Graphs and Combinatorics (1992) 8: 309–312

Graphs and Combinatorics

The Uniformity Lemma for Hypergraphs*

P. Frankl[1] and V. Rödl[2]

[1] CNRS, 20 rue de Tournon, 75006 Paris, France
[2] Emory University, Atlanta, GA 30322, USA

Abstract. In 1973, E. Szemeredi proved a theorem which found numerous applications in extremal combinatorial problems—The Uniformity Lemma for Graphs. Here we consider an extension of Szemeredi's theorem to r-uniform hypergraphs.

In [3] Szemeredi proved the theorem which found a lot of applications in graph theory – the Uniformity Lemma for graphs. Here we give an extension to r-graphs. We hope that this will prove to be nearly as useful as Szemeredi's theorem. So far, we have found two applications: proof of a conjecture of Erdös concerning Turan-type problems [1, 2] and giving an alternative condition for quasirandomness (Chung, F.R.K., Graham, R.L., Quasi-random hypergraphs, preprint, 1989). Proof of these applications will be a subject of a subsequent paper.

Let $\mathbf{G} = (V, E)$ be a graph and A, $B \subset V$ be a pair of disjoint subsets of V. The density (A, B) is the fraction $d(A, B) = e(A, B)/|A||B|$, where $e(A, B)$ is the number of edges with one endpoint in A and second in B and $|A|$, $|B|$ denote the cardinalities of A and B respectively. A pair (A, B) is called δ-uniform (with respect to $\mathbf{G}$) if for every $A' \subset A$ $B' \subset B$, $|A'| > \delta|A|$, $|B'| > \delta|B|$, $|d(A, B) - d(A', B')| < \delta$ holds. The partition $V = C_0 \cup C_1 \cup \cdots \cup C_l$ is called δ-uniform if:

i) $|C_0| < \delta|V|$
ii) $|C_1| = |C_2| = \cdots = |C_l|$
iii) all but $\delta\binom{l}{2}$ of the pairs (C_i, C_j) are δ-uniform $1 \le i < j \le l$.

For every $\delta > 0$ and positive integer m, there exist positive integers $n_0(\delta, m)$ and $m_0(\delta, m)$ such that every graph with at least $n_0(\delta, m)$ vertices has a δ-uniform partition into l-classes, where l is an integer satisfying $m < l < m_0(\delta, l)$.

Here we extend this lemma to r-uniform hypergraphs—(r-graphs). Our proof utilizes similar idea as that of Szemeredi, but uses somewhat different concepts.

Now we introduce some further notation: Let $\mathbf{H}$ be an r-uniform hypergraph with vertex set X. Let $X = X_1 \cup X_2 \cup \cdots \cup X_v$ be a partition. Suppose now that for

* The work on this paper was done while the authors were visiting AT&T Bell Laboratories in 1985

every $1 \le i_1 < i_2 < \cdots < i_{r-1} \le v$, $\mathbf{I} = \{i_1, i_2, \ldots, i_{r-1}\}$ the complete $r-1$ partite graph $\mathbf{K}(X_{i_1}, X_{i_2}, \ldots, X_{i_{r-1}}) = \bigcup_{\alpha=1}^{m} \mathbf{G}_{\mathbf{I}}^{\alpha}$ is partitioned into m subgraphs $G_{\mathbf{I}}^{\alpha}$. For $1 \le i_1 < i_2 < \cdots < i_r \le v$, $\mathbf{J} = \{i_1, i_2, \ldots, i_r\}$ and $1 \le \alpha_1, \alpha_2, \ldots, \alpha_r \le m$ let $\mathbf{R}(J, \alpha_1, \alpha_2, \ldots, \alpha_r)$ be the collection of complete $(r-1)$-graphs on r-points (viewed as r element sets) in the union of r, $(r-1)$-partite graphs $\mathbf{G}_{J-\{i_1\}}^{\alpha_1}, \mathbf{G}_{J-\{i_2\}}^{\alpha_2} \ldots \mathbf{G}_{J-\{i_r\}}^{\alpha_r}$. Clearly, we can view $R(J, \alpha_1, \alpha_2, \ldots, \alpha_r)$ as an edge set of r-graph.

Set now $\mathbf{H}(X_{i_1}, X_{i_2}, \ldots, X_{i_r}) = \{H \in \mathbf{H}, |H \cap X_j| = 1, j = i_1, i_2, \ldots, i_r\}$ and define the density of $\mathbf{H}$ with respect to $r-1$ partite graphs $\mathbf{G}_{I_j}^{\alpha_j}$, $j = 1, 2, \ldots, r$, where $I_j = J - \{i_j\}$ by $d\ (\mathbf{G}_{I_j}^{\alpha_j}, j = 1, \ldots, r) = \dfrac{|\mathbf{H}(X_{i_1}, \ldots, X_{i_r}) \cap \mathbf{R}(J, \alpha_1, \ldots, \alpha_r)|}{|\mathbf{R}(J, \alpha_1, \ldots, \alpha_r)|}$ if $\mathbf{R}(J, \alpha_1 \ldots \alpha_r) \ne \varnothing$. If $\mathbf{R}(J, \alpha_1 \ldots \alpha_r) = \varnothing$ we set $d(\mathbf{G}_{I_j}^{\alpha_j}, 1 \le j \le r) = 0$. We set also $r(J, \alpha_1, \alpha_2, \ldots, \alpha_r) = |\mathbf{R}(J, \alpha_1 \ldots \alpha_r)|$.

Finally, we say that an r-tuple $(\mathbf{G}_{I_j}^{\alpha_j}, 1 \le j \le r)$ is δ-uniform (with respect to $\mathbf{H}$) if for all subgraphs $\mathbf{F}_{I_j}^{\alpha_j} \subset \mathbf{G}_{I_j}^{\alpha_j}$, $j = 1, 2, \ldots, r$ satisfying $r(\mathbf{F}_{I_j}^{\alpha_j}, 1 \le j \le r) > \delta r(\mathbf{G}_{I_j}^{\alpha_j}, 1 \le j \le r)$ one has $|d(\mathbf{F}_{I_j}^{\alpha_j}, 1 \le j \le r) - d(G_{I_j}^{\alpha_j}, 1 \le j < r)| < \delta$.

A system $\left\{\mathbf{G}_I^{\alpha}, I \in \binom{[1, v]}{r-1}, \alpha = 1, 2, \ldots, m\right\}$ is called δ-uniform if all but δ-proportion of r-tuples $(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$, $x_{i_j} \in X_{i_j}$ form a complete r-gon in some δ-uniform r-tuple $(\mathbf{G}_{I_j}^{\alpha_j}, j = 1, 2, \ldots, r)$. Now we are ready to state the uniformity lemma.

Let $0 < \delta < 1$ and t, and v be positive integers then there exist $t_0(\delta, t)$ and $n(\delta, t)$ such that for every r-uniform hypergraph $\mathbf{H}$ with vertex set X partitioned into partition classes $X_1 \cup \cdots \cup X_v$, $|X_i| > n(\delta, t)$ there exists a δ-uniform system $\left\{\mathbf{G}_I^{\alpha}, \alpha = 1, 2, \ldots, m, I \in \binom{[1, v]}{r-1}\right\}$, where $t \le m \le t_0(\delta, t)$.

Note that if $|X_1| = |X_2| = \cdots = |X_v|$, similarly to Szemeredi's uniformity lemma one can achieve here that the same statement is true with $m+1$ classes $\mathbf{G}_I^{\alpha}$, $\alpha = 0, 1, \ldots, m$ for every $I \in \binom{[1, v]}{r-1}$ such that $|\mathbf{G}_I^0| < \delta \prod_{i \in I} |X_i|$ and $|\mathbf{G}_I^1| = |\mathbf{G}_I^2| = \cdots = |\mathbf{G}_I^m|$.

We start with one auxiliary claim which we shall state without proof and we will use later:

Let μ_b, d_b, $b \in A$ be positive reals satisfying $\sum_{b \in A} \mu_b = 1$. Set $d = \sum_{b \in A} \mu_b d_b$. Let $B \subset A$ be a subset such that $\sum_{b \in B} \mu_b \ge \delta$ and $\sum_{b \in B} \mu_b d_b \ge \delta(d + \delta)$. Then $\sum_{b \in A} \mu_b d_b^2 \ge d^2 + \dfrac{\delta^3}{1 - \delta}$.

Let $X = X_1 \cup \cdots \cup X_v$ be an arbitrary partition of the vertex set of $\mathbf{H}$ with large (this will be specified later) partition classes. For every $r-1$ tuple $1 \le i_1 < i_2 < \cdots < i_{r-1} \le v$, $I = \{i_1, i_2, \ldots, i_{r-1}\}$ set $\mathbf{K}(I) = \mathbf{K}(X_{i_1}, X_{i_2}, \ldots, X_{i_{r-1}})$ and let $\{\mathbf{G}_I^{\alpha}\} = \{\mathbf{G}_I^{\alpha}, I \subset \{1, 2, \ldots, v\}, |I| = r-1, \alpha = 1, \ldots, t\}$ be an arbitrary system of $(r-1)$ partite $(r-1)$ graphs such that $\bigcup_{\alpha=1}^{t} \mathbf{G}_I^{\alpha} = \mathbf{K}(I)$.

If we are lucky this system is δ-uniform and we are done. Suppose that this is not the case. For an r-tuple $J = \{i_1, i_2, \dots, i_r\}$ set $I_j = J - \{i_j\}$ $d(J, \alpha_1, \alpha_2, \dots, \alpha_r) = d(\mathbf{G}_{I_j}^{\alpha_j}, j = 1, 2, \dots, r)$.

Also, if $a = (\alpha_1, \alpha_2, \dots, \alpha_r)$, $1 \leq \alpha_i \leq t$ then we set $\lambda_{J,a} = \dfrac{r(J,a)}{\sum_K \left\{ \prod_{j \in K} |X_j|, K \in \binom{[1,v]}{r} \right\}}$ to be the ratio of the number of r-tuples in $\mathbf{R}(J, \alpha_1 \dots \alpha_r)$ to the number of all r-tuples which intersect each of the sets X_i in at most one element.

We define the smoothness of the present partition by $\sigma(\{\mathbf{G}_I^\alpha\}) = \sum \lambda_{J,a}(d(J,a))^2$ where the summation is made over all $\left(\text{at most } t^r \binom{v}{r} \text{ classes of } a \text{ and } J.\right)$

Note that smoothness is a weighted average of squares of densities and therefore it never exceeds 1. For each nonuniform r-tuple $(\mathbf{G}_{I_i}^{\alpha_i}, i \in J)$ choose $(r-1)$-partite $(r-1)$-graphs $\mathbf{F}_{I_1}^{\alpha_1} \subset \mathbf{G}_{I_1}^{\alpha_1}, \mathbf{F}_{I_2}^{\alpha_2} \subset \mathbf{G}_{I_2}^{\alpha_2}, \dots, \mathbf{F}_{I_r}^{\alpha_r} \subset \mathbf{G}_{I_r}^{\alpha_r}$ such that $r(\mathbf{F}_{I_j}^{\alpha_j}, j = 1, 2, \dots, r) > \delta r(\mathbf{G}_{I_j}^{\alpha_j}, j = 1, 2, \dots, r)$ and $|d(J, \alpha_1, \dots, \alpha_r) - d(\mathbf{F}_{I_j}^{\alpha_j}, j = 1, 2, \dots, r)| \geq \delta$.

For each $r-1$ tuple I there are at most $(v - r + 1)t^r$ different $\mathbf{F}_I$'s occuring in the nonuniform r-tuples, i.e., for which the second equation holds. Let $\mathbf{F}_I^\mu$, $\mu = 1, 2, \dots, q(I)$ $(1 \leq q(I) \leq (v - r + 1)t^r)$ be the list of all these $(r-1)$-partite $(r-1)$-graphs. We have clearly $\mathbf{F}_I^\mu \subset \mathbf{K}(X_{i_1}, \dots, X_{i_{r-1}})$, where $I = \{i_1 \dots i_{r-1}\}$. Let us consider the Venn diagram of the system $\{\mathbf{F}_I^\mu, 1 \leq \mu \leq q(I)\} \cup \{\mathbf{G}_I^\alpha, 1 \leq \alpha \leq t\}$, in the set $\mathbf{K}(X_{i_1} \dots X_{i_{r-1}})$ i.e., the coarsest partition $\mathbf{K}(X_{i_1} \dots X_{i_{r-1}}) = \mathbf{E}_I^1 \cup \dots \cup \mathbf{E}_I^{p(I)}$ satisfying either $\mathbf{E}_I^\nu \subset \mathbf{F}_I^\mu \cap \mathbf{G}_I^\alpha$ or $\mathbf{E}_I^\nu \cap \mathbf{F}_I^\mu \cap \mathbf{G}_I^\alpha \neq \emptyset$ for all ν, μ and α. Clearly $p(I) \leq t 2^{q(I)}$ holds.

For $J = \{i_1, \dots, i_r\}$, $I_j = J - \{i_j\}$ and for two r-tuples $a = (\alpha_1 \dots \alpha_r)$, $b = (\beta_1, \dots, \beta_r)$, $1 \leq \alpha_i \leq t$, $1 \leq \beta_i \leq p(I)$ write $b < a$ if $\mathbf{E}_{I_j}^{\beta_j} \subset \mathbf{G}_{I_j}^{\alpha_j}$ for every $j = 1, 2, \dots, r$. Set also $\mu_b = \dfrac{r(\mathbf{E}_{I_j}^{\beta_j}, 1 \leq j \leq r)}{r(\mathbf{G}_{I_j}^{\alpha_j}, 1 \leq j \leq r)}$ and $d_b = d(\mathbf{E}_{I_j}^{\beta_j}, 1 \leq j \leq r)$.

We will now compare the smoothness of the new system $\{\mathbf{E}_I^\beta\} = \left\{\mathbf{E}_{I_\beta}, I \in \binom{[1,v]}{r}, 1 \leq \beta \leq p(I)\right\}$ and old system $\{\mathbf{G}_I^\alpha\} = \left\{\mathbf{G}_I^\alpha, I \in \binom{[1,v]}{r}, 1 \leq \alpha \leq t\right\}$. We have $d(J,a) = d(\mathbf{G}_{I_j}^{\alpha_j}, 1 \leq j \leq r) = \sum_b \{\mu_b d_b; b < (\alpha_1, \dots, \alpha_r)\}$ and clearly $\sum_{b<a} \mu_b = 1$.

According to this we have clearly $d(J,a)^2 \leq \sum_{b<a} \mu_b d_b^2$. However, if the r-tuple $(\mathbf{G}_{I_1}^{\alpha_1}, \mathbf{G}_{I_2}^{\alpha_2}, \dots, \mathbf{G}_{I_r}^{\alpha_r})$ is not δ-uniform, then by the Claim we have moreover $\sum_{b<a} \mu_b d_b^2 \geq d(J,a)^2 + \dfrac{\delta^3}{1-\delta}$. On the other hand from the assumption that the old system $\{\mathbf{G}_I^\alpha\}$ is not δ-uniform i.e., that $\sum \lambda_{J,a} \geq \delta$ we infer that $\sigma(\mathbf{E}_I^\beta) = \sum \lambda_{J,a}\left(\sum_{b<a} \mu_b d_b^2\right) \geq \sigma(\{\mathbf{G}_I^\alpha\}) + \dfrac{\delta^4}{1-\delta}$ holds. Thus, on assumption that $\{\mathbf{G}_I^\alpha\}$ is not δ-uniform we get a system, the smoothness of which is larger and the difference is at least $\dfrac{\delta^4}{1-\delta}$.

If the new partition is δ-uniform then we are done, otherwise, we repeat this procedure replacing $\{\mathbf{G}_I^\alpha\}$ by $\{\mathbf{E}_I^\beta\}$ and instead of $\{\mathbf{E}_I^\beta\}$ we obtain a new system as long as we get a δ-uniform system. This can be repeated, however, at most $\frac{1-\delta}{\delta^4}$ times, otherwise, we obtain a partition with smoothness bigger than 1. Thus, if $n > n(\delta, t)$ is sufficiently large, we have to arrive to a system which is δ-uniform. □

References

1. Erdös, P., Frankl, P., Rödl, V. (1986): The Asymptotic Number of Graphs not Containing a Fixed Subgraph and a Problem for Hypergraphs Having no Exponent. Graphs and Combinatorics **2**, 113–121
2. Frankl, P., Füredi, Z. (1987): Exact Solution of Some Turan-Type Problems, JCT A **45**, 226–262
3. Szemeredi, E. (1976): Regular Partitions of Graphs, Proc. Colloq. Int. CNRS, pp. 399–401. Paris CNRS

Received: August 10, 1989
Revised: October 24, 1990