and go on to the step 2.

Remarks. 1. In Algorithm 4.1, the solution of the IP is sought in steps whose lengths are determined by Lemma 4.4 as follows. Assume that L for $x \in (0, a)$ and $\{r^a_l\}_{l \geq 1}$ have been found. Construct $\widetilde{L} \in \mathrm{P}^a_\varkappa$ so that $\widetilde{r}^a_l = r^a_l$, $l \geq 1$. Put

$$\alpha = \sup\left\{b > a : \lim \mathrm{P}_{s1}(\rho, b)\rho^l = 0,\ \ l \geq 1,\ \ s = \overline{1, N}\right\}.$$

Then Lemma 4.4 implies that $\widetilde{L} \in \mathrm{P}^\alpha_\varkappa$, i.e., we found L for $x \in (a, \alpha)$.

2. If the conditions of Theorem 4.1 are not satisfied, then a solution of the IP will not be unique. Indeed, let $T < \infty$, $n = 3$, $\sigma_{\xi 0} = \sigma_{\xi T} = 3 - \xi$. Consider the following IP: given the WF's $\{\mathfrak{M}_{12}(\lambda), \mathfrak{M}_{23}(\lambda)\}$, construct the coefficients $p_0(x)$ and $p_1(x)$. So, $N = 2$, $\varkappa = \{2, 3\}$, and $I = \{(1,2), (2,3)\}$. It is easy to see that in this case the information condition is not satisfied, i.e., the set $\{\mathfrak{M}_{12}(\lambda), \mathfrak{M}_{23}(\lambda)\}$ of the WF's is not a $P_\varkappa$-system. It was shown in Sec. 3.3 that a solution of this IP is not unique even in the class of analytic coefficients.

3. Theorem 4.1 remains in force when the condition $A^1_l \# 0$ is replaced by the condition $A^2_l \# 0$. Then, Algorithm 4.1 can be replaced by the simpler Algorithm 4.2.

Algorithm 4.2. (1) Take $a = 0$.

(2) Compute $\{r^a_l\}_{l \geq 1}$. For this we do successively for $l = 1, 2, \ldots$ the following operations: construct $\widetilde{L} \in \mathrm{P}^a_\varkappa$ such that $\widetilde{r}^a_\mu = r^a_\mu$, $\mu = \overline{1, l-1}$, and find r^a_l from (4.3) for $k = 2$.

(3) Construct L for $x \in (a, \alpha)$ by (4.12).

(4) If $\alpha < T$, then we put $a := \alpha$ and go on to the step 2.

Algorithm 4.2 is simpler than Algorithm 4.1 since it does not require to compute the functions $\mathfrak{N}^T_\xi(\lambda, a)$ and $Q^T_{\xi\nu}(\lambda, a)$ at each step.

4. Theorem 4.1 remains in force when the condition of piecewise analyticity is replaced by a more general condition, ensuring that an asymptotics for the integral (4.9) exists.

4.3. Particular cases.

Case 1. We study the IP of recovering a single coefficient of the DE (2.1) from one WF. For definiteness, let $n = 2q$, $\sigma_{\xi 0} = \xi - 1$. Take a fixed integer $\varkappa$ $(2 \leq \varkappa \leq n)$.

Problem 4.3. From the WF $\mathfrak{M}_{12}(\lambda)$ and coefficients $p_\nu(x)$, $\nu \neq n - \varkappa$, find the function $p_{n-\varkappa}(x)$.

In this case $N = 1$. The information condition $A_l \# 0$, $l \geq 1$, for Problem 4.3 is fulfilled (see [95]). Furthermore, it is obvious that $A_l^2 \# 0$, $l \geq 1$. Thus, the following theorem is proved.

Theorem 4.2. *Problem* 4.3 *has a unique solution in the class* $p_{n-\varkappa}(x) \in PA[0,T)$ (*the rest of* $p_k(x) \in \mathcal{L}(0,T)$, $k \neq n - \varkappa$). *The solution of Problem* 4.3 *can be found by applying Algorithm* 4.2.

Case 2. We consider the IP of recovering all coefficients of the DE (2.1) from the first row of the WM.

Problem 4.4. Find the DE (2.1) from the WF's $\{\mathfrak{M}_{1k}(\lambda)\}_{k=\overline{2,n}}$.

It is shown in [96] that the set $\{\mathfrak{M}_{1k}(\lambda)\}_{k=\overline{2,n}}$ of the WF's is a $P_\varkappa$-system for $\varkappa = \{2, \dots, n\}$, and $A_l \# 0$, $A_l^1 \# 0$, $l \geq 1$. Therefore, from Theorem 4.1 we obtain the following theorem.

Theorem 4.3. *Problem* 4.4 *has a unique solution in the class* $p_k(x) \in PA[0,T)$, $k = \overline{0, n-2}$. *This solution can be found by Algorithm* 4.1.

The counterexample from Sec. 3.3 shows that there are no $P_\varkappa$-systems for $\varkappa = \{2, \dots, n\}$ with the exeption of the first row of the WM.

Remark. Let $T < \infty$ and let G_k, $k = \overline{0, n-1}$, denote the boundary value problems for (2.1) with the conditions $y^{(k)}(0) = y(T) = \cdots = y^{(n-2)}(T) = 0$. It was shown in Sec. 3 (see Lemma 3.10) that the specification of each WF $\mathfrak{M}_{1k}(\lambda)$ is equivalent to the specification of two spectra for problems G_0 and G_{k-1}. Hence, for $T < \infty$ Problem 4.3 consists in finding one of the coefficients of the DE from the two spectra of G_0 and G_1, and Problem 4.4 consists in finding the DE (2.1) from the system of n spectra of G_k, $k = \overline{0, n-1}$. L. Sakhnovich was the first who investigated an IP of this type. In [77], he proved a uniqueness theorem for recovering of the two-term operator $l_1 y = y^{(n)} + p_0(x) y$ from the system of n spectra of G_k, $k = \overline{0, n-1}$, in the class of entire functions. The same result is established in [88] in the class of piecewise analytic functions. The transformation operator method is used in [77, 88]. Thus, Theorems 4.2 and 4.3 essentially strengthen the results from [77, 88]. We note that an IP for the two-term operator l_1 in another formulation was considered in [43].

Case 3. We consider the IP of recovering a self-adjoint DE from the spectral function. For $n = 2$, this IP was studied by Marchenko [60, 61], Gel'fand and Levitan [33], and for higher-order DE by L. Sakhnovich [78, 79] and Khachatryan [44]. In particular, in [44] the transformation operator method for $n > 2$ is used to prove a uniqueness theorem in the class of analytic in a certain sector functions.

The IP of recovering the self-adjoint operator from the spectral function can be reduced to the IP from the WF's. So we can obtain a uniqueness theorem and an algorithm for the solution of the IP from the spectral function in the class of piecewise-analytic coefficients. For brevity, we consider only the case in which $n = 4$, $U_{\xi a}(y) = y^{(\xi-1)}(a)$, $a = 0, T$.

Let $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ be solutions of the DE

$$ly \equiv y^{(4)} + P_2(x) y'' + p_1(x) y' + p_0(x) y = \lambda y, \qquad 0 \leq x \leq T \leq \infty, \tag{4.13}$$

with the conditions $\psi_k^{(\xi-1)}(0, \lambda) = \delta_{\xi, 3-k}$, $k, \xi = 1, 2$, and also $\psi_k(T, \lambda) = \psi_k(T, \lambda) = 0$ for $T < \infty$ and $\psi_k(x, \lambda) = O(1)$, $x \to \infty$, for $T = \infty$. Denote $M(\lambda) = [M_{kr}(\lambda)]_{k,r=1,2}$, $M_{kr} = \psi_k^{(r+1)}(0, \lambda)$. It is known [69] that if the DE (4.13) is self-adjoint, then the specification of $M(\lambda)$ is equivalent to the specification of the spectral function $\sigma(\lambda) = [\sigma_{kr}(\lambda)]_{k,r=1,2}$ of the DE with the conditions $y(0) = y'(0) = 0$ (and $y(T) = y'(T) = 0$ for $T < \infty$). The IP is formulated as follows: given the matrix $M(\lambda)$ construct the DE (4.13).

Denote $d(\alpha) = \left[d_{kr}(\alpha, 0),\ d_{kr}(\alpha+1, 1),\ d_{kr}(\alpha+2, 2)\right]_{k,r=1,2}$, where $d_{kr}(\alpha, \nu) = 1 + i^{\alpha+3\nu+k+r-1} - (1 + i)^{\alpha+1}(i^{r-1} - i^{k+3\nu-1})$. It is easy to see that the information condition for this IP has a form $d(\alpha) \# 0$,

$\alpha \geq 0$, and it is clearly satisfied. Therefore, applying to this IP the method of standard models we obtain that the specification of $M(\lambda)$ uniquely determines the DE (4.13) in the class $p_k(x) \in PA[0,T)$. In particular, if the DE (4.13) is self-adjoint, then the specification of the spectral function $\sigma(\lambda)$ uniquely determines the DE in the class $PA[0,T)$.

4.4. An inverse problem of the elasticity theory. The problem of determining the dimensions of the transverse cross-sections of a beam from the given frequencies of its natural vibrations is examined. Frequency spectra are indicated which determine the dimenions of the transverse cross-sections of the beam uniquely, an effective procedure is presented for solving the IP, and a uniqueness theorem is proved. The method of standard models is used to solve the IP.

Consider the DE describing beam vibrations in the form

$$(h^{\mu}(x)y'')'' = \lambda h(x)y, \qquad 0 \leq x \leq T. \tag{4.14}$$

Here $h(x)$ is a function characterizing the beam transverse section, and $\mu = 1, 2, 3$ is a fixed number. We will assume that the function $h(x)$ is absolutely continuous in the segment $[0,T]$ and $h(x) > 0$, $h(0) = 1$. The IP for (4.14) in the case $\mu = 2$ (similar transverse sections) was investigated in [5] in determining small changes in the beam trasverse sections for given small changes in a finite number of its natural vibration frequencies.

Let $\{\lambda_{kj}\}_{k\geq 1}$, $j = 1, 2$, be the eigenvalues of boundary value problems Q_j for (4.14) with the boundary conditions

$$y(0) = y^{(j)}(0) = y(T) = y'(T) = 0.$$

The IP is formulated as follows.

Problem 4.5. Find the function $h(x)$, $x \in [0,T]$, from the given spectra $\{\lambda_{kj}\}_{k\geq 1, j=1,2}$.

Let us show that this IP can be reduced to the IP of recovering the DE (4.14) from the WF. Let $\Phi(x,\lambda)$ be a solution of (4.14) under the conditions $\Phi(0,\lambda) = \Phi(T,\lambda) = \Phi'(T,\lambda) = 0$, $\Phi'(0,\lambda) = 1$. We set $\mathfrak{M}(\lambda) = \Phi''(0,\lambda)$. The function $\mathfrak{M}(\lambda)$ is called the WF for (4.14). Let the functions $C_\nu(x,\lambda)$, $\nu = \overline{0,3}$, be solutions of (4.14) under the initial conditions $C_\nu^{(\mu)}(0,\lambda) = \delta_{\nu\mu}$, $\nu, \mu = \overline{0,3}$. Denote

$$\Delta_j(\lambda) = C_{3-j}(T,\lambda)C_3'(T,\lambda) - C_3(T,\lambda)C_{3-j}'(T,\lambda), \qquad j = 1, 2.$$

Then $\Phi(x,\lambda) = (\Delta_1(\lambda))^{-1} \det\left[C_\nu(x,\lambda), C_\nu(T,\lambda), C_\nu'(T,\lambda)\right]_{\nu=1,2,3}$, and hence $\mathfrak{M}(\lambda) = -(\Delta_1(\lambda))^{-1}\Delta_2(\lambda)$.

The eigenvalues $\{\lambda_{kj}\}_{k\geq 1, j=1,2}$ of the boundary value problems Q_j coincide with the zeros of the entire functions $\Delta_j(\lambda)$. As in Sec. 3 (see Sec. 3.4), it is easy to see that the functions $\Delta_j(\lambda)$ are uniquely determined by their zeros. Hence the specification of the spectra $\{\lambda_{kj}\}_{k\geq 1, j=1,2}$ uniquely determines the WF $\mathfrak{M}(\lambda)$. Thus, Problem 4.5 is reduced to the following IP.

Problem 4.6. Given the WF $\mathfrak{M}(\lambda)$, find $h(x)$, $x \in [0,T]$.

We shall solve Problem 4.6 by the method of standard models. Let $\lambda = \rho^4$. For $|\lambda| \to \infty$, $\arg\lambda = \varphi \neq 0$, $\rho \in S$, $x \in [0,T]$, the following asymptotic formula is valid:

$$\Phi^{(\nu)}(x,\lambda) = \rho^{\nu-1} \sum_{\xi=1}^{2} \left(R_\xi \gamma'(x)\right)^{\nu} g_{\xi 0}(x) \exp\left(\rho R_\xi \gamma(x)\right)\left(1 + O(\rho^{-1})\right), \tag{4.15}$$

where

$$\gamma(x) = \int_0^x \left(h(t)\right)^{\frac{1-\mu}{4}} dt.$$

The functions $g_{\xi 0}(x)$ are absolutely continuous, and $g_{\xi 0}(x) \neq 0$, $g_{10}(0) = -g_{20}(0) = (R_1 - R_2)^{-1}$. In particular, $\mathfrak{M}(\lambda) = \rho(R_1 + R_2)(1 + O(\rho^{-1}))$.

Lemma 4.5. *Let $p(x) = h^{\mu}(x)$. The following relation holds:*

$$\int_0^T \Big(\widehat{h}(x)\lambda\Phi(x,\lambda)\widetilde{\Phi}(x,\lambda) - \widehat{p}(x)\Phi''(x,\lambda)\widetilde{\Phi}''(x,\lambda)\Big)dx = \widehat{\mathfrak{M}}(\lambda). \tag{4.16}$$

Proof. Denote $l_\lambda y = (p(x)y'')'' = \lambda h(x)y$, $\mathcal{L}(y,z) = (py'')'z - py''z' + py'z'' - y(pz'')'$. Then

$$\int_0^T l_\lambda y(x)\cdot z(x)\,dx = \Big|_0^t \mathcal{L}(y(x),z(x)) + \int_0^T y(x)l_\lambda z(x)\,dx. \tag{4.17}$$

Using (4.17) and the equality $l_\lambda\Phi(x,\lambda) = \widetilde{l}_\lambda\widetilde{\Phi}(x,\lambda) = 0$, we obtain

$$\int_0^T (l_\lambda - \widetilde{l}_\lambda)\Phi(x,\lambda)\cdot\widetilde{\Phi}(x,\lambda)\,dx = -\Big|_0^T \widetilde{\mathcal{L}}\big(\Phi(x,\lambda),\widetilde{\Phi}(x,\lambda)\big) - \int_0^T \Phi(x,\lambda)\widetilde{l}_\lambda\widetilde{\Phi}(x,\lambda)\,dx$$

$$= \Phi'(0,\lambda)\widetilde{\Phi}''(0,\lambda) - \Phi''(0,\lambda)\widetilde{\Phi}'(0,\lambda) = -\widehat{\mathfrak{M}}(\lambda).$$

On the other hand, integrating by parts, we have

$$\int_0^T (l_\lambda - \widetilde{l}_\lambda)\Phi(x,\lambda)\cdot\widetilde{\Phi}(x,\lambda)\,dx = \Big|_0^T \Big(\big(\widehat{p}(x)\Phi''(x,\lambda)\big)'\widetilde{\Phi}(x,\lambda) - \widehat{p}(x)\Phi''(x,\lambda)\widetilde{\Phi}'(x,\lambda)\Big)$$

$$+ \int_0^T \Big(\widehat{p}(x)\Phi''(x,\lambda)\widetilde{\Phi}''(x,\lambda) - \lambda\widehat{h}(x)\Phi(x,\lambda)\widetilde{\Phi}(x,\lambda)\Big)\,dx.$$

Since the substitution vanishes, we obtain (4.16).

Lemma 4.6. *Consider the integral*

$$J(z) = \int_0^T f(x)H(x,z)\,dx, \tag{4.18}$$

$$f(x)\in C[0,T],\quad f(x)\sim f_\alpha\frac{x^\alpha}{\alpha!}\quad (x\to+0),\qquad h(x,z) = \exp(-za(x))\left(1+\frac{\xi(x,z)}{z}\right),$$

$$a(x)\in C^1[0,T],\qquad 0<a(x_1)<a(x_2)\quad (0<x_1<x_2),$$

$$a^{(\nu)}(x)\sim a_0x^{1-\nu}\quad (x\to+0,\quad \nu=0,1),\qquad a'(x)>0,$$

where the function $\xi(x,z)$ is continuous and bounded for $x\in[0,T]$, $z\in Q=\big\{z:\arg z\in[-\frac{\pi}{2}+\delta_0,\ \frac{\pi}{2}-\delta_0]$, $\delta_0>0\big\}$. Then, as $z\to\infty$, $z\in Q$,

$$J(z)\sim f_\alpha(a_0z)^{-\alpha-1}. \tag{4.19}$$

Proof. The function $t=a(x)$ has the inverse $x=b(t)$, where $b(t)\in C^1[0,T_1]$, $T_1=a(T)$, $b(t)>0$ $(t>0)$ and $b^{(\nu)}(t)\sim a_0^{-1}t^{1-\nu}$, $\nu=0,1$, as $t\to+0$. Let us make the change of variable $t=a(x)$ in the integral in (4.18). We obtain

$$J(z) = \int_0^{T_1} g(t)\exp(-zt)\int_{(} 1+z^{-1}\xi(b(t),z)\big)\,dt, \tag{4.20}$$

where $g(t) = f(b(t))b'(t)$. It is clear that for $t\to+0$

$$g(t)\sim f_\alpha(a_0)^{-\alpha-1}\frac{t^\alpha}{\alpha!}.$$

Applying Lemma 4.1 to (4.20), we obtain (4.19).

Denote

$$A_\alpha = (R_1 - R_2)^{-2} \sum_{\xi,s=1}^{2} \frac{(-1)^{\xi+s}(1 - \mu R_\xi^2 R_s^2)}{(R_\xi + R_s)^{\alpha+1}}, \qquad \alpha \geq 1.$$

Let us show that $A_\alpha \neq 0$ for all $\alpha \geq 1$. For definiteness, we put $\arg\rho \in (0, \frac{\pi}{4})$, i.e., $\{R_1, R_2\} = \{-1, i\}$. Then

$$A_\alpha = -(R_1 - R_2)^{-2}(-2)^{-\alpha-1} a_\alpha,$$

where $a_\alpha = (\mu - 1)(1 + i^{\alpha+1}) + 2(\mu + 1)(1 + i)^{\alpha+1}$. Since $|1 + i^{\alpha+1}| \leq 2$, $|1 + i|^{\alpha+1} = (\sqrt{2})^{\alpha+1}$, it follows that $a_\alpha \neq 0$, $\alpha \geq 1$. Hence $A_\alpha \neq 0$ for all $\alpha \geq 1$.

Lemma 4.7. *As $x \to 0$ let $\widehat{h}(x) \sim \widehat{h}_\alpha(\alpha!)^{-1}x^\alpha$. Then as $|\lambda| \to \infty$, $\arg\lambda = \varphi \neq 0$ there exists a finite limit $I_\alpha = \lim \rho^{\alpha-1}\widehat{\mathfrak{M}}(\lambda)$, and*

$$A_\alpha \widehat{h}_\alpha = I_\alpha. \tag{4.21}$$

Proof. Since $p(x) = h^\mu(x)$, and by virtue of the conditions of the lemma, we have, as $x \to +0$, $\widehat{p}(x) \sim \mu\widehat{h}_\alpha(\alpha!)^{-1}x^\alpha$. Using the asytmptotic formulas (4.15) and Lemma 4.6 we find, as $|\lambda| \to \infty$, $\arg\lambda = \varphi \neq 0$, $\rho \in S$:

$$\int_0^T \widehat{h}(x)\lambda\Phi(x,\lambda)\widetilde{\Phi}(x,\lambda)\,dx \sim \rho^{1-\alpha}\widehat{h}_\alpha(R_1 - R_2)^{-2} \sum_{\xi,s=1}^{2} \frac{(-1)^{\xi+s}}{(R_\xi + R_s)^{\alpha+1}},$$

$$\int_0^T \widehat{p}(x)\Phi''(x,\lambda)\widetilde{\Phi}''(x,\lambda)\,dx \sim \rho^{1-\alpha}\mu\widehat{h}_\alpha(R_1 - R_2)^{-2} \sum_{\xi,s=1}^{2} \frac{(-1)^{\xi+s}R_\xi^2 R_s^2}{(R_\xi + R_s)^{\alpha+1}}.$$

Substituting the expressions obtained in (4.16), we obtain the assertion of the lemma.

From the facts presented above we have the following theorem.

Theorem 4.4. *Problem 4.6 has a unique solution in the class $h(x) \in A[0,T]$. This solution can be found according to the following algorithm:*

1. *we calculate $h_\alpha = h^{(\alpha)}(0)$, $\alpha \geq 0$, $h_0 = 1$. For this we successively perform operations for $\alpha = 1, 2, \ldots$: we construct the function $\widetilde{h}(x) \in A[0,T]$, $\widetilde{h}(x) > 0$ such that $\widetilde{h}^{(\nu)}(0) = h_\nu$, $\nu = \overline{0, \alpha - 1}$, and arbitrary in the rest, and we calculate h_α from (4.21);*
2. *we construct the function $h(x)$ from the formula*

$$h(x) = \sum_{\alpha=0}^{\infty} h_\alpha \frac{x^\alpha}{\alpha!}, \qquad 0 < x < R,$$

where

$$R = \left(\overline{\lim_{\alpha\to\infty}} \sqrt[\alpha]{\frac{|h_\alpha|}{\alpha!}} \right)^{-1}.$$

If $R < T$, then for $R < x < T$ the function $h(x)$ is constructed by analytic continuation.

We note that the IP in the class of piecewise-analytic functions can be solved in an analogous manner.

4.5. Nonlinear differential equations. Consider the nonlinear DE

$$-y''(x) + q(x)y(x) + p(x)y^2(x) = \lambda y(x), \qquad x \geq 0, \tag{4.22}$$

where $q(x),\ p(x)\in\mathcal{L}(0,\infty)$ are complex-valued functions. The nonlinear term qualitatively modifies the study of the IP. In this section, we formulate and solve the IP for the model nonlinear DE (4.22). The utilized method can be applied to IP's for a wide class of nonlinear equations.

Let us construct a Iost-type solution of (4.22). Let $\lambda=\rho^2$, $\operatorname{Im}\rho\ge 0$, and $G=\{\rho:\operatorname{Im}\rho\ge 0,\ |\rho|\ge \|q\|_{\mathcal{L}(0,\infty)}+4\|p\|_{\mathcal{L}(0,\infty)}$. We introduce the functions $\{z_k(x,\rho)\}_{k\ge 0}$ by the recurrent formulas $z_0(x,\rho)=1$,

$$z_{k+1}(x,\rho)=1+\frac{1}{2i\rho}\int_x^\infty\big(\exp(2i\rho(t-x))-1\big)\big(q(t)z_k(t,\rho)+\exp(i\rho t)p(t)z_k^2(t,\rho)\big)\,dt,\qquad k\ge 0.$$

The estimates $|z_k(x,\rho)|\le 2$ and $|z_{k+1}(x,\rho)-z_k(x,\rho)|\le 2^{-k-1}$ are valid for $x\ge 0$, $\rho\in G$; therefore the series $z(x,\rho)=1+\sum\limits_{k=0}^{\infty}\big(z_{k+1}(x,\rho)-z_k(x,\rho)\big)$ is absolutely and uniformly convergent for $\rho\in G$, $x\ge 0$. In addition, $|z(x,\rho)|\le 2$, $\lim\limits_{x\to\infty}z(x,\rho)=1$ (uniformly with respect to $\rho\in G$), and $z(x,\rho)=1+O(\rho^{-1})$ (uniformly with respect to $x\ge 0$) as $|\rho|\to\infty$. The function $\varphi(x,\rho)=z(x,\rho)\exp(i\rho x)$ is a solution of the integral equation

$$\varphi(x,\rho)=\exp(i\rho x)+\frac{1}{2i\rho}\int_x^\infty\big(\exp(i\rho(t-x))-\exp(i\rho(x-t))\big)\big(q(t)\varphi(t,\rho)+p(t)\varphi^2(t,\rho)\big)\,dt,$$

and therefore φ is a solution of (4.22), φ is regular with respect to $\rho\in G$ for each fixed $x\ge 0$, and $\lim\limits_{x\to\infty}\varphi(x,\rho)\exp(-i\rho x)=1$, $\varphi(x,\rho)=\exp(i\rho x)(1+O(\rho^{-1}))$, $|\rho|\to\infty$. Denote $\mathfrak{N}_j(\rho)=\varphi^{(j-1)}(0,\rho)$, $j=1,2$. The IP is formulated as follows.

Problem 4.7. Given the functions $\mathfrak{N}_j(\rho)$, $j=1,2$, and $p(x)$, find the function $q(x)$.

We give the solution of Problem 4.7 for the case in which $q(x)$ and $p(x)$ are analytic for $x\ge 0$. Denote by M the set of analytic functions $f(x)$ for $x\ge 0$ such that $f^{(j)}(x)\in\mathcal{L}(0,\infty)$ for all $j\ge 0$.

Lemma 4.8. *Let $p(x),\ q(x)\in M$. Then the asymptotic formulas*

$$\varphi^{(\nu)}(x,\rho)=\sum_{s=0}^{\infty}\exp\big((s+1)i\rho x\big)(i\rho)^{\nu}\sum_{k=2s}^{\infty}(i\rho)^{-k}\sum_{\mu=0}^{\nu}C_\nu^\mu(s+1)^{\nu-\mu}g_{k-\mu,s}^{(\mu)}(x),\qquad \nu=\overline{0,2},\tag{4.23}$$

$$\mathfrak{N}_j(\rho)=(i\rho)^{j-1}\sum_{k=0}^{\infty}(i\rho)^{-k}\mathfrak{N}_{kj},\qquad \mathfrak{N}_{0j}=1,\quad j=1,2,\qquad g_{00}(x)=1\tag{4.24}$$

are valid as $|\rho|\to\infty$, $\rho\in G$. The functions $g_{ks}(x)$ are analytic for $x\ge 0$, and $g_{ks}(x)q(x)\in M$, $g_{ks}(x)p(x)\in M$. Moreover, the following relations hold:

$$\begin{aligned}(s^2+2s)g_{k+2,s}(x)+2(s+1)g'_{k+1,s}(x)+g''_{ks}(x)&=q(x)g_{ks}(x)\\&+p(x)\sum_{j=0}^{k-2s+2}\sum_{i=1}^{s}g_{j+2i-2,i-1}(x)g_{k-j-2i+2,s-i}(x),\qquad s\ge 0,\quad k\ge 2s-2,\end{aligned}\tag{4.25}$$

$$\mathfrak{N}_{k1}=\sum_{s=0}^{[k/2]}g_{ks}(0),\qquad \mathfrak{N}_{k2}=\sum_{s=0}^{[k/2]}\big((s+1)g_{ks}(0)+g'_{k-1,s}(0)\big).\tag{4.26}$$

Here and below, $g_{ks}(x)\equiv 0$ for $k<2s$.

Proof. Let us show that

$$z(x,\rho)=\sum_{k=0}^{N}(i\rho)^{-k}\sum_{s=0}^{[k/2]}g_{ks}(x)\exp(is\rho x)+\rho^{-N-1}\xi_{N+1}(x,\rho)\tag{4.27}$$

for any $N \geq 0$, where $[\cdot]$ denotes the greatest integer in the number, and $|\xi_{N+1}(x,\rho)| \leq C_{N+1}$ for all $x \geq 0$, $\rho \in G$. We prove this fact by induction on N. For $N = 0$, (4.27) is obvious. Assume that this formula holds for some $N = N_0$. By construction, the function $z(x,\rho)$ is a solution of the integral equation

$$z(x,\rho) = 1 + \frac{1}{2i\rho}\int\limits_x^\infty \big(\exp(2i\rho(t-x)) - 1\big)\big(q(t)z(t,\rho) + p(t)z^2(t,\rho)\exp(i\rho t)\big)\,dt. \tag{4.28}$$

For $N = N_0$, we have

$$z^2(x,\rho) = \sum_{k=0}^{N}(i\rho)^{-k}\sum_{s=0}^{[k/2]} h_{ks}(x)\exp(is\rho x) + \rho^{-N-1}\varkappa_{N+1}(x,\rho), \tag{4.29}$$

where the functions $h_{ks}(x)$ are analytic for $x \geq 0$ and $h_{ks}(x)p(x) \in M$, $|\varkappa_{N+1}(x,\rho)| \leq C^0_{N+1}$ for all $x \geq 0$, $\rho \in G$. Then, substituting (4.27) and (4.29) into the right-hand side of (4.28), we obtain

$$\begin{aligned} z(x,\rho) = 1 + \sum_{k=0}^{N} 2^{-1}(i\rho)^{-k-1}\sum_{s=0}^{[k/2]}\int\limits_x^\infty &\big(\exp(2i\rho(t-x)) - 1\big) \\ &\times\big(q(t)g_{ks}(t)\exp(is\rho t) + p(t)h_{ks}(t)\exp((s+1)i\rho t)\big)\,dt \\ + \frac{1}{2i}\rho^{-N-2}\int\limits_x^\infty &\big(\exp(2i\rho(t-x)) - 1\big)\big(q(t)\xi_{N+1}(t,\rho) + p(t)\varkappa_{N+1}(t,\rho)\exp(i\rho t)\big)\,dt. \end{aligned}$$

Integrating by parts the kth term $N_0 - k + 1$ times we obtain (4.27) for $N = N_0 + 1$.

Thus, the asymptotic formula

$$z(x,\rho) = \sum_{k=0}^{\infty}(i\rho)^{-k}\sum_{s=0}^{[k/2]} g_{ks}(x)\exp(is\rho x)$$

holds uniformly with respect to $x \geq 0$ as $|\rho| \to \infty$, $\rho \in G$. Therefore (4.23) is proved for $\nu = 0$. If $\nu > 0$, then the consideration is similar. Since $\mathfrak{N}_j(\rho) = \varphi^{(j-1)}(0,\rho)$, then (4.24) and (4.26) are obvious corollaries of (4.23). Now by substituting (4.23) into (4.22) and equating the coefficients for $\rho^{-k}\exp(i\rho(s+1))$, we obtain (4.25). The lemma is proved.

Differentiate (4.25) ν times with respect to x and set $x = 0$. We obtain

$$\begin{aligned} (s^2+2s)g^{(\nu)}_{k+2,s}(0) + 2(s+1)g^{(\nu+1)}_{k+1,s}(0) + g^{(\nu+2)}_{ks}(0) = \sum_{j=0}^{\nu} C^j_\nu q^{(j)}(0)g^{(\nu-j)}_{ks}(0) \\ + \sum_{m=0}^{\nu} C^m_\nu p^{(\nu-m)}(0)\sum_{j=0}^{k-2s+2}\sum_{i=1}^{s}\sum_{\mu=0}^{m} C^\mu_m g^{(\mu)}_{j+2i-2,i-1}(0)g^{(m-\mu)}_{k-j-2i+2,s-i}(0), \qquad s \geq 0, \quad k \geq 2s-2, \quad \nu \geq 0. \end{aligned} \tag{4.30}$$

For $l \geq 0$ we consider linear algebraic systems X_l that consist of (4.26) for $k = l+2$ and (4.30) for $s = \overline{0, [(l+2)/2]}$, $k + \nu = l$, $\nu \geq 0$, $k \geq \max(0, 2s-2)$, with respect to the unknowns $q^{(l)}(0)$, $g^{(l-k+2)}_{ks}(0)$, $s = \overline{0, [(l+2)/2]}$, $k = \overline{\max(1,2s), l+2}$. Since for each $l \geq 0$ the matrix of the system X_l is triangular with nonzero elements on the main diagonal, the determinants of the systems X_l are nonzero. By solving the systems X_l for $l = 0, 1, 2, \ldots$, we find $q^{(l)}(0)$, $g^{(\nu)}_{ks}(0)$, and, consequently, the function $q(x)$. Therefore, the following statement holds.

Theorem 4.5. *Let $p(x)$, $q(x) \in M$. Then the solution of Problem 4.7 is unique and can be found by the following algorithm:*

(1) *calculate $g_{10} = \mathfrak{N}_{11} = \mathfrak{N}_{12}$;*

(2) *for $l = 0, 1, 2, \ldots$ successively solve the linear algebraic systems X_l and find $q^{(l)}(0)$, $g_{ks}^{(l-k+2)}(0)$, $s = \overline{0, [(l+2)/2]}$, $k = \overline{\max(1, 2s), l+2}$;*

(3) *construct the function $q(x)$ by the formula*

$$q(x) = \sum_{l=0}^{\infty} q^{(l)}(x) \frac{x^l}{l!}.$$

5. Differential Operators with Locally Integrable Coefficients

We investigate here the IP for the non-self-adjoint differential operator (2.1) on the half-line with locally integrable analytic coefficients from the so-called generalized Weyl functions. To solve the IP we use connections with an IP for partial differential equations, and also use the Riemann–Fage formula [28] for the solution of the Cauchy problem for higher-order partial differential equations.

5.1. Distributions. Let us introduce the space of generalized functions (distributions) by analogy with [62]. Let D be the set of all integrable entire functions of exponential type on the real axis, with ordinary operations of addition and multiplication by complex numbers and with the following convergence: $z_k(\mu)$ is said to converge to $z(\mu)$ if the types σ_k of the functions $z_k(\mu)$ are bounded ($\sup \sigma_k < \infty$) and $\|z_k(\mu) - z(\mu)\|_{\mathcal{L}(-\infty,\infty)} \to 0$ as $k \to \infty$. The linear manifold D with such convergence is our space of test functions.

Definition 5.1. All additive, homogeneous and continuous functionals $\langle z(\mu), R \rangle$ defined on D are called generalized functions (GF). The set of GF is denoted by D'. The sequence of GF $R_k \in D'$ converges to $R \in D'$ if $\lim_{k\to\infty} \langle z(\mu), R_k \rangle = \langle z(\mu), R \rangle$ for any $z(\mu) \in D$. A GF $R \in D'$ is called regular if it is determined by the following formula:

$$\langle z(\mu), R \rangle = \int_{-\infty}^{\infty} z(\mu) R(\mu)\, d\mu, \qquad R(\mu) \in \mathcal{L}_\infty.$$

Definition 5.2. Let the function $f(t)$ be locally integrable for $t > 0$ (i.e., it is integrable on every finite segment $[0, \sigma]$). A GF $L_f(\mu) \in D'$ defined by the equality

$$\langle z(\mu), L_f(\mu) \rangle \stackrel{\text{def}}{=} \int_0^{\infty} f(t)\, dt \int_{-\infty}^{\infty} z(\mu) \exp(i\mu t)\, d\mu, \qquad z(\mu) \in D, \tag{5.1}$$

is called the generalized Fourier–Laplace transformation for the function $f(t)$.

Since $z(\mu) \in D$ in (5.1), $z(\mu) \in \mathcal{L}_2(-\infty, \infty)$. Therefore, by virtue of the Paley–Wiener theorem, the function

$$\int_{-\infty}^{\infty} z(\mu) \exp(i\mu t)\, dt$$

is continuous and finite. Consequently, the integral in (5.1) exists. We note that $f(t) \in \mathcal{L}(0, \infty)$ implies

$$\langle z(\mu), L_f(\mu) \rangle = \int_{-\infty}^{\infty} z(\mu)\, d\mu \int_0^{\infty} f(t) \exp(i\mu t)\, dt.$$

Consequently, in this case, $L_f(\mu)$ is a regular GF and coincides with the ordinary Fourier–Laplace transformation for the function $f(t)$.

Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos\mu x}{\mu^2}\exp(i\mu t)\,d\mu=\begin{cases}x-t, & t<x,\\ 0, & t>x,\end{cases}$$

the following inversion formula takes place:

$$\int_0^x (x-t)f(t)\,dt=\left\langle\frac{1}{\pi}\cdot\frac{1-\cos\mu x}{\mu^2},L_f(\mu)\right\rangle. \tag{5.2}$$

5.2. Solution of the IP for third-order DO's. In this section, in order to simplify calculations, we give formulation and solution of the IP for DO's of the third order. The general case of arbitrary order operators will be described, in brief form, in Sec. 5.3.

Let us consider the DE

$$ly\equiv y'''+p_1(x)y'+p_0(x)y=\lambda y=(i\mu)^3y,\qquad x>0. \tag{5.3}$$

Denote $q_0(x)=-p_1(x)$, $q_1(x)=p_0(x)-p_1'(x)$, $B=\left\{x:|\arg x|<\frac{\pi}{6}\right\}$, $R_k=\exp\left(2(k-1)\frac{\pi i}{3}\right)$, $k=\overline{1,3}$, and assume that the functions $q_\nu(x)$ are regular for $x\in B$, $x=0$, and continuous in $\overline{B}$. Let us consider the following integral equation:

$$\begin{aligned}Q(x,s)=Q_1(x,s)&+\sum_{\nu=0}^{1}\left(-\frac{1}{3}\int_0^s\frac{(s-u)^\nu}{\nu!}du\int_0^x q_\nu(t)Q(t,u)\,dt\right.\\&+\sum_{k=2}^{3}\frac{R_k^{2-\nu}}{3(1-R_k)}\int_0^s du\int_0^{s-u}\frac{(s-u-\xi)^\nu}{\nu!}\left(q_\nu\left(\frac{\xi}{1-R_k}+x\right)Q\left(\frac{\xi}{1-R_k}+x,u\right)\right.\\&\left.\left.-q_\nu\left(\frac{\xi}{1-R_k}\right)Q\left(\frac{\xi}{1-R_k},u\right)\right)d\xi\right),\end{aligned} \tag{5.4}$$

where

$$Q_1(x,s)=\sum_{\nu=0}^{1}\left(-\frac{1}{3}\cdot\frac{s^\nu}{\nu!}\int_0^x q_\nu(t)\,dt+\sum_{k=2}^{3}\frac{R_k^{2-\nu}}{3(1-R_k)}\int_0^s\frac{(s-\xi)^\nu}{\nu!}\left(q_\nu\left(\frac{\xi}{1-R_k}+x\right)-q_\nu\left(\frac{\xi}{1-R_k}\right)\right)d\xi\right). \tag{5.5}$$

By the method of successive approximations it is easy to show that in the domain $s\geq 0$, $x\in\overline{B}$, the integral equation (5.4) has a unique solution $Q(x,s)$. The function $Q(x,s)$ is continuous and, for any fixed $s\geq 0$, is regular with respect to $x\in B$.

Moreover, if the functions $q_\nu(x)$ are regular for $|x|<\delta$, then the function $Q(x,s)$ is regular in the domain $\mathcal{F}_\delta=\left\{(x,s):|x|<\delta,\ |s|<\sqrt{3}\delta,\ \left|x+s(1-R_k)^{-1}\right|<\delta,\ k=2,3\right\}$.

Let the function $Q(x,s)$ be a solution of (5.4). Denote $u(x,t)=Q(x,t-x)$, $0\leq x\leq t<\infty$; $u(x,t)=0$, $t<x$, and consider the GF

$$\Phi(x,\mu)=\exp(i\mu x)+L_u(\mu), \tag{5.6}$$

i.e.,

$$\langle z(\mu),\Phi(x,\mu)\rangle=\int_{-\infty}^{\infty}z(\mu)\exp(i\mu x)\,d\mu+\int_x^{\infty}u(x,t)\,dt\int_{-\infty}^{\infty}z(\mu)\exp(i\mu t)\,d\mu,\qquad z(\mu)\in D. \tag{5.7}$$

Put also

$$\langle z(\mu), (i\mu)^3\Phi(x,\mu)\rangle = \langle (i\mu)^3 z(\mu), \Phi(x,\mu)\rangle,$$

$$\langle z(\mu), \Phi^{(j)}(x,\mu)\rangle = \frac{d^j}{dx^j}\langle z(\mu), \Phi(x,\mu)\rangle, \qquad j=\overline{1,3},$$

for $z(\mu)\in D$, $\mu^3 z(\mu)\in \mathcal{L}(-\infty,\infty)$.

Theorem 5.1. *The following relations hold:*

$$l\Phi(x,\mu) - (i\mu)^3\Phi(x,\mu) = 0, \qquad \Phi(0,\mu)=1.$$

Proof. Equality (5.4) can be transformed to the following form:

$$Q(x,s) = \sum_{\nu=0}^{1}\sum_{k=2}^{3}\frac{R_K^{2-\nu}}{3}\left(\int\limits_0^{x+\frac{s}{1-R_k}} \frac{\big(s-(1-R_k)(\eta-x)\big)^\nu}{\nu!} q_\nu(\eta)\,d\eta\right.$$
$$-\int\limits_0^{\frac{s}{1-R_k}} \frac{\big(s-(1-R_k)\eta\big)^\nu}{\nu!} q_\nu(\eta)\,d\eta + \int\limits_0^s du \int\limits_0^{x+\frac{s-u}{1-R_k}} \frac{\big((s-u)-(1-R_k)(\eta-x)\big)^\nu}{\nu!} q_\nu(\eta)Q(\eta,u)\,d\eta \tag{5.8}$$
$$\left.-\int\limits_0^s du \int\limits_0^{\frac{s-u}{1-R_k}} \frac{\big((s-u)-(1-R_k)\eta\big)^\nu}{\nu!} q_\nu(\eta)Q(\eta,u)\,d\eta\right).$$

Indeed, first we can transform $Q_1(x,s)$. Via the changes of variables in the right-hand part of (5.5), $\eta = \xi(1-R_k)^{-1}+x$ and $\eta=\xi(1-R_k)^{-1}$ respectively, we obtain

$$Q_1(x,s) = \sum_{\nu=0}^{1}\left(-\frac{1}{3}\frac{s^\nu}{\nu!}\int\limits_0^x q_\nu(t)\,dt + \sum_{k=2}^{3}\frac{R_k^{2-\nu}}{3}\int\limits_x^{x+\frac{s}{1-R_k}} \frac{\big(s-(1-R_k)(\eta-x)\big)^\nu}{\nu!} q_\nu(\eta)\,d\eta\right.$$
$$\left.-\int\limits_0^{\frac{s}{1-R_k}} \frac{\big(s-(1-R_k)\eta\big)^\nu}{\nu!} q_\nu(\eta)\,d\eta\right). \tag{5.9}$$

Using the regularity of the integrand, we can make the change

$$\int\limits_x^{x+\frac{s}{1-R_k}} = \int\limits_0^{x+\frac{s}{1-R_k}} - \int\limits_0^x .$$

Since

$$\sum_{k=1}^{3} R_k^j = 0, \qquad j=1,2,$$

the integrals $\int\limits_0^x$ can be canceled, and (5.9) has the form

$$Q_1(x,s) = \sum_{\nu=0}^{1}\sum_{k=2}^{3}\frac{R_k^{2-\nu}}{3}\left(\int\limits_0^{x+\frac{s}{1-R_k}} \frac{\big(s-(1-R_k)(\eta-x)\big)^\nu}{\nu!} q_\nu(\eta)\,d\eta - \int\limits_0^{\frac{s}{1-R_k}} \frac{\big(s-(1-R_k)\eta\big)^\nu}{\nu!} q_\nu(\eta)\,d\eta\right).$$

The rest terms in the right-hand part of (5.4) can be transformed in analogous way.

Differentiating (5.8) we obtain

$$Q_x(x,s) = \sum_{k=2}^{3} \left(\frac{R_k^2}{3} q_0 \left(x + \frac{s}{1-R_k} \right) + \frac{R_k(1-R_k)}{3} \int\limits_0^{x+\frac{s}{1-R_k}} q_1(\eta)\, d\eta \right.$$

$$\left. + \frac{R_k^2}{3} \int\limits_0^s q_0 \left(x + \frac{s-u}{1-R_k} \right) Q \left(x + \frac{s-u}{1-R_k}, u \right) du + \frac{R_k(1-R_k)}{3} \int\limits_0^s du \int\limits_0^{x+\frac{s-u}{1-R_k}} q_1(\eta) Q(\eta, u)\, d\eta \right), \quad (5.10)$$

$$Q_s(x,s) = \sum_{k=2}^{3} \left(\frac{R_k^2}{3(1-R_k)} q_0 \left(x + \frac{s}{1-R_k} \right) + \frac{R_k}{3} \int\limits_0^{x+\frac{s}{1-R_k}} q_1(\eta)\, d\eta - \frac{R_k^2}{3(1-R_k)} q_0 \left(\frac{s}{1-R_k} \right) \right.$$

$$- \frac{R_k}{3} \int\limits_0^{\frac{s}{1-R_k}} q_1(\eta)\, d\eta + \frac{R_k^2}{3(1-R_k)} \int\limits_0^s q_0 \left(x + \frac{s-u}{1-R_k} \right) Q \left(x + \frac{s-u}{1-R_k}, u \right) du$$

$$+ \frac{r_k^3}{3} \int\limits_0^s du \int\limits_0^{x+\frac{s-u}{1-R_k}} q_1(\eta) Q(\eta, u)\, d\eta - \frac{R_k^2}{3(1-R_k)} \int\limits_0^s q_0 \left(\frac{s-u}{1-R_k} \right) Q \left(\frac{s-u}{1-R_k}, u \right) du$$

$$\left. - \frac{r_k^3}{3} \int\limits_0^s du \int\limits_0^{\frac{s-u}{1-R_k}} q_1(\eta) Q(\eta, u)\, d\eta + \frac{R_k^2}{3} \int\limits_0^x q_0(\eta) Q(\eta, s)\, d\eta \right), \quad (5.11)$$

$$Q_{xx}(x,s) = \sum_{k=2}^{3} \left(\frac{R_k^2}{3} q_0' \left(x + \frac{s}{1-R_k} \right) + \frac{R_k(1-R_k)}{3} q_1 \left(x + \frac{s}{1-R_k} \right) \right.$$

$$+ \frac{R_k^2}{3} \int\limits_0^s \left(q_0' \left(x + \frac{s-u}{1-R_k} \right) Q \left(x + \frac{s-u}{1-R_k}, u \right) + q_0 \left(x + \frac{s-u}{1-R_k} \right) Q_x \left(x + \frac{s-u}{1-R_k}, u \right) \right) du$$

$$\left. + \frac{R_k(1-R_k)}{3} \int\limits_0^s q_1 \left(x + \frac{s-u}{1-R_k} \right) Q \left(x + \frac{s-u}{1-R_k}, u \right) du \right), \quad (5.12)$$

$$Q_{xs}(x,s) = \sum_{k=2}^{3} \left(\frac{R_k^2}{3(1-R_k)} q_0' \left(x + \frac{s}{1-R_k} \right) + \frac{R_k}{3} q_1 \left(x + \frac{s}{1-R_k} \right) \right.$$

$$+ \frac{R_k^2}{3(1-R_k)} \int\limits_0^s \left(q_0' \left(x + \frac{s-u}{1-R_k} \right) Q \left(x + \frac{s-u}{1-R_k}, u \right) + q_0 \left(x + \frac{s-u}{1-R_k} \right) Q_x \left(x + \frac{s-u}{1-R_k}, u \right) \right) du$$

$$\left. + \frac{R_k}{3} \int\limits_0^s q_1 \left(x + \frac{s-u}{1-R_k} \right) Q \left(x + \frac{s-u}{1-R_k}, u \right) du + \frac{R_k^2}{3} q_0(x) Q(x,s) \right) \quad (5.13)$$

From this we obtain

$$\frac{\partial^3 Q(x,s)}{\partial x^3} - 3\frac{\partial^3 Q(x,s)}{\partial x^2 \partial s} + 3\frac{\partial^3 Q(x,s)}{\partial x \partial s^2} = -p_0(x)Q(x,s) + p_1(x)\left(\frac{\partial Q(x,s)}{\partial s} - \frac{\partial Q(x,s)}{\partial x}\right). \tag{5.14}$$

Moreover, it follows from (5.8) and (5.11) that

$$\begin{cases} Q(0,s) = 0, \qquad Q(x,0) = \dfrac{1}{3}\displaystyle\int_0^x p_1(t)\,dt, \\ Q_s(x,s)_{|s=0} = \dfrac{1}{3}\left(p_1(x) - p_1(0) - \displaystyle\int_0^x \big(p_0(t) - p_1(t)Q(t,0)\big)\,dt\right). \end{cases} \tag{5.15}$$

Since $u(x,t) = Q(x, t-x)$, $0 \le x \le t$, (5.14) and (5.15) imply

$$\frac{\partial^3 u(x,t)}{\partial t^3} + \frac{\partial^3 u(x,t)}{\partial x^3} + p_1(x)\frac{\partial u(x,t)}{\partial x} + p_0(x)u(x,t) = 0, \tag{5.16}$$

$$u(0,t) = 0, \tag{5.17}$$

$$\begin{cases} u(x,x) = \dfrac{1}{3}\displaystyle\int_0^x p_1(t)\,dt, \\ u_x(x,t)_{|t=x} = \dfrac{1}{3}\left(p_1(0) + \displaystyle\int_0^x \big(p_0(\xi) - p_1(\xi)u(\xi,\xi)\big)\,d\xi\right). \end{cases} \tag{5.18}$$

Consequently,

$$3\frac{d}{dx}u(x,x) = p_1(x), \qquad 3\frac{d}{dx}\left(u_x(x,t)_{|t=x}\right) + p_1(x)u(x,x) = p_0(x). \tag{5.19}$$

Further, using (5.7), we calculate

$$\langle z(\mu), l\Phi(x,\mu)\rangle = \int_{-\infty}^{\infty} z(\mu)\Bigg((i\mu)^3 - (i\mu)^2 u(x,x) + i\mu\left(p_1(x) - 2\frac{d}{dx}u(x,x) - u_x(x,t)_{|t=x}\right)$$

$$+\left(p_0(x) - p_1(x)u(x,x) - \frac{d^2}{dx^2}u(x,x) - \frac{d}{dx}\left(u_x(x,t)_{|t=x}\right) - u_{xx}(x,t)_{|t=x}\right)\Bigg)\exp(i\mu x)\,d\mu$$

$$+\int_x^{\infty}\left(\frac{\partial^3 u(x,t)}{\partial x^3} + p_1(x)\frac{\partial u(x,t)}{\partial x} + p_0(x)u(x,t)\right)dt\int_{-\infty}^{\infty} z(\mu)\exp(i\mu t)\,d\mu.$$

On the other hand, the integration by parts gives

$$\langle z(\mu), -(i\mu)^3\Phi(x,\mu)\rangle = \int_{-\infty}^{\infty}(-i\mu)^3 z(\mu)\exp(i\mu x)\,d\mu + \int_x^{\infty} u(x,t)\,dt\int_{-\infty}^{\infty}(-i\mu)^3 z(\mu)\exp(i\mu t)\,d\mu$$

$$= \int_{-\infty}^{\infty} z(\mu)\Big(-(i\mu)^3 + (i\mu)^2 u(x,x) - (i\mu)u)t(x,t)_{|t=x} + u_{tt}(x,t)_{|t=x}\Big)\exp(i\mu x)\,d\mu \tag{5.21}$$

$$+\int_x^{\infty}\frac{\partial^3 u(x,t)}{\partial t^3}\,dt\int_{-\infty}^{\infty} z(\mu)\exp(i\mu t)\,d\mu.$$

Since

$$u_x(x,t)_{|t=x} + u_t(x,t)_{|t=x} = \frac{d}{dx}u(x,x),$$

$$\frac{d^2}{dx^2}u(x,x) + u_{xx}(x,t)_{|t=x} - u_{tt}(x,t)_{|t=x} = 2\frac{d}{dx}\Big(u_x(x,t)_{|t=x}\Big),$$

from (5.16), (5.19), (5.20), and (5.21) it follows that

$$\langle z(\mu), l\Phi(x,\mu) - (i\mu)^3\Phi(x,\mu)\rangle$$

$$= \int_{-\infty}^{\infty} z(\mu)\left((i\mu)\left(p_1(x) - 3\frac{d}{dx}u(x,x)\right) + \left(p_0(x) - 3\frac{d}{dx}\big(u_x(x,t)_{|t=x}\big) - p_1(x)u(x,x)\right)\right)\exp(i\mu x)\,d\mu$$

$$+ \int_x^{\infty}\left(\frac{\partial^3 u(x,t)}{\partial t^3} + \frac{\partial^3 u(x,t)}{\partial x^3} + p_1(x)\frac{\partial u(x,t)}{\partial x} + p_0(x)u(x,t)\right)dt\int_{-\infty}^{\infty} z(\mu)\exp(i\mu t)\,d\mu = 0.$$

From (5.7) for $x = 0$ and (5.17) we obtain

$$\langle z(\mu), \Phi(0,\mu)\rangle = \int_{-\infty}^{\infty} z(\mu)\,d\mu,$$

i.e., $\Phi(0,\mu) = 1$. Theorem 5.1 is proved.

Definition 5.3. The GF $\Phi(x,\mu)$ is called the Weyl generalized solution of the DE (5.3), and the functions $\mathfrak{M}_\nu(\mu) = \Phi^{(\nu)}(0,\mu)$, $\nu = 1,2$, are called the Weyl generalized functions (WGF).

Note that if $\mathrm{P}_\infty \stackrel{\text{def}}{=} \int_0^\infty r(y)\,dy < \infty$, $r(y) \stackrel{\text{def}}{=} \max_\nu \sup_{\substack{\operatorname{Re} x = y\\ y\in B}} |q_\nu(x)|$, then $|u(x,t)| < C\exp(Ct)$, and the function

$$\Phi(x,\mu) = \exp(i\mu x) + \int_x^\infty u(x,t)\exp(i\mu t)\,dt, \qquad \arg\mu \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right),$$

is the ordinary Weyl solution.

The IP for the DE (5.3) can be formulated as follows:

Problem 5.1. Given the WGF's $\{\mathfrak{M}_\nu(\mu)\}_{\nu=1,2}$, construct the functions $\{p_k(x)\}_{k=0,1}$.

For this IP let us prove the uniqueness theorem.

Theorem 5.2. *If $\mathfrak{M}_\nu(\mu) = \widetilde{\mathfrak{M}}_\nu(\mu)$, $\nu = 1,2$, then $p_k(x) = \tilde{p}_k(x)$, $x \geq 0$, $k = 0,1$.*

Proof. We denote

$$h_\nu(t) = \frac{\partial^\nu}{\partial x^\nu}u(x,t)_{|x=0}, \qquad \nu = 1,2, \quad t \geq 0.$$

Taking into account (5.18), from (5.7) we deduce that

$$\mathfrak{M}_1(\mu) = (i\mu) + L_{h_1}(\mu), \qquad \mathfrak{M}_2(\mu) = (i\mu)^2 - 2h_1(0) + L_{h_2}(\mu).$$

Using the inversion formula (5.2), we calculate

$$\begin{cases} h_1(t) = \dfrac{d^2}{dt^2}\left\langle \dfrac{1}{\pi}\cdot\dfrac{1-\cos\mu t}{\mu^2}, \mathfrak{M}_1(\mu) - i\mu\right\rangle, \\[2ex] h_2(t) = \dfrac{d^2}{dt^2}\left\langle \dfrac{1}{\pi}\cdot\dfrac{1-\cos\mu t}{\mu^2}, \mathfrak{M}_2(\mu) - (i\mu)^2 + 2h_1(0)\right\rangle. \end{cases} \tag{5.22}$$

If the conditions of the theorem are fulfilled, we obtain from (5.22) that $h_\nu(t) \equiv \widetilde{h}_\nu(t)$, $\nu = 1,2$. But (5.16) and (5.17) imply

$$\frac{\partial^3 \widehat{u}(x,t)}{\partial t^3} + \frac{\partial^3 \widehat{u}(x,t)}{\partial x^3} + p_1(x)\frac{\partial \widehat{u}(x,t)}{\partial x} + p_0(x)\widehat{u}(x,t) + \widehat{p}_1(x)\frac{\partial \widetilde{u}(x,t)}{\partial x} + \widehat{p}_0(x)\widetilde{u}(x,t) = 0,$$
$$\left.\frac{\partial^\nu \widehat{u}(x,t)}{\partial x^\nu}\right|_{x=0} = 0, \quad \nu = 0,1,2.$$

For this Cauchy problem we use the Riemann–Fage formula (see [28]) in the vicinity of the point $x = t = 0$, and obtain:

$$\widehat{u}(x,t) = -\int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} \Big(\sum_{\nu=0}^{1} \widehat{p}_\nu(\xi_3)\widetilde{u}_\nu(\xi_3, -t + x + (R_2 - R_1)\xi_1 + (R_3 - R_2)\xi_2 - R_3\xi_3)\Big)$$
$$\times V\big(0,0,\xi_3, x-\xi_1, \xi_1-\xi_2, \xi_2\big)\, d\xi_3,$$

where V is the Riemann–Fage function, $u_\nu(x,t) = \dfrac{\partial^\nu}{\partial x^\nu} u(x,t)$. By changing the order of integration, we obtain

$$\widehat{u}(x,t) = \int_0^x \sum_{\nu=0}^{1} \widehat{p}_\nu(\xi) B_\nu(x,t,\xi)\, d\xi, \tag{5.23}$$

where

$$B_\nu(x,t,\xi) = -\int_\xi^x d\xi_1 \int_\xi^{\xi_1} \widetilde{u}_\nu(\xi, -t + x + (R_2 - R_1)\xi_1 + (R_3 - R_2)\xi_2 - R_3\xi) V\big(0,0,\xi, x-\xi_1, \xi_1-\xi_2, \xi_2\big)\, d\xi_2,$$

$\nu = 0,1$. Since

$$\left.\frac{\partial^{i+j} B_\nu(x,t,\xi)}{\partial x^i \partial t^j}\right|_{\xi=x} = 0, \qquad i,j = 0,1,$$

it follows from (5.23) that

$$\frac{\partial^{i+j} \widehat{u}(x,t)}{\partial x^i \partial t^j} = \int_0^x \sum_{\nu=0}^{1} \widehat{p}_\nu(\xi) \frac{\partial^{i+j} B_\nu(x,t,\xi)}{\partial x^i \partial t^j}\, d\xi, \qquad i,j = 0,1,2. \tag{5.24}$$

Using (5.19), we obtain

$$\widehat{p}_1(x) = 3\frac{d}{dx}\widehat{u}(x,x),$$
$$\widehat{p}_0(x) = 3\frac{d}{dx}\big(\widehat{u}_x(x,t)_{|t=x}\big) + p_1(x)\widehat{u}(x,x) + 3\widetilde{u}(x,x)\frac{d}{dx}\widehat{u}(x,x).$$

From this and from (5.24) follows that

$$\widehat{p}_k(x) = \int_0^x \sum_{\nu=0}^{1} A_{k\nu}(x,\xi)\widehat{p}_\nu(\xi)\, d\xi, \qquad k = 0,1,$$

where

$$A_{1\nu} = 3\frac{d}{dx} B_\nu(x,t,\xi),$$
$$A_{0\nu} = 3\frac{d}{dx}\left(\left.\frac{\partial B_\nu(x,t,\xi)}{\partial x}\right|_{t=x}\right) + p_1(x)B_\nu(x, x\xi) + 3\widetilde{u}(x,x)\frac{d}{dx}B_\nu(x,x,\xi).$$

Consequently, $\widehat{p}_k(x) = 0$, $k = 0, 1$. Theorem 5.2 is proved.

Solution of the IP. Let the WGF $\mathfrak{M}_\nu(\mu)$, $\nu = 1, 2$, of the DE (5.3) be given. We construct the functions $h_\nu(t)$, $\nu = 1, 2$, by (5.22). Since $u(x,t) = Q(x, t - x)$, $Q(0, s) = 0$, we have

$$h_1(s) = Q_x(0, s), \qquad h_2(s) = Q_{xx}(0, s) - 2Q_{xs}(0, s). \tag{5.25}$$

Note that the functions $h_\nu(s)$ are regular at the point $s = 0$. More precisely, if the functions $q_k(x)$ are regular for $|x| < \delta$, then the functions $h_\nu(s)$ are regular for $|s| < \sqrt{3}\delta$.

Denote $q_0^1(x) = q_0'(x)$, $Q^1(x,s) = Q_x(x,s)$. Using (5.25), (5.10), (5.12), and (5.13), we calculate

$$\begin{aligned} h_{2-\nu}^{(\nu)} = \frac{1}{3}\sum_{k=2}^{3}\Bigg(&\frac{R_k^{\nu+1}}{1-R_k} q_0^1\left(\frac{s}{1-R_k}\right) + R_k^\nu q_1\left(\frac{s}{1-R_k}\right) \\ &+ \frac{R_k^{\nu+1}}{1-R_k}\int_0^s \left(q_0^1\left(\frac{s-u}{1-R_k}\right) Q\left(\frac{s-u}{1-R_k}, u\right) + q_0\left(\frac{s-u}{1-R_k}\right) Q^1\left(\frac{s-u}{1-R_k}, u\right)\right) du \\ &+ R_k^\nu \int_0^s q_1\left(\frac{s-u}{1-R_k}\right) Q\left(\frac{s-u}{1-R_k}, u\right) du\Bigg), \qquad \nu = 0, 1. \end{aligned} \tag{5.26}$$

Having solved (5.26) with respect to the functions $q_0^1(x)$ and $q_1(x)$, we obtain

$$q_1(x) - \sum_{k=1}^{3} R_k^2 \int_0^x \Big(q_0^1(u) Q\big(u, (1-R_k)(x-u)\big) + q_0(u) Q^1\big(u, (1-R_k)(x-u)\big)\Big) du = I_1(x), \tag{5.27}$$

$$q_0^1(x) + 3\sum_{k=2}^{3} R_k^2 \int_0^x q_1(u) Q\big(u, (1-R_k)(x-u)\big)\, du = I_2(x), \tag{5.28}$$

where

$$I_1(x) = 3\sum_{k=2}^{3} (R_k^2 - R_k)^{-1}\Big(R_k h_1'\big((1-R_k)x\big) - h_2\big((1-R_k)x\big)\Big),$$

$$I_2(x) = -3\sum_{k=2}^{3} R_k^2\Big(R_k h_1'\big((1-R_k)x\big) - h_2\big((1-R_k)x\big)\Big).$$

Since $q_0(0) = -p_1(0) = -3h_1(0)$, we obtain

$$q_0(x) = -3h_1(0) + \int_0^x q_0^1(u)\, du. \tag{5.29}$$

Equality (5.10) can then be rewritten as follows:

$$\begin{aligned} Q^1(x,s) = \frac{1}{3}\sum_{k=2}^{3}\Bigg(&R_k^2\left(-3h_1(0) + \int_0^{x+\frac{s}{1-R_k}} q_0^1(u)\, du\right) + R_k(1-R_k)\int_0^{x+\frac{s}{1-R_k}} q_1(\eta)\, d\eta \\ &+ R_k^2 \int_0^s q_0\left(x + \frac{s-u}{1-R_k}\right) Q\left(x + \frac{s-u}{1-R_k}, u\right) du + R_k(1-R_k)\int_0^s du \int_0^{x+\frac{s-u}{1-R_k}} q_1(\eta) Q(\eta, u)\, d\eta\Bigg). \end{aligned} \tag{5.30}$$

Let us consider the system of nonlinear integral equations (5.8) and (5.27)–(5.30) with respect to the functions $q_0(x)$, $q_0^1(x)$, $q_1(x)$, $Q(x,s)$, and $Q^1(x,s)$. We solve this system by the method of successive approximations in sufficiently small neighbourhood of the point $x = s = 0$. Then we obtain the following obvious result.

Lemma 5.1. *Let, for a fixed $\delta > 0$, the functions $h_\nu(s)$, $\nu = 1, 2$, be regular for $|s| < \sqrt{3}\delta$. Then there exist unique functions $q_0(x)$, $q_0^1(x)$, and $q_1(x)$, which are regular for $|x| < \delta$, and unique functions $Q(x,s)$ and $Q^1(x,s)$, which are regular in $\mathcal{F}_\delta$, where all the functions satisfy the system* (5.8), (5.27)–(5.30), *i.e., the system* (5.8), (5.27)–(5.30) *is uniquely solvable in a neighborhood of the point $x = s = 0$. The solution of the system* (5.8), (5.27)–(5.30) *can be found by the method of successive approximations, and $q_0^1(x) = q_0'(x)$, $Q^1(x,s) = Q_x(x,s)$.*

Thus we can construct the solution of the IP via the following algorithm.

Algorithm 5.1. The WGF $\mathfrak{M}_\nu(\mu)$, $\nu = 1, 2$, of the DE (5.3) are given.

(1) By formulas (5.22) we construct the functions $h_\nu(t)$, $\nu = 1, 2$.
(2) Having solved the system (5.8), (5.27)–(5.30), we find the functions $q_0(x)$, $q_0^1(x)$, $q_1(x)$, $Q(x,s)$, and $Q^1(x,s)$.
(3) We construct the functions $p_1(x) = -q_0(x)$, $p_0(x) = q_1(x) = q_0^1(x)$, $|x| < \delta$.
(4) By means of analytic continuation we obtain the functions $p_0(x)$ and $p_1(x)$ for $x > 0$.

5.3. Similar results are also valid for the DE of an arbitrary order

$$ly \equiv y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)} = \lambda y = (i\mu)^n y, \qquad x > 0. \tag{5.31}$$

In this case,

$$q_\nu(x) = \sum_{j=0}^{\nu} (-1)^{(j+1)} C_{n-2-j}^{\nu-j} p_{n-2-j}^{(\nu-j)}(x), \qquad \nu = \overline{0, n-2},$$

$$B = \left\{x : |\arg x| < \frac{\pi}{2} - \frac{\pi}{n}\right\}, \qquad R_k = \exp\left(\frac{2(k-1)\pi i}{n}\right), \quad k = \overline{1,n}.$$

The function $Q(x,s)$ can be determined by the integral equation

$$Q(x,s) = Q_1(x,s) + \sum_{\nu=0}^{n-2}\left(-\frac{1}{n}\int_0^s \frac{(s-u)^\nu}{\nu!}\,du \int_0^x q_\nu(t) Q(t,u)\,dt\right.$$
$$+\sum_{k=2}^{n} \frac{R_k^{n-1-\nu}}{n(1-R_k)} \int_0^s du \int_0^{s-u} \frac{(s-u-\xi)^\nu}{\nu!} \left(q_\nu\left(\frac{\xi}{1-R_k} + x\right) Q\left(\frac{\xi}{1-R_k} + x, u\right)\right.$$
$$\left.\left. - q_\nu\left(\frac{\xi}{1-R_k}\right) Q\left(\frac{\xi}{1-R_k}, u\right)\right) d\xi\right),$$

where

$$Q_1(x,s)=\sum_{\nu=0}^{n-2}\left(-\frac{1}{n}\cdot\frac{s^\nu}{\nu!}\int\limits_0^x q_\nu(t)\,dt\right.$$

$$\left.+\sum_{k=2}^{n}\frac{R_k^{n-1-\nu}}{n(1-R_k)}\int\limits_0^s\frac{(s-\xi)^\nu}{\nu!}\left(q_\nu\left(\frac{\xi}{1-R_k}+x\right)-q_\nu\left(\frac{\xi}{1-R_k}\right)\right)d\xi\right).$$

The function $u(x,t)$ defined by the equalities $u(x,t)=Q(x,t-x)$, $1\le x\le t<\infty$, $u(x,t)=0$, $t<x$, satisfies the following relations:

$$\frac{\partial^n u(x,t)}{\partial x^n}+\sum_{\nu=0}^{n-2}p_\nu(x)\frac{\partial^\nu u(x,t)}{\partial x^\nu}=(-1)^n\frac{\partial^n u(x,t)}{\partial t^n},\qquad u(0,t)=0,$$

$$\sum_{m=1}^{n-j}C_{n-m}^j\frac{d^{n-m-j}}{dx^{n-m-j}}u_{m-1}(x,x)+\sum_{\nu=j+1}^{n-2}p_\nu(x)\sum_{m=1}^{\nu-j}C_{\nu-m}^j\frac{d^{\nu-m-j}}{dx^{\nu-m-j}}u_{m-1}(x,x)$$

$$+(-1)^{n-j}\left.\frac{\partial^{n-j-1}u(x,t)}{\partial t^{n-j-1}}\right|_{t=x}=p_j(x),\qquad j=\overline{0,n-2},$$

where

$$u_\nu(x,t)=\frac{\partial^\nu}{\partial x^\nu}u(x,t),\qquad c_k^j=\frac{k!}{j!(k-j)!}.$$

The Weyl generalized solution $\Phi(x,\mu)$ is defined by the formula $\Phi(x,\mu)=\exp(i\mu x)+L_u(\mu)$, and $l\Phi(x,\mu)-(i\mu)^n\Phi(x,\mu)=0$, $\Phi(0,\mu)=1$. The functions $\mathfrak{M}_\nu(\mu)=\Phi^{(\nu)}(0,\mu)$, $\nu=\overline{1,n-1}$, are called the WGF of (5.31).

The IP is formulated here as follows: find the coefficients $\{p_k(x)\}_{k=\overline{0,n-2}}$ of (5.31) via the given WGF $\{\mathfrak{M}_\nu(\mu)\}_{\nu=\overline{1,n-1}}$. Solution of this IP can be obtained in exactly the same way as in the case $n=3$.

PART 2

HIGHER-ORDER DIFFERENTIAL OPERATORS WITH A SINGULARITY

6. Differential Operators on the Half-Line

6.1. Fundamental systems of solutions. Let us consider the DE

$$ly\equiv y^{(n)}+\sum_{j=0}^{n-2}\left(\frac{\nu_j}{x^{n-j}}+q_j(x)\right)y^{(j)}=\lambda y\tag{6.1}$$

on the half-line $x>0$. Let $\mu_1,\dots,\mu_n$ be the roots of the characteristic polynomial

$$\delta(\mu)=\sum_{j=0}^{n}\nu_j\prod_{k=0}^{j-1}(\mu-k),\qquad \nu_n=1,\quad \nu_{n-1}=0.$$

It is clear that $\mu_1+\dots+\mu_n=n(n-1)/2$. For definiteness, we assume that $\mu_k-\mu_j\ne sn$ ($s=0,\pm1,\pm2,\dots$), $\operatorname{Re}\mu_1<\dots<\operatorname{Re}\mu_n$, $\mu_k\ne 0,1,\dots,n-3$ (other cases require minor modifications). Let

$\theta_n = n-1-\mathrm{Re}(\mu_n-\mu_1)$. Denote $q_{0j}(x) = q_j(x)$ for $x \geq 1$, and $q_{0x}(x) = q_j(x)x^{\min(\theta_n - j, 0)}$ for $x \leq 1$, and assume that $q_{0j}(x) \in \mathcal{L}(0,\infty)$, $j = \overline{0, n-2}$.

In this section, we construct special FSS's for the DE (6.1) and use them to investigate the IP. The presence of a singularity in the DE introduces essential qualitative modifications in the investigation of the operator. Basic difficulties arise when $n > 2$. In the construction of the special FSS's for (6.1) the elementary solutions of the simplest equation are no longer exponentials, but functions that are generalizations of the Hankel solutions of the Bessel equation. An important and technically difficult problem is the determination of the asymptotic behaviour of the Stokes multipliers for the constructed FSS's. Using properties of the FSS's and the Stokes multipliers, we introduce and study the WS's and the WM for (6.1), and investigate the IP: to construct the operator l from its WM.

We mention that DE's with singularities arise in various areas of mathematics as well as in applications. In addition, various DE's with a turning point, for example, the equation

$$z^{(n)}(t) = \lambda r(t)z(t), \quad t > 0; \qquad r(t) \sim \alpha t^{\gamma}, \quad t \to +0, \quad \gamma > 0,$$

and other more general equations, can be reduced to (6.1). We also note that for $n = 2$ IP's for operators with a singularity have been studied by several authors (see, for example, [18, 32, 82]).

First of all, we consider the DE

$$l_0 y \equiv y^{(n)} + \sum_{j=0}^{n-2} \frac{\nu_j}{x^{n-j}} y^{(j)} = y. \tag{6.2}$$

Let $x = r\exp(i\varphi)$, $r > 0$, $\varphi \in (-\pi, \pi]$, $x^{\mu} = \exp\big(\mu(\ln r + i\varphi)\big)$, and Π_- be the x-plane with a cut along the semiaxis $x \leq 0$. Take numbers c_{j0}, $j = \overline{1,n}$, from the condition

$$\prod_{j=1}^{n} c_{j0} = \left(\det\left[\mu_j^{\nu-1}\right]_{j,\nu=\overline{1,n}}\right)^{-1}.$$

Then the functions

$$C_j(x) = x^{\mu_j} \sum_{k=1}^{\infty} c_{jk} x^{nk}, \qquad c_{jk} = c_{j0} \cdot \left(\prod_{s=1}^{k} \Delta(\mu_j + sn)\right)^{-1} \tag{6.3}$$

are solutions of (6.2), and $\det\left[C_j^{(\nu-1)}(x)\right]_{j,\nu=\overline{1,n}} \equiv 1$. Furthermore, the functions $C_j(x)$ are regular in Π_-.

Denote $\varepsilon_k = \exp\left(\dfrac{2\pi i(k-1)}{n}\right)$, $S_\nu = \left\{x : \arg x \in \left(\dfrac{\nu\pi}{n}, \dfrac{(\nu+1)\pi}{n}\right)\right\}$, $S_1^* = \overline{S}_{n-1}$, $S_k^* = \overline{S}_{n-2k+1} \cup \overline{S}_{n-2k+2}$, $k = \overline{2,n}$; $Q_k = \left\{x : \arg x \in \left[\max\left(-\pi, (-2k+2)\dfrac{\pi}{n}\right), \min\left(\pi, (2n-2k+2)\dfrac{\pi}{n}\right)\right]\right\}$, $k = \overline{1,n}$. For $x \in S_k^*$ there are solutions of (6.2) $e_k(x)$, $k = \overline{1,n}$, of the form $e_k^{(\nu-1)}(x) = \varepsilon_k^{\nu} \exp(\varepsilon_k x) z_{k\nu}(x)$, $\nu = \overline{0, n-1}$, where $z_{k\nu}(x)$ are solutions of the integral equations

$$z_{k\nu}(x) + 1 + \frac{1}{n} \int_x^{\infty} \left(\sum_{j=1}^{n} \varepsilon_j^{\nu+1} \varepsilon_k^{-\nu} \exp\left((\varepsilon_k - \varepsilon_j)(t-x)\right)\right) \left(\sum_{m=0}^{n-2} \nu_m \varepsilon_k^m t^{m-n} z_{km}(t)\right) dt$$

(here $\arg t = \arg x$, $|t| > |x|$). Using the FSS $\{C_j(x)\}_{j=\overline{1,n}}$ we can write

$$e_k(x) = \sum_{j=1}^{n} \beta_{kj}^0 C_j(x). \tag{6.4}$$

In particular, this gives the analytic continuation for $e_k(x)$ on Π_-.

Lemma 6.1. *The system* $\{e_k(x)\}_{k=\overline{1,n}}$, $x \in \Pi_-$, *is a FSS of* (6.2), *and*

$$\det\left[e_k^{(\nu-1)}(x)\right]_{k,\nu=\overline{1,n}} = \det\left[\varepsilon_k^{\nu-1}\right]_{k,\nu=\overline{1,n}}.$$

The asymptotics

$$e_k^{(\nu-1)}(x) = \varepsilon_k^{\nu-1}\exp(\varepsilon_k x)\big(1+O(x^{-1})\big), \qquad |x|\to\infty, \quad x\in Q_k, \tag{6.5}$$

are valid.

We observe that the asymptotics (6.5) holds in the sectors Q_k which are wider than the sectors S_k^*. Next we obtain connections between the Stokes multipliers β_{kj}^0.

Lemma 6.2.

$$\beta_{kj}^0 = \beta_{1j}^0\varepsilon_k^{\mu_j}, \qquad j,k=\overline{1,n}, \tag{6.6}$$

$$\prod_{j=1}^n \beta_{1j}^0 = \left(\det\left[\varepsilon_k^{\mu_j}\right]_{k,j=\overline{1,n}}\right)^{-1}\det\left[\varepsilon_k^{j-1}\right]_{k,j=\overline{1,n}} \neq 0. \tag{6.7}$$

Indeed, for $\arg x\in(-\pi,\pi-2\pi s/n)$, by virtue of (6.3) and (6.4), we have

$$e_k(\varepsilon^s x) = \sum_{j=1}^n \beta_{kj}^0(\varepsilon^s)^{\mu_j}C_j(x). \tag{6.8}$$

It is easily seen from the construction of the functions $e_k(x)$ that $e_1(\varepsilon^s x) = e_{k+1}(x)$. Substituting (6.4) and (6.8) in this equality and comparing the corresponding coefficients, we obtain (6.6). After this, (6.7) becomes obvious.

Now we consider the DE

$$l_0 y = \lambda y = \rho^n y, \qquad x>0. \tag{6.9}$$

It is evident that if $y(x)$ is a solution of (6.2), then $y(\rho x)$ satisfies (6.9). Define $C_j(x,\lambda)$ by

$$C_j(x,\lambda) = \rho^{-\mu_j}C_j(\rho x) = x^{\mu_j}\sum_{k=0}^\infty c_{jk}(\rho x)^{nk}.$$

The functions $C_j(x,\lambda)$ are entire in λ, and $\det\left[C_j^{(\nu-1)}(x,\lambda)\right]_{j,\nu=\overline{1,n}} \equiv 1$. From Lemmas 6.1 and 6.2 we get the following theorem.

Theorem 6.1. *In each sector* $S_{k_0} = \{\rho : \arg\rho \in (k_0\pi/n,\ (k_0+1)\pi/n\}$ *Eq.* (6.9) *has a FSS* $B_0 = \{y_k(x,\rho)\}_{k=\overline{1,n}}$ *such that* $y_k(x,\rho) = y_k(\rho x)$,

$$\left|y_k^{(\nu)}(x,\rho)(\rho R_k)^{-\nu}\exp(\rho R_k x) - 1\right| \le M_0\big(|r|x\big)^{-1}, \qquad \rho\in\overline{S}_{k_0}, \quad |\rho|x\ge 1, \quad \nu=\overline{0,n-1}, \tag{6.10}$$

$$\det\left[y_k^{(\nu-1)}(x,\rho)\right]_{k,\nu=\overline{1,n}} \equiv \rho^{\frac{n(n-1)}{2}}\Omega, \qquad \Omega \stackrel{\text{def}}{=} \det\left[R_k^{\nu-1}\right]_{k,\nu=\overline{1,n}} \neq 0, \tag{6.11}$$

$$y_k(x,\rho) = \sum_{j=1}^n b_{kj}^0\rho^{\mu_j}C_j(x,\lambda), \qquad b_{kj}^0 = \beta_j^0 R_k^{\mu_j}, \quad \beta_j^0\neq 0, \tag{6.12}$$

where the constant M_0 *depends only on* $\{\nu_j\}$.

The functions $y_k(x,\rho)$ are analogs of the Hankel functions for the Bessel equation. Denote

$$C_j^*(x,\lambda) = \det\left[C_k^{(\nu)}(x,\lambda)\right]_{\nu=\overline{0,n-2},k=\overline{1,n}\setminus n-j+1},$$

$$y_j^*(x,\rho) = (-1)^{n-j}\left(\rho^{\frac{(n-1)(n-2)}{2}}\Omega\right)^{-1}\det\left[y_k^{(\nu)}(x,\rho)\right]_{\nu=\overline{0,n-2},k=\overline{1,n}\setminus j},$$

$$F_{k\nu}(\rho x)=\begin{cases} R_k^\nu \exp(\rho R_k x), & |\rho|x>1,\\ (\rho x)^{\mu_1-\nu}, & |\rho|x\le 1,\end{cases} \qquad F_k^*(\rho x)=\begin{cases}\exp(-\rho R_k x), & |\rho|x>1,\\ (\rho x)^{n-1-\mu_n}, & |\rho|x\le 1,\end{cases}$$

$$u_{k\nu}^0(x,\rho)=y_k^{(\nu)}(x,\rho)\big(\rho^\nu F_{k\nu}(\rho x)\big)^{-1}, \qquad u_k^{0,*}(x,\rho)=y_k^*(x,\rho)\big(F_k^*(\rho x)\big)^{-1},$$

$$g(x,t,\lambda)=\sum_{j=1}^n(-1)^{n-j}C_j(x,\lambda)C_{n-j+1}^*(t,\lambda)=\rho^{1-n}\sum_{j=1}^n y_j(x,\rho)y_j^*(t,\rho).$$

The function $g(x,t,\lambda)$ is the Green function of the Cauchy problem $l_0y-\lambda y=f(x)$, $y^{(\nu)}(0)=0$, $\nu=\overline{0,n-1}$. Using (6.10)–(6.12), we obtain

$$\big|u_{k\nu}^0(x,\rho)\big|\le M_1, \qquad \big|u_k^{0,*}(x,\rho)\big|\le M_1, \qquad x\ge 0, \quad \rho\in\overline{S}_{k_0}, \tag{6.13}$$

$$\big|C_j^{(\nu)}(x,\lambda)\big|\le M_2\big|x^{\mu_j-\nu}\big|, \tag{6.14}$$

$$\left|\frac{\partial^\nu}{\partial x^\nu}g(x,t,\lambda)\right|\le M_2\sum_{j=1}^n\big|x^{\mu_j-\nu}t^{n-1-\mu_j}\big|, \qquad |\rho x|\le C_0, \quad t\le x,$$

where M_1 depends on $\{\nu_j\}$, and M_2 depends on $\{\nu_j\}$ and C_0.

Let us now construct FSS's of Eq. (6.1). Denote

$$J(\rho)=\sum_{m=0}^{n-2}|\rho|^{\mathrm{Re}(\mu_1-\mu_n)}\int_0^{|\rho|^{-1}}t^{\theta_n-m}\big|q_m(t)\big|\,dt+|\rho|^{m-n+1}\int_{|\rho|^{-1}}^{\infty}\big|q_m(t)\big|\,dt.$$

Lemma 6.3.

$$J(\rho)\le\frac{Q}{|\rho|}, \qquad |\rho|\ge 1, \qquad Q\overset{\text{def}}{=}\sum_{m=0}^{n-2}\int_0\infty\big|q_{0m}(t)\big|\,dt.$$

We construct the functions $S_j(x,\lambda)$, $j=\overline{1,n}$, from the system of integral equations

$$S_j^{(\nu)}(x,\lambda)=C_j^{(\nu)}(x,\lambda)-\int_0^x\frac{\partial^\nu}{\partial x^\nu}g(x,t,\lambda)\left(\sum_{m=0}^{n-2}q_m(t)S_j^{(m)}(t,\lambda)\right)dt, \qquad \nu=\overline{0,n-1}. \tag{6.15}$$

By (6.14), system (6.15) has a unique solution; moreover the functions $S_j^{(\nu)}(x,\lambda)$ are entire in λ for each $x>0$, the functions $\{S_j(x,\lambda)\}_{j=\overline{1,n}}$ form an FSS for (6.1), $\det\big[S_j^{(\nu-1)}(x,\lambda)\big]_{j,\nu=\overline{1,n}}\equiv 1$, and

$$S_j^{(\nu)}(x,\lambda)=O(x^{\mu_j-\nu}), \qquad \big(S_j(x,\lambda)-C_j(x,\lambda)\big)x^{-\mu_j}=o(x^{\mu_n-\mu_1}), \quad x\to 0. \tag{6.16}$$

Let $S_{k_0,\alpha}=\{\rho:\rho\in S_{k_0},|\rho|>\alpha\}$, $\rho_0=2M_1Q+1$. For $k=\overline{1,n}$, $\rho\in\overline{S}_{k_0,\rho_0}$ consider the system of integral equations

$$u_{k\nu}(x,\rho)=u_{k\nu}^0(x,\rho)+\sum_{m=0}^{n-2}\int_0^\infty A_{k\nu m}(x,t,\rho)u_{km}(t,\rho)\,dt, \qquad x\ge 0, \quad \nu=\overline{0,n-1}, \tag{6.17}$$

where

$$A_{k\nu m}(x,t,\rho)=\frac{q_m(t)F_{km}(\rho t)}{\rho^{n-1-m}F_{k\nu}(\rho x)}\begin{cases}-\sum\limits_{j=1}^k F_{j\nu}(\rho x)u_{j\nu}^0(x,\rho)F_j^*(\rho t)u_j^{0,*}(t,\rho), & t\le x,\\ \sum\limits_{j=k+1}^n F_{j\nu}(\rho x)u_{j\nu}^0(x,\rho)F_j^*(\rho t)u_j^{0,*}(t,\rho), & t>x.\end{cases}$$

Using (6.13) and Lemma 6.3, we obtain

$$\sum_{m=0}^{n-2}\int_0^\infty \big|A_{k\nu m}(x,t,\rho)\big|\,dt \le M_1 J(\rho) \le M_1 Q|\rho|^{-1}.$$

Consequently, system (6.17) with $\rho\in\overline{S}_{k_0,\rho_0}$ has a unique solution and, uniformly in $x\ge 0$,

$$u_{k\nu}(x,\rho)-u^0_{k\nu}(x,\rho)=O(\rho^{-1}),\qquad \rho\in\overline{S}_{k_0,\rho_0}. \tag{6.18}$$

Theorem 6.2. *For $x>0$, $\rho\in S_{k_0,\rho_0}$ there exists an FSS of (6.1), $B=\{Y_k(x,\rho)\}_{k=\overline{1,n}}$ of the form $Y_k^{(\nu)}(x,\rho)=\rho^\nu F_{k\nu}(\rho x)u_{k\nu}(x,\rho)$, where the functions $u_{k\nu}(x,\rho)$ are solutions of (6.17), and (6.18) is true. The functions $Y_k^{(\nu)}(x,\rho)$, considered for each $x>0$, are regular in $\rho\in S_{k_0,\rho_0}$, continuous in $\rho\in\overline{S}_{k_0,\rho_0}$, and*

$$\det\big[Y_k^{(\nu-1)}(x,\rho)\big]_{k,\nu=\overline{1,n}}=\rho^{\frac{n(n-1)}{2}}\Omega\big(1+O(\rho^{-1})\big)\qquad \text{as}\quad |\rho|\to\infty.$$

The functions $Y_k(x,\rho)$ satisfy the equality

$$Y_k(x,\rho)=y_k(x,\rho)-\rho^{1-n}\int_0^x\left(\sum_{j=1}^k y_j(x,\rho)y_j^*(t,\rho)\right)\left(\sum_{m=0}^{n-2}q_m(t)Y_k^{(m)}(t,\rho)\right)dt$$
$$+\rho^{1-n}\int_x^\infty\left(\sum_{j=k+1}^n y_j(x,\rho)y_j^*(t,\rho)\right)\left(\sum_{m=0}^{n-2}q_m(t)Y_k^{(m)}(t,\rho)\right)dt.$$

Moreover, one has a representation

$$Y_k(x,\rho)=\sum_{j=1}^n b_{kj}(\rho)S_j(x,\lambda), \tag{6.19}$$

where

$$b_{kj}(\rho)=b^0_{kj}\rho^{\mu_j}\big(1+O(\rho^{-1})\big),\qquad |\rho|\to\infty,\quad \rho\in\overline{S}_{k_0}. \tag{6.20}$$

The only part of the theorem that needs a proof is the asymptotic formula (6.20). Let ρ be fixed, $x\le|\rho|^{-1}$. Then (6.12) and (6.19) become

$$\begin{cases} u^0_{k0}(x,\rho)=\displaystyle\sum_{j=1}^n b^0_{kj}(\rho x)^{\mu_j-\mu_1}\widehat{C}_j(x,\lambda),\\ u_{k0}(x,\rho)=\displaystyle\sum_{j=1}^n b_{kj}(\rho)\rho^{-\mu_1}x^{\mu_j-\mu_1}\widehat{S}_j(x,\lambda),\end{cases} \tag{6.21}$$

where $\widehat{C}_j(x,\lambda)=x^{-\mu_j}C_j(x,\lambda)$, $\widehat{S}_j(x,\lambda)=x^{-\mu_j}S_j(x,\lambda)$, $\widehat{S}_j(0,\lambda)=\widehat{C}_j(0,\lambda)=c_{j0}\ne 0$. It follows from (6.21) that

$$u_{k0}(x,\rho)-u^0_{k0}(x,\rho)=\sum_{j=1}^n\big(b_{kj}(\rho)\rho^{-\mu_1}-b^0_{kj}\rho^{\mu_j-\mu_1}\big)x^{\mu_j-\mu_1}\widehat{S}_j(x,\lambda)$$
$$+\sum_{j=1}^n b^0_{kj}(\rho x)^{\mu_j-\mu_1}\big(\widehat{S}_j(x,\lambda)-\widehat{C}_j(x,\lambda)\big). \tag{6.22}$$

Denote

$$\begin{cases} \mathcal{F}_{k1}(x,\rho)=u_{k0}(x,\rho)-u^0_{k0}(x,\rho),\\ \mathcal{F}_{k,s+1}(x,\rho)=\Big(\mathcal{F}_{ks}(x,\rho)-\mathcal{F}_{ks}(0,\rho)\widehat{S}_s(x,\lambda)c_{s0}^{-1}\Big)x^{\mu_s-\mu_{s+1}},\qquad s=\overline{1,n-1}.\end{cases} \tag{6.23}$$

Lemma 6.4.

$$\Big(b_{ks}(\rho)\rho^{-\mu_1} - b^0_{ks}\rho^{\mu_s-\mu_1}\Big)c_{s0} = \mathcal{F}_{ks}(0,\rho), \qquad s=\overline{1,n}, \tag{6.24}$$

$$\mathcal{F}_{ks}(x,\rho) = \left(\Big(u_{k0}(x,\rho) - u^0_{k0}(x,\rho)\Big) - \sum_{j=1}^{s-1}\Big(b_{kj}(\rho)\rho^{-\mu_1} - b^0_{kj}\rho^{\mu_j-\mu_1}\Big)x^{\mu_j-\mu_1}\widehat{S}_j(x,\lambda)\right)x^{\mu_1-\mu_s}, \qquad s=\overline{1,n}. \tag{6.25}$$

Proof. When $s=1$ equality (6.24) follows from (6.22) for $x=0$, while (6.25) is obviously true. Assume now that (6.24) and (6.25) have been proved for $s=1,\dots,N-1$. Then

$$\left(\Big(u_{k0}(x,\rho) - u^0_{k0}(x,\rho)\Big) - \sum_{j=1}^{N-1}\Big(b_{kj}(\rho)\rho^{\mu_1} - b^0_{kj}\rho^{\mu_j-\mu_1}\Big)x^{\mu_j-\mu_1}\widehat{S}_j(x,\lambda)\right)x^{\mu_1-\mu_N}$$

$$= \left(\Big(u_{k0}(x,\rho) - u^0_{k0}(x,\rho)\Big) - \sum_{j=1}^{N-2}\Big(b_{kj}(\rho)\rho^{\mu_1} - b^0_{kj}\rho^{\mu_j-\mu_1}\Big)x^{\mu_j-\mu_1}\widehat{S}_j(x,\lambda)\right)x^{\mu_1-\mu_{N-1}}x^{\mu_{N-1}-\mu_N}$$

$$-\Big(b_{k,N-1}(\rho)\rho^{\mu_1} - b^0_{k,N-1}\rho^{\mu_{N-1}-\mu_1}\Big)\widehat{S}_{N-1}(x,\lambda)x^{\mu_{N-1}-\mu_N} = \mathcal{F}_{kN}(x,\rho),$$

which gives (6.25) for $s=N$. We now write (6.22) as

$$\mathcal{F}_{kN}(x,\rho) = \sum_{j=1}^{n}\Big(b_{kj}(\rho)\rho^{-\mu_1} - b^0_{kj}\rho^{\mu_j-\mu_1}\Big)x^{\mu_j-\mu_N}\widehat{S}_j(x,\lambda)$$
$$+\sum_{j=1}^{n} b^0_{kj}(\rho x)^{\mu_j-\mu_1}\big(\widehat{S}_j(x,\lambda) - \widehat{C}_j(x,\lambda)\big)x^{\mu_1-\mu_N}.$$

Hence, using (6.16), we obtain $\mathcal{F}_{kN}(0,\rho) = \Big(b_{kN}(\rho)\rho^{-\mu_1} - b^0_{kN}\rho^{\mu_N-\mu_1}\Big)c_{N0}$, which gives (6.24) for $s=N$ and completes the proof of Lemma 6.4.

Now write (6.17) for $\nu=0$ as

$$\mathcal{F}_{k1}(x,\rho) = \rho^{1-n}\left(-\int_0^x\left(\sum_{j=1}^{n} u^0_{j0}(x,\rho)u^{0,*}_j(t,\rho)\right)(\rho t)^{n-1-\mu_n}V_k(t,\rho)\,dt + \int_0^\infty\left(\sum_{j=k+1}^{n} u^0_{j0}(x,\rho)u^{0,*}_j(t,\rho)F^*_j(\rho t)\right)v_k(t,\rho)\,dt\right), \tag{6.26}$$

where

$$V_k(t,\rho) = \sum_{m=0}^{n-2} q_m(t)\rho^m F_{km}(\rho t)u_{km}(t,\rho).$$

Since for $t\le x\le|\rho|^{-1}$ we have

$$\sum_{j=1}^{n} u^0_{j0}(x,\rho)u^{0,*}_j(t,\rho) = \rho^{\mu_n-\mu_1}x^{-\mu_1}t^{1-n+\mu_n}g(x,t,\lambda),$$

it follows from (6.14) that

$$\left|\sum_{j=1}^{n} u_{j0}^{0} 9x,\rho) u_j^{0,*}(t,\rho)\right| \le M_3 \left|(\rho x)^{\mu_n-\mu_1}\right|, \qquad 0 \le t \le x \le |\rho|^{-1}. \tag{6.27}$$

Lemma 6.5.

$$\mathcal{F}_{ks}(0,\rho) = \rho^{\mu_s-\mu_1-n+1} c_{s0} \int_0^{\infty} \left(\sum_{j=k+1}^{n} b_{js}^{0} F_j^*(\rho t) u_j^{0,*}(t,\rho) \right) V_k(t,\rho)\, dt, \tag{6.28}$$

$$\mathcal{F}_{ks}(x,\rho) = \rho^{1-n} \left(- x^{\mu_1-\mu_s} \int_0^{x} \left(\sum_{j=1}^{n} u_{j0}^{0}(x,\rho) u_j^{0,*}(t,\rho) \right) (\rho t)^{n-1-\mu_n} V_k(t,\rho)\, dt \right.$$

$$- \sum_{l=1}^{s-1} x^{\mu_l-\mu_s} \int_0^{\infty} \left(\sum_{j=k+1}^{n} b_{jl}^{0} \rho^{\mu_l-\mu_1} \big(\widehat{S}_l(x,\lambda) - \widehat{C}_l * x,\lambda)\big) F_j^*(\rho t) u_j^{0,*}(t,\rho) \right) V_k(t,\rho)\, dt \tag{6.29}$$

$$\left. + \int_0^{\infty} \left(\sum_{j=k+1}^{n} \left(\sum_{\xi=s}^{n} b_{j\xi}^{0} \rho^{\mu_\xi-\mu_1} x^{\mu_\xi-\mu_s} \widehat{C}_\xi(x,\lambda) \right) F_j^*(\rho t) u_j^{0,*}(t,\rho) \right) V_k(t,\rho)\, dt \right), \qquad x \le |\rho|^{-1}.$$

Proof. For $s=1$, (6.28) and (6.29) follow from (6.26), in view of (6.21). Assume now that (6.28) and (6.29) have been proved for $s = 1, \dots, N$. Then, using (6.23), we obtain

$$\mathcal{F}_{k,N+1}(x,\rho) = \Big(\mathcal{F}_{kN}(x,\rho) - \mathcal{F}_{kN}(0,\rho) \widehat{S}_N(x,\lambda) x_{N0}^{-1} \Big) x^{\mu_N-\mu_{N+1}}$$

$$= -\rho^{1-n} \left(-x^{\mu_1-\mu_{N+1}} \int_0^{x} \left(\sum_{j=1}^{n} u_{j0}^{0}(x,\rho) u_j^{0,*}(t,\rho) \right) (\rho t)^{n-1-\mu_n} V_k(t,\rho)\, dt \right.$$

$$- \sum_{l=1}^{N-1} x^{\mu_l-\mu_{N+1}} \int_0^{\infty} \left(\sum_{j=k+1}^{n} b_{jl}^{n} \rho^{\mu_l-\mu_1} \big(\widehat{S}_l(x,\lambda) - \widehat{C}_l(x,\lambda)\big) F_j^*(\rho t) u_j^{0,*}(t,\rho) \right) V_k(t,\rho)\, dt$$

$$+ x^{\mu_N-\mu_{N+1}} \int_0^{\infty} \left(\sum_{j=k+1}^{n} \left(\sum_{\xi=N}^{n} b_{j\xi}^{0} \rho^{\mu_\xi-\mu_1} x^{\mu_\xi-\mu_N} \widehat{C}_\xi(x,\lambda) - b_{jN}^{0} \rho^{\mu_N-\mu_1} \widehat{C}_N(x,\lambda) \right.\right.$$

$$\left.\left.\left. - b_{jN}^{0} \rho^{\mu_N-\mu_1} \big(\widehat{S}_N(x,\lambda) - \widehat{C}_N(x,\lambda)\big) \right) F_j^*(\rho t) u_j^{0,*}(t,\rho) \right) V_k(t,\rho)\, dt \right)$$

$$= \rho^{1-n} \left(- x^{\mu_1-\mu_{N+1}} \int_0^{x} \left(\sum_{j=1}^{n} u_{j0}^{0}(x,\rho) u_j^{0,*}(t,\rho) \right) (\rho t)^{n-1-\mu_n} V_k(t,\rho)\, dt \right.$$

$$- \sum_{l=1}^{N} x^{\mu_l-\mu_{N+1}} \int_0^{\infty} \left(\sum_{j=k+1}^{n} b_{jl}^{0} \rho^{\mu_l-\mu_1} \big(\widehat{S}_l(x,\lambda) - \widehat{C}_l(x,\lambda)\big) F_j^*(\rho t) u_j^{0,*}(t,\rho) \right) V_k(t,\rho)\, dt$$

$$+\int_0^\infty \left(\sum_{j=k+1}^n \left(\sum_{\xi=N+1}^n b^0_{j\xi} \rho^{\mu_\xi - \mu_1} x^{\mu_\xi - \mu_{N+1}} \widehat{C}_\xi(x,\lambda) \right) F^*_j(\rho t) u^{0,*}_j(t,\rho) \right) V_k(t,\rho)\, dt \Bigg),$$

giving (6.29) for $s = N+1$. We now let $x \to 0$ in (6.29) for $s = N+1$. Using (6.27), we obtain (6.28) for $s = N+1$. This proves Lemma 6.3.

It follows from (6.24) and (6.28) that

$$b_{ks}(\rho)\rho^{-\mu_s} - b^0_{ks} = \rho^{1-n} \int_0^\infty \left(\sum_{j=k+1}^n b^0_{js} F^*_j(\rho t) u^{0;j}_j(t,\rho) \right) V_k(t,\rho)\, dt. \tag{6.30}$$

Using (6.30), (6.13), (6.18), and Lemma 6.3, we obtain

$$b_{ks}(\rho)\rho^{-\mu_s} - b^0_{ks} = O(J(\rho)) = O(\rho^{-1}), \qquad |\rho| \to \infty, \quad \rho \in S_{k_0},$$

i.e., (6.20) is valid. Theorem 6.2 is proved.

Note that, as a consequence of (6.10) and (6.18),

$$\left| Y^{(\nu)}_k(x,\rho)(\rho R_k)^{-1} \exp(-\rho R_k x) - 1 \right| \le M_4 |\rho|^{-1}, \qquad x \ge 1, \quad \rho \in \overline{S}_{k_0,\rho_0}. \tag{6.31}$$

6.2. The Weyl matrix. Solution of the inverse problem. Let the functions $\Phi_m(x,\lambda)$, $m = \overline{1,n}$, be solutions of (6.1) satisfying the conditions $\Phi_m(x,\lambda) \sim c_{m0} x^{\mu_m}$, $x \to 0$; $\Phi_m(x,\lambda) = O\big(\exp(\rho R_m x)\big)$, $x \to \infty$, $\rho \in S_{k_0}$. We call $\Phi_m(x,\lambda)$ the WS's for (6.1). Let $\{Y_k(x,\rho)\}_{k=\overline{1,n}}$ be the FSS B of (6.1) in S_{k_0,ρ_0}. We will look for the WS's in the form

$$\Phi_m(x,\lambda) = \sum_{k=1}^n a_{mn}(\rho) Y_k(x,\rho) = \sum_{j=1}^n S_j(x,\lambda) \sum_{k=1}^n b_{kj}(\rho) a_{mk}(\rho).$$

The conditions imposed on the WS's, combined with (6.16) and (6.31), imply that for $|\rho| \ge 2M_4$

$$a_{mk}(\rho) = 0, \quad k > m; \qquad \sum_{k=1}^m b_{kj}(\rho) a_{mk}(\rho) = \delta_{jm}, \quad j = \overline{1,m}.$$

Hence we obtain

$$\Phi_m(x,\lambda) = \sum_{k=1}^m a_{mk}(\rho) Y_k(x,\rho) = S_m(x,\lambda) + sum^n_{j=m+1} \mathfrak{M}_{mj}(\lambda) S_j(x,\lambda), \tag{6.32}$$

$$a_{mk}(\rho) = (-1)^{m-k} \big(\Delta_{mm}(\rho)\big)^{-1} \det\big[b_{\xi\nu}(\rho)\big]_{\xi=\overline{1,m}\setminus k;\nu=\overline{1,m-1}}, \tag{6.33}$$

$$\mathfrak{M}_{mj}(\lambda) = \sum_{k=1}^m b_{kj}(\rho) a_{mk}(\rho) = \big(\Delta_{mm}(\rho)\big)^{-1} \Delta_{mj}(\rho), \qquad j > m, \tag{6.34}$$

where $\delta_{mj}(\rho) = \det\big[b_{k\nu}(\rho)\big]_{k=\overline{1,m};\nu=\overline{1,m-1},j}$. Denote $\Delta^0_{00} = 1$, $\Delta^0_{mj} = \det\big[b^0_{k\nu}\big]_{k=\overline{1,m};\nu=\overline{1,m-1}\setminus j}$, $j \ge m \ge 1$, $a^0_{mk} = \det\big[b^0_{\xi\nu}\big]_{\xi=\overline{1,m}\setminus k;\nu=\overline{1,m-1}}(-1)^{m-k}\big(\Delta^0_{mm}\big)^{-1}$, and $\Pi_{\pm 1}$ is the λ-plane with the cut $\pm\lambda \ge 0$. Since $b^0_{k\nu} = \beta^0_\nu R^{\mu_\nu}_k$, we have $\Delta^0_{mj} \ne 0$. Cleary, $a^0_{mm} = \big(\Delta^0_{mm}\big)^{-1} \Delta^0_{m-1,m-1} \ne 0$. Using (6.20), (6.32)–(6.34), we see that for $|\rho| \to \infty$, $\rho \in S_{k_0}$, $\arg\rho = \varphi$,

$$a_{mk}(\rho) = \rho^{-\mu_m}\big(a^0_{mk} + O(\rho^{-1})\big), \tag{6.35}$$

$$\mathfrak{M}_{mj}(\lambda) = \rho^{\mu_j - \mu_m} \mathfrak{M}^0_{mj}\big(1 + O(\rho^{-1})\big), \qquad \mathfrak{M}^0_{mj} = \big(\Delta^0_{mm}\big)^{-1} \Delta^0_{mj} \ne 0, \tag{6.36}$$

$$\Phi^{(\nu)}_m(x,\lambda) = \rho^{-\mu_m} a^0_{mm} (\rho R_m)^\nu \exp(\rho R_m x)\big(1 + O(\rho^{-1})\big) \tag{6.37}$$

for every fixed $x > 0$.

Repeating the preceding arguments for the FSS $B_{\alpha m} = \{y_{mk}(x,\rho)\}_{k=\overline{1,n}}$, we obtain

$$y_{mk}(x,\rho) = \sum_{j=1}^{n} B_{mkj}(\rho) S_j(x,\lambda),$$

$$\mathfrak{M}_{mj}(\lambda) = \left(\Delta^1_{mm}(\rho)\right)^{-1} \Delta^1_{mj}(\rho), \qquad \Delta^1_{mj}(\rho) = \det\left[B_{mkj}(\rho)\right]_{k=\overline{1,m};\nu=\overline{1,m-1},j}.$$

Denote $G = \left\{\rho : \arg\rho \in \left(\frac{((-1)^{n-m}-1)\pi}{2n}, \frac{((-1)^{n-m}+3\pi}{2n}\right)\right\}$. The domain G is the union of two sectors with the same $\{R_\xi\}_{\xi=\overline{1,m}}$. Consequently, the functions $\Delta^1_{mj}(\rho)$ are regular for $\rho \in G$, $|\rho| > \rho_\alpha$, and continuous for $\rho \in \overline{G}$, $|\rho| \geq \rho_\alpha$. We have thus proved

Theorem 6.3. *The WS's $\Phi_m(x,\lambda)$ can be written as*

$$\Phi_m(x,\lambda) = S_m(x,\lambda) + \sum_{j=m+1}^{n} \mathfrak{M}_{mj}(\lambda) S_j(x,\lambda), \tag{6.38}$$

where the functions $\mathfrak{M}_{mj}(\lambda)$ are regular in $\Pi_{(-1)^{n-m}}$ with the exception of an at most countable bounded set of poles Λ'_{mj}, and continuous in $\overline{\Pi}_{(-1)^{n-m}}$ with the exception of bounded sets Λ_{mj}. The WS's $\{\Phi_m(x,\lambda)\}_{m=\overline{1,n}}$ form an FSS for (6.1), such that $\det\left[\Phi_m^{(\nu-1)}(x,\lambda)\right]_{m,\nu=\overline{1,n}} \equiv 1$. For $|\rho| \to \infty$, $\rho \in S_{k_0}$, $\arg\rho = \varphi$, and fixed $x > 0$, we have the asymptotic formulas (6.36) and (6.37).

The functions $\mathfrak{M}_{mj}(\lambda)$ are called the WF's, and the matrix $\mathfrak{M}(\lambda) = \left[\mathfrak{M}_{mj}(\lambda)\right]_{m,j=\overline{1,n}}$, $\mathfrak{M}_{mj}(\lambda) = \delta_{mj}$ $(m \geq j)$ is called the WM for l.

The IP is formulated as follows: given the WM $\mathfrak{M}(\lambda)$ construct the DO l.

Let us prove the uniqueness theorem for the solution of the IP.

Theorem 6.4. *If $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$, then $l = \tilde{l}$.*

Proof. Denote $S(x,\lambda) = \left[S_j^{(\nu)}(x,\lambda)\right]$, $\Phi(x,\lambda) = \left[\Phi_j^{(\nu)}(x,\lambda)\right]$. (6.38) becomes

$$\Phi(x,\lambda) = S(x,\lambda)\mathfrak{M}^T(\lambda). \tag{6.39}$$

Moreover, $\det\Phi(x,\lambda) = \det S(x,\lambda) \equiv 1$. Define a matrix $P(x,\lambda) = \left[P_{jk}(x,\lambda)\right]_{j,k=\overline{1,n}}$ by the formula $P(x,\lambda) = \Phi(x,\lambda)\left(\widetilde{\Phi}(x,\lambda)\right)^{-1}$ or

$$P_{jk}(x,\lambda) = \det\left[\widetilde{\Phi}_m(x,\lambda), \ldots, \widetilde{\Phi}_m^{(k-2)}(x,\lambda), \Phi_m^{(j-1)}(x,\lambda), \widetilde{\Phi}_m^{(k)}(x,\lambda), \ldots, \widetilde{\Phi}_m^{(n-1)}(x,\lambda)\right]_{m=\overline{1,n}}. \tag{6.40}$$

From (6.40) and the asymptotic properties of the WS's $\Phi_m(x,\lambda)$ and $\widetilde{\Phi}_m(x,\lambda)$ we see that for a fixed $x > 0$ and $|\lambda| \to \infty$

$$P_{jk}(x,\lambda) = O(\rho^{j-k}), \qquad P_{1k}(x,\lambda) - \delta_{1k} = O(\rho^{-1}). \tag{6.41}$$

Using (6.39), we transform the matrix $P(x,\lambda)$ as follows:

$$P(x,\lambda) = \Phi(x,\lambda)\left(\widetilde{\Phi}(x,\lambda)\right)^{-1} = S(x,\lambda)\left(\widetilde{S}(x,\lambda)\right)^{-1}.$$

Hence we conclude that for each fixed $x > 0$ the functions $P_{jk}(x,\lambda)$ are entire in λ. Using (6.41) and the Liouville theorem, we obtain $P_{11}(x,\lambda) \equiv 1$, $P_{1k}(x,\lambda) \equiv 0$, $k = \overline{2,n}$. But then $\Phi_m(x,\lambda) \equiv \widetilde{\Phi}_m(x,\lambda)$ for all x, λ, m, and hence $l = \tilde{l}$. Theorem 6.4 is proved.

Using the results obtained above, and the contour integral method, one can obtain an algorithm for the solution of the IP from the WM, along with necessary and sufficient conditions of its solvability, in analogous manner as in Sec. 2.

7.1. The asymptotics of the spectrum. The inverse problem. Consider now the DE (6.1) on a finite interval $x \in (0, T)$. As in Sec. 6, let $\mu_1, \dots, \mu_n$ be the roots of the characteristic polynomial. For definiteness, we will assume that $n = 2m$, $\mu_k \neq \mu_j \pmod n$, $\operatorname{Re}\mu_1 < \cdots < \operatorname{Re}\mu_n$, $\mu_k \neq 0, 1, \dots, n-3$. Denote $\theta_n = n - 1 - \operatorname{Re}(\mu_n - \mu_1)$. We assume that the functions $q_j^{(\nu)}$, $\nu = \overline{0, j-1}$, are absolutely continuous and $q_j^{(\nu)}(x) \cdot x^{\theta_n - j + \nu} \in \mathcal{L}(0, T)$, $\nu = \overline{0, j}$.

In this section, we study the boundary value problem L for the DE (6.1), $x \in (0, T)$, with the boundary conditions

$$y(x) = O(x^{\mu_{m+1}}), \qquad x \to 0,$$

$$V_p(y) \equiv y^{\tau_p}(T) + \sum_{k=0}^{\tau_p - 1} v_{pk} y^{(k)}(T) = 0, \qquad p = \overline{1, m}, \quad 0 \leq \tau_p \leq n-1, \quad \tau_p \neq \tau_s \quad (p \neq s).$$

Theorems on the completeness and expansion in eigenfunctions and associated functions of L are obtained. The equiconvergence theorem is provided, and the IP is studied. Our consideration essentially uses the results obtained in Sec. 6, where special FSS's of (6.1) are constructed and analytical and asymptotic properties of the Stokes multipliers are investigated.

Denote $\Delta(\lambda) = \det\left[V_p(S_j(x, \lambda))\right]_{p=\overline{1,m};\, j=\overline{m+1,n}}$, where $\left[S_j(x, \lambda)\right]_{j=\overline{1,n}}$ is the FSS of (6.1) constructed in Sec. 6. The zeros of $\Delta(\lambda)$ coincide with the eigenvalues of L.

Theorem 7.1. *The boundary value problem L has a countable set $\{\lambda_l\}$ of eigenvalues, and for $l \to \infty$*

$$\lambda_l = (-1)^m \left((l + \theta)\pi T^{-1} + O(l^{-1})\right)^n, \qquad \theta = \theta(\mu_1, \dots, \mu_n).$$

All eigenvalues, starting with some eigenvalue, are simple.

Let us define LF V_p for $p = \overline{m+1, n}$. Let the functions $\Phi_k(x, \lambda)$, $k = \overline{1, n}$, be solutions of (6.1) under the conditions $\Phi_k(x, \lambda) \sim c_k x^{\mu_k}$, $x \to 0$, and $V_p(\Phi_k) = 0$, $p = \overline{1, n-k}$. Then $\det\left[\Phi_k^{(\nu-1)}(x, \lambda)\right]_{k,\nu=\overline{1,n}} \equiv 1$ and

$$\Phi_k(x, \lambda) = S_k(x, \lambda) + \sum_{j=k+1}^{n} \mathfrak{M}_{kj}(\lambda) S_j(x, \lambda).$$

The functions $\Phi_k(x, \lambda)$ are called the WS's, and the matrix $\mathfrak{M}(\lambda) = \left[\mathfrak{M}_{mk}(\lambda)\right]_{m,k=\overline{1,n}}$, $\mathfrak{M}_{mk}(\lambda) = \delta_{mk}$, $m \geq k$, is the WM for the operator l. The WS's and WM are convenient for studying both direct and inverse problems for l. Let us formulate the uniqueness theorem for the solution of the IP using the WM.

Theorem 7.2. *If $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$, then $l = \tilde{l}$, $V_p = \tilde{V}_p$.*

7.2. The Green function. Let us define the Green function $G(x, t, \lambda)$ for L using WS's of L and L^*,

$$G(x, t, \lambda) = \begin{cases} \displaystyle\sum_{k=m+1}^{n} (-1)^{k-1} \Phi_{n-k+1}(x, \lambda) \overline{\Phi_k^*(t, \overline{\lambda})}, & x \geq t, \\ \displaystyle\sum_{k=1}^{m} (-1)^k \Phi_{n-k+1}(x, \lambda) \overline{\Phi_k^*(t, \overline{\lambda})}, & x \leq t. \end{cases}$$

We put $\mu_k^* = n - 1 - \overline{\mu}_{n-k+1}$.

Lemma 7.1. *Let $f(t) t^{\mu^*_{m+1}} \in \mathcal{L}(0, T)$, $\Delta(\lambda) \neq 0$. Put*

$$y(x) = \int_0^T G(x, t, \lambda) f(t)\, dt.$$

Then

$$ly - \lambda y = f, \qquad y(x) = o(x^{\mu_m}), \quad x \to 0;$$
$$V_p(y) = 0, \quad p = \overline{1,m}.$$

The converse is also valid. We note that the function $G(x,t,\lambda)$ is meromorphic with respect to λ with poles at the points $\lambda = \lambda_l$.

Lemma 7.2. *Let λ_l be a simple zero of the function $\Delta(\lambda)$. Then*

$$\operatorname*{res}_{\lambda=\lambda_l} G(x,t,\lambda) = -\alpha_l \varphi_l(x)\overline{\varphi_l^*(t)}, \qquad \alpha_l = \left(\int_0^T \varphi_l(t)\overline{\varphi_l^*(t)}\,dt\right)^{-1},$$

where $\varphi_l(x)$ and $\varphi_l^*(x)$ are eigenfunctions of L and L^*, respectively.

Let $\lambda = \rho^n$, $\lambda_l^0 = (-1)^m\big((l+\theta)\pi T^{-1}\big)^n$, $\lambda_l^0 = (\rho_l^0)^n$, $\varepsilon_0 > 0$. Denote $G_0 = \{\rho : |\rho - \rho_l^0| \geq \varepsilon_0\}$.

Theorem 7.3. *Put*

$$y_{\nu j}(x,\lambda) = \int_0^T \frac{\partial^{\nu+j} G(x,t,\lambda)}{\partial x^\nu \partial t^j} f(t)\,dt,$$

where $f(t)t^{\varkappa} \in \mathcal{L}(0,T)$ for $\varkappa \leq \operatorname{Re}\mu_{m+1}^* - j$. Then for $\rho \in G_0$, $|\rho| \geq \rho^0$ and $0 < x \leq T$, we have

$$\big|y_{\nu j}(x,\lambda)\big| \leq \omega(\rho)|\rho|^{\nu+j-m+1+\langle\varkappa\rangle}, \qquad |\rho|x \geq 1,$$
$$\big|y_{\nu j}(x,\lambda)\big| \leq \omega(\rho)\big|x^{\mu_{m+1}-\nu}\big|\Big(\big|\rho^{j-\mu_m^*+\varkappa}\big| + \Omega\big|x^{\mu_m^*-j-\varkappa}\big|\Big), \qquad |\rho|x \leq 1,$$

where $\langle\varkappa\rangle = \max(\varkappa, 0)$,

$$\Omega = \begin{cases} 0, & \varkappa \leq \operatorname{Re}\mu_m^* - j, \\ 1, & \varkappa > \operatorname{Re}\mu_m^* - j, \end{cases}$$

and $\omega(\rho) = o(1)$ as $|\rho| \to \infty$.

7.3. Completeness, expansion and equiconvergence theorems. Applying Theorem 7.3 and the method of contour integration, we come to completeness, expansion and equiconvergence theorems. Let α be a real number and $1 \leq p < \infty$. Consider the Banach spaces $\Phi_{\alpha,p}\{f(x) : f(x)x^{-a} \in \mathcal{L}_p(0,T)\}$ with the norm $\|f\|_{\alpha,p} = \|f(x)x^{-a}\|_{\mathcal{L}_p(0,T)}$.

Lemma 7.3. *For $1 \leq s \leq p < \infty$, $\beta - \alpha < \frac{1}{s} - \frac{1}{p}$, the space $\Phi_{\alpha,p}$ is densely embedded into $\Phi_{\beta,s}$.*

Let us intoduce notations

$$\psi = \operatorname{Re}\mu_{m+1}, \qquad \varphi = \min(0, -\operatorname{Re}\mu_m), \qquad \gamma = \min(0, \operatorname{Re}\mu_{m+1}^*), \qquad \eta = \max(0, -\operatorname{Re}\mu_{m+1}).$$

Theorem 7.4. *The system of eigenfunctions and associated functions of boundary value problem L is complete in the space $\Phi_{\beta,s}$ for $1 \leq s < \infty$, $\beta < \psi + \frac{1}{s}$.*

Corollary. *The system of eigenfunctions and associated functions of L is complete in $\mathcal{L}_s(0,T)$ for* $\operatorname{Re}\mu_{m+1} > -\frac{1}{s}$.

Theorem 7.5. *Let a function $f(t)$ be such that $f(t)t^{\varphi}$ is absolutely continuous on $[0,T]$, $f(t)t^{\varphi-1} \in \mathcal{L}(0,T)$, and if $\tau_1 \cdot \dots \cdot \tau_m = 0$, then $f(T) = 0$. Put*

$$y(x,\lambda) = \int_0^T G(x,t,\lambda) f(t)\,dt.$$

Then

$$\lim_{N\to\infty}\max_{0\le x\le T}\left|x^\eta\left(\frac{1}{2\pi i}\int_{\Gamma_N} y(x,\lambda)\,d\lambda+f(x)\right)\right|=0,$$

where $\Gamma_N=\{\lambda:|\lambda|=r_N\}$ *are circles of radii* $r_N\to\infty$ *at a positive distance from the spectrum of* L. *In particular, if* L *has a simple spectrum, then*

$$\lim_{N\to\infty}\max_{0\le x\le T}\left|x^\eta\left(f(x)-\sum_{l=1}^{N}\alpha_l\varphi_l(x)\int_0^T f(t)\overline{\varphi_l^*(t)}\,dt\right)\right|=0,\qquad \alpha_l=\left(\int_0^T\varphi_l(t)\overline{\varphi_l^*(t)}\,dt\right)^{-1}.$$

In conclusion, we formulate the equiconvergence theorem for L and $\widetilde{L}$ on the whole segment $[0,T]$.

Theorem 7.6. *Let* $f(t)t^\gamma\in\mathcal{L}(0,T)$. *Put* $\widehat{y}(x,\lambda)=\int\limits_0^T\widehat{G}(x,t,\lambda)f(t)\,dt$. *Then*

$$\lim_{N\to\infty}\max_{0\le x\le T}\left|x^\eta\frac{1}{2\pi i}\int_{\Gamma_N}\widehat{y}(x,\lambda)\,d\lambda\right|=0.$$

In particular, if L *and* $\widetilde{L}$ *have simple spectra, then*

$$\lim_{N\to\infty}\max_{0\le x\le T}\left|x^\eta\sum_{l=1}^{N}\left(\alpha_l\varphi_l(x)\int_0^T f(t)\overline{\varphi_l^*(t)}\,dt-\widetilde{\alpha}_l\widetilde{\varphi}_l(x)\int_0^T f(t)\overline{\widetilde{\varphi}_l^*(t)}\,dt\right)\right|=0.$$

PART 3

NONLOCAL INVERSE PROBLEMS

8. An Inverse Problem for Integro-Differential Operators

In this section, perturbation of the Sturm–Liouville operator by a Volterra integral operator is considered. The presence af an "aftereffect" in a mathematical model produced qualitative changes in the study of the IP. The main result of the section are expressed by Theorems 8.1 and 8.3. Note that the IP for integro-differential operators in various formulations has been studied in [23, 58, 91, 92]. Among other things, in [91] a connection is pointed out between the IP under consideration here and the completeness of the eigen- and associated functions of a bundle of fourth-order integro-differential operators.

8.1. The uniqueness theorem. Let $\{\lambda_m\}_{n\ge1}$ be the eigenvalues of a boundary value problem $L=L(q,M)$ of the form

$$ly(x)\equiv-y''(x)+q(x)y(x)+\int_0^x M(x-t)y(t)\,dt=\lambda y(x)=\rho^2y(x), \tag{8.1}$$

$$y(0)=y(\pi)=0. \tag{8.2}$$

Consider the following problem.

Problem 8.1. Given the function $q(x)$ and the spectrum $\{\lambda_n\}_{n\ge1}$, find the function $M(x)$.

Put

$$M_0(x) = (\pi - x)M(x), \qquad M_1(x) = \int_0^x M(t)\,dt, \qquad Q(x) = M_0(x) - M_1(x).$$

We shall assume that $q(x),\ Q(x) \in \mathcal{L}_2(0,\pi),\ M_k(x) \in \mathcal{L}(0,\pi),\ k = 0,1$.

Let $S(x,\lambda)$ be the solution of (8.1) under the initial conditions $S(0,\lambda) = 0,\ S'(0,\lambda) = 1$. Denote $\Delta(\lambda) = S(\pi,\lambda)$. The eigenvalues $\{\lambda_n\}_{n\geq 1}$ of L coincide with the zeros of $\Delta(\lambda)$ and as $n \to \infty$

$$\rho_n = \sqrt{\lambda_n} = n + \frac{A_1}{n} + \frac{\varkappa_n}{n}, \qquad \{\varkappa_n\} \in l_2, \qquad A_1 = \frac{1}{2\pi}\int_0^\pi q(t)\,dt. \tag{8.3}$$

The folowing assertions could be proved by well-known methods.

Lemma 8.1. *The representation*

$$S(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_0^x K(x,t)\frac{\sin\rho t}{\rho}\,dt \tag{8.4}$$

holds, where $K(x,t)$ is a continuous function, and $K(x,0) = 0$.

Lemma 8.2. *The function $\Delta(\lambda)$ is uniquely determined by its zeros, and*

$$\Delta(\lambda) = \pi\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2}. \tag{8.5}$$

We shall now prove the uniqueness theorem for the solution of Problem 8.1. Let $\{\widetilde{\lambda}_n\}_{n\geq 1}$ be the eigenvalues of the boundary value problem $\widetilde{L} = L(1,\widetilde{M})$.

Theorem 8.1. *If $\lambda_n = \widetilde{\lambda}_n,\ n \geq 1$, then $M(x) \overset{\text{def}}{=} \widetilde{M}(x),\ x \in [0,\pi]$.*

Proof. Let the function $S^*(x,\lambda)$ be the solution of the equation

$$l^* z \equiv -z''(x) + q(x)z(x) + \int_x^\pi M(t-x)z(t)\,dt = \lambda z(x)$$

under the conditions $S^*(\pi,\lambda) = 0,\ S^{*\prime}(\pi,\lambda) = -1$. Put $\Delta^*(\lambda) = S^*(0,\lambda)$. Then

$$\int_0^\pi S^*(x,\lambda)\,dx\int_0^x \widehat{M}(x-t)\widetilde{S}(t,\lambda)\,dt = \int_0^\pi l^*S^*(x,\lambda)\cdot\widetilde{S}(x,\lambda)\,dx - \int_0^\pi S^*(x,\lambda)\cdot\widetilde{l}\widetilde{S}(x,\lambda)\,dx$$
$$+\Big|_0^\pi\Big(\widetilde{S}(x,\lambda)S^{*\prime}(x,\lambda) - \widetilde{S}'(x,\lambda)S^*(x,\lambda)\Big) = \Delta^*(\lambda) - \widetilde{\Delta}(\lambda).$$

For $\widetilde{l} = l$ we have $\Delta^*(\lambda) \equiv \Delta(\lambda)$, and consequently

$$\int_0^\pi S^*(x,\lambda)\,dx\int_0^x \widehat{M}(x-t)\widetilde{S}(t,\lambda)\,dt = \widehat{\Delta}(\lambda). \tag{8.6}$$

Transform (8.6) into

$$\int_0^\pi \widehat{M}(x)\,dx\int_x^\pi S^*(t,\lambda)\widetilde{S}(t-x,\lambda)\,dt = \widehat{\Delta}(\lambda). \tag{8.7}$$

Denote $w(x,\lambda) = S^*(\pi - x, \lambda)$, $N(x) = M(\pi - x)$,

$$\varphi(x,\lambda) = \int_0^x w(t,\lambda)\widetilde{S}(x-t,\lambda)\,dt. \tag{8.8}$$

Then (8.7) takes the form

$$\int_0^\pi \widehat{N}(x)\varphi(x,\lambda)\,d\xi = \widehat{\Delta}(\lambda). \tag{8.9}$$

Lemma 8.3. *The representation*

$$\varphi(x,\lambda) = \frac{1}{2\rho^2}\left(-x\cos\rho x + \int_0^x V(x,t)\cos\rho t\,dt\right) \tag{8.10}$$

holds, where $V(x,t)$ is a continuous function.

Indeed, since $w(x,\lambda) = S^*(\pi - x,\lambda)$, the function $w(x,\lambda)$ is the solution of the Cauchy problem

$$-w''(x,\lambda) + q(\pi - x)w(x,\lambda) + \int_0^x M(x-t)w(t,\lambda)\,dt = \lambda w(x,\lambda),$$

$$w(0,\lambda) = 0, \qquad w'(0,\lambda) = 1.$$

Therefore, by Lemma 8.1, the representation

$$w(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_0^x K^0(x,\tau)\frac{\sin\rho t}{\rho}\,dt, \tag{8.11}$$

holds, where $K^0(x,t)$ is a continuous function. Substituting (8.4) and (8.11) into (8.8), we obtain (8.10).

Let us return to the proof of Theorem 8.1. Since $\lambda_n = \widetilde{\lambda}_n$, $n \ge 1$, we have, by Lemma 8.2, $\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda)$. Then, substituting (8.10) into (8.9), we obtain

$$\int_0^\pi \cos\rho x\left(-x\widehat{N}(x) + \int_x^\pi V(t,x)\widehat{N}(t)\,dt\right)dx \equiv 0,$$

and consequently,

$$-x\widehat{N}(x) + \int_x^\pi V(t,x)\widehat{N}(t)\,dt = 0. \tag{8.12}$$

For each fixed $\varepsilon > 0$, (8.12) is a Volterra homogeneous integral equation of the second kind in the interval (ε,π). Consequently $\widehat{N}(x) = 0$ a.e. in (ε,π) and, since ε is arbitrary, in the whole interval $(0,\pi)$. Thus, $M(x) = \widetilde{M}(x)$ a.e. in $(0,\pi)$, and the theorem is proved.

8.2. Solution of the inverse problem. Relation (8.9) also makes possible to obtain an algorithm for solving Problem 8.1 in the case when $M(x) \in PA[0,\pi]$. Consider $L(q,M)$ and $L(q,\widetilde{M})$, and assume that $q(x) \in \mathcal{L}_2(0,\pi)$, $M(x)$ and $\widetilde{M}(x) \in PA$. Let for some fixed $a > 0$

$$\begin{aligned} \widehat{N}(x) &= 0, \qquad x \in (a,\pi),\\ \widehat{N}(x) &\sim \widehat{N}_\alpha^a(\alpha!)^{-1}(a-x)^\alpha, \qquad x \to a-0. \end{aligned} \tag{8.13}$$

It follows from (8.10) that as $|\rho| \to \infty$, $\arg\rho \in [\delta, \pi-\delta]$, $x \in (\varepsilon,\pi)$, $\delta > 0$, $\varepsilon > 0$, the asymptotic formula

$$\varphi(x,\lambda) = -x(4\rho^2)^{-1}\exp(-i\rho x)\bigl(1 + O(\rho^{-1})\bigr) \tag{8.14}$$

holds. Furthermore, it follows from (8.10) that

$$|\varphi(x,\lambda)| < C\big|\rho^{-2}\exp(-i\rho x)\big|, \qquad x\in[0,\pi], \quad \operatorname{Im}\rho\ge 0. \tag{8.15}$$

Using (8.15) we obtain the estimate

$$\left|\int_0^{\varepsilon} \widehat{N}(x)\varphi(x,\lambda)\,dx\right| < C\big|\rho^{-2}\exp(-i\rho\varepsilon)\big|, \qquad \operatorname{Im}\rho\ge 0. \tag{8.16}$$

Using (8.13), (8.14) and Lemma 4.1, we obtain for $|\rho|\to\infty$, $\arg\rho\in[\delta,\pi-\delta]$

$$\int_{\varepsilon}^{a} \widehat{N}(x)\varphi(x,\lambda)\,dx = \frac{a}{4(-i\rho)^{\alpha+3}}\exp(-i\rho a)\big(\widehat{N}_\alpha^a + o(1)\big). \tag{8.17}$$

Since $\widehat{N}(x)=0$ for $x\in(a,\pi)$, from (8.9), (8.16), and (8.17) follows that as $|\rho|\to\infty$, $\arg\rho\in[\delta,\pi-\delta]$,

$$\widehat{\Delta}(\lambda) = \frac{a}{4}(-i\rho)^{-\alpha-3}\exp(-i\rho a)\big(\widehat{N}_\alpha^a + o(1)\big),$$

and consequently

$$\widehat{N}_\alpha^a = \frac{4}{a}\lim \widehat{\Delta}(\lambda)(-i\rho)^{\alpha+3}\exp(i\rho a), \qquad |\rho|\to\infty, \quad \arg\rho\in[\delta,\pi-\delta]. \tag{8.18}$$

Thus we have proved the following theorem.

Theorem 8.2. *Let $\{\lambda_n\}_{n\ge1}$ be the eigenvalues of $L(q,M)$, where $q(x)\in\mathcal{L}_2(0,\pi)$, $M(x)\in PA$. Then the solution of Problem 8.1 can be found by the following algorithm:*

(1) *From $\{\lambda_n\}_{n\ge1}$ construct the function $\Delta(\lambda)$ by formula (8.5).*
(2) *Take $a=\pi$.*
(3) *For $\alpha=0,1,2,\dots$ carry out successively the operations: construct a function $\widetilde{M}(x)\in PA$ so that $\widehat{N}(x)=0$, $x\in(a,\pi)$; $\widehat{N}^{(k)}(a-0)=0$, $k=\overline{1,\alpha-1}$, and find $N_\alpha^a=(-1)^\alpha N^{(\alpha)}(a-0)$ from (8.18).*
(4) *Construct $N(x)$ for $x\in(a^+,a)$ by the formula*

$$N(x)=\sum_{\alpha=0}^{\infty} N_\alpha^a\frac{(a-x)^\alpha}{\alpha!}.$$

(5) *If $a^+>0$, set $a:=a^+$ and pass to step (3).*

We shall now investigate the question of solving Problem 8.1 "in the small," and the question of stability. First, let us prove an auxiliary assertion.

Lemma 8.4. *Consider in a Banach space B the nonlinear equation*

$$r = f + \sum_{j=2}^{\infty}\psi_j(r), \tag{8.19}$$

$$\|\psi_j(r)\| \le \big(C\|r\|\big)^j, \qquad \|\psi_j(r)-\psi_j(r^*)\| \le \|r-r^*\|\big(C\max(\|r\|,\|r^*\|)\big)^{j-1}.$$

There exists $\delta>0$ such that if $\|f\|<\delta$, then in the ball $\|r\|\le 2\delta$ Eq. (8.19) has a unique solution $r\in B$, for which $\|r\|\le 2\|f\|$.

Proof. Assume that $C\ge1$. Put

$$\psi(r)=\sum_{j=2}^{\infty}\psi_j(r), \qquad C_0=2C^2, \quad \delta=(4C_0)^{-1}.$$

If $\|r\|,\ \|r^*\| \le (2C_0)^{-1}$, then

$$\left\{\begin{aligned} \|\psi(r)\| &\le \sum_{j=2}^{\infty} (C\|r\|)^j \le C_0\|r\|^2 \le \frac{1}{2}\|r\|, \\ \|\psi(r) - \psi(r^*)\| &\le \|r - r^*\| \sum_{j=2}^{\infty} \left(C(2C_0)^{-1}\right)^{j-1} \le \frac{1}{2}\|r - r^*\|. \end{aligned}\right. \tag{8.20}$$

Let $\|f\| \le \delta$; construct $r_0 = f$, $r_{k+1} = f + \psi(r_k)$, $k \ge 0$. By induction, using (8.20), we obtain the estimates

$$\|r_k\| \le 2\|f\|, \qquad \|r_{k+1} - r_k\| \le 2^{-k-1}\|f\|, \qquad k \ge 0.$$

Consequently, the series

$$r = r_0 + \sum_{k=0}^{\infty} (r_{k+1} - r_k)$$

converges to the solution of (8.19), and $\|r\| \le 2\|f\|$. Lemma 8.4 is proved.

Theorem 8.3. *For the boundary value problem $L = L(q, M)$ with the spectrum $\{\lambda_n\}_{n\ge1}$, there exists $\delta > 0$ (which depends on L) such that if the numbers $\{\widetilde{\lambda}\}_{n\ge1}$ satisfy the condition*

$$\Lambda \stackrel{\text{def}}{=} \left(\sum_{n=1}^{\infty} |\lambda_n - \widetilde{\lambda}_n|^2\right)^{\frac{1}{2}} < \delta,$$

then there exists a unique $\widetilde{L} = L(q, \widetilde{M})$, for which the numbers $\{\widetilde{\lambda}_n\}_{n\ge1}$ are the eigevalues, and

$$\|Q(x) - \widetilde{Q}(x)\|_{\mathcal{L}_2(0,\pi)} \le C\Lambda,$$
$$\|M_k(x) - \widetilde{M}_k(x)\|_{\mathcal{L}(0,\pi)} \le C\Lambda, \qquad k = 0, 1.$$

Here and below, C denotes various constants dependent on L.

Proof. For brevity, we confine ourselves to the case when all the eigenvalues are simple. The Cauchy problem $ly(x) - \lambda y(x) + f(x) = 0$, $y(0) = y'(0) = 0$ has a unique solution

$$y(x) = \int_0^x g(x, t, \lambda) f(t)\, dt,$$

where $g(x, t, \lambda)$ is the Green function satisfying the relations

$$-g_{xx}(x, t, \lambda) + q(x)g(x, t, \lambda) - \lambda g(x, t, \lambda) + \int_t^x M(x - \tau) g(\tau, t, \lambda)\, d\tau = 0, \qquad x > t,$$
$$g(t, t, \lambda) = 0, \qquad g_x(x, t, \lambda)_{|x=t} = 1.$$

Denote

$$G(x, t, \lambda) = g_t(x, t, \lambda), \qquad y_n(x) = S(x, \widetilde{\lambda}_n), \qquad \varepsilon_n = n^2 \Delta(\widetilde{\lambda}_n),$$
$$v_n(x, t) = \begin{cases} w'(\pi - x - t, \widetilde{\lambda}_n), & 0 < t < \pi - x, \\ 0, & \pi - x < t < \pi, \end{cases}$$
$$G_n(x, t, s) = \begin{cases} G(x, s + t, \widetilde{\lambda}_n), & s + t \le x, \\ 0, & s + t > x, \end{cases}$$

$$\varphi_n(x) = \int_0^x w(t, \widetilde{\lambda}_n) S(x - t, \widetilde{\lambda}_n)\, dt, \qquad \xi_n(x) = \int_0^\pi v(x, t) y_n(t)\, dt,$$

$$\psi_n(x) = \frac{n}{x}\varphi_n'(x), \qquad \psi_{n0}(x) = \frac{n}{2\widetilde{\rho}_n}\sin\widetilde{\rho}_n x, \qquad \eta_n(x) = \frac{n}{\pi - x}\overline{\xi_n(x)}.$$

Let W_2^1 be the space of functions $f(x)$ absolutely continuous on $[0,\pi]$ and such that $f'(x) \in \mathcal{L}_2(0,\pi)$, with the norm $\|f\|_{W_2^1} = \|f\|_{\mathcal{L}_2(0,\pi)} + \|f'\|_{\mathcal{L}_2(0,\pi)}$, and let $W_{20}^1 = \{f(x) \in W_2^1 : f(0) = f(\pi) = 0\}$.

Lemma 8.5. *The functions $\{\psi_n(x)\}_{n\ge1}$ constitute a Riesz basis in $\mathcal{L}_2(0,\pi)$ and the biorthogonal basis $\{\psi_n^*(x)\}_{n\ge1}$ possesses the following properties:*

(1) $\psi_n^*(x) \in W_{20}^1$,
(2) $|\psi_n^*(x)| < C$, $n \ge 1$, $x \in [0,\pi]$,
(3) *for any* $\{\theta_n\} \in L_2$

$$\theta(x) \overset{\text{def}}{=} \sum_{n=1}^\infty \frac{\theta_n}{n}\psi_n^*(x) \in W_{20}^1, \qquad \|\theta(x)\|_{W_2^1} < C\left(\sum_{n=1}^\infty |\theta_n|^2\right)^{\frac{1}{2}}.$$

To prove, we shall use the well-known results for the Sturm–Liouville IP. Since $\Lambda < \infty$, from (8.3) follows that

$$\widetilde{\rho}_n = \sqrt{\widetilde{\lambda}_n} = n + \frac{A_1}{n} + \frac{\widetilde{\varkappa}_n}{n}, \qquad \{\widetilde{\varkappa}_n\} \in l_2. \tag{8.21}$$

Consequently, there exists a function $\widetilde{q}(x)$ (not unique) such that the numbers $\{\widetilde{\lambda}_n\}_{n\ge1}$ are the eigenvalues of the Sturm–Liouville boundary value problem

$$-y'' + \widetilde{q}(x) y = \lambda y, \qquad y(0) = y(\pi) = 0. \tag{8.22}$$

Let $\widetilde{s}_n(x)$ be the eigenfunctions of (8.22) normalized by the condition $\widetilde{s}_n'(0) = n/2$. The functions $\{\widetilde{s}_n(x)\}_{n\ge1}$ constitute a Riesz basis in $\mathcal{L}_2(0,\pi)$, and

$$\int_0^\pi \widetilde{s}_n(x)\widetilde{s}_m(x)\, dx = \delta_{nm}\widetilde{\alpha}_n. \tag{8.23}$$

Using Lemma 8.1, we obtain

$$\widetilde{s}_n(x) = \psi_{n0}(x) + \int_0^x \widetilde{K}(x,t)\psi_{n0}(t)\, dt, \qquad \widetilde{K}(x,0) = 0. \tag{8.24}$$

In particular, from (8.24), (8.23), and (8.21) follows that

$$\widetilde{s}_n(x) = \frac{1}{2}\sin nx + O\left(\frac{1}{n}\right), \quad \widetilde{\alpha}_n = \int_0^\pi \widetilde{s}_n^2(x)\, dx = \frac{\pi}{8} + O\left(\frac{1}{n}\right), \qquad n \to \infty.$$

Due to (8.24), the functions $\{\psi_{n0}(x)\}_{n\ge1}$ constitute a Riesz basis in $\mathcal{L}_2(0,\pi)$. Denote

$$\psi_{n0}^{**}(x) = \widetilde{s}_n(x) + \int_x^\pi \widetilde{K}(t,x)\widetilde{s}_n(t)\, dt. \tag{8.25}$$

From (8.23)–(8.25) follows that

$$\begin{aligned}\int\limits_0^\pi \psi_{n0}(x)\psi_{m0}^{**}(x)\,dx &= \int\limits_0^\pi \psi_{n0}(x)\left(\widetilde{s}_m(x)+\int\limits_x^\pi \widetilde{K}(t,x)\widetilde{s}_m(t)\,dt\right)dx\\ &= \int\limits_0^\pi \widetilde{s}_m(x)\left(\psi_{n0}(x)+\int\limits_0^x \widetilde{K}(x,t)\psi_{n0}(t)\,dt\right) = \int\limits_0^\pi \widetilde{s}_n(x)\widetilde{s}_m(x)\,dx = \delta_{nm}\widetilde{\alpha}_n.\end{aligned} \tag{8.26}$$

Further, we compute

$$\psi_n(x) = \frac{n}{x}\int\limits_0^x w(t,\widetilde{\lambda}_n)S'(x-t,\widetilde{\lambda}_n)\,dt. \tag{8.27}$$

Since

$$S'(x,\lambda) = \cos\rho x + \int\limits_0^x K^1(x,t)\cos\rho t\,dt, \tag{8.28}$$

we obtain, substituting (8.28) and (8.11) into (8.27), as in the proof of Lemma 8.3,

$$\psi_n(x) = \psi_{n0}(x) + \int\limits_0^x V_0(x,t)\psi_{n0}(t)\,dt, \tag{8.29}$$

where $V_0(x,t)$ is a continuous function, $V_0(x,0)=0$. Solving the integral equation (8.29), we find

$$\psi_{n0}(x) = \psi_n(x) + \int\limits_0^x V_1(x,t)\psi_n(t)\,dt, \qquad V_1(x,0)=0. \tag{8.30}$$

Consider the functions

$$\psi_n^{**}(x) = \psi_{n0}^{**}(x) + \int\limits_x^\pi V_1(t,x)\psi_{n0}^{**}(t)\,dt. \tag{8.31}$$

From (8.26), (8.30), and (8.31) follows that

$$\int\limits_0^\pi \psi_n(x)\psi_m^{**}(x)\,dx = \delta_{nm}\widetilde{\alpha}_n. \tag{8.32}$$

By virtue of (8.29) and (8.32), the functions $\{\psi_n(x)\}_{n\geq 1}$ constitute a Riesz basis in $\mathcal{L}_2(0,\pi)$, and the biorthogonal basis $\{\psi_n^*(x)\}_{n\geq 1}$ has the form $\psi_n^*(x) = \overline{\widetilde{\alpha}_n^{-1}\psi_n^{**}(x)}$. Substituting (8.25) into (8.31), we have

$$\psi_n^{**}(x) = \widetilde{s}_n(x) + \int\limits_x^\pi V_1^0(t,x)\widetilde{s}_n(t)\,dt, \qquad V_1^0(t,0)=0.$$

Hence we obtain the required properties of the biorthogonal basis. Lemma 8.5 is proved.

Since $\eta_n(x) = \psi_n(\pi - x)$, Lemma 8.5 implies

Corollary 8.1. *The functions $\{\eta_n(x)\}_{n\geq 1}$ constitute a Riesz basis in $\mathcal{L}_2(0,\pi)$, and the biorthogonal basis $\{\chi_n(x)\}_{n\geq 1}$ possesses the properties:*

(1) $\chi_n(x) \in W_{20}^1$,
(2) $|\chi_n(x) < C,\ n \geq 1,\ x\in[0,\pi]$,

(3) *for any* $\{Q_n\} \in l_2$

$$\theta(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\theta_n}{n} \chi_n(x) \in W_{20}^1, \qquad \|\theta(x)\|_{W_2^1} < C \left(\sum_{n=1}^{\infty} |\theta_n|^2 \right)^{\frac{1}{2}}.$$

Let us return to proving Theorem 8.3. Put

$$\varepsilon() = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n} \chi_n(x). \tag{8.33}$$

Using Lemma 8.1, the relations $\Delta(\lambda) = S(\pi, \lambda)$, $\Delta(\lambda_n) = 0$, and formulas (8.3), (8.21), we obtain the estimate $|\varepsilon_n| = n^2 \big|\Delta(\widetilde{\lambda}_n) - \Delta(\lambda_n)\big| \le C|\lambda_n - \widetilde{\lambda}_n|$. Now by Corollary 8.1 we have $\varepsilon(x) \in W_{20}^1$, $\|\varepsilon(x)\|_{W_2^1} \le C\Lambda$.

Consider in W_{20}^1 the nonlinear equation

$$r = \varepsilon + \sum_{j=2}^{\infty} \psi_j(r), \tag{8.34}$$

where $\varepsilon(x)$ is defined by (8.33), and the operators $z_j = \psi_j(r)$ act from W_{20}^1 to W_{20}^1 according to the formula

$$z_j(x) = -\sum_{n=1}^{\infty} \Bigg(\underbrace{\int_0^{\pi} \cdots \int_0^{\pi}}_{j} r(t_1) \cdots r(t_j) B_{nj}(t_1, \dots, t_j)\, dt_1 \cdots dt_j \Bigg) \chi_n(x),$$

$$B_{nj}(t_1, \dots, t_j) = \frac{n}{(\pi - t_1) \cdots (\pi - \tau_j)} \underbrace{\int_0^{\pi} \cdots \int_0^{\pi}}_{j} v_n(t_1, s_1) G_n(s_1, t_2, s_2) \cdots G_n(s_{j-1}, t_j, s_j) y_n(s_j)\, ds_1 \cdots ds_j,$$

and

$$\big\|\psi_j(r)\big\|_{W_2^1} \le \Big(C\|r\|_{W_2^1}\Big)^j,$$

$$\big\|\psi_j(r) - \psi_j(r^*)\big\|_{W_2^1} \le \|r - r^*\|_{W_2^1} \Big(C \max\big(\|r\|_{W_2^1}, \|r^*\|_{W_2^1}\big)\Big)^{j-1}.$$

By Lemma 8.4, there exists $\delta > 0$ such that for $\Lambda < \delta$ Eq. (8.34) has a solution $r(x) \in W_{20}^1$, $\|r(x)\|_{W_2^1} \le C\Lambda$. Put $\widetilde{M}(x) = M(x) - \big((\pi - x)^{-1} r(x)\big)'$, and consider the boundary value problem $\widetilde{L} = L(q, \widetilde{M})$. Clearly $\widetilde{Q}(x) = Q(x) - r'(x) \in \mathcal{L}_2(0, \pi)$, $\big\|Q(x) - \widetilde{Q}(x)\big\|_{\mathcal{L}_2(0,\pi)} \le C\Lambda$. Since

$$\widehat{M}_1(x) = -\frac{1}{\pi - x} \int_x^{\pi} \widehat{Q}(t)\, dt, \qquad \widehat{M}_0(x) = \widehat{Q}(x) + \widehat{M}_1(x),$$

we have $\big\|M_k(x) - \widetilde{M}_k(x)\big\|_{\mathcal{L}(0,\pi)} \le C\Lambda$, $k = 0, 1$.

It remains to show that the numbers $\{\widetilde{\lambda}_n\}_{n \ge 1}$ are the eigenvalues of the problem $\widetilde{L}$. To do this, consider the functions $\widetilde{y}_n(x)$ which are solutions of the integral equations

$$\widetilde{y}_n(x) = y_n(x) + \int_0^{\pi} \widehat{M}_1(t)\, dt \int_0^{\pi} G_n(x, t, s) \widetilde{y}_n(s)\, ds, \tag{8.35}$$

or, which is the same,

$$\widetilde{y}_n(x) = y_n(x) + \int\limits_0^x \widehat{M}_1(t)\,dt \int\limits_0^{x-t} G(x, s+t, \widetilde{\lambda}_n)\widetilde{y}_n(s)\,ds. \tag{8.36}$$

After integration by parts, (8.36) takes the form

$$\widetilde{y}_n(x) = y_n(x) - \int\limits_0^x \widehat{M}_1(t)\,dt \int\limits_t^x g(x, s, \widetilde{\lambda}_n)\widetilde{y}_n'(s-t)\,ds.$$

Reverse the integration order,

$$\widetilde{y}_n() = y_n(x) - \int\limits_0^x g(x, t, \widetilde{\lambda}_n)\,dt \int\limits_0^t \widehat{M}_1(s)\widetilde{y}_n'(t-s)\,ds.$$

Integrate by parts,

$$\widetilde{y}_n(x) = y_n(x) - \int\limits_0^x g(x, t, \widetilde{\lambda}_n)\,dt \int\limits_0^t \widehat{M}(t-s)\widetilde{y}_n(s)\,ds. \tag{8.37}$$

It follows from (8.37) that

$$l\big(\widetilde{y}_n(x) - y_n(x)\big) = \int\limits_0^x \widehat{M}(t-s)\widetilde{y}_n(s)\,ds = (l - \widetilde{l})\widetilde{y}_n(x),$$

and consequently, $\widetilde{l}\widetilde{y}_n(x) = \widetilde{\lambda}_n\widetilde{y}_n(x)$, $\widetilde{y}_n(0) = 0$, $\widetilde{y}_n'(0) = 1$. Since the solution of the Cauchy problem is unique, we have $\widetilde{y}_n(x) = \widetilde{S}(x, \lambda_n)$.

Write (8.7) in the form

$$\int\limits_0^\pi \widehat{M}_1(x)\,dx \int\limits_0^{\pi-x} w(\pi - x - t, \lambda)\widetilde{S}(t, \lambda)\,dt = \widehat{\Delta}(\lambda).$$

Integrating by parts, we obtain for $\lambda = \widetilde{\lambda}_n$

$$\int\limits_0^\pi \widehat{M}(x)\,dx \int\limits_0^\pi v_n(x, t)\widetilde{y}_n(t)\,dt = \widehat{\Delta}(\widetilde{\lambda}_n). \tag{8.38}$$

Solving (8.35) by the method of successive approximations, we have

$$\widetilde{y}_n(x) = y_n(x) + Y_n(x), \tag{8.39}$$

$$Y_n(x) = \sum_{j=1}^\infty \underbrace{\int\limits_0^\pi \cdots \int\limits_0^\pi}_{j} \widehat{M}_1(t_1)\cdots\widehat{M}_1(t_j) \Bigg(\underbrace{\int\limits_0^\pi \cdots \int\limits_0^\pi}_{j} G_n(x, t_1, s_1)$$

$$\times\, G_n(s_1, t_2, s_2)\cdots G_n(s_{j-1}, t_j, s_j)y_n(s_j)\,ds_1\cdots ds_j \Bigg)\,dt_1\cdots dt_j.$$

Further, multiplying (8.34) by $\overline{\eta_n(x)}$ and integrating from 0 to π, we obtain

$$\int_0^\pi r(x)\overline{\eta_n(x)}\,dx + \sum_{j=2}^{\infty} \underbrace{\int_0^\pi \cdots \int_0^\pi}_{j} r(t_1)\cdots r(t_j) B_{nj}(t_1,\ldots,t_j)\,dt_1\cdots dt_j = \frac{\varepsilon_n}{n}. \tag{8.40}$$

Since $r(x) = (\pi - x)\widehat{M}_1(x)$, $\overline{\eta_n(x)} = n(\pi - x)^{-1}\xi_n(x)$, we can transform (8.40) to the form

$$\int_0^\pi \widehat{M}_1(x)\xi_n(x)\,dx + \sum_{j=2}^{\infty} \underbrace{\int_0^\pi \cdots \int_0^\pi}_{j} \widehat{M}_1(t_1)\cdots \widehat{M}_1(t_j) \Bigg(\underbrace{\int_0^\pi \cdots \int_0^\pi}_{j} v_n(t_1, s_1)$$

$$\times G_n(s_1, t_2, s_2)\cdots G_n(s_{j-1}, t_j, s_j) y_n(s_j)\,ds_1\cdots ds_j \Bigg)\,dt_1\cdots dt_j = \frac{\varepsilon_n}{n^2}.$$

Hence, taking (8.39) into account, we obtain

$$\int_0^\pi \widehat{M}_1(x)\,dx \int_0^\pi v_n(x,t)\widetilde{y}_n(t)\,dt = \Delta(\widetilde{\lambda}_n). \tag{8.41}$$

Comparing (8.38) with (8.41), we find that $\widetilde{\Delta}(\widetilde{\lambda}_n) = 0$. Hence the numbers $\{\widetilde{\lambda}_n\}_{n\ge 1}$ are the eigenvalues of the boundary value problem $\widetilde{L}$. Theorem 8.3 is proved.

9. One-Dimensional Perturbations of Integral Volterra Operators

9.1. Formulations of the results. In this section, we investigate the IP for the integral operator $A = A(M, g, v)$ of the form

$$Af = \int_0^x M(x,t) f(t)\,dt + g(x) \int_0^\pi f(t) v(t)\,dt, \qquad 0 \le x \le \pi. \tag{9.1}$$

Let $M(x,t,\lambda)$ denote the kernel of the integral operator $M_\lambda = (E - \lambda M)^{-1} M$, where E is the identity operator and $Mf = \int_0^x M(x,t) f(t)\,dt$. Let us set

$$g(x,\lambda) = g(x) + \lambda \int_0^x M(x,t,\lambda) g(t)\,dt. \tag{9.2}$$

Then the characteristic numbers $\{\lambda_k\}$ of A coincide with the zeros of the function

$$\mathcal{L}(\lambda) = 1 - \lambda \int_0^\pi v(x) g(x,\lambda)\,dx, \tag{9.3}$$

which is called the characteristic function of A. The eigen- and associated functions $g_k(x)$ of the operator have the form

$$g_{k+\nu}(x) = \frac{\partial^\nu}{\partial \lambda^\nu} g(x,\lambda)_{|\lambda=\lambda_k}, \qquad \nu = \overline{0, r_k - 1},$$

if r_k is the multiplicity of λ_k ($\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r_k-1}$). Let $\beta_k = g_k(\pi)$. We will call the set of the numbers $\{\lambda_k, \beta_k\}$ the spectral data of A.

We consider the following IP's.

Problem 9.1. Given the spectrum $\{\lambda_k\}$ and the functions $M(x,t)$, $g(x)$, construct the function $v(x)$.

Problem 9.2. Given the spectral data $\{\lambda_k, \beta_k\}$ and the function $M(x,t)$, construct the functions $g(x)$ and $v(x)$.

Let the function $M(x,t)$ satisfy the following condition (condition M_1): *the functions* $\dfrac{\partial^{\nu+j}}{\partial x^\nu \partial t^j} M(x,t)$, $\nu, j = 0, 1$, *are continuous for* $0 \le t \le x \le \pi$, *and* $M(x,x) = -i$, $\dfrac{\partial}{\partial x} M(x,t)_{|t=x} = 0$.

Then the operator $D = M^{-1}$ has the form

$$Dy = iy'(x) + \int\limits_0^x H(x,t) y(t)\, dt, \qquad y(0) = 0,$$

where $H(x,t)$ is a continuous function for $0 \le t \le x \le \pi$.

Definition. We will write $A \in \Lambda_{00}^{(1)}$, if the function $M(x,t)$ satisfies the condition M_1, the functions $g(x)$ and $v(x)$ are absolutely continuous for $0 \le x \le \pi$, $g'(x)$ and $v'(x) \in \mathcal{L}_2(0,\pi)$ and $a_0 b_0 \ne 0$, where

$$\begin{cases} a_0 = 1 + ig(0)v(0) + \int\limits_0^\pi v(\tau) \left(ig'(\tau) + \int\limits_0^\tau H(\tau,s) g(s)\, ds \right) d\tau, \\ b_0 = ig(0)v(\pi). \end{cases} \tag{9.4}$$

For simplicity, we solve Problems 9.1 and 9.2 for operators of the class $\Lambda_{00}^{(1)}$.

Theorem 9.1. *Let* $A \in \Lambda_{00}^{(1)}$. *Then the spectral data* $\{\lambda_k, \beta_k\}$, $k = 0, \pm 1, \pm 2, \ldots$, *of the operator* A *have the form*

$$\begin{cases} \lambda_k = 2k + \alpha + \varkappa_k, \qquad \beta_k = \alpha_1 + \varkappa_{k1}, \\ \lambda_k \ne 0, \quad \alpha_1 \ne 0, \quad \{\varkappa_k\}, \{\varkappa_{k1}\} \in l_2. \end{cases} \tag{9.5}$$

Theorem 9.2. *Let the functions* $M(x,t)$ *and* $g(x)$ *be given such that* $M(x,t)$ *satisfies the condition* M_1, $g(x)$ *is absolutely continuous,* $g'(x) \in \mathcal{L}_2(0,\pi)$, $g(0) \ne 0$. *Further, let the numbers* λ_k, $k = 0, \pm 1, \pm 2, \ldots$, *are of the form* $\lambda_k = 2k + \alpha + \varkappa_k$, $\lambda_k \ne 0$, $\{\varkappa_k\} \in l_2$. *Then there exists a unique operator* $A(M,g,v) \in \Lambda_{00}^{(1)}$ *for which* $\{\lambda_k\}$ *are the characteristic numbers.*

Theorem 9.3. *If a function* $M(x,t)$ *satisfying the condition* M_1 *and numbers* $\{\lambda_k, \beta_k\}$, $k = 0, \pm 1, \pm 2, \ldots$, *of the form* (9.5) *are given, then there exists a unique operator* $A(M,g,v) \in \Lambda_{00}^{(1)}$ *for which* $\{\lambda_k, \beta_k\}$ *are the spectral data.*

9.2. Proofs of the theorems. Let us first formulate several auxiliary assertions.

Lemma 9.1. *Let the numbers* $\{\lambda_k\}$, $k = 0, \pm 1, \pm 2, \ldots$, *of the form* $\lambda_k = 2k + \alpha + \varkappa_k$, $\lambda_k \ne 0$, $\{\varkappa_k\} \in l_2$ *be given. Denote*

$$\mathcal{L}(\lambda) = \exp(p\lambda) \prod_{k=-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\frac{\lambda}{\lambda_k}\right), \tag{9.6}$$

where

$$p = p_0 + \sum_{k=-\infty}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_k^0} \right), \qquad p_0 = i\pi \exp(i\alpha\pi), \qquad \lambda_k^0 = 2k + \alpha$$

(*the case where* α *is an even integer brings insignificant changes*). *Then the following representation is valid for* $\mathcal{L}(\lambda)$:

$$\mathcal{L}(\lambda) = \gamma\Big(1 - \exp\big(i(\alpha - \lambda)\pi\big)\Big) + \int_0^\pi w(t)\exp(-i\lambda t)\,dt, \tag{9.7}$$

$$\gamma = \prod_{k=-\infty}^{\infty} \frac{\lambda_k^0}{\lambda_k}, \qquad w(t) \in \mathcal{L}_2(0,\pi).$$

Proof. The function $\mathcal{L}_0(\lambda) = 1 - \exp\big(i(\alpha-\lambda)\pi\big)$ has zeros $\{\lambda_k^0\}$ and admits the representation

$$\mathcal{L}_0(\lambda) = \exp(p_0\lambda)\prod_{k=-\infty}^{\infty}\left(1 - \frac{\lambda}{\lambda_k^0}\right)\exp\left(\frac{\lambda}{\lambda_k^0}\right).$$

Therefore,

$$\mathcal{L}(\lambda) = \gamma\mathcal{L}_0(\lambda)F(\lambda), \qquad F(\lambda) = \prod_{k=-\infty}^{\infty}\left(1 + \frac{\varkappa_k}{\lambda_k^0 - \lambda}\right). \tag{9.8}$$

Let us show that $|F(\lambda)| < C_\delta$ in the domain $G_\delta = \big\{\lambda : |\lambda - \lambda_k^0| \ge \delta\big\}$ for a fixed $\delta > 0$. We choose an integer N such that $|\varkappa_k| \le \frac{\delta}{2}$ for $|k| \ge N$. Then, for $\lambda \in G_\delta$

$$F(\lambda) = \exp\big(H_N(\lambda)\big)\prod_{|k|<N}\left(1 + \frac{\varkappa_k}{\lambda_k^0 - \lambda}\right), \tag{9.9}$$

where

$$H_N(\lambda) = \sum_{|k|\ge N}\ln\left(1 + \frac{\varkappa_k}{\lambda_k^0 - \lambda}\right) = \sum_{|k|\ge N}\frac{\varkappa_k}{\lambda_k^0 - \lambda}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu+1}\left(\frac{\varkappa_k}{\lambda_k^0 - \lambda}\right)^\nu.$$

Since

$$\big|H_N(\lambda)\big| \le \sum_{|k|\ge N}\frac{|\varkappa_k|}{|\lambda_k^0 - \lambda|}\sum_{\nu=0}^{\infty}\frac{1}{2^\nu} \le C\left(\sum_{|k|\ge N}\frac{1}{\big|2k + \alpha - \lambda\big|}\right)^{\frac{1}{2}},$$

from (9.9) follows that $|F(\lambda)| < C_\delta$ for $\lambda \in G_\delta$.

Further, it follows from (9.8) that

$$\mathcal{L}(\lambda_n^0) = -\gamma\varkappa_n b_n\left(\frac{d}{d\lambda}\mathcal{L}_0(\lambda)\right)_{|\lambda=\lambda_n^0}, \qquad b_n = \prod_{\substack{k=-\infty\\ k\ne n}}^{\infty}\left(1 + \frac{\varkappa_k}{\lambda_k^0 - \lambda_n^0}\right),$$

i.e., $\big\{\mathcal{L}(\lambda_n^0)\big\} \in l_2$. We consider the function

$$\Delta(\lambda) = \mathcal{L}(\lambda) - \gamma\mathcal{L}_0(\lambda). \tag{9.10}$$

Denote $\theta_n = \Delta(\lambda_n^0)$. It is obvious that $\{\theta_n\} \in l_2$. Let us construct a function $w(t) \in \mathcal{L}_2(0,\pi)$ such that $\theta_n = \int_0^\pi w(t)\exp(-i\lambda_n^0 t)\,dt$. Let us consider the function $\theta(\lambda) = \int_0^\pi w(t)\exp(-i\lambda t)\,dt$ and set $S(\lambda) = \big(\mathcal{L}_0(\lambda)\big)^{-1}\big(\theta(\lambda) - \Delta(\lambda)\big)$. The function $S(\lambda)$ is entire of λ. We have $\big|\mathcal{L}_0(\lambda)\big| > C\big(1 + \exp(\operatorname{Im}\lambda\pi)\big)$ in the domain G_δ. From (9.8) and (9.10) follows that $\Delta(\lambda) = \gamma\mathcal{L}_0(\lambda)\big(F(\lambda) - 1\big)$. Using the maximum modulus principle for analytic functions, we see that $S(\lambda)$ is bounded, hence, $S(\lambda) \equiv C$. Since $\lim S(ix) = 0$ for $x \to -\infty$, we have $C = 0$, and we arrive at (9.7). Lemma 9.1 is proved.

By the method of successive approximations one can prove the following assertion.

Lemma 9.2. *The integral equation*

$$P(x,t,\alpha) = i\int_{\alpha}^{x-t+\alpha} H(t+\xi,\xi)\,d\xi + i\int_{\alpha}^{x-t+\alpha} ds\int_0^t H(s+t,s+\xi)P(s+\xi-\alpha,\xi,\alpha)\,d\xi, \qquad (9.11)$$
$$0\le t\le x\le \pi-\alpha, \quad 0\le\alpha-\le\pi,$$

has a unique solution $P(x,t,\alpha)$ *and the functions* $P(x,t,\alpha)$, $\frac{\partial}{\partial x}P(x,t,\alpha)$, $\frac{\partial}{\partial\alpha}P(x,t,\alpha)$ *are continuous with respect to all the variables.*

Lemma 9.3. *Let* $A\in\Lambda_{00}^{(1)}$. *Then the following statements are valid:*

(1) *The characteristic function* $\mathcal{L}(\lambda)$ *of* A *has the form*

$$\mathcal{L}(\lambda) = 1-\lambda\int_0^{\pi} m(t)\exp\big(-i\lambda(\pi-t)\big)\,dt, \qquad (9.12)$$

where

$$m(t) = g(0)u(t) + \int_0^t u(\tau)Q(t,\tau)\,d\tau, \qquad u(t)=v(\pi-t),$$
$$Q(t,\tau) = \frac{d}{dt}\left(g(t-\tau) + \int_0^{t-\tau} P\big(\pi-t+s,s,t-\tau-s\big)g(t-\tau-s)\,ds\right). \qquad (9.13)$$

The function $P(x,t,\alpha)$ *is the solution of* (9.11). *The function* $m(t)$ *is continuous,* $m'(t)\in\mathcal{L}_2(0,\pi)$, $1+im(\pi)=a_0$, $m(0)=-ib_0$, *where* a_0,b_0 *have the form* (9.4).

(2) *The following representation is valid for* $g(\pi,\lambda)$;

$$g(\pi,\lambda) = g(0)\exp(-i\lambda\pi) + \int_0^{\pi}\gamma(t)\exp(-i\lambda t)\,dt, \qquad (9.14)$$

where $\gamma(t)=\mu'(t)\in\mathcal{L}_2(0,\pi)$,

$$\mu(\pi-t) = -g(t) - \int_0^t P\big(\pi-t,t-\tau,\tau\big)g(\tau)\,d\tau. \qquad (9.15)$$

Proof. It is clear that $M_\lambda^{-1}y = Dy-\lambda y$, $y(0)=0$, and the function $z(x)=M(x+\alpha,\alpha,\lambda)$ is the solution of the Cauchy problem

$$iz'(x) + \int_0^x H(x+\alpha,t+\alpha)z(t)\,dt = \lambda z(x), \qquad z(0)=-i, \quad 0\le x\le\pi-\alpha \qquad (9.16)$$

for fixed $\alpha\in[0,\pi]$. Consequently,

$$M(x+\alpha,\alpha,\lambda) = -i\left(\exp(-i\lambda x) + \int_0^x P(x,t,\alpha)\exp\big(-i\lambda(x-t)\big)\,dt\right), \qquad (9.17)$$

since the right-hand side of (9.17) is also a solution of the Cauchy problem (9.16). We substitute (9.17) in (9.2) and obtain

$$g(x,\lambda) = g(x) - i\lambda \int_0^x \exp(-i\lambda t)\left(g(x-t) + \int_0^{x-t} P\big(t+\tau,\tau,x-t-\tau\big)g(x-t-\tau)\,d\tau\right) dt. \tag{9.18}$$

From this we obtain (9.14). Further, we substitute (9.18) in (9.3) and obtain (9.12), where $m(t)$ is defined by (9.13) and is a continuous function. Let us show that $m'(t) \in \mathcal{L}_2(0,\pi)$. For this, we write $m(t)$ in the form

$$m(t) = g(0)u(t) + \int_0^t u(t-\tau)R(t,\tau)\,d\tau,$$

$$R(t,\tau) = g'(\tau) + P\big(\pi-t+\tau,\tau,0\big)g(0) + \int_0^\tau g'(\tau-s)P\big(\pi-t+s,s,\tau-s\big)\,ds$$

$$+ \int_0^\tau g(\tau-s)\widetilde{P}\big(\pi-t+s,s,\tau-s\big)\,ds,$$

$$\widetilde{P}(x,t,\alpha) = \left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial x}\right) P(x,t,\alpha).$$

By virtue of (9.11), we obtain

$$\widetilde{P}\big(\pi-t+s,s,\tau-s\big) = -iH(\tau,\tau-s) - i\int_{\tau-s}^{\pi-t+\tau-s} d\eta \int_0^s H(\eta+s,\eta+\xi)\widetilde{P}\big(\eta+\xi-\tau+s,\xi,\tau-s\big)\,d\xi.$$

Thus, $R(t,\tau)$ is continuously differentiable with respect to t, hence, $m'(t) \in \mathcal{L}_2(0,\pi)$. The lemma is proved.

Proof of Theorem 9.1. By virtue of Lemma 9.3, the characteristic function $\mathcal{L}(\lambda)$ of the operator A has the form

$$\mathcal{L}(\lambda) = a_0 - b_0\exp(-i\lambda\pi) + \int_0^\pi w(t)\exp(-i\lambda t)\,dt = a_0\mathcal{L}_0(\lambda) + \mathcal{L}_1(\lambda), \tag{9.19}$$

where

$$\mathcal{L}_0(\lambda) = 1 - \exp\big(i(\alpha-\lambda)\pi\big), \qquad \mathcal{L}_1(\lambda) = \int_0^\pi w(t)\exp(-i\lambda t)\,dt,$$

$$\exp(-i\alpha\pi) = a_0 b_0^{-1}, \qquad w(t) = i\frac{d}{dt}m(\pi-t) \in \mathcal{L}_2(0,\pi).$$

The estimate $|\mathcal{L}_0(\lambda)| > C\big(1+\exp(\operatorname{Im}\lambda\pi)\big)$ is valid in the domain $G_\delta = \{\lambda : |\lambda-\lambda_k^0| \geq \delta\}$, where $\lambda_k^0 = 2k+\alpha$ and, consequently, $|a_0\mathcal{L}_0(\lambda)| > |\mathcal{L}_1(\lambda)|$ for sufficiently large $|\lambda|$. Therefore, by the Rouche theorem, $2N+1$ zeros λ_k, $k = 0, \pm1, \dots, \pm N$, of $\mathcal{L}(\lambda)$, lie inside the contour $\Gamma_N = \{\lambda : |\lambda-\alpha| = 2N+1\}$ for sufficiently large N, and exactly one zero λ_k of $\mathcal{L}(\lambda)$ lies inside the contour $\gamma_k(\delta) = \{\lambda : |\lambda-\lambda_k^0| = \delta\}$ for sufficiently large λ_k, i.e., $\lambda_k = 2k+\alpha+\varkappa_k$, $\varkappa_k = o(1)$. Substituting this expression in (9.19), we obtain $\{\varkappa_k\} \in l_2$. Using (9.14), we now easily obtain the desired asymptotic formula for β_k. The theorem is proved.

Proof of Theorem 9.2. From given numbers $\{\lambda_k\}$ we construct $\mathcal{L}(\lambda)$ by (9.6). According to Lemma 9.1, the representation (9.7) is valid for $\mathcal{L}(\lambda)$. Let us set

$$m(t) = -ib_0 + i\int_0^t w(\pi - t)\,d\tau, \qquad b_0 = \gamma\exp(i\alpha\pi).$$

Further, let $u(t)$ be a solution of (9.13). It is clear that $u(t)$ is continuous, $u'(t) \in \mathcal{L}_2(0,\pi)$, and $u(0) \neq 0$. Denote $v(t) = u(\pi - t)$ and consider the operator $A(M, g, v)$ of the form (9.1). Let $\mathcal{L}^*(\lambda)$ be the characteristic function of A. Then, as in the proof of Lemma 9.3, we obtain

$$\mathcal{L}^*(\lambda) = 1 - \lambda\int_0^\pi m(t)\exp\big(-i\lambda(\pi - t)\big)\,dt,$$

or, after integrating by parts,

$$\mathcal{L}^*(\lambda) = 1 + im(\pi) - im(0)\exp(-i\lambda\pi) + \int_0^\pi w(t)\exp(-i\lambda t)\,dt.$$

Comparing this equality with (9.7) and taking into account the relations $\mathcal{L}(0) = \mathcal{L}^*(0) = 1$, $im(0) = \gamma\exp(i\alpha\pi)$, we get $\mathcal{L}^*(\lambda) \equiv \mathcal{L}(\lambda)$, $1 + im(\pi) = \gamma$, and consequently, $A \in \Lambda_{00}^{(1)}$, and $\{\lambda_k\}$ is the spectrum of A. If it is assumed that there exists an operator $A(M, g, \widetilde{v}) \in \Lambda_{00}^{(1)}$ with the same spectrum $\{\lambda_k\}$, then it would follow from Lemma 9.3 and the uniqueness of the solution of the integral equation (9.13) that $v(t) = \widetilde{v}(t)$, $t \in [0,\pi]$. The theorem is proved.

Proof of Theorem 9.3. For simplicity, we confine ourselves to the case where all λ_k are different. As in the proof of Theorem 9.2, we construct $\mathcal{L}(\lambda)$, $m(t)$, and $P(x,t,\alpha)$ from given $M(x,t)$ and $\{\lambda_k\}$. Denote $\mu_k = \lambda_k - \alpha$, $g = \alpha_1\exp(i\alpha\pi)$, $\widetilde{\beta}_k = \beta_k - g\exp(-i\lambda_k\pi)$. It is clear that $\{\widetilde{\beta}_k\} \in l_2$. The system of the functions $\exp(-i\mu_k t)$ forms a Riesz basis in $\mathcal{L}_2(0,\pi)$, since it is complete and quadratically close to the orthogonal basis $\exp(-2kit)$. Let $h(t) \in \mathcal{L}_2(0,\pi)$ be such that

$$\widetilde{\beta}_k = \int_0^\pi h(t)\exp(-i\mu_k t)\,dt$$

and set

$$\mu(t) = -g - \int_t^\pi h(\tau)\exp(i\alpha\tau)\,d\tau.$$

Let the function $g(t)$ be a solution of (9.15). It is clear that $g(t)$ is continuous, $g'(t) \in \mathcal{L}_2(0,\pi)$, and $g(0) = g \neq 0$. As in Theorem 9.2, we now find the function $v(t)$. Thus, we construct the operator $A(M, g, v)$ of the form (9.1), and the numbers $\{\lambda_k, \beta_k\}$ are the spectral data of A. As in Theorem 9.2, the uniqueness follows obviously from Lemma 9.3. In the case of multiple λ_k, the system of the functions $t^\nu\exp(-i\mu_k t)$, $\nu = \overline{0, r_k - 1}$, where r_k is the multiplicity of λ_k, is a Riesz basis. The theorem is proved.

Remark. Results, analogous to the above ones, hold also for other classes of operators, e.g., for the operators $A \in \Lambda_{\nu\mu}^{(m)}$, $\max(\nu,\mu) < m$, whose characteristic functions have the form

$$\mathcal{L}(\lambda) = \sum_{k=0}^{m-1}\lambda^{-k}\Big(a_k - b_k\exp\big(-i\lambda\pi\big)\Big) + \lambda^{1-m}\int_0^\pi w_m(t)\exp\big(-i\lambda t\big)\,dt,$$

$$w_m(t) \in \mathcal{L}_2(0,\pi), \qquad a_\nu b_\mu \neq 0, \quad a_k = b_j = 0, \quad k = \overline{0, \nu - 1}, \quad j = \overline{0, \mu - 1}.$$

Let us observe that similar results are also valid for the case where M^{-1} is an integro-differential operator of second order.

9.3. Connections with IP's for DO's. In spite of the qualitative difference of the above-considered problems from the IP's for DO's, there are connections between them. In this section, by the example of the Borg theorem [16] we show how the IP for DO's can be reduced to Problem 9.1. For this, we give here a general uniqueness theorem for the solution of Problem 9.1.

Let us consider an operator A of the form (9.1) under the assumption that the function $M(x,t)$ is the Hilbert–Schmidt kernel and $g(x)$, $v(x) \in \mathcal{L}_2(0,\pi)$.

Theorem 9.4. *Let the system of the eigen- and associated functions $g_k(x)$ of the operator $A(M,g,v)$ be complete in $\mathcal{L}_2(0,\pi)$, and let $\{\lambda_k\}$ and $\{\widetilde{\lambda}_k\}$ be the spectra of $A = A(M,g,v)$ and $\widetilde{A} = A(M,g,\widetilde{v})$ respectively. If $\lambda_k = \widetilde{\lambda}_k$ for all k, then $v(x) = \widetilde{v}(x)$ a.e. on $[0,\pi]$.*

Indeed, under the conditions of the theorem, from (9.3) follows that

$$\int_0^\pi \big(v(x) - \widetilde{v}(x)\big) g(x,\lambda)\, dx = \lambda^{-1}\big(\widetilde{\mathcal{L}}(\lambda) - \mathcal{L}(\lambda)\big).$$

Therefore $\int_0^\pi \big(v(x) - \widetilde{v}(x)\big) g_k(x)\, dx = 0$, and, consequently, $v(x) = \widetilde{v}(x)$ a.e. on $[0,\pi]$.

Let us consider the boundary value problems $L_i = L\big(q(x), h, H_i\big)$, $i = 1,2$:

$$\begin{gathered} -y'' + q(x)y = \lambda y, \qquad q'(x) \in \mathcal{L}_2(0,\pi), \\ y'(0) - hy(0) = y'(\pi) + H_i y(\pi) = 0, \qquad H_1 \neq H_2. \end{gathered} \tag{9.20}$$

Let the functions $\varphi(x,\lambda)$ and $\psi_i(x,\lambda)$ be the solutions of (9.20) under the initial conditions $\varphi(0,\lambda) = \psi_i(\pi,\lambda) = 1$, $\varphi'(0,\lambda) = h$, $\psi_i'(\pi,\lambda) = -H_i$, and let $M(x,t,\lambda)$ be the Green function of the operator $y'' - q(x)y - \lambda y$, $y(0) = y'(0) = 0$. Then the eigenvalues $\{\mu_{ni}\}_{n\geq 0}$ of L_i are the zeros of the functions $\Delta_i(\lambda) = \psi_i'(0,\lambda) - h\psi_i(0,\lambda)$, and the functions $\Delta_i(\lambda)$ are determined uniquely by their zeros. It is known that if a function $G(x,t)$ satisfies the conditions

$$\begin{cases} \dfrac{\partial^2 G(x,t)}{\partial x^2} - q(x)G(x,t) = \dfrac{\partial^2 G(x,t)}{\partial t^2} - \widetilde{q}(x)G(x,t), \qquad 0 \leq t \leq x \leq \pi, \\ G(x,x) = h = \dfrac{1}{2}\displaystyle\int_0^x \big(q9t) - \widetilde{q}(t)\big)\, dt, \qquad \left(\dfrac{\partial G(x,t)}{\partial t} - \widetilde{h}G(x,t)\right)\Bigg|_{t=0} = 0, \end{cases} \tag{9.21}$$

then

$$\varphi(x,\lambda) = (E+G)\widetilde{\varphi}(x,\lambda), \qquad M_\lambda(E+G) = (E+G)\widetilde{M}_\lambda, \tag{9.22}$$

where

$$(E+G)f = F(x) + \int_0^x G(x,t)f(t)\, dt, \qquad M_\lambda f = \int_0^x M(x,t,\lambda)f(t)\, dt.$$

Let us consider the family of the operators $L_{\alpha,i}\big(q(x), h, H_2, H_2\big)$:

$$\begin{gathered} L_{\alpha,i}y = y'' - q(x)y + \alpha y, \qquad -\infty < \alpha < \infty, \\ y'(0) - hy(0) = y'(\pi) + H_i y(\pi) = 0. \end{gathered}$$

The inverse operators $A_{\alpha,i} = L_{\alpha,i}^{-1}$ have the form

$$A_{\alpha,i}f = \int_0^x M(x,t,\alpha)f(t)\,dt + \frac{\varphi(x,\alpha)}{\Delta_i(\alpha)}\int_0^\pi \psi_i(t,\alpha)f(t)\,dt,$$

and $\{\mu_{ni} - \alpha\}$ is the spectrum of $A_{\alpha,i}$. Analogously, the operators

$$\widetilde{A}_{\alpha,i}f = \int_0^x \widetilde{M}(x,t,\alpha)f(t)\,dt + \frac{\widetilde{\varphi}(x,\alpha)}{\widetilde{\Delta}_i(\alpha)}\int_0^\pi \widetilde{\psi}_i(t,\alpha)f(t)\,dt,$$

are inverse to the operators $L_{\alpha,i}\big(\widetilde{q}(x),\widetilde{h},\widetilde{H}_1,\widetilde{H}_2\big)$ and have the spectrum $\{\widetilde{\mu}_{ni} - \alpha\}$. Now we show that the Borg theorem [16] can be obtained as a corollary of Theorem 9.4.

Theorem (Borg). *If $\mu_{ni} = \widetilde{\mu}_{ni}$, $i = \overline{1,2}$, then $q(x) = \widetilde{q}(x)$, $h = \widetilde{h}$, $H_k = \widetilde{H}_k$.*

Proof. Let $G(x,t)$ satisfy (9.21). Denote $B_{\alpha,i} = (E+G)^{-1}A_{\alpha,i}(E+G)$. Then, using (9.22), we obtain

$$B_{\alpha,i}f = \int_0^x \widetilde{M}(x,t,\alpha)f(t)\,dt + \frac{\widetilde{\varphi}(x,\alpha)}{\widetilde{\Delta}_i(\alpha)}\int_0^\pi v_i(x,\alpha)f(t)\,dt,$$

where

$$v_i(x,\alpha) = (E+G^*)\psi_i(x,\alpha), \qquad (E+G^*)f = f(x) + \int_x^\pi G(t,x)f(t)\,dt.$$

Under the conditions of the theorem, the operators $\widetilde{A}_{\alpha,i}$ and $B_{\alpha,i}$ have identical spectra and, consequently, by Theorem 9.4 we have $\widetilde{\psi}_i(x,\alpha) = (E+G^*)\psi_i(x,\alpha)$. Since

$$\varphi(x,\alpha) = (H_1 - H_2)^{-1}\big(\Delta_2(\alpha)\psi_1(x,\alpha) - \Delta_1(\alpha)\psi_2(x,\alpha)\big),$$

we have $\widetilde{\varphi}(x,\alpha) = (E+G^*)\varphi(x,\alpha)$, which, together with (9.22), gives $(E+G^*) = (E+G)^{-1}$. This is possible only in the case where $G(x,t) \equiv 0$. Consequently, $q(x) = \widetilde{q}(x)$, $h = \widetilde{h}$, $H_k = \widetilde{H}_k$. The theorem is proved.

PART 4

NONLINEAR INTEGRABLE DIFFERENTIAL EQUATIONS AND THE INVERSE PROBLEM METHOD

10. A Mixed Problem for the Boussinesq Equation

We study a mixed problem for the nonlinear Boussinesq equation on the half-line. An algorithm for the solution and necessary and sufficient conditions of solvability of this problem are obtained, and uniqueness is proved.

Let us consider the following problem:

$$u_t = i(2v_x - u_{xx}), \qquad v_t = i\left(v_{xx} - \frac{2}{3}u_{xxx} - \frac{2}{3}uu_x\right), \qquad x > 0, \quad t > 0, \tag{10.1}$$

$$u|_{t=0} = u_0(x), \qquad v|_{t=0} = v_0(x), \qquad u_0(x), v_0(x) \in \mathcal{L}(0,\infty), \tag{10.2}$$

$$u|_{x=0} = u_1(t), \qquad u_x|_{x=0} = u_2(t), \qquad v|_{x=0} = v_1(t), \qquad v_x|_{x=0} = v_2(t). \tag{10.3}$$

System (10.1), after elimination of $v(x,t)$, reduces to the Boussinesq equation

$$3u_{tt} = u_{xxxx} + 2(u^2)_{xx}.$$

In this section, the mixed problem (10.1)–(10.3) is solved by the inverse problem method. For this we use the results of the IP for third-order DO's on the half-line by its WM, obtained in Sec. 2. We note that in [3, 13, 68, 80], using the IP for second-order equations, evolution of spectral data for difference and differential nonlinear equations on the half-line is obtained, and in [21, 47] the Boussinesq equation on the line is studied by the inverse scattering method.

Let $D = \{(x,t) : x \geq 0,\ t \geq 0\}$, and let J_n be the set of functions $z(x,t)$ such that functions $\dfrac{\partial^{j+k}}{\partial x^j \partial t^k} z(x,t)$, $0 \leq j + 2k \leq n$, are continuous in D, integrable on the half-line $x \in (0,\infty)$ for any fixed $t > 0$, and $x\left(\dfrac{\partial^n z}{\partial x^n}\right) \in \mathcal{L}(0,\infty)$. We shall write $\{u(x,t), v(x,t)\} \in M$, if $u(x,t) \in J_3$, $v(x,t) \in J_2$. We denote by A_{ij}, B_{ij}, ..., elements of matrices A, B, ..., where i is the number of the row, and j is the number of the column.

10.1. Auxiliary statements. Let $\{u(x,t), v(x,t)\} \in M$. For a fixed $t \geq 0$, we consider the DE with respect to x

$$ly \equiv y''' + uy' + vy = \lambda y = \rho^3 y. \tag{10.4}$$

Let $\Phi(x,t,\lambda) = [\Phi_k^{(j-1)}(x,t,\lambda)]_{j,k=\overline{1,3}}$, where $\Phi_k(x,t,\lambda)$ is the solution of (10.4) under the conditions $\Phi_k^{(j-1)}(0,t,\lambda) = \delta_{jk}$, $j = \overline{1,k}$, $\Phi_k(x,t,\lambda) = O\big(\exp(\rho r_k x)\big)$, $x \to \infty$. Here r_k are the roots of the equation $r^3 - 1 = 0$ such that

$$\operatorname{Re}\rho r_1 < \operatorname{Re}\rho r_2 < \operatorname{Re}\rho r_3. \tag{10.5}$$

We set $\mathfrak{M}(t,\lambda) = \Phi(0,t,\lambda)$, i.e.,

$$\mathfrak{M}^T(t,\lambda) = [\mathfrak{M}_{kj}(t,\lambda)]_{k,j=\overline{1,3}}, \qquad \mathfrak{M}_{kj}(t,\lambda) = \delta_{kj} \quad (j \leq k), \tag{10.6}$$

where $\mathfrak{M}_{kj}(t,\lambda) = \Phi_k^{(j-1)}(0,t,\lambda)$, $k < j$. Functions $\mathfrak{M}_{kj}(t,\lambda)$ are the WF's, and the matrix $\mathfrak{M}(t,\lambda)$ is the WM for l.

Let $\mathfrak{M}^* = \mathfrak{M}^{-1}$, $\Phi_j^* = \Phi_1\Phi'_{j+1} - \Phi_{j+1}\Phi'_1$, $j = 1,2$. It is clear that the functions Φ_j^* are solutions of the equation

$$l^* z \equiv -z''' - (uz)' + vz = \lambda z.$$

For fixed $t > 0$, $k = \overline{1,3}$, $j = \left[\dfrac{k+1}{2}\right]$, $p = \left[\dfrac{5-k}{2}\right]$ we consider the functions

$$\psi_k(x,\lambda) = \big(\psi_{k1}(x,\lambda), \psi_{k2}(x,\lambda)\big) = \left(\Phi_j(x,t,\lambda)\Phi_p^*(x,t,\lambda),\ -\int_x^\infty \Phi'_j(s,t,\lambda)\Phi_p^*(s,t,\lambda)\,ds\right),$$

where $[\cdot]$ denotes the greatest integer in the number. In [103], the following completeness theorem is proved.

Theorem 10.1. *If*

$$\int_0^\infty \psi_k(x,\lambda) f(x)\, fx = 0, \qquad k = \overline{1,3}, \tag{10.7}$$

$$f(x) = \big(f_1(x), f_2(x)\big)^T \in \mathcal{L}(0,\infty),$$

then $f(x) = 0$ *a.e.*

Let us now denote

$$G(x,t,\lambda)=\begin{bmatrix}0&1&0\\0&0&1\\\lambda-v&-u&0\end{bmatrix},\qquad F(x,t,\lambda)=i\begin{bmatrix}\frac{2}{3}u&0&1\\\lambda-v+\frac{2}{3}u_x&-\frac{1}{3}u&0\\\frac{2}{3}u_{xx}-v_x&\lambda-v+\frac{1}{3}u_x&-\frac{1}{3}u\end{bmatrix},$$

$$Q=G_t-F_x+GF-FG,\qquad q=[Q_1,Q_2]^T,$$

$$Q_1-0v_t+i\left(v_{xx}-\frac{2}{3}u_{xxx}-\frac{2}{3}uu_x\right),\qquad Q_2=-u_t+i(2v_x-u_{xx}).$$

Then

$$Q(x,t)=\begin{bmatrix}0&0&0\\0&0&0\\Q_1&Q_2&0\end{bmatrix},$$

i.e., system (10.1) is equivalent to the equality $Q=0$. We define the matrices $W(x,t,\lambda)$ and $S(x,t,\lambda)$ from the relations

$$W_x=G(x,t,\lambda)W,\qquad W|_{x=0}=E, \tag{10.8}$$

$$S_t=F(x,t,\lambda)S,\qquad S|_{t=0}=E, \tag{10.9}$$

where $E=[\delta_{jk}]_{j,k=\overline{1,3}}$ is the identity matrix. Then

$$\Phi(x,t,\lambda)=W(x,t,\lambda)\mathfrak{M}(t,\lambda). \tag{10.10}$$

Consider the matrices

$$C^0(t,\lambda)=\Big(\mathfrak{M}_t^*(t,\lambda)+\mathfrak{M}^*(t,\lambda)F^0(t,\lambda)\Big)\mathfrak{M}(t,\lambda),\qquad F^0(t,\lambda)=F(0,t,\lambda), \tag{10.11}$$

$$d(x,t,\lambda)=F^0(t,\lambda)-\int_0^x W^{-1}(s,t,\lambda)Q(s,t)W(s,t,\lambda)\,ds, \tag{10.12}$$

$$C(x,t,\lambda)=C^0(t,\lambda)-\int_0^x \Phi^{-1}(s,t,\lambda)Q(s,t)\Phi(s,t,\lambda)\,ds. \tag{10.13}$$

Lemma 10.1. *The following equality holds:*

$$\Phi_t(x,t,\lambda)=F(x,t,\lambda)\Phi(x,t,\lambda)-\Phi(x,t,\lambda)C(x,t,\lambda). \tag{10.14}$$

Proof. By virtue of (10.8), we have $\big(W_t-FW\big)_x-G(W_t-FW\big)=QW$, $\big(W_t-FW\big)\big|_{x=0}=-F^0(t,\lambda)$. Consequently

$$W_t(x,t,\lambda)=F(x,t,\lambda)W(x,t,\lambda)-W(x,t,\lambda)d(x,t,\lambda). \tag{10.15}$$

Hence, according to (10.10), we obtain (10.14).

Lemma 10.2. *The following relations are valid:*

$$C^0_{kj}(t,\lambda)=(-1)^{k-1}\int_0^\infty \varphi_{k+j-2}(x,t,\lambda)q(x,t)\,dx,\qquad 1\le j<k\le 3, \tag{10.16}$$

where $\varphi_k=(\psi_{k1},\psi'_{k2})$, $k=\overline{1,3}$.

Proof. Rewriting (10.13) in coordinates and using properties of the functions Φ_k we obtain in particular that with fixed t, λ

$$C_{kj}(x,t,\lambda) = O(1), \quad k \geq j; \qquad C_{12}(x,t,\lambda) = O\Big(\exp\big(\rho(r_2 - r_2)x\big)\Big), \quad x \to \infty, \tag{10.17}$$

$$C_{kj}(x,t,\lambda) = C^0_{kj}(t,\lambda) + (-1)^k \int\limits_0^x \varphi_{k+j-2}(s,t,\lambda) q(s,t)\, ds, \qquad 1 \leq j < k \leq 3, \tag{10.18}$$

and

$$\Phi_k^{(j-1)}(x,t,\lambda) = \rho^{1-k} \sum_{m=1}^{k} (\rho r_m)^{j-1} \exp(\rho r_m x)\big(a_{km} + o(1)\big), \qquad |r|x \to \infty, \quad a_{mm} \neq 0.$$

Now, by virtue of (10.14),

$$\sum_{m=1}^{3} \Phi_m^{(j-1)}(x,t,\lambda) C_{mk}(x,t,\lambda) = -\frac{\partial}{\partial t}\Phi_k^{(j-1)}(x,t,\lambda) + \sum_{m=1}^{3} F_{jm}(x,t,\lambda)\Phi_k^{(m-1)}(x,t,\lambda).$$

Hence, using (10.17) we calculate with fixed t, λ ($\operatorname{Im}\lambda \neq 0$)

$$\lim_{x\to\infty} C_{kj}(x,t,\lambda) = 0, \qquad j < k.$$

Together with (10.18) it implies (10.16). Lemma 10.2 is proved.

Rewriting (10.11) in coordinates, substituting into (10.16) and solving with respect to $\dfrac{\partial}{\partial t}\mathfrak{M}_{jk}(t,\lambda)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\mathfrak{M}_{jk} = \sum_{m=0}^{3-j} \mathfrak{M}_{j,j+m}\Big(F^0_{k,j+m} - F^0_{j,j+m}\mathfrak{M}_{jk} + \delta_{j2}F^0_{1,2+m}\big(-\mathfrak{M}_{13} + \mathfrak{M}_{12}\mathfrak{M}_{23}\big)\Big) \\ + (-1)^k \int\limits_0^\infty \big(\varphi_{k+j-2} - \delta_{k-j,2}\mathfrak{M}_{23}\varphi_1\big) q\, dx, \qquad 0 \leq j < k \leq 3. \end{aligned} \tag{10.19}$$

10.2. Solution of the problem (10.1)-(10.3). In the following theorem evolution of the WM with respect to t is obtained.

Theorem 10.2. *Let $\{u(x,t), v(x,t)\}$ be the solution of the problem* (10.1) – –(10.3). *We denote $u_3(x,t) = u_{xx}\big|_{x=0}$ and $\mathfrak{M}^0_{jk}(\lambda) = \mathfrak{M}_{jk}(0,\lambda)$ are the WF for $\{u_0(x),\ v_0(x)\}$ and*

$$\widetilde{F}(t,\lambda) = i \begin{bmatrix} \frac{2}{3}u_1(t) & 0 & 1 \\ \lambda - v_1(t) + \frac{2}{3}u_2(t) & -\frac{1}{3}u_1(t) & 0 \\ \frac{2}{3}u_3(t) - v_2(t) & \lambda - v_1(t) + \frac{1}{3}u_2(t) & -\frac{1}{3}u_1(t) \end{bmatrix}. \tag{10.20}$$

Let the matrix $R(t,\lambda)$ be the solution of the Cauchy problem

$$R_t(t,\lambda) = -R(t,\lambda)\widetilde{F}(t,\lambda), \qquad R\big|_{t=0} = E. \tag{10.21}$$

We define

$$\begin{cases} \Delta_k(t,\lambda) = R_{3k}(t,\lambda) - \mathfrak{M}^0_{23}(\lambda)R_{2k}(t,\lambda) + \Big(\mathfrak{M}^0_{12}(\lambda)\mathfrak{M}^0_{23}(\lambda) - \mathfrak{M}^0_{13}(\lambda)\Big)R_{1k}(t,\lambda), \\ \Delta_{mk}(t,\lambda) = \det\Big[R_{jp}(y,\lambda) - \mathfrak{M}^0_{1j}(\lambda)R_{1p}(t,\lambda)\Big]_{j=2,3;p=m,k}. \end{cases} \tag{10.22}$$

Then

$$\mathfrak{M}_{12}(t,\lambda) = -\frac{\Delta_{13}(t,\lambda)}{\Delta_{23}(t,\lambda)}, \qquad \mathfrak{M}_{13}(t,\lambda) = -\frac{\Delta_{21}(t,\lambda)}{\Delta_{23}(t,\lambda)}, \qquad \mathfrak{M}_{23}(t,\lambda) = -\frac{\Delta_{2}(t,\lambda)}{\Delta_{3}(t,\lambda)}, \tag{10.23}$$

$$\frac{\partial}{\partial t}\mathfrak{M}_{jk} = \sum_{m=0}^{3-j} \mathfrak{M}_{j,j+m}\Big(\widetilde{F}_{k,j+m} - \widetilde{F}_{j,j+m}\mathfrak{M}_{jk} + \delta_{j2}\widetilde{F}_{1,2+m}\big(-\mathfrak{M}_{13} + \mathfrak{M}_{12}\mathfrak{M}_{23}\big)\Big), \qquad 0 \le j < k \le 3. \tag{10.24}$$

Proof. Since $\{u(x,t), v(x,t)\}$ is the solution of (10.1)–(10.3), then $Q(x,t) = 0$, $q(x,t) = 0$, $\widetilde{F}(t,\lambda) = F^0(t,\lambda)$. Consequently, by virtue of (10.12)–(10.16), we have

$$C^0_{kj}(t,\lambda) = 0, \qquad 1 \le j < k \le 3, \tag{10.25}$$

$$W_t(x,t,\lambda) = F(x,t,\lambda)W(x,t,\lambda) - W(x,t,\lambda)F^0(t,\lambda), \tag{10.26}$$

$$\Phi_t(x,t,\lambda) = F(x,t,\lambda)\Phi(x,t,\lambda) - \Phi(x,t,\lambda)C^0(t,\lambda). \tag{10.27}$$

It follows from (10.26), in accordance with (10.9), (10.10), and (10.21) that

$$W(x,t,\lambda) = S(x,t,\lambda)W(x,0,\lambda)R(t,\lambda) \quad \text{and} \quad \Phi(x,t,\lambda) = S(x,t,\lambda)\Phi(x,0,\lambda)B(t,\lambda), \tag{10.28}$$

where

$$B(t,\lambda) = \mathfrak{M}^*(0,\lambda)R(t,\lambda)\mathfrak{M}(t,\lambda). \tag{10.29}$$

Differentiating (10.28) with respect to t and comparing with (10.27), we obtain

$$B_t(t,\lambda) = -B(t,\lambda)C^0(t,\lambda), \qquad B(0,\lambda) = E.$$

Hence, from (10.25) we find $B_{kj}(t,\lambda) = 0$, $j < k$. Rewriting now (10.29) in coordinates for $j < k$ and solving with respect to $\mathfrak{M}_{jk}(t,\lambda)$, we obtain (10.23). Equalities (10.24) follow from (10.19), since $F^0(t,\lambda) = \widetilde{F}(t,\lambda)$, $q(x,t) = 0$. We note that (10.24) can be obtained directly from (10.23) by differentiating with respect to t. Theorem 10.2 is proved.

Using evolution relations (10.23) and the solution of the IP for equation (10.4), we obtain the following algorithm for the solution of the mixed problem (10.1)–(10.3).

Algorithm 10.1. For $x \ge 0$, $t \ge 0$ continuous functions $u_0(x)$, $v_0(x)$, $u_1(x)$, $v_1(x)$, $u_2(x)$, $v_2(x)$ are given. Let $u_0(x)$, $v_0(x) \in \mathcal{L}(0,\infty)$, $u_0(0) = u_1(0)$, $v_0(0) = v_1(0)$, and $\frac{\partial}{\partial t}u_1(t)$ be continuous. We then:

(1) compute the function $u_3(t) = 2v_2(t) + i\frac{\partial}{\partial t}u_1(t)$;
(2) find the WF's $\mathfrak{M}^0_{kj}(\lambda)$, $1 \le k < j \le 3$ for $\{u_0, v_0\}$;
(3) find the matrix $R(t,\lambda)$ from (10.20) and (10.21);
(4) compute the matrix $\mathfrak{M}(t,\lambda)$ using formulas (10.6), (10.22), and (10.23);
(5) find the functions $\{u(x,t), v(x,t)\}$ by solving the IP by the method described in Sec. 2.

Let us now find the conditions of existence of the solution of (10.1)–(10.3). The following theorem shows that existence of the solution of (10.1)(10.3) is equivalent to solvability of the corresponding IP.

Theorem 10.3. *Let the matrix $\mathfrak{M}(t,\lambda)$ be constructed from the given functions u_j, v_j, $j = \overline{0,2}$, according to steps* (1)–(4) *of Algorithm* 10.1. *We assume that there exist functions $\{u(x,t),\ v(x,t)\} \in M$ for which $\mathfrak{M}(t,\lambda)$ is the WM. Then $\{u(x,t),\ v(x,t)\}$ is the solution for* (10.1)–(10.3).

Proof. From (10.23) with $t = 0$ we find $\mathfrak{M}_{jk}(0,\lambda) = \mathfrak{M}^0_{jk}(\lambda)$, $j < k$. The coefficients of the DE (10.4) are uniquely determined from its WM. Then we have $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, i.e., $u(x,t)$, $v(x,t)$ satisfy the initial conditions (10.2).

Differentiating (10.23) with respect to t, we obtain (10.24). Comparing (10.24) with (10.19), we obtain

$$\begin{cases} h_{21}(t) + \big(h_{22}(t) - h_{11}(t)\big)\mathfrak{M}_{12}(t,\lambda) + T_1(t,\lambda) = h_{32}(t) - T_3(t,\lambda) = 0, \\ h_{31}(t) + h_{32}(t)\mathfrak{M}_{12}(t,\lambda) + \big(h_{33}(t) - h_{11}(t)\big)\mathfrak{M}_{13}(t,\lambda) - T_2(t,\lambda) = 0, \end{cases} \tag{10.30}$$

where

$$h(t) = F^0(t,\lambda) - \widetilde{F}(t,\lambda),$$

$$T_k(t,\lambda) = \int_0^\infty \Big(\varphi_k(x,t,\lambda) - \delta_{k2}\mathfrak{M}_{23}(t,\lambda)\varphi_1(x,t,\lambda)\Big)q(x,t)\,dx.$$

Using the above-mentioned asymptotic properties of the functions Φ_k, we compute with fixed $t > 0$

$$\mathfrak{M}_{jk}(t,\lambda) = (-1)^{j-1}(\rho r_{2j-1})^{k-j}\big(1 + o(1)\big), \qquad j < k,$$
$$T_k(t,\lambda) = o(1), \qquad |\lambda| \to \infty.$$

Then (10.30) yields

$$h_{32}(t) = h_{22}(t) - h_{11}(t) = h_{33}(t) - h_{11}(t) = h_{21}(t) = h_{31}(t) = 0,$$

and consequently

$$\begin{cases} u(0,t) = u_1(t), \qquad u_x(0,t) = u_2(t), \qquad v(0,t) = v_1(t), \\ \dfrac{2}{3}\big(u_{xx}(0,t) - u_3(t)\big) - \big(v_x(0,t) - v_2(t)\big) = 0, \end{cases} \tag{10.31}$$

$$\int_0^\infty \varphi_k(x,t,\lambda)q(x,t)\,dx = 0, \qquad k = \overline{1,3}. \tag{10.32}$$

From (10.32) we have

$$\int_0^\infty \psi_k(x,\lambda)f(x)\,dx + f^0\psi_{k2}(0,\lambda) = 0, \qquad k = \overline{1,3}, \tag{10.33}$$

with fixed $t > 0$, where $f(x) = \left[Q_1(x,t), \dfrac{\partial}{\partial x}Q_2(x,t)\right]^T$, $f^0 = Q_2(0,t)$. If $|\lambda| \to \infty$, then from (10.33) one gets $f^0 = 0$, i.e.,

$$\int_0^\infty \psi_k(x,\lambda)f(x)\,dx = 0, \qquad k = \overline{1,3}.$$

Hence, by virtue of Theorem 10.1, we conclude that $f(x) = 0$ or $Q_1(x,t) = Q_2(x,t) = 0$. Thus, the functions $\{u(x,t),\ v(x,t)\}$ are the solutions of system (10.1). Finally, since

$$u_{xx}(0,t) = 2v_x(0,t) + iu_t(0,t), \qquad u_3(t) = 2v_2(t) + i\frac{\partial}{\partial t}u_1(t),$$

then in accordance with (10.31) we obtain $v_x(0,t) = v_2(t)$. Theorem 10.3 is proved.

It follows that problem (10.1)–(10.3) has a solution if and only if the solution of the corresponding IP exists. Necessary and sufficient conditions of solvability of the IP from the WM and algorithm for the solution are given in Sec. 2.

11. Integrable Dynamical Systems Connected with Higher-Order Difference Operators

For a fixed $q \geq 1$ we consider the following Cauchy problem for the nonlinear semi-infinite system of nonlinear DE's:

$$\dot{a}_{nj}(t) = a_{n1}(t)a_{n+1,j-1}(t) - a_{n+j-1,1}(t)a_{n,j-1}(t), \tag{11.1}$$

$$a_{nj}(0) = a^0_{nj}. \tag{11.2}$$

Here $-q+1 \leq j \leq 1$, $n \geq q$ for $j = 1$, and $n \geq q - j$ for $j \leq 0$; $a_{n,-q} = 1$, $a_{q-1,1} = 0$, $a_{q+j-1,-j} = 0$ for $j = \overline{1, q-1}$; a^0_{nj} are complex numbers, and $a^0_{nj} \neq 0$. System (11.1) is a difference analog for equations like the KdV equation, and it is equivalent to the Lax equation $\dot{L} = [A, L]$, where $A = [a_{n1}\delta_{n,j-1}]_{n,j \geq q}$, $L = [a_{n,j-n}]_{n,j\geq q}$, $a_{nj} = 0$ for $j > 1$ and $j < -q$. Thus, integration of the Cauchy problem (11.1)–(11.2) is connected with investigation of the spectral properties and the solution of the IP for higher-order difference operators:

$$(ly)_n = \sum_{\mu=0}^{q+1} a_{n,\mu-q} y_{n+\mu-q}, \qquad a_{n1} \neq 0, \quad a_{n,-q} = 1. \tag{11.3}$$

For $q = 1$, system (11.1) is the Toda chain, which has been studied fairly completely (see [2, 13, 59, 86] and references therein). Things are more complicated for $q > 1$, and integrable dynamical systems connected with higher-order difference operators have not been investigated enough. In this direction, we mention the papers [14–15], in which important integrable systems are pointed out, connected with two-term difference operators of the form (11.3) for $q > 1$, where $a_{n,-j} = 0$ for $j = \overline{0, q-1}$.

In the first part of this section, we provide the solution of the IP for the difference operators (11.3). Here we place no restrictions on the growth of the coefficients a_{nj} at infinity. As the main spectral characteristic, we introduce and study the WM for the difference operator. In the second part of the section, we obtain the solution of the Cauchy problem (11.1)–(11.2) by the inverse problem method. We find the evolution of the WM with respect to t, provide an algorithm for the solution of the problem, and obtain necessary and sufficient conditions for solvability of the problem (11.1)–(11.2) in the classes of analytic and meromorphic functions.

11.1. For a fixed $q \geq 1$, we consider the difference equation

$$(ly)_n \equiv \sum_{\mu=0}^{q+1} a_{n,\mu-q} y_{n+\mu-q} = \lambda y_n, \qquad n \geq q, \tag{11.4}$$

where $y = [y_n]_{n\geq 0}$, a_{nj} are complex numbers, $a_{n,-q} = 1$, $a_{n1} \neq 0$ for $n \geq q$, and $a_{n,-j} = 0$ for $n - q + 1 \leq j \leq q - 1$, $q \leq n \leq 2q - 2$.

Denote by A_R (M_R) the set of analytic (meromorphic) for $|t| < R$ functions. Let $A = \bigcup_{R>0} A_R$, $M = \bigcup_{R>0} M_R$, and let A' be the set of sequences $\{\alpha_k\}_{k\geq 1}$ such that $\alpha_k = O\left(\left(\frac{k}{\delta}\right)^k\right)$ for a certain $\delta > 0$. We shall write $\{f_k(t)\}_{k\geq 1} \in A$ (A^0) if there exists $R > 0$ such that $f_k(t) \in A_R$ $(f_k(t) \neq 0$, $|t| < R)$ for all k.

Let Λ be the set of polynomials of the form $F(\lambda) = \sum_k F_k \lambda^k$, $-i \leq k \leq j$ $(i, j \geq 0$ depends on the polynomial). Denote by $\mathcal{F}$ the set of linear functionals on Λ. We call the elements of $\mathcal{F}$ generalized functions (GF's). If $P \in \mathcal{F}$, the numbers $P_{k+1} = (\lambda^k, P)$ are called the moments of P. Here $(\cdot, P)$ denotes the action of P. It is clear that the GF $P \in \mathcal{F}$ is uniquely determined by its moments via the formula $\big(F(\lambda), P\big) = \sum_k F_k P_{k+1}$, $F(\lambda) \in \Lambda$. It is convenient to represent a GF $P \in \mathcal{F}$ in the form of the formal series $P(\lambda) = \sum_k \dfrac{P_k}{\lambda^k}$. A GF $P \in \mathcal{F}$ can be multiplied by elements of Λ using the formula $(F(\lambda), G(\lambda)P) = (F(\lambda)G(\lambda), P)$, $F(\lambda)$, $G(\lambda) \in \Lambda$. Denote by $\mathcal{F}_l$ the set of GF's $P \in \mathcal{F}$ for which $P_k = 0$ for $k < l$. If $\{P_k\}_{k\geq 1} \in A'$, we can define the GF $\sigma(\lambda, t) = \exp(\lambda t)P(\lambda)$ with the moments

$$\sigma_k(t) = \sum_{j=0}^{\infty} P_{k+j}\frac{t^j}{j!}.$$

Let $\Phi_n(\lambda) = \left[\Phi_n^i(\lambda)\right]^T_{i=\overline{1,q}}$, $n \geq 0$, be the solutions of (11.4) under the conditions

$$\Phi_n^i = \delta_{i,n+1} \quad (n = \overline{0, q-1}); \qquad \Phi_n^i(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\lambda^k}\Phi_{kn}^i \in \mathcal{F}_0, \quad n \geq q.$$

Here T denotes transposition, i.e., $\left[\Phi_n^i(\lambda)\right]^T_{i=\overline{1,q}}$ is a row vector. The solutions $\Phi_n^i(\lambda)$ exist, are unique, and can be constructed from the relations

$$\Phi_{k+1,n}^i = \sum_{\mu=0}^{q+1} a_{n,\mu-q}\Phi_{k,n+\mu-q}^i, \qquad n \geq q, \quad i = \overline{1,q}, \quad k \geq 0,$$

$$\Phi_{0n}^u = 0, \quad n \geq q; \qquad \Phi_{0n}^i = \delta_{i,n+1}, \quad \Phi_{kn}^i = 0, \quad k \geq 1, \quad n = \overline{0, q-1}.$$

We introduce the WM $\mathfrak{M}(\lambda) = \left[\mathfrak{M}^i(\lambda)\right]^T_{i=\overline{1,q}}$ by the formula $\mathfrak{M}^i(\lambda) = \Phi_q^i(\lambda)$. The IP is formulated as follows: given the WM $\mathfrak{M}(\lambda)$, construct the operator l. Denote

$$\Delta_k = \det[\mu_{in}]_{i,n=\overline{0,k}}, \qquad \mu_{in} = \left(1, \lambda^i\mathfrak{M}(\lambda)R_n(\lambda)\right), \tag{11.5}$$

where $R_n(\lambda) = \left[R_n^i(\lambda)\right]_{i=\overline{1,q}}$, $R_{qs+m}^i(\lambda) = \delta_{i,m+1}\lambda^s$, $m = \overline{0, q-1}$. In particular, $\mu_{00} = 1$, $\mu_{in} = 0$ for $0 \leq i < n \leq q-1$.

Lemma 11.1. *$\Delta_k \neq 0$ for all $k \geq 0$.*

Denote by M^0 the set of matrices $\mathfrak{M}(\lambda) = \left[\mathfrak{M}^i(\lambda)\right]^T_{i=\overline{1,q}}$, $\mathfrak{M}^i(\lambda) \in \mathcal{F}_1$, for which $\mu_{00} = 1$, $\mu_{in} = 0$ for $0 \leq i < n \leq q-1$.

Theorem 11.1. *For a matrix $\mathfrak{M}(\lambda) \in M^0$ to be the WM for l of the form* (11.4), *it is necessary and sufficient that $\Delta_k \neq 0$ for all $k \geq 0$. The operator l is uniquely determined by the WM and can be found by the following algorithm.*

Algorithm 11.1. *Given a matrix $\mathfrak{M}(\lambda) \in M^0$.*

(1) *Construct c_{ik}, $0 \leq i \leq k$, by the formula*

$$c_{ik} = \Delta_k^{-1}(-1)^{k-i}\det[\mu_{jn}]_{j=\overline{0,k}\setminus i;n=\overline{0,k-1}}, \qquad c_{00} = 1.$$

(2) *Compute a_{nj} by the formula*

$$a_{nj} = \left(c_{n+j-q,n+j-q}\right)^{-1}\left(c_{n+j-q-1,n-q} - \sum_{\mu=j+1}^{q} a_{n\mu}c_{n+j-q,n+\mu-q}\right).$$

11.2. Consider the Cauchy problem (11.1)–(11.2). Let $\overset{\circ}{\mathfrak{M}}(\lambda) = \left[\overset{\circ}{\mathfrak{M}}{}^i(\lambda)\right]^T_{i=\overline{1,q}}$ be the WM for the operator l_0 of the form (11.3), constructed from the initial data $\{a_{nj}^0\}$. Let $\left\{\overset{\circ}{\mathfrak{M}}{}_j^i\right\}_{j\geq 1}$ be the moments of $\overset{\circ}{\mathfrak{M}}{}^i(\lambda)$. Suppose that there exists an analytic solution $\{a_{nj}(t)\} \in A$ at $t = 0$ of the Cauchy problem (11.1)–(11.2). We consider the corresponding difference operator $l = l(t)$ of the form (11.4). In particular, it follows from (11.1) that $a_{n1}(t) \neq 0$, and consequently, $\Delta_k(t) \neq 0$, $k \geq 0$. In the following theorem evolution of the WM with respect to t is obtained.

Theorem 11.2. *We have*

$$\begin{cases} \overset{\circ}{\mathfrak{M}}{}^1 = (\lambda - a_{q0})\mathfrak{M}^1 - 1, \\ \overset{\circ}{\mathfrak{M}}{}^i = (\lambda - a_{q0})\mathfrak{M}^i - a_{q+i-2,1}\mathfrak{M}^{i-1}, \qquad i = \overline{2,q}. \end{cases} \tag{11.6}$$

Let us now integrate the evolution equations (11.6) with the initial conditions $\mathfrak{M}^i(0,\lambda) = \overset{\circ}{\mathfrak{M}}{}^i(\lambda)$. We note that in addition to $\mathfrak{M}^i(t,\lambda)$, the functions $a_{q0}(t)$, $a_{q+i-2,1}(t)$, $i = \overline{2,q}$, in (11.6) are also unknown. Denote $B(t) = \exp\left(\int\limits_0^t a_{q0}(s)\,ds\right)$.

Theorem 11.3. *We have*

$$B(t)\mathfrak{M}^1(t,\lambda) = \exp(\lambda t)\overset{\circ}{\mathfrak{M}}{}^1(\lambda) - \int\limits_0^t \exp\big(\lambda(t-\tau)\big)B(\tau)\,d\tau, \tag{11.7}$$

$$B(t)\mathfrak{M}^i(t,\lambda) = \exp(\lambda t)\overset{\circ}{\mathfrak{M}}{}^i(\lambda) - \int\limits_0^t \exp\big(\lambda(t-\tau)\big)B(\tau)a_{q+i-2,1}(\tau)\mathfrak{M}^{i-1}(\tau,\lambda)\,d\tau, \qquad i = \overline{2,q}, \tag{11.8}$$

$$B(t) = \sum_{j=0}^{\infty} \overset{\circ}{\mathfrak{M}}{}^1_{j+1}\frac{t^j}{j!}, \tag{11.9}$$

$$a_{q+i-2,1}(t) = \left(B(t)\prod_{j=0}^{i-3} a_{q+j,1}(t)\right)^{-1} \sum_{j=0}^{\infty} \overset{\circ}{\mathfrak{M}}{}^i_{j+i}\frac{t^j}{j!}, \qquad i = \overline{2,q}, \tag{11.10}$$

where $\{\overset{\circ}{\mathfrak{M}}{}^i_j\}_{j\geq 1} \in A'$, $i = \overline{1,q}$.

Thus, we obtain the following algorithm for the solution of the problem (11.1)-(11.2) by the inverse problem method.

Algorithm 11.2. Let $\{a^0_{nj}\}$, $a^0_{n1} \neq 0$ be given. We then

(1) construct $\{\overset{\circ}{\mathfrak{M}}{}^i_j\}_{j\geq 1}$, $i = \overline{1,q}$;
(2) compute $B(t)$, $a_{q+i-2,1}(t)$, $i = \overline{2,q}$, by (11.9) and (11.10);
(3) find $\mathfrak{M}^i(t,\lambda)$, $i = \overline{1,q}$, by (11.7)-(11.8), and calculate $\Delta_k(t)$, $k \geq 0$, by (11.5);
(4) construct the functions $\{a_{nj}(t)\}$ by solving the IP with the help of Algorithm 11.1.

Remark. Algorithm 11.2 also works when $a_{nj}(t) \in M$, i.e., in the class of meromorphic functions.

Theorem 11.4. *For the Cauchy problem* (11.1)–(11.2) *to have a solution $a_{nj}(t) \in M$ it is necessary and sufficient that $\{\overset{\circ}{\mathfrak{M}}{}^i_j\}_{j\geq 1} \in A'$, $i = \overline{1,q}$. This solution is unique and can be constructed with Algorithm* 11.2. *In addition, $a_{nj}(t) \in M_R$ if and only if*

$$\mathfrak{N}_i(t) \overset{\text{def}}{=} \sum_{j=0}^{\infty} \overset{\circ}{\mathfrak{M}}{}^i_{j+i}\frac{t^j}{j!} \in M_R, \qquad i = \overline{1,q}.$$

The following theorem gives necessary and sufficient conditions for solvability in the narrower classes A and A_R.

Theorem 11.5. *For the Cauchy problem* (11.1)–(11.2) *to have a solution* $\{a_{nj}(t)\} \in A$ *it is necessary and sufficient that* $\{\overset{\circ}{\mathfrak{M}}{}_j^i\}_{j\geq 1} \in A'$, $i = \overline{1,q}$, *and* $\{\Delta_k(t)\}_{k\geq 0} \in A^0$. *In addition,* $a_{nj}(t) \in A_R$ *if and only if* $\mathfrak{N}_i(t) \in A_R$, $\mathfrak{N}(t) \neq 0$, $\Delta_k(t) \neq 0$, $|t| < R$, $i = \overline{1,q}$, $k \geq 0$.

REFERENCES

1. V. A. Ambarzumian, "Über eine Frage der Eigenwerttheorie," *Zs. F. Phys.*, **53**, 690–695 (1929).
2. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia (1981).
3. M. J. Ablowitz and H. Segur, "The inverse scattering transform: semi-infinite interval," *J. Math. Phys.*, **16**, 1054–1056 (1975).
4. Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Gordon and Breach, New York (1963).
5. L. Y. Ainola, "An inverse problem on eigenoscillations of elastic covers," *Appl. Math. Mech.*, **2**, 358–365 (1971).
6. T. Aktosun, "Scattering and inverse scattering for a second-order differential equation," *J. Math. Phys.*, **34**, 1619–1634 (1993).
7. L. E. Andersson, "Inverse eigenvalue problems for a Sturm–Liouville equation in impendance form," In: *Inverse Problems*, Vol. 4 (1988), pp. 929-971.
8. Y. E. Anikonov, *Methods of Investigating Multi-Dimensional Inverse Problems for Differential Equations* [in Russian], Novosibirsk (1978).
9. E. A. Baranova, "On the recovery of higher-order ordinary differential operators from their spectra," *Dokl. Akad. Nauk SSSR*, **205**, 1271–1273 (1972).
10. R. Beals, "The inverse problem for ordinary differential operators on the line," *Amer. J. Math.*, **107**, 281–366 (1985).
11. R. Beals, P. Deift, and C. Tomei, "Direct and inverse scattering on the line," In: *Math. Surveys and Monographs*, Vol. 28, Amer. Math. Soc., Providence, Rhode Island (1988).
12. Y. M. Berezanskii, "On a uniqueness theorem in the inverse spectral problem for the Schrödinger equation," *Tr. Mosk. Mat. Obshch.*, **7**, 3–51 (1958).
13. Y. M. Berezanskii, "Integration of nonlinear difference equations by the inverse problem method," *Dokl. Akad. Nauk SSSR*, **281**, 16–19 (1985).
14. O. I. Bogoyavlenskii, "Some constructions of integrable dynamic systems," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **51**, 737–766 (1987).
15. O. I. Bogoyavlenskii, "Integrable dynamic systems connected with the KdV equation," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **51**, 1123–1141 (1987).
16. G. Borg, "Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe," *Acta Math.*, **78**, 1–96 (1946).
17. A. L. Bukhgeim, *Introduction to the Inverse Problem Theory* [in Russian], Novosibirsk (1988).
18. R. Carlson, "Inverse spectral theory for some singular Sturm–Liouville problems," *J. Diff. Equations*, **106**, 121–140 (1993).
19. P. J. Caudrey, "The inverse problem for the third-order equation," *Phys. Lett.*, **79A**, 264–268 (1980).
20. P. Deift P. and E. Trubowitz, "Inverse scattering on the line," *Comm. Pure Appl. Math.*, **32**, 121–251 (1979).

21. P. Deift, C. Tomei, and E. Trubowitz, "Inverse scattering and the Boussinesq equation," *Comm. Pure Appl. Math.*, **35**, 567–628 (1982).

22. P. Deift and X. Zhou, "Direct and inverse scattering on the line with arbitrary singularities," *Comm. Pure Appl. Math.*, **44**, No. 5, 485–533 (1991).

23. M. S. Eremin, "An inverse problem for a second-order integro-differential equation with a singularity," *Differents. Uravn.*, **24**, 350–351 (1988).

24. L. D. Faddeev, "On a connection the S-matrix with the potential for the one-dimensional Schrödinger operator," *Dokl. Akad. Nauk SSSR*, **121**, 63–66 (1958).

25. L. D. Faddeev, "Properties of the S-matrix of the one-dimensional Schrödinger equation," *Tr. Mat. Inst. Steklov*, **73**, 139-166 (1964).

26. L. D. Faddeev and L. A. Takhtadjan, *Hamilton Approach in the Soliton Theory* [in Russian], Nauka, Moscow (1986).

27. M. K. Fage, "Integral representations of operator-valued analytic functions of one independent variable," *Tr. Mosk. Mat. Obshch.*, **8**, 3–48 (1959).

28. M. K. Fage, "Solution of certain Cauchy problem via enlargement of number of independent variables," *Mat. Sb.*, **46 (88)**, 261–290 (1958).

29. G. Gardner, J. Green, M. Kruskal, and R. Miura, "A method for solving the Korteveg–de Vries equation," *Phys. Rev. Lett.*, **19**, 1095–1098 (1967).

30. F. D. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford and Addison-Wesley, Reading, Mass. (1966).

31. M. G. Gasymov and B. M. Levitan, "Determination of a differential equation by two of its spectra," *Usp. Mat. Nauk*, **19**, No. 2, 3–63 (1964).

32. M. G. Gasymov, "Determination of Sturm–Liouville equation with a singular point from two spectra," *Dokl. Akad. Nauk SSSR*, **161**, 274–276 (1965).

33. I. M. Gel'fand and B. M. Levitan, "On the determination of a differential equation from its spectral function," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **15**, 309–360 (1951).

34. B. Grebern and R. Weder, "Reconstruction of a potential on the line that is a priori known on the half-line," *SIAM J. Appl. Math.*, **55**, No. 1, 242–254 (1995).

35. O. H. Hald, "Discontinuous inverse eigenvalue problems," *Comm. Pure Appl. Math.*, **37**, 539–577 (1984).

36. H. Hochstadt, "The inverse Sturm–Liouville problem," *Comm. Pure Appl. Math.*, **26**, 715–729 (1973).

37. H. Hochstadt, "On the well-posedness of the inverse Sturm–Liouville problems," *J. Diff. Eq.*, **23**, No. 3, 402–413 (1977).

38. A. P. Hromov, "Transformation operators for differential equations of arbitrary orders," In: *Differential Equations and Function Theory* [in Russian], Vol. 3, Saratov Univ., Saratov (1971), pp. 10–24.

39. J. Kay and H. Moses, "The determination of the scattering potential from the spectral measure function. I," *Nuovo Cim.*, **2**, 917–961 (1955).

40. J. Kay and H. Moses, "The determination of the scattering potential from the spectral measure function. II," *Nuovo Cim.*, **3**, 56–84 (1956).

41. J. Kay and H. Moses, "The determination of the scattering potential from the spectral measure function. III," *Nuovo Cim.*, **3**, 276–304 (1956).

42. D. J. Kaup, "On the inverse scattering problem for cubic eigenvalue problems," *Stud. Appl. Math.*, **62**, 189–216 (1980).

43. I. G. Khachatryan, "The recovery of a differential equation from its spectrum," *Funkts. Anal. Prilozhen.*, **10**, No. 1, 93–94 (1976).
44. I. G. Khachatryan, "On some inverse problems for higher-order differential operators on the half-line," *Funkts. Anal. Prilozhen.*, **17**, No. 1, 40–52 (1983).
45. I. G. Khachatryan, "Necessary and sufficient conditions of solvability of the inverse scattering problem for higher-order differential equations on the half-line," *Dokl. Akad. Nauk Armyan. SSR*, **77**, No. 2, 55–58 (1983).
46. I. G. Khachatryan, "On transformation operators for higher-order differential equations," *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.*, **13**, No. 3, 215–237 (1978).
47. I. G. Khachatryan, "On some solutions of the nonlinear string equation," *Dokl. Akad. Nauk Armyan. SSR*, **78**, 108–112 (1984).
48. M. G. Krein, "Solution of the inverse Sturm–Liouville problem," *Dokl. Akad. Nauk SSSR*, **76**, 21–24 (1951).
49. M. G. Krein, "On a method of effective solution of the inverse problem," *Dokl. Akad. Nauk SSSR*, **94**, 987–990 (1954).
50. M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskii, *Ill-posed Problems in Mathematical Physics and Analysis*, Nauka, Moscow (1980).
51. P. Lax, "Integrals of nonlinear equations of evolution and solitary waves," *Comm. Pure Appl. Math.*, **21**, 467–490 (1968).
52. Z. L. Leibenzon, "The inverse problem of spectral analysis for higher-order ordinary differential operators," *Tr. Mosk. Mat. Obshch.*, **15**, 70–144 (1966).
53. Z. L. Leibenson, "Spectral expansions of transformations of systems of boundary value problems," *Tr. Mosk. Mat. Obshch.*, **25**, 15–58 (1971).
54. A. F. Leont'ev A growth estimate for the solution of a differential equation for large parameter values and its application to some problems of function theory, *Sib. Mat. Zh.*, **1**, 456–487 (1960).
55. A. F. Leont'ev, *Entire Functions. Exponential Series*, Nauka, Moscow (1983).
56. N. Levinson, "The inverse Sturm–Liouville problem," *Math. Tidsskr.*, **13**, 25–30 (1949).
57. B. M. Levitan, *Inverse Sturm–Liouville Problems*, VNU Sci. Press, Utrecht (1987).
58. M. M. Malamud, "On certain inverse problems," In: *Boundary Value Problems of Mathematical Physics*, Kiev (1979), pp. 116–124.
59. S. V. Manakov, S. P. Novikov, L. P. Pitaevskii, and V. E. Zakharov, *The Soliton Theory. The Inverse Problem Method*, Nauka, Moscow (1980).
60. V. A. Marchenko, "Some questions in the theory of a second-order differential operator," *Dokl. Akad. Nauk SSSR*, **72**, 457–460 (1950).
61. V. A. Marchenko, "Some questions in the theory of linear differential operators," *Tr. Mosk. Mat. Obshch.*, **1**, 327–420 (1952).
62. V. A. Marchenko, *The spectral theory of Sturm–Liouville operators*, Naukova Dumka, Kiev (1972).
63. V. A. Marchenko, *Sturm–Liouville operators and their applications*, Birkhauser (1986).
64. V. I. Matsaev, "The existence of operator transformations for higher-order differential operators," *Dokl. Akad. Nauk SSSR*, **130**, 499–502 (1960).
65. J. R. McLaughlin, "Analytical methods for recovering coefficients in differential equations from spectral data," *SIAM Rev.*, **28**, 53–72 (1986).
66. J. R. McLaughlin, "An inverse eigenvalue problem of order four," *SIAM J. Math. Anal.*, **7**, No. 5, 646–661 (1976).

67. A. Mizutani, "On the inverse Sturm–Liouville problem," *J. Fac. Sci. Univ. Tokio, Sec. IA, Math.*, **31**, 319–350 (1984).

68. H. E. Moses, "A solution of the Korteveg–de Vries equation in a half-space bounded by a wall," *J. Math. Phys.*, **17**, 73–75 (1976).

69. M. A. Naimark, *Linear Differential Operators, Parts I, II*, Ungar, New York (1967, 1968).

70. L. P. Nizhnik, *Inverse Scattering Problems for Hyperbolic Equations*, Naukova Dumka, Kiev (1991).

71. J. Poschel and E. Trubowitz, *Inverse Spectral Theory*, Academic Press, New York (1987).

72. I. I. Privalov, *Introduction to the Theory of Functions of a Complex Variable*, Nauka, Moscow (1967).

73. F. S. Rofe-Beketov, "The spectral matrix and the inverse Sturm–Liouville problem," In: *Theory of Functions, Functional Analysis, and Their Applications* [Russian], Vol. 4, Kharkov (1967), pp. 189–197.

74. V. G. Romanov, *Inverse Problems in Mathematical Physics*, Nauka, Moscow (1984).

75. W. Rundell and P. E. Sacks, "The reconstruction of Sturm–Liouville operators," *Inverse Problems*, **8**, 457–482 (1992).

76. V. A. Sadovnichii, "On some formulations of inverse spectral problems," *Usp. Mat. Nauk*, **38**, No. 5, 132 (1983).

77. L. A. Sakhnovich, "The inverse problem for differential operators of order $n > 2$ with analytic coefficients," *Mat. Sb.*, **46 (88)**, 61–76 (1958).

78. L. A. Sakhnovich, "The transformation operator method for higher-order equations," *Mat. Sb.*, **55 (97)**, 347–360 (1961).

79. L. A. Sakhnovich, "On an inverse problem for fourth-order equations," *Mat. Sb.*, **56 (98)**, 137–146 (1962).

80. L. A. Sakhnovich, "Evolution of spectral data and nonlinear equations," *Ukr. Math. Zh.*, **40**, 533–535 (1988).

81. K. Shadan and P. Sabatier, *Inverse Problems in Quantum Scattering*, Springer-Verlag, New York (1989).

82. V. V. Stashevskaya, "On inverse problems of spectral analysis for a certain class of differential equations," *Dokl. Akad. Nauk SSSR*, **93**, 409–412 (1953).

83. V. A. Strakhov, "On some questions of the inverse problem theory for differential operators," *Mat. Zametki*, **21**, No. 2, 151–160 (1977).

84. V. V. Sukhanov, "The inverse problem for a self-adjoint differential operator on the line," *Mat. Sb.*, **137 (179)**, 242–259 (1988).

85. A. N. Tikhonov, "On uniqueness of the solution of a electro-reconnaissance problem," *Dokl. Akad. Nauk SSSR*, **69**, 797–800 (1949).

86. M. Toda, *Theory of Nonlinear Latties*, Springer-Verlag, Berlin (1981).

87. V. A. Yurko, *On an inverse problem for higher-order differential operators*, deposited at All-Russian Institute for Scientific and Technical Information, 1975, No. 312–75.

88. V. A. Yurko, "The inverse problem for differential operators of even order," In: *Differential Equations and Numerical Analysis,* Part 2 [in Russian], Vol. 6, Saratov Univ., Saratov (1976), pp. 55–66.

89. V. A. Yurko, "On stability of recovering the Sturm–Liouville operators," In: *Differential Equations and Theory of Functions* [in Russian], Vol. 3, Saratov Univ., Saratov (1980), pp. 113–124.

90. V. A. Yurko, "Reconstruction of fourth-order differential operators.," *Diff. Uravneniya*, **19**, 2016–2017 (1983).

91. V. A. Yurko, "On reconstruction of integrodifferential operators from their spectra," In: *Theory of Functions and Approximations, Proc. of Saratov Winter School,* Part 2 [in Russian], Saratov (1983), pp. 156–160.
92. V. A. Yurko, "An inverse problem for integrodifferential operators of the first order," In: *Functional Analysis* [Russian], Ul'yanovsk (1984), pp. 144–151.
93. V. A. Yurko, "Boundary value problems with a parameter in the boundary conditions," *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.*, **19**, No. 5, 398–409 (1984).
94. V. A. Yurko, "An inverse problem for integral operators," *Mat. Zametki*, **37**, No. 5, 690–701 (1985).
95. V. A. Yurko, "Uniqueness of the reconstruction of binomial differential operators from two spectra," *Mat. Zametki*, **43**, No. 3, 356–364 (1988).
96. V. A. Yurko, *Inverse Problem for Differential Operators* [in Russian], Saratov Univ., Saratov (1989).
97. V. A. Yurko, "Reconstruction of higher-order differential operators," *Diff. Uravneniya*, **25**, No. 9, 1540–1550 (1989).
98. V. A. Yurko, "Recovery of differential operators from the Weyl matrix," *Dokl. Akad. Nauk SSSR*, **313**, 1368–1372 (1990).
99. V. A. Yurko, "A problem in elasticity theory," *Prikl. Mat. Mekh.*, **54**, 998–1002 (1990).
100. V. A. Yurko, "Recovery of non-self-adjoint differential operators on the half-line from the Weyl matrix," *Mat. Sb.*, **182**, No. 3, 431–456 (1991).
101. V. A. Yurko, "An inverse problem for integro-differential operators," *Mat. Zametki*, **50**, No. 5, 134–146 (1991).
102. V. A. Yurko, "An inverse problem for differntial operators on the half-line," *Izv. Vuzov, Ser. Mat.*, **12**, 67–76 (1991).
103. V. A. Yurko, "Solution of the Boussinesq equation on the half-line by the inverse problem method," *Inverse Problems*, **7**, 727–738 (1991).
104. V. A. Yurko, "Inverse problem for differential equations with a singularity," *Diff. Uravneniya*, **28**, 1355–1362 (1992).
105. V. A. Yurko, "Inverse problem for self-adjoint differential operators on the half-line," *Dokl. Rossiisk. Akad. Nauk*, **333**, No. 4, 449-451 (1993).
106. V. A. Yurko, "On higher-order differential operators with a singular point," *Inverse Problems*, **9**, 495–502 (1993).
107. V. A. Yurko, "On determination of self-adjoint differential operators on the half-line," *Mat. Zametki*, **57**, No. 3, 451–462 (1995).
108. V. A. Yurko, "On higher-order difference operators," *J. Difference Eq. Appl.*, **1**, 347–352 (1995).
109. V. A. Yurko, "On integration of nonlinear dynamical systems by the inverse problem method," *Mat. Zametki*, **57**, No. 6, 945–949 (1995).
110. V. A. Yurko, "On inverse problems for nonlinear differential equations," *Diff. Uravneniya*, **31**, No. 10, 1685-1686 (1995).
111. V. A. Yurko, "On higher-order differential operators with a regular singularity," *Mat. Sb.*, **186**, No. 6, 133–160 (1995).
112. V. A. Yurko, "An inverse problems for operators of a triangular structure," *Results in Mathematics*, **30**, Nos. 3/4, 346–373 (1996).