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Markov Traces and II_1 Factors in Conformal Field Theory

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Abstract. Using the duality equations of Moore and Seiberg we define for every primary field in a Rational Conformal Field Theory a proper Markov trace and hence a knot invariant. Next we define two nested algebras and show, using results of Ocneanu, how the position of the smaller algebra in the larger one reproduces part of the duality data. A new method for constructing Rational Conformal Field Theories is proposed.

1. Introduction

In the past few years several attempts have been made to find the basic underlying principles and structures governing Rational Conformal Field Theories (RCFT). In one approach, quantum groups are proposed as the underlying algebraic structure of RCFT [21]. In [21] the philosophy is that the quantum group can be seen as the centralizer of a representation of the braid group. This approach is in particular successful for WZW models, where one can compute braid matrices using the analogue of $6j$-symbols. The result of this construction for arbitrary RCFT is, however, unclear.

In another approach, Rational Conformal Field Theories are seen to be intimately related with three-dimensional topological field theories [16, 3]. Here, the Hilbert space associated to a constant time slice with charges in the three-dimensional theory is equal to the space of conformal blocks of a RCFT. The observables of the three-dimensional theory are knotted links whose expectation values can also be computed (as we will show) from RCFT.

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In this paper we will take a look at these two approaches from a somewhat different angle. Instead of quantum groups we will end up with inclusions of certain II_1 factors. These are infinite dimensional algebras that can be obtained by taking a certain limit of finite dimensional ones. They arise as algebras of paths on a graph constructed from the fusion rules, and a primary field ϕ. The graph is closely related to the fusion graph, but not necessarily identical to it. For instance for a field ϕ with the fusion rule $\phi^2 = \mathbf{1} + \phi$ the graph is the Dynkin graph A_4.

An outline of the contents and the results of this paper is as follows:

In Sect. 2 we will give a review of the duality relations that govern RCFT. Using these it will be shown in Sect. 3 how one can obtain link invariants from arbitrary Rational Conformal Field Theories, by construction of a proper Markov trace. Some examples will be given where the invariant is equivalent to some well-known knot invariant. In particular this shows that there exists a well-defined three-dimensional topological field theory, where the expectation values of links agree with the link invariant obtained from RCFT. One could in principle use this to properly define expectation values of graphs as well, as has been done for Chern-Simons theories in [14], and more recently for arbitrary RCFT in [5].

In Sects. 4–6 we will explain the relation between II_1 factors and RCFT, using Ocneanu's path algebras [9]. The algebras presented in those sections have the properties that their representation theory coincides with part of the fusion rules, and that the intertwiners between these representations are (up to a normalization) braiding matrices. In the case that the special chosen field ϕ is self-conjugate, our construction should give the same resulting algebras as in [21], suggesting a close relation between quantum groups and path algebras. The precise relation is, however, unclear, and must presumably be sought along the lines of Witten's work [15].

As a by-product of our graphic representation of the string algebras we find in Sect. 7 a relation between the positive half of the Virasoro algebra and the Temperley-Lieb algebra. These results are also valid for certain statistical mechanical models, because we can define an IRF model based on the same string algebras, where the Boltzmann weights are braiding matrices. In this context the elements of the string algebras can be seen as transfer matrices.

The final part of this paper consists of a study of the reverse process, namely constructing Rational Conformal Field Theories out of inclusions of factors. We establish some necessary (but, unfortunately, not sufficient) conditions for inclusions to produce Rational Conformal Field Theories, and present some examples.

2. Duality in CFT

Rational Conformal Field Theories are conformal field theories in which the Hilbert space decomposes into a finite sum of irreducible representations of the (maximally extended) chiral algebra $\mathscr{A}_L \otimes \mathscr{A}_R$,

$$\mathscr{H} = \bigoplus_{i,\bar{i}} \mathscr{H}_i \otimes \mathscr{H}_{\bar{i}}\,.$$

The physical correlation functions in such a theory can be expressed in terms of finite sums of holomorphic times antiholomorphic functions, which are called the conformal blocks. Whereas these conformal blocks are multivalued functions, the

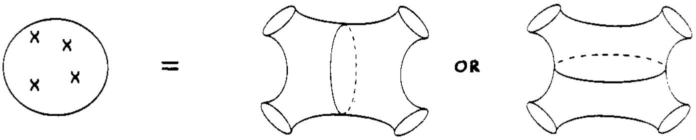
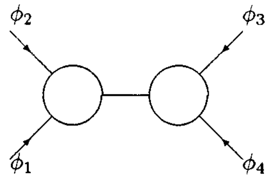


Fig. 1. Different sewing procedures for the 4 punctured sphere

physical correlation functions are constructed out of the conformal blocks in a monodromy invariant way. Graphically, we can represent a n-point conformal block $\mathscr{F}_{\phi_1,\dots,\phi_n}$ as a skeleton diagram. For example, a 4-point conformal block on a genus two surface can be represented as



The number of blocks can be easily computed from the fusion rules N^k_{ij}, which for the above case gives

$$\text{number of blocks} = \sum_p \mathrm{Tr}(N^{\phi_1}N^{\phi_2}N_p)\,\mathrm{Tr}(N^pN^{\phi_3}N^{\phi_4}),$$

where $(N^p)_{ij} = N^p_{ij}$.

The idea here is that a punctured Riemann surface can be formed by sewing a number of trinions (i.e. three holed spheres). This sewing procedure gives the different conformal blocks when one sums over the intermediate states in the channel that is formed by the sewn holes. Of course, the same punctured Riemann surface can be obtained by different sewing procedures. For example, the four punctured sphere can be obtained from two different sewing procedures, as shown in Fig. 1. These different sewing procedures give rise to different conformal blocks. Now, the basic axiom of duality in Conformal Field Theory [1] assures that the vector space spanned by the conformal blocks is independent of the sewing procedure. This means that the conformal blocks obtained from one sewing procedure are linear combinations of conformal blocks obtained from another. The matrices representing these linear transformations are called "duality matrices."

Moore and Seiberg [2] have shown that the duality data of a Conformal Field Theory are contained in the braiding and fusion matrices and the modular matrix $S(j)$ (see below). Furthermore, they have proven that the conditions on these duality matrices, stemming from the requirement of duality and modular covariance on arbitrary genus, can be represented by a finite number of equations, the polynomial equations. We will review these polynomial equations below.

The basic duality data for genus zero are contained in the braid matrix $\mathrm{B}_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}(\varepsilon)$ or the fusion matrix $\mathrm{F}_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}$, which are defined in the following

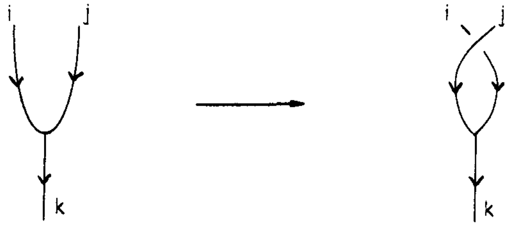


Fig. 2. Braiding on a simple conformal block

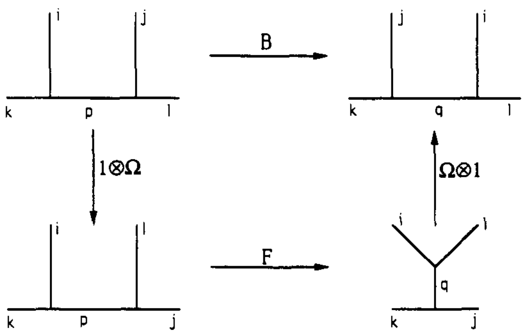
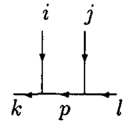
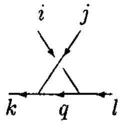


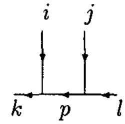
Fig. 3. Proof of Eq. (2)

picture

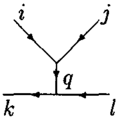


$$= \sum_q B_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}(+)$$





$$= \sum_q F_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}$$



($\varepsilon = \pm$ depending how the braid is performed). A particular simple example of the braid matrix, which we will denote by Ω^k_{ij}, is given in Fig. 2. We always can make a "choice of gauge" such that Ω^k_{ij} becomes (Δ_i denotes the conformal weight of the primary i)

$$\Omega^k_{ij}(\varepsilon) = B_{ji}\begin{bmatrix} i & j \\ k & 0 \end{bmatrix}(\varepsilon) = \varepsilon^k_{ij} e^{i\pi\varepsilon(\Delta_i + \Delta_j - \Delta_k)}, \tag{2.1}$$

where ε^k_{ij} (not to be confused with the ε which denotes the orientation of the braid) can be ± 1. For WZW models we can use the fact that for representations ε^k_{ij} is $-/+$ depending on whether k appears (anti)symmetrically in the tensor product of i and j. In more general situations ε^k_{ij} has to be determined consistently from the polynomial equations. Note that if $i=j$ we also have $B(\varepsilon) = F^{-1}\Omega(\varepsilon)F$, which implies that in this case $B(\varepsilon)$ has the same eigenvalues as $\Omega(\varepsilon)$.

The braid and fusion matrices are not independent, in fact

$$B(\varepsilon) = (\Omega(-\varepsilon)\otimes 1)F(1\otimes\Omega(\varepsilon)) \tag{2.2}$$

which can be easily deduced when one applies the simple moves shown in Fig. 3.

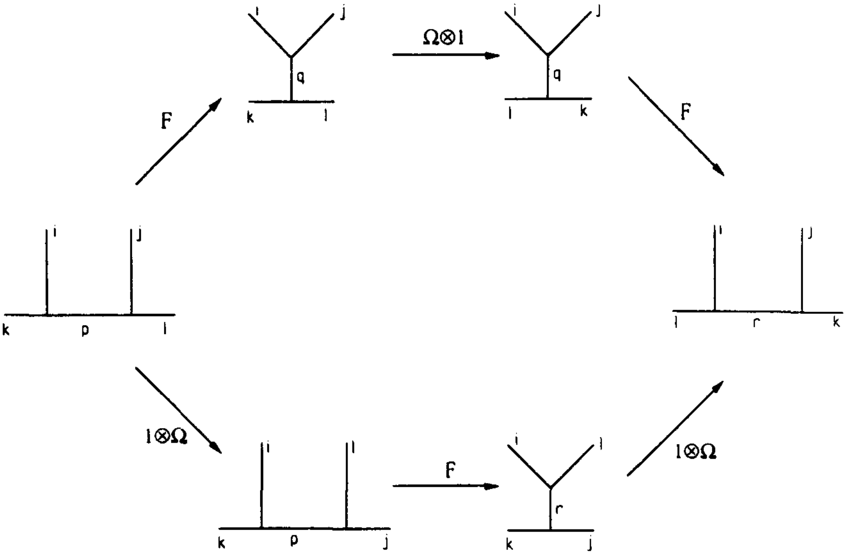


Fig. 4. Proof of the hexagon identity

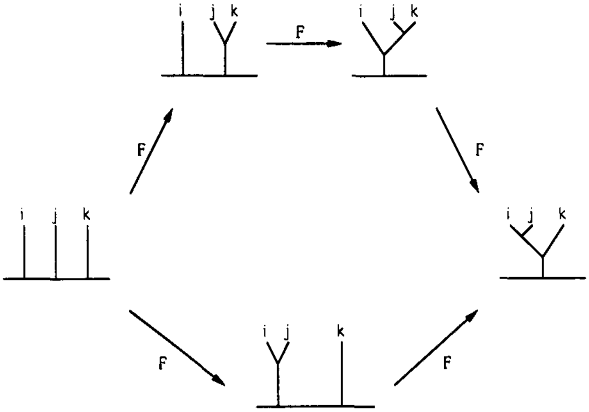


Fig. 5. Proof of the pentagon identity

From (2.2) we have $B(\varepsilon)\circ B(-\varepsilon)=\mathbf{1}$, which is obvious. Furthermore, since $\Omega^*(\varepsilon)=\Omega(-\varepsilon)$ and $F^*=F^\vee$, we have $B^*(\varepsilon)=B^\vee(-\varepsilon)$, where $B^\vee$ denotes the braid matrix with the fields ϕ_i replaced by their duals $\phi_i^\vee$ (recall that the dual $\phi^\vee$ of a field ϕ is the unique field with which ϕ has the fusion rule $\phi\times\phi^\vee=\mathbf{1}+\ldots$). One can easily prove that this implies that the braid matrix $B(\varepsilon)$ is in fact unitary.

Applying a series of B and F moves on special conformal blocks one can easily derive lots of identities for the $B_{pq}\begin{bmatrix} i & j \\ k & l\end{bmatrix}$ and $F_{pq}\begin{bmatrix} i & j \\ k & l\end{bmatrix}$. The results of Moore and Seiberg guarantee that all these identities are in fact equivalent to just two identities, which we will now derive. The first is called the hexagon identity and is expressed graphically in Fig. 4. In terms of the fusion matrix it reads

$$F(\Omega(\varepsilon)\otimes\mathbf{1})F=(\mathbf{1}\otimes\Omega(\varepsilon))F(\mathbf{1}\otimes\Omega(\varepsilon))\,. \tag{2.3}$$

The second fundamental identity is called the pentagon identity. Its graphic derivation is given in Fig. 5, which gives the following expression in terms of fusion

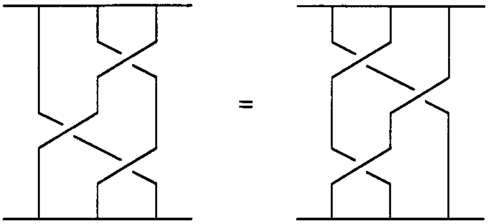


Fig. 6. Proof of the Yang-Baxter equation

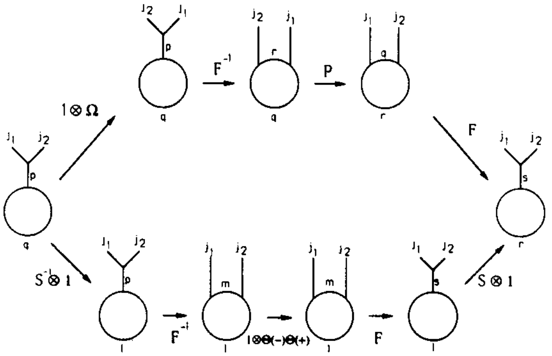


Fig. 7. Proof of the genus one identity

matrices:

$$F_{23}F_{12}F_{23}=P_{23}F_{13}F_{12}, \tag{2.4}$$

where P is the permutation operator. Using (2.3) and the connection between the B and F matrix as in (2.2), we can rewrite (2.4) as

$$B_{12}(\varepsilon)B_{23}(\varepsilon)B_{12}(\varepsilon)=B_{23}(\varepsilon)B_{12}(\varepsilon)B_{23}(\varepsilon) \tag{2.5}$$

whose graphic interpretation is given in Fig. 6. Equation (2.5) is the Yang-Baxter equation and is due to the fact that the B matrices form a representation of the braid group.

In addition to the genus zero equations which we have derived above, there are of course duality constraints from higher genus. One of the surprising results of Moore and Seiberg is that the only new fundamental duality equations come from genus one. We will now derive these equations. First, the new duality data in genus one are given by the modular matrices $S(j)$ and T, where $S(j)$ represents the behavior of the one point functions on the torus under the transformation $\tau\to-1/\tau$, and T equals $T_{ij}=\delta_{ij}e^{2i\pi(\Delta_i-c/24)}$. Since the modular matrices $S(j)$ and T should form a (projective) representation of the modular group, we have the following two identities

$$S(j)TS(j)=T^{-1}S(j)T^{-1}, \tag{2.6}$$

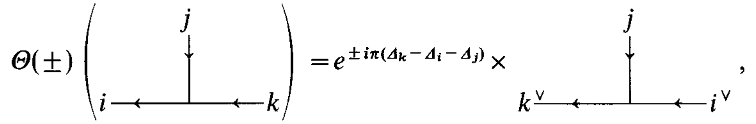
$$S^2(j)=\pm e^{-i\pi\Delta_j}C, \tag{2.7}$$

where C is the charge conjugation matrix $C_{ij}=N_{ij}^0$, which maps the field ϕ_i to its dual $\phi_i^\vee$.

Besides these two identities we have one more genus one relation, which can be represented pictorially as in Fig. 7, and which gives the following constraint on the duality matrices:

$$(S\otimes 1)F(1\otimes\Theta(-)\Theta(+))F^{-1}(S^{-1}\otimes 1)=FPF^{-1}(1\otimes\Omega(-)). \tag{2.8}$$

The idea [6] of Fig. 7 is that one inserts a primary p on the torus, where p is some field contained in the operator product expansion of j_1 and j_2. Then one "defuses" to get an insertion of j_1 and j_2 instead of p. Subsequently, one transports j_2 along the a or b cycle of the torus, and fuses it again with j_1 to get an insertion of some other field s. The two processes of transporting along the a or b cycle are related via the modular transformation $S:\tau\to -1/\tau$. Schematically, $SaS^{-1}=b$. Performing the moves explained here one arrives at (2.8), where $\Theta(\pm)$ acts as



so for the case of Fig. 7 we have $\mathbf{1}\otimes\Theta(-)\Theta(+)=e^{2i\pi(\Delta_i-\Delta_p)}$.

Note that if we take $j_1=j_2=j$ and $p=s=0$ in Fig. 7 this implies [6, 2],

$$N_{ij}^k=\sum_m S_{im}\frac{S_{km}^\dagger}{S_{0m}}S_{jm}, \tag{2.9}$$

where

$$\frac{S_{ij}}{S_{00}}=\frac{S_{0i}}{S_{00}}\frac{S_{0j}}{S_{00}}\sum_m B_{0m}\begin{bmatrix} i & j \\ i & j^\vee \end{bmatrix}(-)B_{m0}\begin{bmatrix} j & i \\ i & j^\vee \end{bmatrix}(-), \tag{2.10}$$

and where we used the "tetrahedron" symmetry

$$F_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}=F_{kj}\begin{bmatrix} i^\vee & q \\ p & l \end{bmatrix}\sqrt{\frac{F_pF_q}{F_kF_j}}, \tag{2.11}$$

$(F_i=S_{0i}/S_{00})$ which can be proven from the pentagon identity (2.4).

From (2.6) and (2.9) we also have

$$\frac{S_{ij}}{S_{00}}=\sum_m\frac{S_{0m}}{S_{00}}N_{ij}^m e^{2i\pi(\Delta_i+\Delta_j-\Delta_m)}. \tag{2.12}$$

So we see that we can make the following consistent "gauge choice":

$$B_{0m}\begin{bmatrix} i & j \\ i & j^\vee \end{bmatrix}(\varepsilon)=\sqrt{\frac{F_m}{F_iF_j}}\,\varepsilon_{ij}^m e^{i\pi\varepsilon(\Delta_m-\Delta_i-\Delta_j)}. \tag{2.13}$$

Taking the tetrahedron symmetry into account this is in fact the only gauge choice consistent with (2.1).

The above Eqs. (2.3–2.4) and (2.6–2.8) are the polynomial equations which encapture the fundamental duality relations of a Conformal Field Theory. In the next section we will explore these polynomial equations to show that we can define for every primary field in a Conformal Field Theory a topological invariant of knots (or more generally links). As we will see, these invariants are intimately connected with so-called Markov traces, which already appeared in a slightly different context in [40]. In the next section we will give a proof of the existence of such traces in Conformal Field Theory using the polynomial equations of Moore and Seiberg.

3. Topological Aspects of CFT

To define a topological invariant of links for Rational Conformal Field Theories we first have to discuss the relation between knots or links and braids. The braid group defined on n strands will be denoted by B_n and is generated by the simple braids $\langle\sigma_1, \ldots, \sigma_{n-1}\rangle$ which satisfy

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \tag{3.1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geqq 2. \tag{3.2}$$

The $B_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}$ encountered in the previous section form a representation of the braid group, and the Yang-Baxter equation (2.5) is a direct consequence of (3.1).

To discuss links in terms of braids, we will take a two dimensional point of view towards links. When one projects a link down to two dimensions to get a knot diagram, as in Fig. 8, the question which diagrams give equivalent links arises. A theorem in knot theory [4] states that knot diagrams give equivalent links when they can be transformed into each other via so-called Reidemeister moves, shown in Fig. 9. A link invariant defined on the level of these diagrams should of course be invariant under these Reidemeister moves to be a true topological invariant.

It is more or less obvious that every link can be obtained by the closure of a braid. Such a closure of a braid α (see Fig. 10) will be denoted by $\hat{\alpha}$. According to a

Fig. 8. A knot diagram

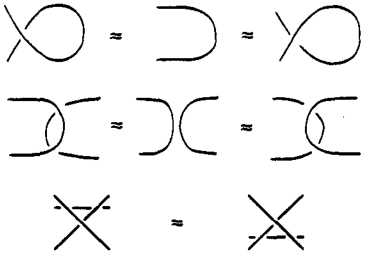


Fig. 9. The Reidemeister moves

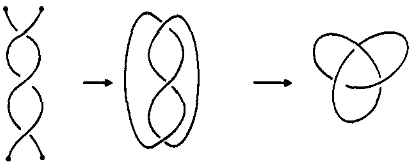


Fig. 10. Closure of a braid

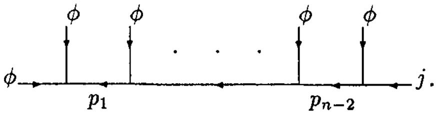
theorem of Markov, this means that an invariant $L(\hat{\alpha})$ defined in terms of braids α, should satisfy the following properties:

$$L(\widehat{\alpha\beta}) = L(\widehat{\beta\alpha}), \qquad \alpha, \beta \in B_n, \tag{3.3}$$

$$L(\widehat{\alpha\sigma_n^{\pm 1}}) = L(\hat{\alpha}), \qquad \alpha \in B_n, \tag{3.4}$$

where the trace property (3.3) is clear when one closes the braid, and (3.4) is the consequence of the first Reidemeister move. Conversely, these properties are sufficient to guarantee that L defines a link invariant.

The representation π of the braid group B_n that we will study here is a representation on special conformal blocks. These are genus zero blocks with n external ϕ lines and one "spectator" field j [21]



Such conformal blocks will be denoted as $\mathscr{F}^{(n)}_{\phi,j}$. The dimension $d^{(n)}_{\phi,j}$ of the vector space spanned by these conformal blocks is easily computed as

$$d^{(n)}_{\phi,j} = (N_\phi^{n-1})_{\phi j}\,.$$

So for a fixed spectator field j the braid matrices $\pi(\sigma_i)$ are elements of $\mathrm{Mat}(d^{(n)}_{\phi,j}, \mathbf{C})$, the space of complex square matrices of dimension $d^{(n)}_{\phi,j}$. We can use this to build finite dimensional C^*-algebras $C^{(n)}_\phi$ as follows:

$$C^{(n)}_\phi = \bigoplus_j \mathrm{Mat}(d^{(n)}_{\phi,j}, \mathbf{C})\,.$$

This gives us a sequence of inclusions of C^*-algebras

$$C^{(2)}_\phi \subset C^{(3)}_\phi \subset \ldots \subset C^{(n)}_\phi \subset \ldots, \tag{3.5}$$

with the inclusion matrix given by N_ϕ [13].

To define a link invariant for Rational Conformal Field Theories we will first look for a so-called Markov trace M_ϕ, which is defined on $C_\phi = \bigcup_n C^{(n)}_\phi$, and satisfies the following properties:

$$M(\pi(1)) = 1\,, \tag{3.6}$$

$$M(\pi(\alpha\beta)) = M(\pi(\beta\alpha)), \qquad \alpha, \beta \in B_n, \tag{3.7}$$

$$M(\pi(\alpha\sigma_n)) = zM(\pi(\alpha)), \qquad \alpha \in B_n, \tag{3.8}$$

$$M(\pi(\alpha\sigma_n^{-1})) = \bar{z}M(\pi(\alpha)), \qquad \alpha \in B_n, \tag{3.9}$$

where z is called the Markov parameter. Once we have such a Markov trace we can easily define a topological invariant out of it, as follows:

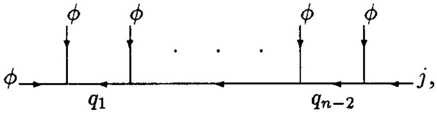
$$L(\hat{\alpha}) = (z\bar{z})^{-n/2} \left(\frac{\bar{z}}{z}\right)^{w(\alpha)/2} M(\pi(\alpha)), \qquad \alpha \in B_n, \tag{3.10}$$

where $w(\alpha)$ is the wraith of the braid α, i.e. the number of overcrossings minus the number of undercrossings in a knot diagram (with the choice that the σ_i generate overcrossings and the σ_i^{-1} undercrossings). Note that we have the following normalization:

$$L\left(\bigcirc\right)=\frac{1}{|z|}. \tag{3.11}$$

The idea now is that in order to study knots or links we first write them as the closure of braids, and then assign numbers to these braids as follows. We perform the same braids on the conformal block of (3.5), which then equals

$$\sum_{q_1,\ldots,q_{n-2}} B^{\phi,j}_{p_1,\ldots,p_{n-2};q_1,\ldots,q_{n-2}}(\alpha)\ \times$$



where $B^{\phi,j}_{p_1,\ldots,p_{n-2};q_1,\ldots,q_{n-2}}(\alpha)$ is a product of the braiding matrices $B_{pq}\begin{bmatrix}\phi & \phi\\ i & j\end{bmatrix}$, so it is a map from the braid group B_n to $C^{(n)}_{\phi}$. Taking the trace inside each $C^{(n)}_{\phi}$ we get another map t^{ϕ}_j from $C^{(n)}_{\phi}$ to $\mathbf{C}$

$$t^{\phi}_j : C^{(n)}_{\phi} \to \mathbf{C},$$
$$t^{\phi}_j(\pi(\alpha)) = \sum_{p_1,\ldots,p_{n-2}} B^{\phi j}_{p_1,\ldots,p_{n-2};p_1,\ldots,p_{n-2}}(\alpha),$$

and $t^{\phi}_j(\pi(\alpha))$ is the number we want to associate with the braid α.

The reason we restricted ourselves to the conformal blocks of (3.5) is that since we want to study links in terms of braids, all the external lines have to be the same, otherwise the braid cannot always be closed.

The final step is to construct out of the numbers $t^{\phi}_j(\pi(\alpha))$ a Markov trace $M_{\phi}(\pi(\alpha))$. A proposal for such a Markov trace is given in [21, 40] (and implicitly in [16]). We will not repeat the arguments leading to this proposal here, but simply state the result

$$M_{\phi}(\pi(\alpha)) = \left(\frac{S_{00}}{S_{0\phi}}\right)^{n} \sum_j \frac{S_{0j}}{S_{00}}\, t^{\phi}_j(\pi(\alpha)). \tag{3.12}$$

Note that due to (2.9) $M_{\phi}(\pi(\mathbf{1}))=1$. We will now prove that this proposal for the Markov trace indeed satisfies the Markov properties (3.7) and (3.8). As one can easily verify the trace property (3.7) is fulfilled due to the fact that we have taken the trace inside each $C^{(n)}_{\phi}$.

To prove the second Markov property (3.8) we first have to determine what it means in terms of the braid matrices. Setting $\alpha=\mathbf{1}$ in (3.8) and evaluating it on $\mathscr{F}^{(2)}_{\phi,j}$ gives for the Markov parameter z

$$z = \left(\frac{S_{00}}{S_{0\phi}}\right)^{2} \sum_j \frac{S_{0j}}{S_{00}}\, N^{j}_{\phi\phi}\varepsilon^{j}_{\phi\phi} e^{i\pi(2\Delta_{\phi}-\Delta_j)}, \tag{3.13}$$

which we will show to be equal to

$$z = \frac{e^{-2i\pi\Delta_{\phi}}}{S_{0\phi}/S_{00}}. \tag{3.14}$$

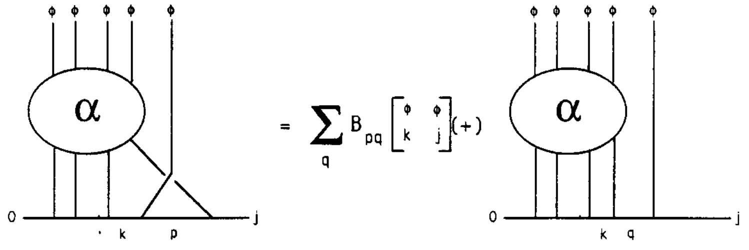


Fig. 11. Working out (3.8) on a general conformal block

The implication of (3.8) for general conformal blocks and general braids α is worked out in Fig. 11. From (3.12) and Fig. 11 we deduce that (3.8) becomes

$$\sum_j \frac{S_{0j}}{S_{00}} B_{pp}\begin{bmatrix}\phi & \phi\\ k & j\end{bmatrix}(+) = z\frac{S_{0\phi}}{S_{00}}\frac{S_{0p}}{S_{00}}N_{pk}^{\phi}. \tag{3.15}$$

Note that for $k=0$ this equation reduces to (3.13), so to show that (3.12) defines a good Markov trace we only have to prove (3.15). Using (2.13) we can rewrite the left-hand side (l.h.s.) of (3.15) as

$$\begin{aligned}\text{l.h.s.} &= \frac{S_{0p}}{S_{00}}\frac{S_{0\phi}}{S_{00}}\sum_j B_{0j}\begin{bmatrix}p & \phi\\ p & \phi^\vee\end{bmatrix}(-)B_{j0}\begin{bmatrix}\phi & p\\ p & \phi^\vee\end{bmatrix}(-)e^{2i\pi(\Delta_j-\Delta_p-\Delta_\phi)}B_{pp}\begin{bmatrix}\phi & \phi\\ k & j\end{bmatrix}(+)\\ &= \frac{S_{0p}}{S_{00}}\frac{S_{0\phi}}{S_{00}}e^{2i\pi(\Delta_p-\Delta_\phi-\Delta_k)}\sum_j B_{0j}\begin{bmatrix}p & \phi\\ p & \phi^\vee\end{bmatrix}(-)B_{pp}\begin{bmatrix}\phi & \phi\\ k & j\end{bmatrix}(-)B_{j0}\begin{bmatrix}\phi & p\\ p & \phi^\vee\end{bmatrix}(-)\\ &= \frac{S_{0p}}{S_{00}}\frac{S_{0\phi}}{S_{00}}e^{2i\pi(\Delta_p-\Delta_\phi-\Delta_k)}\Omega_{p\phi}^{k}(-)\Omega_{p\phi}^{k}(-)B_{00}\begin{bmatrix}\phi & \phi\\ \phi & \phi^\vee\end{bmatrix}(-)\\ &= \frac{S_{0p}}{S_{00}}e^{-2i\pi\Delta_\phi}N_{pk}^{\phi},\end{aligned} \tag{3.16}$$

where going from the first to the second line we used (2.2) and from the second to the third we used the Yang-Baxter equation. So we have proven that (3.12) indeed satisfies the Markov properties with the Markov parameter z given by (3.14).

We thus have produced for every primary field ϕ a link invariant given by

$$L_\phi(\hat{\alpha}) = e^{2i\pi\Delta_\phi w(\alpha)}\sum_j \frac{S_{0j}}{S_{00}}t_j^{\phi}(\pi(\alpha)) \tag{3.17}$$

with (3.11) replaced by

$$L_\phi\left(\bigcirc\right) = F_\phi. \tag{3.18}$$

If we specialize the above invariant to the case of a $SU(N)_k$ WZW model, with ϕ corresponding to the fundamental representation, $\phi=\square$, our invariant is in fact the Jones polynomial [7]. This can be proven as follows. The fundamental representation for $SU(N)$ has the "fusion" rule

$$\square\times\square = \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square\,.$$

The weights of the fields appearing in this product are given by

$$\Delta_{\Box}=\frac{(N-1)(N+1)}{2N(k+N)}; \quad \Delta_{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}=\frac{(N-2)(N+1)}{N(k+N)}; \quad \Delta_{\Box\Box}=\frac{(N-1)(N+2)}{N(k+N)},$$

which implies that the eigenvalue equation for the braid matrices $\pi(\sigma_i)$ becomes

$$(q^{-1/2N}\pi(\sigma_i)+\sqrt{q})(q^{-1/2N}\pi(\sigma_i)-1/\sqrt{q})=0 \tag{3.19}$$

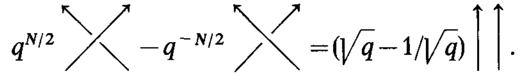
(here $q=e^{\frac{2i\pi}{k+N}}$) which (after some renormalization) is the Hecke relation.

From (3.19) we can derive the following property of the link invariant (3.17):

$$q^{N/2}L_{\Box}^{-}-q^{-N/2}L_{\Box}^{+}=\left(\sqrt{q}-\frac{1}{\sqrt{q}}\right)L_{\Box}^{0}, \tag{3.20}$$

where $L_{\Box}^{+}$ stands for the value of a link with at some point an overcrossing, $L_{\Box}^{-}$ is the value of the same link with the overcrossing replaced by an undercrossing and $L_{\Box}^{0}$ is the value of the link with the crossing removed.

Graphically, we can represent (3.20) as the "skein" relation



$$q^{N/2} \quad - q^{-N/2} \quad = (\sqrt{q}-1/\sqrt{q}) \quad . \tag{3.21}$$

This skein relation can be used to disentangle the knot. Together with the normalization (3.18) it completely determines the polynomial $L_{\Box}$. This polynomial equals the Jones polynomial as given in [16], since the skein relation we derived here is identical to that of [16]. In a similar way we can prove that for $\phi=\Box$ in $SO(N)_k$ or $Sp(2N)_k$ WZW models, our invariant is equivalent to the Kauffman invariant [8]. In fact, we can use (3.17) to construct many new knot polynomials, namely one for every primary field of an arbitrary RCFT (and not just for WZW models, which would give the same polynomials as Witten derived from (2+1)-dimensional Chern Simons theory). Although we should note that in practice the evaluation of the braid matrices appearing in (3.12) can become quite cumbersome.

Before we close this section we return to the issue of inclusions of C^*-algebras, as given in (3.5). We will argue that we can complete $\pi(B_\infty)=C_\phi=\bigcup_n C_\phi^{(n)}$ such that it becomes a so-called II_1 factor. First we will review the definition of a II_1 factor.

An algebra A is a factor if:

- A is a von Neumann algebra, i.e. an algebra of bounded operators on a Hilbert space $\mathscr{H}$, such that it contains the identity, it is closed under taking adjoints, and it is closed in the ultraweak topology[1].
- the center of A is trivial.

It is of type II_1 if it is infinite dimensional and admits a finite normalized trace $\mathrm{tr}: A\to\mathbf{C}$ such that

$$\begin{aligned} &\mathrm{tr}(1)=1, \\ &\mathrm{tr}(ab)=\mathrm{tr}(ba), \quad a,b\in A, \\ &\mathrm{tr}(a^*a)\geqq 0, \quad a\in A. \end{aligned} \tag{3.22}$$

This trace is always unique.

---

[1] This means if $\psi_1, \psi_2\in\mathscr{H}$, $a_n\in A$, $a\in B(\mathscr{H})$ and $\langle\psi_1|a_n\psi_2\rangle\to\langle\psi_1|a\psi_2\rangle$ then $a\in A$ as well

Jones has shown [10] how to associate to a II$_1$ factor M and a subfactor N, a number $[M:N]$, called the index, which measures "how many times N fits into M," similar to the index $[G:H]$ for finite groups. The index need not be an integer however.

There is one more property we will need: a factor is hyperfinite if it contains a dense increasing sequence of finite dimensional sub *-algebras $A_1 \subset A_2 \subset \ldots \subset A$. Up to isomorphism there is only one hyperfinite II$_1$ factor [11] usually denoted by $\mathscr{R}$. In a sense $\mathscr{R}$ is the smallest possible II$_1$ factor [12]: Any II$_1$ subfactor of $\mathscr{R}$ is again isomorphic to $\mathscr{R}$, and any II$_1$ factor contains $\mathscr{R}$. Another property of $\mathscr{R}$ is that the range of the index $[\mathscr{R}:\mathscr{R}']$, where $\mathscr{R}'$ runs over all possible subfactors of $\mathscr{R}$, equals [10]

$$[\mathscr{R}:\mathscr{R}'] \in \left\{4\cos^2\frac{\pi}{n}\right\}_{n\geqq 3} \cup [4, +\infty]. \tag{3.23}$$

Using the Markov trace M_ϕ we can define an inner product on $\pi(B_\infty) = C_\phi$,

$$\langle x|y\rangle = M_\phi(x^* y). \tag{3.24}$$

Using this inner product we can take the weak closure of $\pi(B_\infty)$. It can be proven, unless ϕ is simple [26], that this closure $\bar{\pi}(B_\infty)$ satisfies all the requirements in the definition of a hyperfinite II$_1$ factor, so we see that it is in fact isomorphic to the hyperfinite II$_1$ factor $\mathscr{R}$.

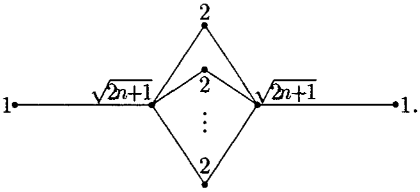
With this factor, naturally comes a subfactor as follows. Take for B'_∞ the braid group generated by the elements $\langle\sigma_2, \sigma_3, \ldots\rangle$, then $\bar{\pi}(B'_\infty)$ is a subfactor of $\bar{\pi}(B_\infty)$. The index of this subfactor can be calculated as

$$[\bar{\pi}(B_\infty):\bar{\pi}(B'_\infty)] = \lim_{n\to\infty} \frac{\sum\limits_j (N_\phi^{n+1})^2_{\phi j}}{\sum\limits_j (N_\phi^n)^2_{\phi j}} = \left(\frac{S_{0\phi}}{S_{00}}\right)^2, \tag{3.25}$$

since $S_{0\phi}/S_{00}$ is the largest eigenvalue of N_ϕ.

For the special value 3 of the index, Jones [34] noted that (for $\phi = \square$ in $SU(2)_4$) $\bar{\pi}(B'_\infty) \subset \bar{\pi}(B_\infty)$, is equivalent to the pair $\mathscr{R}^{D_3} \subset \mathscr{R}^{\mathbf{Z}_2}$, where $\mathscr{R}^G$ denotes the set of fixed points of $\mathscr{R}$ under an outer action (i.e. not of the form gxg^{-1}) of the finite group G. Furthermore, at this value of the index the link invariant (3.17), in this case the Jones polynomial, is equal to $\pm i$ times a power of $\sqrt{3}$, for any link L. A similar situation occurs for index equal to 5. Here, ($\phi = \square$ in $Sp(4)_2$) $\bar{\pi}(B'_\infty) \subset \bar{\pi}(B_\infty)$ can be described as $\mathscr{R}^{D_5} \subset \mathscr{R}^{\mathbf{Z}_2}$, and the link invariant (3.17), now called the Kauffman invariant, is equal to $\pm i$ times a power of $\sqrt{5}$.

To understand these peculiarities we will consider the more general situation of $\mathscr{R}^{D_{2p+1}} \subset \mathscr{R}^{\mathbf{Z}_2}$, and show that similar things happen here, thereby generalizing the results of [34]. The "principal graph" (see Sect. 9) one gets for $\mathscr{R}^{D_{2p+1}} \subset \mathscr{R}^{\mathbf{Z}_2}$ is



(3.26)

As will be explained in Sect. 9, to get an inclusion of factors from RCFT which is equivalent to $\mathscr{R}^{D_{2p+1}} \subset \mathscr{R}^{\mathbf{Z}_2}$, we have to find a primary field which has (3.26) as its fusion graph. Such a primary Φ is given by the field which corresponds to the $\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}$ representation (p blocks) in a $SO(2p+1)_2$ WZW theory, except for $p=1$ for which $\Phi = \square$ in $SU(2)_4$ and $p=2$ for which $\Phi = \square$ in $Sp(4)_2$.

The conformal weight of the field Φ is given by $\Delta_\Phi = \frac{p}{8}$, and the weights of the fields which appear in the product of Φ with itself are given by

$$\Delta_j = \frac{(p-j)(p+j+1)}{4p+2}, \quad j=0,\ldots,p.$$

The group theoretical factor ε for these models is

$$\varepsilon^j_{\Phi\Phi} = i^{(j^2+j)}, \quad j=0,\ldots,p.$$

From this we deduce that the eigenvalue equation for the braid matrices $\pi(\sigma_i)$ becomes

$$\prod_{j=0}^{p} (i^{-p/2}\pi(\sigma_i) - i^{(j^2+j)}\omega^{-1/4(p-j)(p+j+1)}) = 0, \tag{3.27}$$

where $\omega = e^{\frac{2i\pi}{2p+1}}$. We can rewrite this product such that it becomes

$$\prod_{j=0}^{p} (i^{p/2}\pi(\sigma_i) + i^{-p^2}\omega^{j^2}) = 0. \tag{3.28}$$

This allows us to take for the σ_i the so-called "Gaussian" representation [34],

$$\pi(\sigma_i) = \frac{i^{-p/2}}{\sqrt{2p+1}} \sum_{j=0}^{2p} \omega^{-j^2} u_i^j,$$

where the u_i satisfy

$$\begin{aligned} &u_i^{2p+1} = \mathbf{1}, \\ &u_i u_{i+1} = \omega^2 u_{i+1} u_i, \\ &u_i u_j = u_j u_i \quad |i-j| \geqq 2, \end{aligned}$$

since the eigenvalue equation for $\pi(\sigma_i)$ with σ_i defined in this way is equivalent to (3.28).

Now it is shown in [34] that the Markov trace evaluated on a braid in the Gaussian representation gives (up to some constant C which is a power of $2p+1$)

$$\sum_{v \in H_1(S;\mathbf{Z}_{2p+1})} \omega^{\langle v,v \rangle}, \tag{3.29}$$

where S is a Seifert surface for the closed braid, and $\langle\ ,\ \rangle$ is the Seifert pairing (for an explanation of these terms, see for example [4]). For the link invariant L_Φ given by (3.17) this implies that, whenever $2p+1$ is prime, $|L_\Phi(\hat{\alpha})|$ equals $C(2p+1)^{v+\mu/2}$, where $(\mu)v$ is the number of (non)zero eigenvalues of the Seifert pairing.

With the Gaussian representation at our disposal, we can now easily show why $\bar{\pi}(B'_\infty) \subset \bar{\pi}(B_\infty)$ is equivalent to $\mathscr{R}^{D_{2p+1}} \subset \mathscr{R}^{\mathbf{Z}_2}$. Following [34] we deduce that the completion of the algebra generated by u_i, denoted by $A = \overline{\mathrm{Alg}(u_1, u_2, \ldots)}$, is isomorphic to $\mathscr{R}$. On A we have the following $\mathbf{Z}_2$-action: $u_i \to u_i^{-1}$, whose fixed points are the $\pi(\sigma_i)$, so $\bar{\pi}(B_\infty) \cong \mathscr{R}^{\mathbf{Z}_2}$. Furthermore, we have an $\mathbf{Z}_{2p+1}$-action on A given by: $u_1 \to \omega u_1$ and $u_i \to u_i$ for $i \geqq 2$, whose fixed point algebra is generated by the $\langle u_2, u_3, \dots \rangle$. Since $D_{2p+1} = \mathbf{Z}_2 \rtimes \mathbf{Z}_{2p+1}$, this implies that the II_1 factor generated by the $\langle \sigma_2, \sigma_3, \dots \rangle$, i.e. $\bar{\pi}(B'_\infty)$, is isomorphic to $\mathscr{R}^{D_{2p+1}}$. So we can finally conclude that $\bar{\pi}(B'_\infty) \subset \bar{\pi}(B_\infty)$ is equivalent to $\mathscr{R}^{D_{2p+1}} \subset \mathscr{R}^{\mathbf{Z}_2}$.

4. II_1 Factors Coming from RCFT

In this section we will define what a coupling system is and how they can be obtained from Rational Conformal Field Theories. Some background material on coupling systems and their relation with inclusions of factors is gathered in Appendix A.

Let $\mathscr{G}$ be an unoriented graph. A path of length n on $\mathscr{G}$ has the obvious meaning. The vertex where a path ξ starts will be denoted by $s(\xi)$ (source), the endpoint by $r(\xi)$ (range). In particular a path of length one is just an edge with an orientation. The reverse $\xi^{\sim}$ of a path ξ is the same path walked along in the opposite direction. If we have two paths ξ_1 and ξ_2, and ξ_2 starts, where ξ_1 ends, $\xi_1 \circ \xi_2$ will stand for the path "first ξ_1 and then ξ_2." The set of all paths of length n starting at x and ending at a vertex y will be denoted by $\mathrm{Path}^n_{x,y}$, the length of a path by $|\xi|$.

A standard finite measure graph is a finite connected graph $\mathscr{G}$ with a distinguished vertex $* = *_{\mathscr{G}}$ adjacent to only one other vertex $**_{\mathscr{G}}$ via one edge, and with a natural $\mathbf{Z}_2$ grading given by the distance of a vertex to $*$. The even vertices will be denoted by $\mathscr{G}_{\mathrm{even}}$, so $* \in \mathscr{G}_{\mathrm{even}}$, and the odd vertices by $\mathscr{G}_{\mathrm{odd}}$. Let Λ be the incidence matrix of $\mathscr{G}$, then by Perron-Frobenius theory Λ has a unique eigenvector with only positive entries and such that its entry at $*$ is 1. The eigenvalue will be denoted by $\|\Lambda\|$, and the components will be labeled by F_x, where x is a vertex of $\mathscr{G}$.

We have the following definition: a local coupling system is a quadruple $(\mathscr{G}, \mathscr{H}, \tau, W)$, where $\mathscr{G}$ and $\mathscr{H}$ are finite standard measure graphs with $\|\Lambda_{\mathscr{G}}\| = \|\Lambda_{\mathscr{H}}\|$. Furthermore τ is an involution on the set of vertices of $\mathscr{G} \cup \mathscr{H}$, satisfying

$$\begin{gathered} \tau(*_{\mathscr{G}}) = *_{\mathscr{G}}, \quad \tau(*_{\mathscr{H}}) = *_{\mathscr{H}}, \quad \tau(**_{\mathscr{G}}) = **_{\mathscr{H}}, \\ \tau(\mathscr{G}_{\mathrm{even}}) = \mathscr{G}_{\mathrm{even}}, \quad \tau(\mathscr{G}_{\mathrm{odd}}) = \mathscr{H}_{\mathrm{odd}}, \quad \tau(\mathscr{H}_{\mathrm{even}}) = \mathscr{H}_{\mathrm{even}}, \\ F_{\tau(x)} = F_x . \end{gathered} \tag{4.1}$$

W is map which associates to any cell (a_1, a_2, a_3, a_4) consisting of four oriented edges with $\tau(r(a_i)) = s(a_{i+1})$, a number $W(a_1, a_2, a_3, a_4) \in \mathbf{C}$ satisfying five axioms which we will give below.

Consider a RCFT and pick a field Φ. We make a graph by taking $2N$ vertices if N is the number of primary fields in the theory, and label them by ϕ_i and ϕ'_j, where i runs from 1 to N. Next we draw $N^j_{\Phi_i}$ edges from ϕ_i to ϕ'_j. Let $\mathscr{G}$ be the connected component of the resulting graph containing the identity operator $\mathbf{1}$, and let $*_{\mathscr{G}} = \mathbf{1}$. Also let $\mathscr{H}$ be the connected component containing $\mathbf{1}'$ and let $*_{\mathscr{H}} = \mathbf{1}'$. From now on we will usually identify ϕ_i and ϕ'_i. We see that $\mathscr{G}$ is the graph obtained by alternatingly fusing with Φ and its dual field $\Phi^{\vee}$, and $\mathscr{H}$ is obtained by the same process, starting however with $\Phi^{\vee}$. Therefore, as graphs $\mathscr{G}$ and $\mathscr{H}$ are identical. Note that not all fields need occur in $\mathscr{G}$ and $\mathscr{H}$. The eigenvalues for the Perron-Frobenius eigenvector are given by $\|\Lambda_{\mathscr{G}}\| = \|\Lambda_{\mathscr{H}}\| = S_{0\Phi}/S_{00}$. The contragradient map is defined by $\tau(\phi_i)=\phi_i^\vee$ or $\tau(\phi_i)=\phi_i'^\vee$ (compare with the charge conjugation matrix C of Sect. 2). This can be done in such a way that it is compatible with the demands stated above in the definition of a coupling system. The Perron-Frobenius eigenvector has components $F_i=S_{0\phi_i}/S_{00}$.

The definition of W for RCFT's is a bit more involved. Fix once and for all an ε, which may be either $+$ or $-$. Consider a cell (a_1, a_2, a_3, a_4) consisting of four edges $((\phi_1,\phi_2^\vee), (\phi_2,\phi_3^\vee), (\phi_3,\phi_4^\vee), (\phi_4,\phi_1^\vee))$. We have to consider four different cases:

$$\phi_1\in\mathscr{G}_{\text{odd}}\to W^{(1)}(a_1,a_2,a_3,a_4)=B_{\phi_2^\vee\phi_4^\vee}\begin{bmatrix}\Phi & \Phi\\ \phi_1 & \phi_3\end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}}, \tag{4.2}$$

$$\phi_1\in\mathscr{G}_{\text{even}}\to W^{(2)}(a_1,a_2,a_3,a_4)=B_{\phi_4^\vee\phi_2^\vee}\begin{bmatrix}\Phi & \Phi^\vee\\ \phi_1 & \phi_3\end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}}, \tag{4.3}$$

$$\phi_1\in\mathscr{H}_{\text{odd}}\to W^{(3)}(a_1,a_2,a_3,a_4)=B_{\phi_2^\vee\phi_4^\vee}\begin{bmatrix}\Phi^\vee & \Phi^\vee\\ \phi_1 & \phi_3\end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}}, \tag{4.4}$$

$$\phi_1\in\mathscr{H}_{\text{even}}\to W^{(4)}(a_1,a_2,a_3,a_4)=B_{\phi_4^\vee\phi_2^\vee}\begin{bmatrix}\Phi^\vee & \Phi\\ \phi_1 & \phi_3\end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}}. \tag{4.5}$$

For $N^k_{\Phi i}>1$ a pair of vertices does not specify an edge and we would have to include into the definition of W also a dependence on couplings ε, which are elements of a $N^k_{\Phi i}$ dimensional vector space. We have suppressed these as they would just complicate the expressions. Furthermore, Ocneanu has defined a notion of equivalence of two coupling systems, stating that two coupling systems are equivalent precisely when they differ by a unitary transformation in the space of couplings, and therefore everything is independent of a choice of basis in the space of couplings.

In order to check the axioms that W has to satisfy, let us recall some of the symmetries of the braid matrices,

$$B_{pq}\begin{bmatrix}j_1 & j_2\\ i & k\end{bmatrix}(\varepsilon)=B_{ik}\begin{bmatrix}j_1^\vee & j_2\\ p & q\end{bmatrix}(-\varepsilon)\sqrt{\frac{F_pF_q}{F_iF_k}}, \tag{4.6}$$

$$B_{pq}\begin{bmatrix}j_1 & j_2\\ i & k\end{bmatrix}(\varepsilon)=B_{p^\vee q^\vee}\begin{bmatrix}j_2 & j_1\\ k^\vee & i^\vee\end{bmatrix}(\varepsilon), \tag{4.7}$$

$$B_{pq}\begin{bmatrix}j_1 & j_2\\ i & k\end{bmatrix}(\varepsilon)=B_{qp}\begin{bmatrix}j_1^\vee & j_2^\vee\\ k & i\end{bmatrix}(\varepsilon), \tag{4.8}$$

where (4.6) is a consequence of (2.2) and (2.11), (4.7) is due to our convention for the conformal blocks and (4.8) is a direct consequence of (4.6) and (4.7).

First we will check the three axioms that Ocneanu calls local [9].

– The first axiom is that of inversion symmetry: for any cell (a_1, a_2, a_3, a_4) we must have

$$W(a_4^\sim, a_3^\sim, a_2^\sim, a_1^\sim)=W(a_1,a_2,a_3,a_4). \tag{4.9}$$

From now on we will assume that

$$(a_1,a_2,a_3,a_4)=((\phi_1,\phi_2^\vee),(\phi_2,\phi_3^\vee),(\phi_3,\phi_4^\vee),(\phi_4,\phi_1^\vee)). \tag{4.10}$$

In order to check the inversion symmetry we would in principle have to distinguish between four cases, depending on whether ϕ_1 is in $\mathscr{G}$ or in $\mathscr{H}$, and whether it is an even or odd vertex. We will just prove it for one case, the other three being completely similar. So assuming $\phi_1 \in \mathscr{G}_{\mathrm{odd}}$, we have

$$\begin{aligned} W(a_4^{\sim}, a_3^{\sim}, a_2^{\sim}, a_1^{\sim}) &= W^{(3)}((\phi_1^{\vee}, \phi_4), (\phi_4^{\vee}, \phi_3), (\phi_3^{\vee}, \phi_2), (\phi_2^{\vee}, \phi_1)) \\ &= B_{\phi_4\phi_2}\begin{bmatrix} \Phi^{\vee} & \Phi^{\vee} \\ \phi_1^{\vee} & \phi_3^{\vee} \end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}} \\ &= B_{\phi_2\phi_4}\begin{bmatrix} \Phi & \Phi \\ \phi_3^{\vee} & \phi_1^{\vee} \end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}} \\ &= B_{\phi_2^{\vee}\phi_4^{\vee}}\begin{bmatrix} \Phi & \Phi \\ \phi_1 & \phi_3 \end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}} \\ &= W^{(1)}(a_1, a_2, a_3, a_4)\,. \end{aligned}$$

– The next axiom is the axiom of rotation symmetry,

$$W(a_2, a_3, a_4, a_1) = W(a_1, a_2, a_3, a_4)^*\,. \tag{4.11}$$

To check this, take for example $\phi_1 \in \mathscr{G}_{\mathrm{even}}$. We have

$$\begin{aligned} W^{(3)}(a_2, a_3, a_4, a_1) &= B_{\phi_3^{\vee}\phi_1^{\vee}}\begin{bmatrix} \Phi^{\vee} & \Phi^{\vee} \\ \phi_2 & \phi_4 \end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_2F_4}{F_1F_3}} \\ &= B_{\phi_2\phi_4}\begin{bmatrix} \Phi & \Phi^{\vee} \\ \phi_3^{\vee} & \phi_1^{\vee} \end{bmatrix}(-\varepsilon)\sqrt{\frac{F_1F_3}{F_2F_4}}\sqrt[4]{\frac{F_2F_4}{F_1F_3}} \\ &= \left(B_{\phi_2^{\vee}\phi_4^{\vee}}\begin{bmatrix} \Phi^{\vee} & \Phi \\ \phi_3 & \phi_1 \end{bmatrix}(\varepsilon)\right)\sqrt[4]{\frac{F_1F_3}{F_2F_4}} \\ &= \left(B_{\phi_4^{\vee}\phi_2^{\vee}}\begin{bmatrix} \Phi & \Phi^{\vee} \\ \phi_1 & \phi_3 \end{bmatrix}(\varepsilon)\sqrt[4]{\frac{F_1F_3}{F_2F_4}}\right)^* \\ &= W^{(2)}(a_1, a_2, a_3, a_4)^*\,, \end{aligned}$$

where in the second line we used $B^*(\varepsilon) = B^{\vee}(-\varepsilon)$, see Sect. 2.

– The third and last local axiom is the axiom of bi-unitarity. This axiom states that the connection is a unitary matrix, after a certain renormalization. In our case that means that we have to check whether the braid matrices in (4.2–4.5), without the normalization factors, are unitary. This fact was already noted in Sect. 2 below Eq. (2.2), and therefore the third axiom is also satisfied.

This completes the proof that the connections obtained from Rational Conformal Field Theories satisfy all the local axioms.

Next we want to prove the two remaining axioms, which are called the global axioms, to make the coupling system a global one. To state these, one needs to extend the definition of W from cells to more general surfaces, using Ocneanu's cell calculus, where the map W is extended to a map defined on contours. A contour consists of four paths $(\xi_1, \xi_2, \xi_3, \xi_4)$ in either $\mathscr{G}$ or $\mathscr{H}$, with $|\xi_1| = |\xi_3|$, $|\xi_2| = |\xi_4|$ and $s(\xi_{i+1}) = \tau(r(\xi_i))$. A surface s is a family of cells $c(i,j) = (c(i,j)_1, c(i,j)_2, c(i,j)_3, c(i,j)_4)$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ having matching walls: $c(i+1,j)_4 = c(i,j)_2^{\sim}$ and $c(i,j+1)_1$

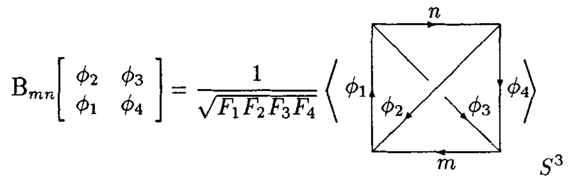


Fig. 12. The braid matrix as the expectation value of a graph

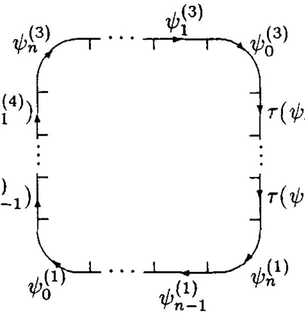


Fig. 13. The boundary of $\Gamma(c)$

$=c(i,j)_{\tilde{3}}$. The boundary of s is a contour $(\xi_1, \xi_2, \xi_3, \xi_4)$ with $\xi_1=c(n,1)\circ\dots\circ c(1,1)$ etc. For a surface s, one defines

$$W(s)=\prod_{i,j} W(c(i,j)) \tag{4.12}$$

and for a contour c,

$$W(c)=\sum_{s} W(s), \tag{4.13}$$

where the sum is taken over all surfaces having boundary c.

To see what these expressions mean in RCFT, observe that $W(c)$ consists of a sum of products of braid matrices. As we have seen, similar expressions are encountered in the computation of knot invariants for RCFT's. So it is tempting to find a knot whose expectation value in 3-d topological field theory equals $W(c)$. However, as it turns out, we need a graph instead of a knot. This is because braid matrices are related to expectation values of graphs rather than knots. The precise relation [14] is given in Fig. 12. From now on we will take $\varepsilon=+$; if one takes $\varepsilon=-$ instead one just has to replace overcrossings by undercrossings and vice versa.

If we have a graph projected onto a plane and a preferred "time" direction, the computation of the expectation value involves a summation over all possible ways to fill in the graph, as explained in [14]. This corresponds precisely to a sum over all surfaces with a fixed boundary as in Eq. (4.13).

Consider now an arbitrary contour $c=(\xi_1, \xi_2, \xi_3, \xi_4)$, and suppose that

$$\xi_1=(\psi_{n-1}^{(1)}, \psi_n^{(1)})\circ(\psi_{n-2}^{(1)}, \psi_{n-1}^{(1)})\circ\dots\circ(\psi_0^{(1)}, \psi_1^{(1)}) \tag{4.14}$$

with similar expressions for ξ_2, ξ_3, and ξ_4. The boundary of our graph will consist of the rectangle with the labeling of the fields as indicated in Fig. 13. Next we have to fill this graph with a set of horizontal and vertical lines in such a way that is

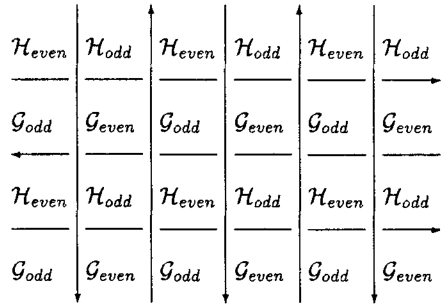


Fig. 14. The interior of $\Gamma(c)$

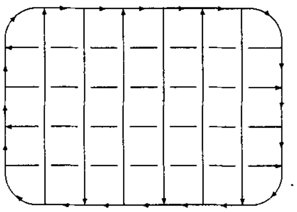


Fig. 15. An example

compatible with the grid depicted in Fig. 14. A typical example we might get in the case $m=4$, $n=6$ and $\psi_0^{(1)} \in \mathscr{G}_{\text{even}}$ is shown in Fig. 15. The unmarked lines in the graph will always represent our special chosen field Φ. Again we might also have to include labels, at every vertex of the graph where three lines meet, to represent the couplings. As remarked before, we will not do this, but the reader should keep in mind that it is always possible to explicitly include the couplings at any stage.

Denote the graph obtained in this way by $\Gamma(c)$. A careful computation of $\langle \Gamma(c) \rangle_{S^3}$ based on the results of Witten [14], using as time direction south-east to north-west, shows that this expectation value precisely equals $W(c)$, up to a normalization factor. This normalization factor can also be computed, where one has to pay special attention to the boundary of the graph (see also [15]). The result is

$$W(c) = F_{\Phi}^{-(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)/4} (F_{s(\xi_1)} F_{s(\xi_2)} F_{s(\xi_3)} F_{s(\xi_4)})^{-1/4} \langle \Gamma(c) \rangle S^3 . \tag{4.15}$$

Using this formula we can now prove that the two global axioms a global coupling system has to fulfill are also valid.

The first one, the parallel transport axiom, states that for any contour c with $r(\xi_i) = s(\xi_i) = *_{\mathscr{G}}$ or $*_{\mathscr{H}}$,

$$W(c) = \delta(\xi_1, \xi_3^{\sim}) \delta(\xi_2, \xi_4^{\sim}) . \tag{4.16}$$

In this case, let us take for example $r(\xi_i) = s(\xi_i) = *_{\mathscr{G}}$, the graph $\Gamma(c)$ consists of two disconnected components, as shown sketchy in Fig. 16. Due to the topological invariance in this theory we can move the two pieces apart and using Eq. (4.15) we find

$$W(c) F_{\Phi}^{(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)/4} = \langle \Gamma_1 \# \Gamma_2 \rangle_{S^3} = \langle \Gamma_1 \rangle_{S^3} \langle \Gamma_2 \rangle_{S^3} . \tag{4.17}$$

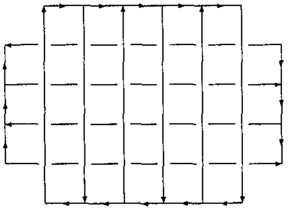
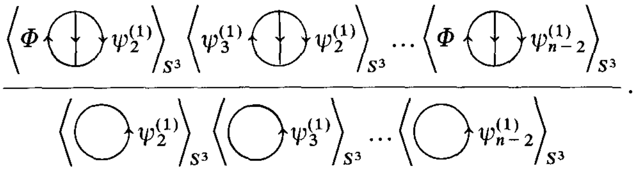


Fig. 16. The first axiom

Expectation values of graphs are topological invariant. One can also prove this directly where invariance under the Reidemeister moves (Fig. 9) is due to the Yang-Baxter equation (2.5), and invariance under moving a line over a vertex where three lines meet is due to the pentagon identity (2.4).

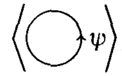
Now it remains to compute $\langle\Gamma_1\rangle_{S^3}$. Using Witten's cutting prescriptions [14, 16] based on the fact that for one-dimensional Hilbert spaces we have $\langle a|b\rangle\langle c|d\rangle=\langle a|d\rangle\langle c|b\rangle$, it follows that

$$\langle\Gamma_1\rangle_{S^3}=\delta(\xi_1,\xi_3^{\sim})$$



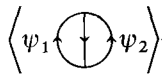
. (4.18)

From Sect. 3 we have (cf. Eq. (3.18))



$$=F_\psi \tag{4.19}$$

which implies



$$=F_{\Phi 0}\begin{bmatrix}\psi_1^{\vee} & \psi_1\\ \psi_2^{\vee} & \psi_2^{\vee}\end{bmatrix}F_{\psi_1}F_{\psi_2}=\sqrt{F_{\psi_1}F_{\psi_2}F_{\Phi}}\,. \tag{4.20}$$

Using this result we find

$$\langle\Gamma_1\rangle_{S^3}=F_{\Phi}^{n/2}\delta(\xi_1,\xi_3^{\sim})=F_{\Phi}^{(|\xi_1|+|\xi_3|)/4}\delta(\xi_1,\xi_3^{\sim})\,. \tag{4.21}$$

Putting everything together the final result is

$$W(c)=\delta(\xi_1,\xi_3^{\sim})\delta(\xi_2,\xi_4^{\sim}) \tag{4.22}$$

as requested.

Finally, the global contragradient axiom states that for any vertex $x\in\mathscr{G}\cup\mathscr{H}$, there is a contour $(\xi_1,\xi_2,\xi_3,\xi_4)$ with $s(\xi_1)=s(\xi_3)=*_{\mathscr{G}}$ or $*_{\mathscr{H}}$, $s(\xi_2)=x$ and $s(\xi_4)=\tau(x)$, and such that $W(c)\neq 0$. For such a contour we find

$$\langle\Gamma(c)\rangle_{S^3}=\frac{1}{F_x}\langle\Gamma_1\rangle_{S^3}\langle\Gamma_2\rangle_{S^3}\,, \tag{4.23}$$

where Γ_1 is given in Fig. 17, for the case $s(\xi_1)=s(\xi_3)=*_{\mathscr{G}}$. Cutting along the dashed line in Fig. 17 shows that

$$\langle\Gamma_1\rangle_{S^3}=\langle\psi_2^{(1)}\dots\psi_{n-1}^{(1)}|\psi_{n-1}^{(3)}\dots\psi_1^{(3)}\rangle\,, \tag{4.24}$$

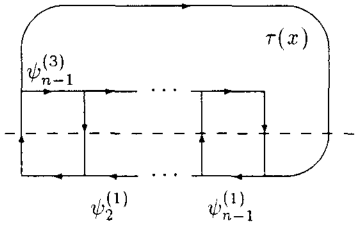


Fig. 17. The contragradient axiom

i.e. the inner product of two states in the Hilbert space of the punctured two-sphere with charges as those occurring along the dashed line. Now if we let the ψ's vary, both states in this inner product run through a basis of this Hilbert space, so it is certainly possible to choose them such that this inner product is nonzero. Therefore, the connection also satisfies the global contragradient axiom, and this completes the proof that the connection obtained from RCFT gives rise to a coupling system.

Due to the one-one relation between coupling systems and irreducible finite index finite depth inclusions of II_1 factors [9], this proves that for every RCFT together with a field Φ there is a corresponding inclusion of such II_1 factors (in fact there are two, as we can take both $\varepsilon = +$ and $\varepsilon = -$, but these two choices need not be inequivalent), see also [41, 42]. An immediate consequence is that we always have

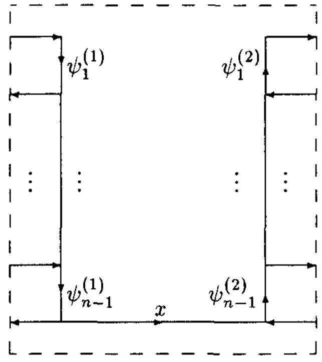
$$\frac{S_{0\Phi}}{S_{00}} \in \left\{\cos\frac{n}{\pi}\right\}_{n\geq 3} \cup [2,\infty]. \tag{4.25}$$

Although this method would enable one to construct many examples of irreducible inclusions of II_1 factors, and maybe even new ones, we will be mainly interested in the reverse process: given an inclusion, when does this correspond to a RCFT? To answer this question we will first take a closer look at the II_1 factors coming from RCFT.

5. The String Algebras

First we construct the string algebra on $\mathscr{G}$. A string is a pair (ξ_1, ξ_2) of paths of length n starting at $*$ and ending at the same vertex x. Write $\xi_1 = (*, \psi^{(1)}_1, \ldots, \psi^{(1)}_{n-1}, x)$ and $\xi_2 = (*, \psi^{(2)}_1, \ldots, \psi^{(2)}_{n-1}, x)$, so that $\psi^{(1)}_1 = \psi^{(2)}_1 = \Phi$. Now define

$$\Omega(\xi_1,\xi_2) = \sqrt{F_x} F_\Phi^{-n/2} \tag{5.1}$$



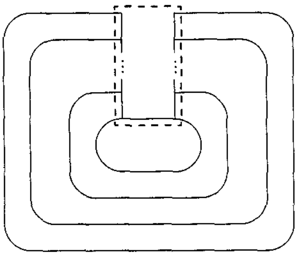


Fig. 18. The closure of Ω

in the case n is even. If n is odd, the definition is the same, only the direction of the four arrows at the bottom of the picture has to be reversed. From now on we will always display one special case, the others being obtainable via a minor modification.

The box in this definition should be seen as a part of a graph embedded in S^3, or alternatively as a state in the Hilbert space of a punctured two-sphere. Given two such Ω's, they can be multiplied with each other: one has to multiply the constants in front and put the graphs next to each other, after which one has to glue them together in an obvious way. The closure of Ω is the graph (including the constant in front) obtained by identifying the in- and outgoing lines as shown in Fig. 18, and will be denoted by $\bar{\Omega}$.

The algebra A_n is the algebra having as basis the $\Omega(\xi_1, \xi_2)$, where (ξ_1, ξ_2) runs over all strings of length n, and with the multiplication rule explained above. The adjoint of $\Omega(\xi_1, \xi_2)$ is equal to $\Omega(\xi_2, \xi_1)$. The algebra A_n constructed this way is isomorphic to the algebra $M_n \cap N'$ occurring in the derived tower [Eq. (A.6)]. Actually, the multiplication rule is very simple. Using the techniques similar as those used in the proof of the global parallel transport axiom [cf. Eq. (4.18)] one can easily derive

$$\Omega(\xi_1, \xi_2)\Omega(\xi_3, \xi_4) = \delta(\xi_2, \xi_3)\Omega(\xi_1, \xi_4). \tag{5.2}$$

A_n is imbedded in A_{n+1} via

$$\Omega(\xi_1, \xi_2) \to \sum_{\zeta} \Omega(\xi_1 \circ \zeta, \xi_2 \circ \zeta), \tag{5.3}$$

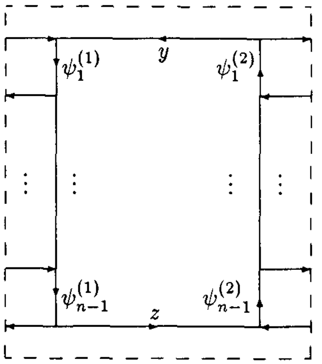
where the sum is over all edges starting at x. A trace on A_n compatible with this imbedding $A_n \to A_{n+1}$ is given by

$$\begin{aligned} \operatorname{tr}(\Omega(\xi_1, \xi_2)) &= F_\Phi^{-n} \langle \overline{\Omega(\xi_1, \xi_2)} \rangle_{S^3} \\ &= F_x F_\Phi^{-n} \delta(\xi_1, \xi_2). \end{aligned} \tag{5.4}$$

This trace can be used to complete $\cup A_n$ into a von Neumann algebra A. This algebra A is in fact a subspace of the space of all conformal blocks. A similar construction works for $\mathscr{H}$ where the paths start at $*_{\mathscr{H}}$. The graphs occurring in the definition of Ω have in this case all the arrows of the in- and outgoing lines reversed. In this way one gets algebras B_n which may be completed into a von Neumann algebra B. The notation used here for the operators $\Omega(\xi_1, \xi_2)$ is more or less similar to the notation used for instance in [17, 40] to label the bases of spaces of intertwiners.

In order to define other operations on the string algebras A_n and B_n, it is convenient to extend the definition (5.1) to two paths starting at a vertex y and ending at a common vertex z. In that case we define

$$\Omega(\xi_1,\xi_2) = \sqrt{F_yF_zF_\Phi^{-n/2}} \qquad (5.5)$$



Again we only have displayed the case $y, z \in \mathscr{G}_{\text{even}}$ or $y, z \in \mathscr{H}_{\text{odd}}$. The definition in the other cases is similar.

Using this extended definition we define a homomorphism $\phi: A_n \to B_{n+1}$ as follows: let $\Omega(\xi_1,\xi_2) \in A_n$ and (ζ_1,ζ_2) be two arbitrary paths on $\mathscr{H}$ of length n starting at $\Phi^\vee$ and having a common range. Let e denote the path of length one on $\mathscr{H}$ from $*_{\mathscr{H}}$ to $\Phi^\vee$. Then ϕ is defined by

$$\phi(\Omega(\xi_1,\xi_2)) = \sum_{\zeta_1,\zeta_2} F_{r(\zeta_1)}^{-1} F_{s(\zeta_1)}^{-1} \langle \overline{\Omega(\xi_1,\xi_2)\Omega(\zeta_1,\zeta_2)} \rangle_{S^3} \Omega(e \circ \zeta_1, e \circ \zeta_2). \qquad (5.6)$$

Define in a similar way a map $\phi: B_n \to A_{n+1}$, and let $\Delta: A_n \to A_{n+2}$ be $\phi \circ \phi$. This is what Ocneanu [9] calls the canonical shift. In fact, Δ plays the role of a generalization of the comultiplication for these string algebras. The map ϕ can be used to define a homomorphism $\phi: A \to B$, and the inclusion of II$_1$ factors belonging to this coupling system is precisely the inclusion $\phi(A) \subset B$.

To see how the even vertices of $\mathscr{G}$ correspond to $A-A$ modules, fix a vertex $x \in \mathscr{G}_{\text{even}}$, and consider a pair (α,β) of paths of length n, having common range, while α starts at $*_{\mathscr{G}}$ and β starts at x. These pairs (α,β), so-called open strings, together form the basis of a linear space $A_n(x)$. Let $\Omega(\xi_1,\xi_2) \in A_n$, then $\Omega(\xi_1,\xi_2)$ acts on (α,β) from the left as follows:

$$\Omega(\xi_1,\xi_2) \cdot (\alpha,\beta) = \delta(\xi_2,\alpha)(\xi_1,\beta). \qquad (5.7)$$

To define the right action we need a generalization of (5.5)[2]

$$(\alpha,\beta) \cdot \Omega(\xi_1,\xi_2) = \sum_{\gamma \in \mathrm{Path}^n_{x,r(\alpha)}} (F_{r(\alpha)}^{-1} F_x^{-1} \langle \overline{\Omega(\xi_1,\xi_2)\Omega(\gamma,\beta)} \rangle_{S^3})(\alpha,\gamma). \qquad (5.8)$$

What goes into this definition is precisely Ocneanu's notion of parallel transport. We see that $A_n(x)$ is a $A_n - A_n$ bimodule and after taking an appropriate completion we get a $A-A$ bimodule $A(x)$. These are irreducible [9]. Therefore we have the interesting result that the irreducible modules of the string algebras correspond to certain primary fields of the underlying RFCT.

If we consider vertices in $\mathscr{G}_{\text{odd}}$ or in $\mathscr{H}$ we must also consider left and right actions of B_n. Again, the expressions are the same as those occurring in (5.7) and (5.8).

---

[2] It is an interesting exercise to check that this right action is indeed compatible with the algebra structure on A_n

6. Tensor Products and the Number of Paths

How do the fusion rules arise in this context? Take for simplicity two even vertices of $\mathscr{G}$, say (x, y), and let (α, β) be a pair of paths starting at respectively x and y having common range. On the linear space $A_n(x, y)$ which has as basis the pairs (α, β) one can again define a left and a right action of A_n, similar as in (5.8),

$$\Omega(\xi_1, \xi_2) \cdot (\alpha, \beta) = \sum_{\gamma \in \mathrm{Path}^{n}_{\mathfrak{x}, r(\alpha)}} (F^{-1}_{r(\alpha)} F^{-1}_{x} \langle \overline{\Omega(\xi_1, \xi_2)\Omega(\alpha, \gamma)} \rangle_{S^3})(\gamma, \beta), \tag{6.1}$$

$$(\alpha, \beta) \cdot \Omega(\xi_1, \xi_2) = \sum_{\gamma \in \mathrm{Path}^{n}_{\mathfrak{y}, r(\alpha)}} (F^{-1}_{r(\alpha)} F^{-1}_{y} \langle \overline{\Omega(\xi_1, \xi_2)\Omega(\gamma, \beta)} \rangle_{S^3})(\alpha, \gamma). \tag{6.2}$$

$A_n(x, y)$ will decompose into irreducible $A_n - A_n$ modules: $A_n(x, y) = \oplus A_n(z)$. We will present some dimensional arguments why we expect that

$$A_n(x, y) = \bigoplus_z N^{\phi_z}_{\phi_x^\vee \phi_y} A_n(z), \tag{6.3}$$

where ϕ_x is the field corresponding to the vertex x etc. Actually, $A_n(x, y)$ is precisely what one finds when one studies the tensor product of the representations $A(x)$ and $A(y)$ using the generalized comultiplication. Therefore, we see that the fusion rules are just the rules for decomposing the tensor product of representations of the string algebras.

Let $f_{ij}(t)$ be the generating function for the number of paths from i to j; that is,

$$f_{ij}(t) = \sum_{k=0}^{\infty} |\mathrm{Path}^k_{i,j}| t^k. \tag{6.4}$$

It is easy to check that $f_{ij}(t) = (\mathbf{1} - t\Lambda_{\mathscr{G}})^{-1}_{ij}$, where $\mathbf{1}$ represents the unit matrix. We would like to check whether

$$f_{ij}(t) = \sum_k N^k_{i^\vee j} f_{0k}(t) \tag{6.5}$$

or equivalently whether

$$g_{ij}(t) = \sum_k N^k_{i^\vee j} g_{0k}(t), \tag{6.6}$$

where

$$g_{ij}(t) = \det(1 - t\Lambda_{\mathscr{G}})(1 - t\Lambda_{\mathscr{G}})^{-1}_{ij}. \tag{6.7}$$

Using

$$N^k_{ij} = \sum_\alpha \frac{S_{\alpha i} S_{\alpha j} S^*_{\alpha k}}{S_{\alpha 0}}$$

and $(S^2)_{ij} = C = \delta_{ij^\vee}$, $SS^* = \mathbf{1}$ one can derive the following expressions for $f_{ij}(t)$ in terms of the modular matrix S

$$f_{ij}(t) = \sum_\alpha S^*_{\alpha i} \frac{1}{1 - t^2 \left|\frac{S_{\alpha\Phi}}{S_{\alpha 0}}\right|^2} S_{\alpha j}, \qquad i, j \in \mathscr{G}_{\text{even}} \text{ or } \mathscr{G}_{\text{odd}}, \tag{6.8}$$

$$f_{ij}(t) = \sum_\alpha S^*_{\alpha i} \frac{t S^*_{\alpha\Phi}/S_{\alpha 0}}{1 - t^2 \left|\frac{S_{\alpha\Phi}}{S_{\alpha 0}}\right|^2} S_{\alpha j}, \qquad i \in \mathscr{G}_{\text{even}}, j \in \mathscr{G}_{\text{odd}}, \tag{6.9}$$

$$f_{ij}(t) = \sum_\alpha S^*_{\alpha i} \frac{t S_{\alpha\Phi}/S_{\alpha 0}}{1 - t^2 \left|\frac{S_{\alpha\Phi}}{S_{\alpha 0}}\right|^2} S_{\alpha j}, \qquad i \in \mathscr{G}_{\text{odd}}, j \in \mathscr{G}_{\text{even}}. \tag{6.10}$$

Similar expressions are valid for $\mathscr{H}$, where $S_{\alpha\Phi}/S_{\alpha 0}$ is replaced by its complex conjugate. The last relation we need in order to put everything together is

$$S^*_{\alpha i}S_{\alpha j}=\sum_k N^k_{i^\vee j}S_{\alpha 0}S_{\alpha k}\,. \tag{6.11}$$

It is now obvious that relation (6.5) is fulfilled, and that we therefore have a perfect agreement with the decomposition rule (6.3), at least as far as dimensions are concerned.

As a side remark, observe that

$$\lim_{t\to S_{00}/S_{0\Phi}}\frac{f_{0i}(t)}{f_{0j}(t)}=\frac{F_i}{F_j}, \tag{6.12}$$

so in a sense F_i measures how many paths there are from $*$ to i. A remarkable fact is that [33]

$$\frac{F_i}{F_j}=\lim_{q\to 1}\frac{\chi_i(q)}{\chi_j(q)}, \tag{6.13}$$

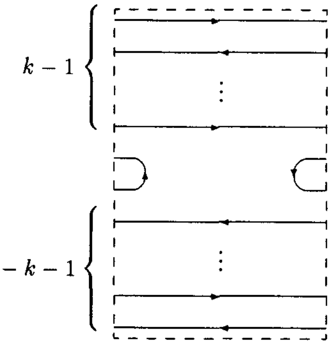
where the character χ_i is the trace of $q^{(L_0-c/24)}$ in the representation corresponding to ϕ_i. We thus see that the number of states in the i^{th} representation grow asymptotically at the same rate relative to each other as the number of paths.

Another way to obtain the fusion rules from path algebras has been studied in [31, 32], by techniques similar to the ones in Sect. 9.

7. Algebras Hidden in the Path Algebras

The projections $e_k\in M_{k+1}$ [see (A.3)] descend to A_{k+1}. Instead of expressing them in terms of the basis (5.1), we will express them directly in terms of (pieces of) a graph. Let $l>k$ then e_k can be represented in A_l as

$$e_k=F_\Phi^{-1}\times$$



. (7.1)

These e_k indeed satisfy a Temperley-Lieb algebra

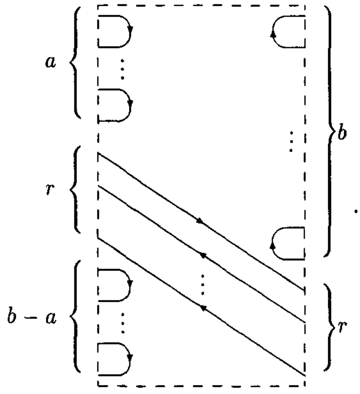
$$\begin{aligned} e_k^2&=e_k\,,\\ e_ke_{k\pm 1}e_k&=F_\Phi^{-2}e_k\,,\\ e_ke_{k'}&=e_{k'}e_k\,,\qquad |k-k'|\geqq 2\,. \end{aligned} \tag{7.2}$$

The last two of these equations follow directly by gluing graphs together, the first one follows from Eq. (4.19). Because $[M:N]=F_\Phi^{-2}$ this algebra is the same as the one appearing in (A.4). Similar pictures to represent the Temperley-Lieb algebra have already been given in [18]. The generators e_k can be expressed in terms of the basis (5.1) of A_l via the identity

$$e_k = \sum_{\xi_3,\xi_4} \langle e_k\overline{\Omega(\xi_4,\xi_3)}\rangle_{S^3}\,\Omega(\xi_3,\xi_4)\,. \tag{7.3}$$

We want to define another set of elements of A_k. These also do not have a very simple expression in terms of the basis (5.1), but can be defined using a graph as for the generators of the Temperley-Lieb algebra. The definition of the element $\Theta^{(r)}_{a,b}$ is (for the case $b \geqq a$)

$$\Theta^{(r)}_{a,b} = F_\Phi^{-\frac{a+b}{2}}$$



. (7.4)

$\Theta^{(r)}_{a,b}$ is an element of A_{r+2b}. The sequence $\{\Theta^{(r)}_{a,b}\}_{r\geqq 0}$ converges to an element of A, the closure of $\cup A_n = A_\infty$. Call this element $\Theta_{a,b}$. These elements satisfy the following algebra, which can be found using (4.19):

$$\Theta_{a,b}\Theta_{c,d} = \Theta_{\max(a,c-b+a),\,\max(d,b+d-c)}\,. \tag{7.5}$$

If we now define for $n \geqq 0$,

$$L_n = \sum_{k=0}^{\infty} \Theta_{k,k+n}\,, \tag{7.6}$$

we find that for n, $m \geqq 0$,

$$[L_n, L_m] = (n-m)L_{n+m}\,, \tag{7.7}$$

i.e. the positive half of the Virasoro algebra! Using graphs we can express these L_n formally in terms of the generators e_k of the Temperley-Lieb algebra (7.2). Let $\tilde{e}_k = F_\Phi e_k$, then

$$L_n = F_\Phi^{-n/2} \sum_{l=0}^{\infty} F_\Phi^{-l}\left(\prod_{k=-\infty}^{-1}(\tilde{e}_{-k+2l}\tilde{e}_{-k+2l+2}\cdots\tilde{e}_{-k+2l+2n-2})\right)(\tilde{e}_{2l-1}\cdots\tilde{e}_3\tilde{e}_1)\,. \tag{7.8}$$

This can be seen as an indication of the suspected relation between the Virasoro algebra and the Temperley-Lieb algebra [19, 20]. It would be interesting to have the negative half of the Virasoro algebra as well, although it seems difficult to express them in a similar way as in (7.6).

8. Reconstruction of RCFT

We would now like to consider the reverse problem: given an irreducible finite index finite depth subfactor of the hyperfinite factor $\mathscr{R}$, when does this inclusion correspond to one obtained from a Rational Conformal Field Theory?

First recall how to get the graphs $\mathscr{G}$ and $\mathscr{H}$ from RCFT. We took $2N$ vertices labeled ϕ_i and ϕ'_j and drew $N^j_{\Phi i}$ edges from ϕ_i to ϕ'_j. Call the resulting graph Γ, which in general will consist of several connected components. If **1** and **1**$'$ are in the same connected component, $\mathscr{G}$ and $\mathscr{H}$ will be the same graph, having a $\mathbf{Z}_2$-automorphism with no fixed points (mapping ϕ_i to ϕ'_i). Otherwise $\mathscr{G}$ and $\mathscr{H}$ will be different but identical graphs. Let $\Gamma_1 \dots \Gamma_r$ denote the other connected components of the graph, i.e. those not containing **1** or **1**$'$. At first sight they could be anything, but in fact the possibilities are quite restricted due to the following

Theorem. *The graphs Γ_i have the following properties:*

$$\mathrm{spec}(\Gamma_i) \subset \mathrm{spec}(\mathscr{G}), \tag{8.1}$$

$$\|\Gamma_i\| = \|\mathscr{G}\|. \tag{8.2}$$

The spectrum of a graph means here the set of eigenvalues of the incidence matrix, not counting multiplicities. Note that the graphs Γ_i also have no loops of odd length, just like $\mathscr{G}$ and $\mathscr{H}$. These conditions on the graphs Γ_i do not determine them completely, but usually only a few possibilities are left. (For more on graph spectra, see e.g. [13, 22].)

To prove the theorem, define the $2N \times 2N$ matrix A

$$A = A_{\mathscr{G}}(\oplus A_{\mathscr{H}}) \oplus A_{\Gamma_1} \oplus \dots \oplus A_{\Gamma_r}, \tag{8.3}$$

and let [3]

$$\omega_j = \arg\left(\frac{S_{\Phi j}}{S_{0j}}\right) \tag{8.4}$$

with the convention that $\arg(0)=0$. Now we define $2N$ eigenvectors of A called $v^{(p)}$; they are defined by the values they take at the vertices corresponding to ϕ_j and ϕ'_j indicated by labels j and j'. The index p takes values in the same set. We put

$$v^{(p)}_j = \frac{S_{jp}}{S_{0p}}, \tag{8.5}$$

$$v^{(p)}_{j'} = e^{i\omega_j/2} \frac{S_{jp}}{S_{0p}}, \tag{8.6}$$

$$v^{(p')}_j = \frac{S_{jp}}{S_{0p}}, \tag{8.7}$$

$$v^{(p')}_{j'} = -e^{i\omega_j/2} \frac{S_{jp}}{S_{0p}}. \tag{8.8}$$

These $2N$ orthogonal eigenvectors all have the value 1 at the vertex $*_{\mathscr{G}}$ corresponding to the identity **1**. Therefore, all eigenvectors $v^{(p)}$ correspond to an eigenvalue of A occurring in $\mathrm{spec}(A_{\mathscr{G}})$. This shows that

$$\mathrm{spec}(A) = \mathrm{spec}(A_{\mathscr{G}}). \tag{8.9}$$

Property (8.1) now follows from

$$\mathrm{spec}(A) = \mathrm{spec}(A_{\mathscr{G}})(\cup \mathrm{spec}(A_{\mathscr{H}})) \cup \mathrm{spec}(A_{\Gamma_1}) \cup \dots \cup \mathrm{spec}(A_{\Gamma_r}) \tag{8.10}$$

---

[3] arg means the argument of a complex number: $\arg(re^{i\phi}) = \phi$

and property (8.2) is a direct consequence of the fact that $S_{0i}/S_{00}>0$, so that $v^{(0)}$ is the Perron-Frobenius eigenvector of Λ.

Part of the reconstruction of a RCFT now goes as follows:

- Start with an inclusion $\mathscr{R}'\subset\mathscr{R}$ and determine $\mathscr{G}$ and $\mathscr{H}$; the first constraint here is that as unlabeled graphs, $\mathscr{G}$ and $\mathscr{H}$ must be identical.
- If $\mathscr{G}$ and $\mathscr{H}$ have a $\mathbf{Z}_2$-automorphism with no fixed points, we may have to omit $\mathscr{H}$ altogether.
- Label the vertices of $\mathscr{G}$ (and $\mathscr{H}$) with ϕ_i and ϕ'_i in a way consistent with how $\mathscr{G}$ (and $\mathscr{H}$) were constructed (i.e. $\tau(\phi_i)=\phi_i^\vee$ or $\phi_i^{\vee\prime}$, $\mathbf{1}=*_{\mathscr{G}}$ $(\mathbf{1}'=*_{\mathscr{H}})$ and there are only edges between primed and unprimed fields).
- Try to determine the S-matrix and check whether $S=S^t$; if this is not true, try to repeat the procedure with extra graphs Γ_i satisfying (8.1) and (8.2).
- Try to determine T from $(ST)^3=S^2$.

In general this procedure will grow more and more complex as we take more graphs Γ_i, so the best thing to do is to use the smallest number of graphs possible. The reason why we expect this to give a well-defined conformal field theory is that in the original graphs $\mathscr{G}$ and $\mathscr{H}$ we automatically have good fusion rules and braiding matrices, and the hope is that they can be extended to the other graphs Γ_i as well. The only severe restrictions here are $S^t=S$ and the fact that $\mathscr{G}$ and $\mathscr{H}$ must be identical as graphs. In the latter case we will call the inclusion factors self-dual, because as paragroups $\mathscr{H}$ can be considered as the dual of $\mathscr{G}$. In the case of finite groups this would restrict us to abelian groups only. Later on we will do some speculation on the meaning of $S'=S$.

Another remark concerns the solution of $(ST)^3=S^2$. This equation only determines the value of the central charge modulo 8 and of the conformal weights modulo 1, but certainly not all possibilities are realized. The two constraints we know of are that the following two numbers must be nonnegative integers:

$$6\left(\frac{N(N-1)}{12}+\sum_{i=0}^{N-1}\left(\frac{c}{24}-\Delta_i\right)\right), \tag{8.11}$$

$$\tfrac{1}{2}M(M-1)+M(\Delta_i+\Delta_j+\Delta_k+\Delta_l)-\sum_s(N^s_{ij}N_{kls}+N^s_{ik}N_{jls}+N^s_{il}N_{jks})\Delta_s, \tag{8.12}$$

where N is the number of primary fields, $M=N^s_{ij}N_{kls}$, and i, j, k, l are arbitrary. These conditions follow from considerations of the characters of RCFT's [23, 24].

9. Examples

We will now give several examples of inclusions of factors of type II_1. We start with inclusions with index smaller than 4, so that the index equals $4\cos^2(\pi/m)$ for some $m\geq 3$. Then $\|\mathscr{G}\|=2\cos(\pi/m)$ and the only possible graphs are the Dynkin diagrams A_n, D_n and E_6, E_7, and E_8 belonging to $m=n+1$, $2n-2$, 12, 18, and 30 respectively. According to Ocneanu, $\mathscr{G}$ cannot be equal to E_7 or D_n with n odd, but the other possibilities do indeed occur. Inclusions producing the Dynkin diagrams A_n can be constructed in terms of the e_k occurring in (7.2).

Given an inclusion we try to find RCFT's, which correspond to this inclusion in the way outlined in the previous sections. However, to prove this correspondence, one would in general also have to compare the connection obtained from the inclusion with the braiding matrices of the Rational Conformal Field Theories. We will not do this, but we believe that this will not cause any problems, for the following reason. Usually, the number of possible connections is very small (at most two if the index is smaller than four [9]), certainly if one identifies the connections that are related to each other by an automorphism of the graphs. Therefore, we think that in the examples that follow, and it is certainly true if the index is smaller than four, the only possible connections are equivalent to those in Eqs. (4.2)–(4.5) with $\varepsilon = +$ or $-$.

Index 1: In this case $\mathscr{G} = \bullet\!-\!\!\!-\!\bullet$, the Dynkin diagram A_2. As $\mathscr{G}$ has a $\mathbf{Z}_2$-automorphism with no fixed points, we may omit $\mathscr{H}$. Assuming there are no further graphs Γ_i, the field identification is $\mathbf{1}\bullet\!-\!\!\!-\!\bullet\mathbf{1}'$ and this corresponds to a holomorphic theory [25]. An example is the $(E_8)_1$ WZW theory. If we do not omit $\mathscr{H}$, a labeling giving a symmetric S-matrix is

$$\mathscr{H}: \mathbf{1}'\bullet\!-\!\!\!-\!\bullet\,\Phi, \qquad \mathscr{G}: \mathbf{1}\bullet\!-\!\!\!-\!\bullet\,\Phi' \tag{9.1}$$

corresponding to $SU(2)_1$. Allowing extra graphs Γ_i, these must all be equal to $\mathscr{G}$, since this is the unique graph with norm one[4]. Examples producing an arbitrary number of Γ_i are rational Gaussian models, and RCFT's having Φ as a simple current [26]. In particular this shows directly that the condition that Φ is a simple current is equivalent to $\Phi\Phi^\vee = \mathbf{1}$, and to $S_{0\Phi}/S_{00} = 1$ as well.

Index 2: $\mathscr{G} = \bullet\!-\!\!\!-\!\bullet\!-\!\!\!-\!\bullet$. We also need $\mathscr{H}$ here. The labeling (no extra Γ's) is

$$\begin{array}{c} \mathbf{1}'\bullet\!-\!\!\!-\!\overset{\Phi}{\bullet}\!-\!\!\!-\!\bullet\,\psi' \\ \mathbf{1}\bullet\!-\!\!\!-\!\underset{\Phi'}{\bullet}\!-\!\!\!-\!\bullet\,\psi \end{array} \tag{9.2}$$

and some corresponding models are the Ising model, $SU(2)_2$ and $(E_8)_2$.

Index $(3+\sqrt{5})/2 = 4\cos^2(\pi/5)$: $\mathscr{G} = \bullet\!-\!\!\!-\!\bullet\!-\!\!\!-\!\bullet\!-\!\!\!-\!\bullet$. Omitting $\mathscr{H}$ gives a theory with fusion rules $\Phi^2 = \mathbf{1} + \Phi$, known from e.g. the Lee-Yang singularity, $(G_2)_1$ and $(F_4)_1$. Including $\mathscr{H}$ gives $SU(2)_3$. The only possible graph Γ_i is the Dynkin diagram A_4. Including one of these gives a situation existing in $SU(3)_2$, reminiscent of the $SU(N)_k \leftrightarrow SU(k)_N$ duality [27].

Index 3: For the first time we have two possibilities for $\mathscr{G}$: either $\mathscr{G} = A_5$ or $\mathscr{G} = D_4$. First, consider $\mathscr{G} = A_5$. How do we find the S-matrix? In general we can use (6.7) to determine the polynomials g_{ij} and then use (6.6) to try to find the fusion rules. Diagonalizing these gives the S-matrix. Another technique is trying to express all fields as polynomials in Φ. Labeling A_5 as

$$\overset{\mathbf{1}}{\bullet}\!-\!\!\!-\!\overset{\Phi}{\bullet}\!-\!\!\!-\!\overset{\psi_2}{\bullet}\!-\!\!\!-\!\overset{\psi_3}{\bullet}\!-\!\!\!-\!\overset{\psi_4}{\bullet} \tag{9.3}$$

gives for instance (assuming $\Phi = \Phi^\vee$) $\Phi^2 = \mathbf{1} + \psi_2$, so $\psi_2 = \Phi^2 - 1$, $\psi_3 = \Phi\psi_2 - \Phi = \Phi^3 - 2\Phi$, and $\psi_4 = \Phi^4 - 3\Phi^2 + 1$. The sequences of polynomials one finds in the case of A_n are Chebyshev polynomials of the second kind. We also must have $\Phi\psi_4 = \psi_3$, giving $\Phi^5 - 4\Phi^3 + 3\Phi = 0$, which is precisely the equation $\det(\Phi\mathbf{1} - A_{\mathscr{G}}) = 0$. We can now consider $\{1, \Phi, \psi_2, \psi_3, \psi_4\}$ as being a basis of the ring $\mathbf{Z}[\Phi]/(\Phi^5 - 4\Phi^3 + 3\Phi)$. Taking the product of two fields and writing it as a sum of basis elements in this ring reproduces the fusion rules. Furthermore,

$$\Phi^5 - 4\Phi^3 + 3\Phi = \Phi(\Phi^2 - 1)(\Phi^2 - 3) \tag{9.4}$$

---

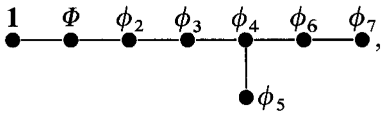
[4] The norm of a graph Γ is $\|A_\Gamma\|$

has roots $\{\pm\sqrt{3}, \pm 1, 0\}$ and computing $\psi_i(\Phi)$ for these values of Φ gives the numbers $S_{\psi_i k}/S_{0k}$ which can be used to compute the S-matrix. However, this method has in general problems if the graph has an automorphism leaving $\mathbf{1}$ invariant.

$\mathscr{G}=\mathscr{H}=A_5$ are graphs obtained from $SU(2)_4$. If $\mathscr{G}=\mathscr{H}=D_4$, there is a problem, because we cannot construct a symmetric S-matrix. Including extra graphs Γ_i which must necessarily also be equal to D_4 according to (8.1) and (8.2) might resolve this problem, but we do not know of any example where this occurs. (This case has also been considered in [37] where it was found to be inconsistent with the duality relations of RCFT.)

Index $4\cos^2(\pi/11)$: $\mathscr{G}=A_{10}$, omitting $\mathscr{H}$ gives $(E_8)_3$ or $(F_4)_2$, including $\mathscr{H}$ gives $SU(2)_9$.

Index $4\cos^2(\pi/30)$: Apart from A_{29}, $\mathscr{G}$ can also be the Dynkin diagram E_8. Let us label the fields as follows



(9.5)

where we have again assumed a self-dual situation, $\Phi^\vee=\Phi$. We can use the technique given above in the index 3 case to try to find the fusion rules belonging to $\mathscr{G}$. Instead of working with Φ as an independent variable, it is more convenient to use $\Phi=\omega+\omega^{-1}$. Computing $\det(\Phi\mathbf{1}-\Lambda_{E_8})=0$ gives

$$f(\omega)=\omega^{16}+\omega^{14}-\omega^{10}-\omega^{8}-\omega^{6}+\omega^{2}+1=0 \tag{9.6}$$

and it is straightforward to express the fields in terms of ω

$$\phi_2(\omega)=\omega^2+1+\omega^{-2}, \qquad \phi_3(\omega)=\omega^3+\omega+\omega^{-1}+\omega^{-3},$$
$$\phi_4(\omega)=\omega^4+\omega^2+1+\omega^{-2}+\omega^{-4}, \qquad \phi_5(\omega)=-\omega^7+\omega^3+\omega+\omega^{-1}+\omega^{-3}-\omega^{-7},$$
$$\phi_6(\omega)=\omega^7+\omega^5+\omega^{-5}+\omega^{-7}, \qquad \phi_7(\omega)=\omega^6+\omega^{-6}.$$

Taking products of these polynomials and using (9.6) to express the result in terms of the fields, gives the following new fusion rules:

$$\phi_2\times\phi_3=\Phi+\phi_3+\phi_5+\phi_6, \qquad \phi_3\times\phi_6=\phi_2+2\phi_4,$$
$$\phi_2\times\phi_4=\phi_2+2\phi_4+\phi_7, \qquad \phi_4\times\phi_4=\mathbf{1}+2\phi_2+3\phi_4+\phi_7,$$
$$\phi_2\times\phi_6=\phi_3+\phi_5+\phi_6, \qquad \phi_4\times\phi_5=\Phi+\phi_3+\phi_5+\phi_6,$$
$$\phi_3\times\phi_3=\mathbf{1}+\phi_2+2\phi_4+\phi_7, \qquad \phi_6\times\phi_6=\mathbf{1}+\phi_2+\phi_4+\phi_7,$$
$$\phi_3\times\phi_5=\phi_2+\phi_4+\phi_7, \qquad \phi_7\times\phi_7=\mathbf{1}+\phi_7.$$

The characters λ_i^j of this fusion algebra can be found by computing $\phi_i(\omega_j)$, where $\omega_j=e^{i\pi k_j/30}$ and $k_j=1,7,11,13,17,19,23,29$. In the case of a RCFT the characters of the fusion algebra are just the numbers S_{ij}/S_{0j}. If we try to compute the S-matrix in this case, we find that there does not exist a symmetric S-matrix. Maybe using extra graphs Γ_i, which must in this case be equal to E_8 as well, it is possible to find an (exotic?) RCFT giving this E_8 diagram.

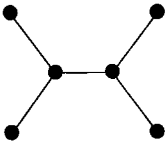
The E_8 case does however exhibit a feature that is shared by all Rational Conformal Field Theories, namely

Theorem. *For any* RCFT, S_{ij}/S_{0j} *is always a finite sum of roots of unity with integer coefficients. The proof of this can be found in Appendix B.*

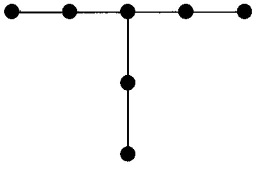
We proceed with the index four case: there are infinitely many graphs with norm 2, namely the $\hat{A}$, $\hat{D}$, and $\hat{E}$ series. The $\hat{A}$ series are ruled out as candidates for $\mathscr{G}$, because they have no distinguished vertex $*$. Subfactors of $\mathscr{R}$ producing graphs of type $\hat{A}$, $\hat{D}$, and $\hat{E}$ can be constructed as follows [13]: realize the hyperfinite factor $\mathscr{R}$ as the completion of $\overset{\infty}{\bigotimes} M_2(\mathbf{C})$, where $M_n(\mathbf{C})$ denotes the algebra of complex $n \times n$ matrices. $SU(2)$ acts on $\mathscr{R}$ by conjugation on every $M_2(\mathbf{C})$, so in particular any finite closed subgroup G of $SU(2)$ acts on $\mathscr{R}$. In the same way we can define an action of G on $\mathscr{R} \otimes M_2(\mathbf{C})$. Consider the inclusion

$$\mathscr{R}^G \subset (\mathscr{R} \otimes M_2(\mathbf{C}))^G, \tag{9.7}$$

where $\mathscr{R}^G$ stands for the elements of $\mathscr{R}$ left invariant by the action of G. Then the principal graph is precisely one of the $\hat{A}$, $\hat{D}$ or $\hat{E}$ series, giving the McKay correspondence between affine Dynkin diagrams and finite subgroups of $SU(2)$. One can obtain the graphs directly from G: take as fusion rules the representation ring of G, and let Φ correspond to the 2-dimensional representation of G obtained by restricting the fundamental representation of $SU(2)$ to G. Then the construction as in Sect. 4 yields the corresponding $\hat{A}$, $\hat{D}$, and $\hat{E}$ graphs.

Take for example the graph $\hat{D}_3$: . Then the graphs Γ_i must be $\hat{A}_4$, $\hat{A}_6$, or $\hat{D}_3$. Omitting $\mathscr{H}$ and including $\Gamma_1 = \hat{A}_4$ gives a situation as in $SU(2)_4$, where Φ is the field corresponding to the spin-1 representation. Including also $\Gamma_2 = \hat{A}_6$ is what happens in a holomorphic D_3 orbifold [25].

For another example take $\mathscr{G} = \mathscr{H} = \hat{E}_6$



(9.8)

We can take $\Gamma_1 = \hat{A}_6$, giving a set of graphs occurring is $SU(3)_3$.

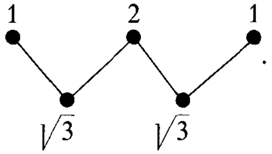
Actually, all possibilities occur in the $c=1$ models that are $SU(2)$ orbifolds [25]. In particular, $SU(2)/D_N$ gives (with an appropriate choice of Φ) $\mathscr{G} = \hat{D}_N$. The total field content of the $SU(2)/D_N$-models is organized as follows: $\mathscr{G} = \mathscr{H} = \hat{D}_N$, and there are $N+1$ extra graphs Γ_i: $\hat{A}_{2N}$ occurs $N-1$ times, and the other two graphs are of type $\hat{A}_4$. The total number of fields is $2 \times (N+3) + (N-1) \times 2N + 2 \times 4 = 2(N^2+7)$, in agreement with the results in [25]. Of course we get twice the number of primary fields, because we are counting primed fields as well as unprimed fields. Using this result one can for instance compute the S-matrix of $SU(2)/D_3$ and show that it is just the tensor product of $SU(2)_1$ and the holomorphic D_3-orbifold.

As a final class of examples, consider the following situation: suppose the finite group G acts outerly on $\mathscr{R}$. Suppose furthermore that G is the semidirect or crossed product of two subgroups H and A, $G=H \rtimes A$, such that A is a normal and abelian subgroup of G. In that case we can start with the inclusion

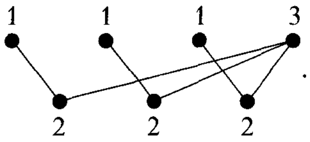
$$\mathscr{R} \rtimes H \subset \mathscr{R} \rtimes G \tag{9.9}$$

or equivalently with $\mathscr{R}^G \subset \mathscr{R}^H$, to try to find Rational Conformal Field Theories, because in this case the graphs $\mathscr{G}$ and $\mathscr{H}$ are equal (cf. [29] and [30, par 8.2]). The even vertices of $\mathscr{G}$ correspond to the irreducible representations of G, the odd vertices correspond to the irreducible representations of H, and the number of edges between a representation π_1 of G and a representation π_2 of H is given by the number of times π_2 occurs in the restriction of π_1 to H. The Perron-Frobenius vector ("the $S_{0\phi}/S_{00}$") has the value $\dim(\pi_1)$ at the vertex corresponding to π_1, and the value $\sqrt{[G:H]}\,\dim(\pi_2)$ at π_2. In particular the index $[\mathscr{R} \rtimes G : \mathscr{R} \rtimes H]=[G:H]$ $=|A|$.

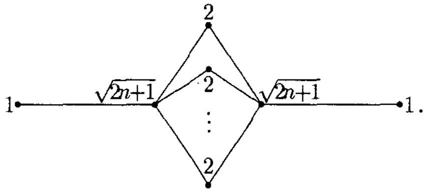
An example of this is $\mathscr{R} \rtimes S_2 \subset \mathscr{R} \rtimes S_3$ giving back the graph A_5



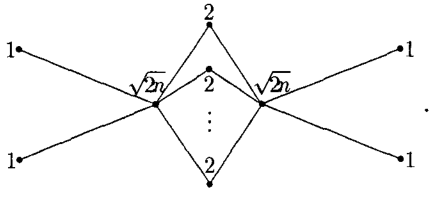
Another example is $\mathscr{R} \rtimes \mathbf{Z}_3 \subset \mathscr{R} \rtimes A_4$. Here A_n is the alternating group on n elements. This inclusion gives back the graph $\hat{E}_6$



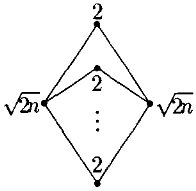
Another set of examples is $\mathscr{R} \rtimes \mathbf{Z}_2 \subset \mathscr{R} \rtimes D_{2n+1}$, where D_{2n+1} is the dihedral group. We have already seen this case in Sect. 3, where it was related to special knot invariants. It has the following graph:



Including apart from $\mathscr{G}$ and $\mathscr{H}$ two graphs Γ_i equal to $\mathscr{G}$ gives a situation occurring in the $\mathbf{Z}_2$ orbifold $\mathscr{A}_{4n+2}/\mathbf{Z}_2$ of the rational Gaussian model $\mathscr{A}_{4n+2}$ [25]. Indeed the total number of fields is $4(n+4)$ which is equal to $2(\frac{1}{2}(4n+2)+7)$ as requested. We can also consider $\mathscr{R} \rtimes \mathbf{Z}_2 \subset \mathscr{R} \rtimes D_{2n}$, which has $\mathscr{G}$ and $\mathscr{H}$ equal to



Adding two graphs which look like



gives the situation of $\mathscr{A}_{4n}/\mathbf{Z}_2$. Again the total number of fields equals $2(n+5)+2(n+2)=2(\frac{1}{2}(4n)+7)$. The inclusions $\mathscr{R}\rtimes\mathbf{Z}_2\subset\mathscr{R}\rtimes D_n$ are also found in the statistical mechanical context in the Fateev-Zamolodchikov model [39].

10. Concluding Remarks

Let us make a few comments on the condition $S=S^t$. We believe that this condition is related to a self-duality of the underlying algebraic structure of Rational Conformal Field Theories, for which S has the interpretation of a sort of generalized Fourier-transform. If for example the fusion rules are those of an abelian group, the S-matrix is symmetric, because the abelian group is self-dual. More generally, suppose the fusion algebra contains the representation ring of a finite group, which happens for instance in holomorphic orbifold theories [25]. In that case, there are also twist fields, needed to make the S-matrix symmetric. The underlying algebraic structure is the quantum double of the algebra of functions on the group, which is self dual [35]. The quantum double of a Hopf algebra A is defined as $A\otimes A^o$ [36], where A^o is the algebra dual to A with the opposite comultiplication. The quantum double is obviously self dual. Also quantum groups and Kac-Moody algebras are in a sense self-dual, because the Borel subalgebras b^- and b^+ are dual to each other. What happens for instance in the $SU(2)/G$ models we have just discussed? Here the underlying algebraic structure must be something like $A\otimes A^o\rtimes\mathbf{Z}_2$, an algebra of dimension $2(\dim G)^2$. Using the analysis of $SU(2)/D_n$ we find that in that case the following isomorphism of algebras

$$A\otimes A^o\rtimes\mathbf{Z}_2\simeq\mathbf{C}^4\oplus(M_2(\mathbf{C}))^{n^2-1}\oplus(M_n(\mathbf{C}))^4\,. \tag{10.1}$$

There are a lot of possible constructions which produce new inclusions from given ones. Some of these seem remarkably similar to certain constructions in Rational Conformal Field Theories

$M_1\subset M_2,\ M_2\subset M_3\Rightarrow M_1\subset M_3$	tensor products
$M_1\subset M_2,\ S\subset M_1\Rightarrow S'\cap M_1\subset S'\cap M_2$	coset construction
$M_1\subset M_2\Rightarrow M_1^G\subset M_2^G$	orbifold construction
$M_1\subset M_2\Rightarrow M_1\rtimes G\subset M_2\rtimes G$	extended algebras

For instance, we have seen that $\mathscr{A}_{2n}/\mathbf{Z}_2$ can be realized as $\mathscr{R}^{D_n}\subset\mathscr{R}^{\mathbf{Z}_2}$, which looks like a $\mathbf{Z}_2$ orbifold of $\mathscr{R}^{\mathbf{Z}_n}\subset\mathscr{R}$, which would then correspond to $\mathscr{A}_{2n}$. However in general there is a problem with models like $\mathscr{A}_n$, as they always give the trivial inclusion $\mathscr{R}\subset\mathscr{R}$ for any choice of field Φ. Our construction seems to forget about any abelian structure present in the theory.

Another remark concerns the central charge. It would be nice to have a simple interpretation of the central charge in terms of subfactors. If we consider the examples of the previous section, then we find a wide variety of central charges in the models giving the same subfactors. The only constraint seems to be that $e^{i\pi c}$ must be in the same ring $\mathbf{Z}[\omega]$ as S_{ij}/S_{0j} and in particular the index $(S_{0\Phi}/S_{00})^2$ (see Appendix B).

A related issue we have not touched upon is the problem of the classification of modular invariants. This can in certain cases also be accomplished using techniques similar to those occurring in our string algebras [38]. However, this technique does not seem to have a direct natural interpretation in the II_1 language.

To conclude, we have established a precise connection between Rational Conformal Field Theories and II_1 factors. It would be very interesting to translate the remarks above into precise conditions on inclusions, thus providing us with a new handle on the wide variety of solutions of the duality equations.

A. Appendix: Inclusions of Factors and Coupling Systems

In [9] Ocneanu has introduced a machinery to study the position of a subalgebra in a larger one. If $A \subset B$ and $A' \subset B'$ are two inclusions of algebras, A and A' have the same position if there is an isomorphism $f: B \to B'$ such that $f(A) = A'$. Associated to such an inclusion is an invariant object called a paragroup. It is invariant in the sense that if A and A' have the same position, the paragroups will be the same as well. In paragroups, the underlying set of a group is replaced by a graph, the group elements are substituted by strings on the graph, and a geometrical connection stands for the composition law.

Of special interest is the case where A and B are II_1 factors, and in particular when they are both isomorphic to the hyperfinite factor $\mathscr{R}$ (see Sect. 3).

Ocneanu [9] has given a complete classification of irreducible subfactors $\mathscr{R}_0$ of $\mathscr{R}$ of finite index and finite depth, in terms of so-called coupling systems, which are particular presentations of paragroups. Here irreducible means that $\mathscr{R}'_0 \cap \mathscr{R} = \mathbf{C}$, that is, the only elements of $\mathscr{R}$ that commute with all of $\mathscr{R}_0$ are the scalar multiples of the identity. What finite depth means will be explained in a moment. In Sect. 4 we have shown how given a RCFT and a particular primary field Φ one can define a coupling system, and hence a subfactor $\mathscr{R}_0$ of $\mathscr{R}$, with index

$$[\mathscr{R}:\mathscr{R}_0] = (S_{0\Phi}/S_{00})^2 \,. \tag{A.1}$$

Let us first explain how to construct a coupling system from an inclusion $N \subset M$ of factors. First of all, one constructs the infinite tower

$$M_0 = N \subset M_1 = M \subset M_2 \subset M_3 \subset \dots \tag{A.2}$$

by iterating the fundamental construction of Jones [10]. Equivalently, one can take $M_{k+1} = M_k \otimes_{M_{k-1}} M_k$, $M_{k+1} = \mathrm{End}_{M_{k-1}}(M_k)$, the endomorphisms of M_k viewed as a right M_{k-1}-module, or $M_{k+1} = \langle M_k, e_k \rangle$, the II_1 factor generated by M_k and e_k on $L^2(M_k, \mathrm{tr})$. This requires some explanation: let tr_k be the faithful normalized trace on M_k, then by $L^2(M_k, \mathrm{tr}_k) = \mathscr{H}$ we mean the Hilbert space obtained by completing M_k with respect to the inner product $\langle x|y \rangle = \mathrm{tr}_k(x^* y)$. The left multiplication of M_k extends to an action of M_k on $L^2(M_k, \mathrm{tr}_k)$, so that M_k is realized as a subalgebra of $B(\mathscr{H})$. Now let e_k be the orthogonal projection [5]

$$e_k : L^2(M_k, \mathrm{tr}_k) \to L^2(M_{k-1}, \mathrm{tr}_{k-1}) \,. \tag{A.3}$$

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[5] The restriction E_k of e_k to M_k is what is called the conditional expectation from M_k to M_{k-1}; for $x \in M_k$ and $y \in M_{k-1}$ we have $\mathrm{tr}_k(xy) = \mathrm{tr}_{k-1}(E_k(x)y)$

Note that tr_{k-1} is equal to the restriction of tr_k to M_{k-1}. These projections e_k satisfy a Temperley-Lieb algebra

$$\begin{aligned} &e_k^2=e_k, \\ &e_k e_{k\pm 1} e_k=\frac{1}{[M:N]} e_k, \\ &e_k e_{k'}=e_{k'} e_k, \quad |k-k'| \geqq 2. \end{aligned} \tag{A.4}$$

Furthermore the tr_k's are Markov traces in the sense that

$$\mathrm{tr}_{k+1}(x e_k)=\frac{1}{[M:N]} \mathrm{tr}_k(x), \quad x \in M_k. \tag{A.5}$$

A nice account of these topics can also be found in the book [13].

Given the tower $M_0 \subset M_1 \subset M_2 \subset \ldots$ one can construct two unoriented bipartite graphs $\mathscr{G}$ and $\mathscr{H}$, i.e. graphs that admit a $\mathbf{Z}_2$-grading of the vertices, so that no two vertices with the same grade are connected via an edge. Equivalently, the graph has no loops of odd length, or it is bicolorable. The even vertices of $\mathscr{G}$ represent the inequivalent irreducible $N-N$ subbimodules of $M_0, M_1, M_2, \ldots$, the odd vertices of $\mathscr{G}$ correspond to the inequivalent irreducible $M-N$ subbimodules, and the even and odd vertices of $\mathscr{H}$ correspond in the same way to irreducible $M-M$ and $N-M$ subbimodules respectively.

The number of edges between an $N-N$ bimodule X and a $M-N$ bimodule Y is given by the number of times X occurs in Y if the left action of M is restricted to N. The number of edges between a $M-M$ and a $N-M$ bimodule is determined similarly.

Furthermore there is a map τ from the set of vertices of $\mathscr{G} \cup \mathscr{H}$ to itself mapping $P-Q$ modules to $Q-P$ modules by interchanging the left and right actions. If P acts on the left on X via $p \cdot x \rightarrow px$ then it acts on the right via $x \cdot p \rightarrow p^* x$. This map τ is called the contragradient map.

The last ingredient of a coupling system is the connection. Given a $N-N$ bimodule X and a $M-M$ bimodule Y, there are two ways to induce X to Y: via $M-N$ and via $N-M$ bimodules. The way in which these two results differ is expressed in terms of a complex number W associated to each set of four bimodules, one of each type. The map W is called the connection.

The inclusion $N \subset M$ is said to be of finite depth if the number of vertices of $\mathscr{G}$ and $\mathscr{H}$ is finite. Actually $\mathscr{G}$ is equal to the principal graph of the derived tower of finite dimensional algebras

$$\partial M / \partial N=N' \cap M_0 \subset N' \cap M_1 \subset N' \cap M_2 \subset \ldots, \tag{A.6}$$

the finiteness of $\mathscr{G}$ means here that the Bratelli diagram for $\partial M / \partial N$ eventually becomes periodic.

To see why a coupling system can be seen as a generalization of group theory consider the example $\mathscr{R} \subset \mathscr{R} \rtimes G$. Here $\mathscr{R} \rtimes G$ means the crossed product of $\mathscr{R}$ by the finite group G: suppose G acts on $\mathscr{R}$ by outer automorphisms ϱ_g and $\varrho_g \varrho_h=\varrho_{gh}$. Then $\mathscr{R} \rtimes G$ has as elements $\sum a_g u_g$, where u_g is unitary, $a_g \in \mathscr{R}$, and $u_g a u_g^*=\varrho_g(a)$. In this case the coupling system reproduces all the information contained in G. The graph $\mathscr{G}$ has one odd vertex and the even vertices are in one-one correspondence with the elements of G, while $\mathscr{H}$ has one odd vertex and one even vertex for every irreducible representation of G. So $\mathscr{H}$ can be considered as being the dual of $\mathscr{G}$.

B. Appendix: A Proof

In this appendix we will supply the proof of

Theorem. *For any* RCFT, S_{ij}/S_{0j} *is always a finite sum of roots of unity with integer coefficients.*

The idea of the proof is to use a famous theorem in algebraic number theory by Kronecker and Weber stating that a field extension of **Q** is contained in a cyclotomic field **Q**[ω] if the extension is normal and has an abelian Galois group [28].

Let L be the field extension of **Q** generated over **Q** by the set $\left\{\frac{S_{ij}}{S_{0j}}\right\}_{i,j}$. The numbers $\frac{S_{ij}}{S_{0j}}$ for fixed i are the roots of the polynomial

$$\det(\lambda 1 - N_i) = 0, \tag{B.1}$$

where N_i is the matrix $(N_i)_{pq} = N^q_{ip}$. Therefore this is a normal field extension of **Q**. Now let g be an element of the Galois group of L, $g \in \mathrm{Gal}(L/\mathbf{Q})$. Because the numbers $\frac{S_{ij}}{S_{0j}}$ are precisely the inequivalent solutions of the fusion rules,

$$\frac{S_{aj}}{S_{0j}} \frac{S_{bj}}{S_{0j}} = \sum_c N^c_{ab} \frac{S_{cj}}{S_{0j}}, \tag{B.2}$$

and the fusion rules are invariant under the action of the Galois group, we must have

$$g\left(\frac{S_{ij}}{S_{0j}}\right) = \frac{S_{ik}}{S_{0k}} \tag{B.3}$$

with k independent of i, so we can put $k = g(j)$. Because $SS^* = 1$ we find

$$\begin{aligned} g\left(\frac{1}{S_{0j}}\right)^2 &= g\left(\sum_i \left(\frac{S_{ij}}{S_{0j}}\right)\left(\frac{S_{ij}}{S_{0j}}\right)^*\right) \\ &= \sum_i \left(\frac{S_{ig(j)}}{S_{0g(j)}}\right)\left(\frac{S_{ig(j)}}{S_{0g(j)}}\right)^* \\ &= \left(\frac{1}{S_{0g(j)}}\right)^2, \end{aligned} \tag{B.4}$$

and combining Eqs. (B.3) and (B.4) yields

$$\begin{aligned} g((S_{ij})^2) &= g\left(\left(\frac{S_{ij}}{S_{0j}}\right)^2 \Big/ \left(\frac{1}{S_{0j}}\right)^2\right) \\ &= \left(\frac{S_{ig(j)}}{S_{0g(j)}}\right)^2 \Big/ \left(\frac{1}{S_{0g(j)}}\right)^2 \\ &= (S_{ig(j)})^2 . \end{aligned} \tag{B.5}$$

We now use the fact that $S = S^t$ so that (B.5) must be symmetric as well; this implies that $(S_{ig(j)})^2 = (S_{g(i)j})^2$ and taking $i = 0$ gives in particular

$$\frac{S_{g(0)j}}{S_{0g(j)}} = \pm 1 . \tag{B.6}$$

Take now arbitrary $g, h \in \mathrm{Gal}(L/\mathbf{Q})$. We have

$$\begin{aligned} gh\left(\frac{S_{ij}}{S_{0j}}\right) &= g\left\{h\left(\left(\frac{S_{ij}}{S_{i0}}\right)\left(\frac{S_{0i}}{S_{00}}\right)\left(\frac{S_{00}}{S_{0j}}\right)\right)\right\} \\ &= g\left(\frac{S_{h(i)j}}{S_{h(i)0}}\,\frac{S_{h(0)i}}{S_{h(0)0}}\,\frac{S_{h(0)0}}{S_{h(0)j}}\right) \\ &= g\left(\frac{S_{h(i)j}}{S_{0j}}\,\frac{S_{0j}}{S_{h(0)j}}\,\frac{S_{h(0)i}}{S_{h(i)0}}\right) \\ &= \frac{S_{h(i)g(j)}}{S_{h(0)g(j)}}\,g\left(\frac{S_{h(0)i}}{S_{h(i)0}}\right). \end{aligned} \tag{B.7}$$

On the other hand

$$\begin{aligned} hg\left(\frac{S_{ij}}{S_{0j}}\right) &= h\left(\frac{S_{ig(j)}}{S_{0g(j)}}\right) \\ &= h\left(\frac{S_{g(j)i}}{S_{0i}}\,\frac{S_{i0}}{S_{00}}\,\frac{S_{00}}{S_{0g(j)}}\right) \\ &= \frac{S_{g(j)h(i)}}{S_{h(0)g(j)}}\,\frac{S_{h(0)i}}{S_{h(i)0}}. \end{aligned} \tag{B.8}$$

Since $g(\pm 1) = \pm 1$ we see from (B.6) that (B.7) and (B.8) are in fact the same. Therefore, the action of gh and hg on L is the same, and we conclude that $\mathrm{Gal}(L/\mathbf{Q})$ is abelian. Applying the theorem of Kronecker and Weber now tells us that $L \subset \mathbf{Q}[\omega]$ for some root of unity ω. Since S_{ij}/S_{0j} is a solution of Eq. (B.1), which is a polynomial with integer coefficients and leading coefficient one, these numbers are also algebraic integers. The subring of algebraic integers of $\mathbf{Q}[\omega]$ is precisely $\mathbf{Z}[\omega]$ [28], and this completes the proof that S_{ij}/S_{0j} is a sum of roots of unity with integer coefficients.

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