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Rational approximation to functions like x^α in integral norms

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§ 1. Introduction

In several papers the degree of rational approximation to the function x^α has been investigated. Let us denote by $R_{n,p}(x^\alpha)$ this degree of approximation in $L^p[0, 1]$ by rationals of order at most n. The most precise results so far have been given by GANELIUS [8] for the case $p=\infty$ and VJAČESLAVOV [14] for the case $0<p\leqq\infty$. However, none of these results are satisfactory unless α is a rational number.

Our result in this direction removes the restriction on α but instead we have to impose new ones on p. Though we can get results also for $p<1$ we shall state our first theorem only for such values on p for which we have perfectly matching estimates both from below and above.

Theorem 1. *Let $1<p<\infty$ and $\alpha>-1/p$. There are positive constants $B=B(\alpha,p)$ and $C=C(\alpha,p)$ such that*

$$B\,|\sin\alpha\pi|\leqq R_{n,p}(x^\alpha)\cdot n^{-1/2p}\exp 2\pi\sqrt{n(\alpha+1/p)}\leqq C$$

for $n=1, 2, \ldots$.

Remark. The estimate from below is included here only for the sake of completeness. It was proved by VJAČESLAVOV in [14]. In the same paper the estimate from above was given only for rational α (with a $C(\alpha,p)$ not depending continously on α). The same phenomenon takes place also in the paper [8] by GANELIUS. Therefore the main conclusion of our theorem is that the algebraic properties of α are *not* important for the degree of approximation.

The main object of our study is, however, the degree of rational approximation in the Hardy spaces H^p to functions analogous to x^α, e.g. $(1-z)^\alpha$.

The method that we shall use is not specially designed for approximating $(1-z)^\alpha$. For instance it can be applied to prove GONČAR's well-known result for approximation of Markov functions. Approximation on the complex unit disc of a function f of the type

$$f(z):=\int_a^b\frac{d\mu(x)}{x-z},\quad\text{where}\quad 1<a<b, \tag{1}$$

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by rationals of order n can be done with an error essentially of the size ϱ^{-n} at most. Here ϱ is the modulus of the ring domain formed by the complement of the union of $|z|\leqq 1$ and $[a, b]$. Without any further information this result is best possible. However, with the additional assumption that $d\mu$ is a positive measure GONČAR showed in [10] that the optimal error is of the size ϱ^{-2n}.

Since the approximation of $(1-z)^\alpha$ can be transformed to approximation of a function of type (1) with a positive measure but with $a=1$, it is natural to look for a joint method for the two situations. Results in this direction were given by JÅRNER in [12] but with conditions on $d\mu$ that exclude $(1-z)^\alpha$.

Before proceeding we introduce some notations. Let U denote the open unit disc and T its boundary. The usual Hardy spaces for U are denoted H^p. For functions $f\in H^p$ we define

$$\|f\|_p := \sup\left\{\left(\frac{1}{2\pi}\int_T |f(rz)|^p |dz|\right)^{1/p} : 0<r<1\right\}$$

with the usual modification for $p=\infty$. The notation $\|f\|_p$ is also used for functions in $L^p(T)$ or $L^p[0, 1]$. It shoud be clear from the function which norm is meant.

In the text the letter C stands for positive constants that are not necessarily the same from time to time. When it is essential we indicate in what sense C is constant or rather on which variables C may depend.

Finally we remark that whenever power functions as z^α occur, we mean the principal branch of the function.

§ 2. The order of approximation in H^p

For each $\mathbf{z}=(z_1, \ldots, z_n)\in \mathbf{C}^n$ we let $\mathcal{R}(\mathbf{z})$ be the class of all rational functions of the form

$$r(z) = p(z)\Big/\prod_{k=1}^{n}(1-z_k z)$$

where p is a polynomial of degree at most n. In the proof of a part of our main theorem we shall need a result on approximation with a weight. Therefore we shall already from the start introduce notations to handle that situation.

For a real number β we let w_β be the function defined by $w_\beta(z):=(1-z)^\beta$ for $z\in U$. We define for $0<p\leqq\infty$ and f such that $fw_\beta\in H^p$

$$\varrho_p(f, \mathbf{z}, \beta) := \inf\{\|(f-r)w_\beta\|_p\colon r\in\mathcal{R}(\mathbf{z}),\ rw_\beta\in H^p\}$$

and

$$\varrho_{n,p}(f, \beta) := \inf\{\varrho_p(f, \mathbf{z}, \beta)\colon \mathbf{z}\in\mathbf{C}^n\}.$$

The case $\beta=0$ is, of course, the most interesting one and then we use just the notations $\varrho_p(f, \mathbf{z})$ and $\varrho_{n,p}(f)$ respectively.

If $p\geqq 1$ then H^p is a Banach space so we can use the Hahn—Banach theorem to see that

$$\varrho_p(f, \mathbf{z}, \beta) = \sup |\Phi(w_\beta f)| \tag{2}$$

where the sup is over all $\Phi\in(H^p)^*$, the dual space of H^p, with dual norm $\|\Phi\|=1$ and such that $\Phi(w_\beta r)=0$ for all $r\in\mathscr{R}(\mathbf{z})$ with the property $w_\beta r\in H^p$.

When $p<1$ the space H^p is not a Banach space but is contained in the Banach space B^p of all functions f analytic in U and with finite norm

$$\|f\|_{B^p} := \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|\,(1-r)^{(1/p)-2}\,dr\,d\theta.$$

This is a result by DUREN, ROMBERG and SHIELDS in [6]. In the same paper they also showed that

$$\|f\|_{B^p} \leqq C(p)\|f\|_p.$$

The corresponding result to (2) for $0<p<1$ is therefore

$$\varrho_p(f, \mathbf{z}, \beta) \geqq C(p) \sup |\Phi(w_\beta f)| \tag{3}$$

with $(H^p)^*$ replaced by $(B^p)^*$. This observation will be used to get estimates from below for $\varrho_p(f, \mathbf{z})$.

For $p<\infty$ every $\Phi\in(H^p)^*$ or $(B^p)^*$ if $p<1$ can be represented in the form

$$\Phi(f) = \lim_{r\to 1-}\int_T f(r\zeta)g(\bar{\zeta})\,d\zeta \tag{4}$$

with $g\in L^1(T)$. If $1\leqq p<\infty$ then $\Phi\in(H^p)^*$ and $\|\Phi\|=1$ if and only if there is a $g\in L^q(T)$ with $p^{-1}+q^{-1}=1$ and $\|g\|_q=1$ such that (4) holds. In the case $0<p<1$ Duren, Romberg and Shields gave a complete description of $(B^p)^*$. Combined with results on the boundary smoothness of analytic functions that one can read in eg. [5] this descriptions shows that there is a $C=C(p)$ such that if

i) $p^{-1}=N+\delta$ with N non-negative integer and $\delta\in(0, 1]$

(ii) there is a $g\in H^\infty$ such that $\|g\|\leqq C$ where

$$\|g\| := \|g\|_\infty + \sup\{|g^{(N+1)}(z)|\,(1-|z|)^{2-\delta}\colon z\in U\}$$

then (4) defines a $\Phi\in(B^p)^*$ with $\|\Phi\|\leqq 1$.

The functions that we shall approximate are transforms of measures. For every finite measure $d\mu$ on [0, 1] we let $\hat{\mu}$ be defined by

(5) $$\hat{\mu}(z) := \int \frac{d\mu(x)}{1-xz}.$$

In order to shorten our notations we introduce a special function.

Notation. For $\alpha+1/p>0$ and $n=1, 2, \ldots$ we let

$$\varepsilon(n, \alpha, p) := n^{1/2p} \exp\left(-\pi \sqrt{2n(\alpha+1/p)}\right).$$

Theorem 2. *Let* $0<p\leqq\infty$ *and* $\alpha+1/p>0$. *Suppose that* $d\mu$ *has the property that* $C_1 w_\alpha(x)\,dx \leqq d\mu(x) \leqq C_2 w_\alpha(x)\,dx$ *for some positive constants* C_1 *and* C_2.

i) *If* p^{-1} *is not an integer then there is a* $C=C(\alpha, p)$ *such that*

$$\varrho_{n,p}(\hat{\mu}) \leqq C\cdot\varepsilon(n, \alpha, p), \quad \textit{for} \quad n = 1, 2, \ldots.$$

ii) *If* $1\leqq p\leqq\infty$ *then there is a* $C=C(\alpha, p)>0$ *such that*

$$\varrho_{n,p}(\hat{\mu}) \geqq C\cdot\varepsilon(n, \alpha, p), \quad \textit{for} \quad n = 1, 2, \ldots.$$

iii) *If* $0<p<1$ *then there is a* $C=C(\alpha, p)>0$ *such that*

$$\varrho_{n,p}(\hat{\mu}) \geqq C n^{(p-1)/p}\cdot\varepsilon(n, \alpha, p) \quad \textit{for} \quad n = 1, 2, \ldots.$$

Remark. The parts i) and ii) give a precise description of $\varrho_{n,p}(\hat{\mu})$ if $1<p<\infty$. For all $p\in(0, \infty]$ we can at least say

$$\lim_{n\to\infty} \left(\varrho_{n,p}(\hat{\mu})\right)^{1/\sqrt{n}} = \exp\left(-\pi\sqrt{2(\alpha+1/p)}\right).$$

The next sections will be devoted to the proof of this theorem. However, we start already here with some general observations. Returning to (4) we find that if p and $d\mu$ satisfy the conditions of the theorem, $\Phi\in(H^p)^*$ and if $\alpha+\beta+1/p>0$ then we have by Fubini's theorem that

(6) $$\Phi(w_\beta\hat{\mu}) = \int_0^1 G_\beta(x)\,d\mu(x)$$

where

(7) $$G_\beta(z) := \frac{1}{2\pi i}\int_T \frac{w_\beta(\zeta)g(\bar{\zeta})}{1-z\zeta}\,d\zeta \quad \text{for} \quad |z|<1.$$

If $1<q<\infty$ and $-1<-\beta q<q-1$ then there is a constant $C(\beta, q)$ such that

(8) $$\|w_{-\beta}G_\beta\|_q \leqq C(\beta, q)\cdot\|g\|_q.$$

This is a generalization of M. Riesz' inquality given by BABENKO [3] that can also be read in [11].

Formula (6) will be central for the proof of the theorem. Let us also remark that the use of the weights w_β is needed only for handling part i) for $p<1$. The proof is simpler when $1<p<\infty$ and then it is sufficient with $\beta=0$.

§ 3. Optimal quadrature in H^q

Suppose $q\in(1, \infty)$ and let $d\mu$ be a positive measure on $(-1, 1)$ such that for some constant C

$$\left|\int f(x)\, d\mu(x)\right| \leqq C\|f\|_q \tag{9}$$

for all functions $f\in H^q$.

We study quadrature formulae of the form

$$I(f) := \int f(x)\, d\mu(x) \approx \sum_{k=1}^{n} \left(a_k f(x_k)+b_k f'(x_k)\right) =: S_n(f) \tag{10}$$

and let the error of the formulae be defined by

$$e_{n,q} := \inf\sup |I(f)-S_n(f)|$$

where the sup is over all f in H^q with $\|f\|_q\leqq 1$ and the inf is over all $a_k, b_k\in \mathbf{R}$, $k=1, \dots, n$, and $-1<x_1<\dots<x_n<1$. Given q and n we say that $S_n(f)$ in (10) is optimal if $e_{n,q}=\sup\{|I(f)-S_n(f)|:\ \|f\|_q\leqq 1\}$.

The following lemma was proved in a work together with BOJANOV [2] in the special case $d\mu=dx$ using results in BOJANOV [4].

Lemma 1. *For each* $q\in(1, \infty)$ *and* $n=1, 2, \dots,$ *there exists an optimal quadrature formula. Furthermore,*

i) *in this formula* $b_k=0$ *for* $k=1, \dots, n$.

ii) *for every Blaschke product* $B_n(x):=\prod_{k=1}^{n}\frac{x-x_k}{1-x_k x}$ *with* $-1<x_1<\dots<x_n<1$ *it holds that*

$$e_{n,q} \leqq \sup\left\{\left|\int f(x)B_n^2(x)\, d\mu(x)\right|:\ \|f\|_q \leqq 1\right\} \tag{11}$$

Proof. Though the proof in [2] was carried out only for the case $d\mu=dx$ it holds in the general case as well if we just replace dx by $d\mu$ in the estimates in [2].

Remark. The result that all the b_k' s vanish will be fundamental for our estimates for $\varrho_{n,p}(\hat{\mu})$.

The next lemma gives an estimate for $e_{n,q}$ by (11) in the special case that we are intrested in.

Lemma 2. *Let $p, q \in (1, \infty)$ be conjugate exponents, i.e. $p^{-1}+q^{-1}=1$. Suppose that $\beta+1/p>0$ and that $d\mu$ is a positive measure with its support on* [0, 1] *such that $d\mu(x) \leqq (1-x)^{\beta} dx$ for $x \in [0, 1]$. Then there is a constant $C=C(\beta, p)$ such that*

$$e_{n,q} \leqq C\varepsilon(n, \beta, p)$$

for $n=1, 2, \ldots$.

Proof. The proof is a slight modification of a result in [1] for $\beta=0$. The condition $\beta+1/p>0$ guarantees that (9) is fulfilled. For $-1<x_1<\ldots<x_n<1$ we find in (11) by Hölder's and Fejér—Riesz' inequalities that

$$e_{n,q} \leqq C\|w_{\beta} B_n^2\|_p \leqq C\|w_{-r}\|_p \cdot \|w_{\beta+r} B_n^2\|_{\infty} \tag{12}$$

for all $r<1/p$, the norms being on [0, 1].

In [1] we used a result by GANELIUS [7, p. 142] to see that for every $R>0$ there is a constant $C=C(R)$ such that if $0 \leqq \beta+r \leqq R$ then the nodes x_k can be choosen so that on [0, 1]

$$\|w_{\beta+r} B_n^2\|_{\infty} \leqq C \exp\left(-\pi \sqrt{2n(\beta+r)}\right).$$

We take $r=(1-n^{-1/2})/p$ in (12) and observe that $0 \leqq \beta+r \leqq \beta+1$ at least if n is large enough. Since $\sqrt{2n(\beta+r)} \geqq \sqrt{2n(\beta+1/p)}-C(\beta, p)$ and $\|w_{-r}\|_p=n^{1/2p}$ we find that

$$e_{n,q} \leqq C\varepsilon(n, \beta, p), \quad \text{for} \quad n=1, 2, \ldots,$$

with a constant $C=C(\beta, p)$.

In our investigations of $\varrho_{n,p}(\hat{\mu})$ we shall also need to know that the nodes $x_1, \ldots, x_n$ that were used in the proof of lemma 2 are essentially optimal for estimating $e_{n,q}$. It seems natural to state this result already now.

Lemma 3. *Let $z_1, \ldots, z_n \in U$ and $B_n(z) := \prod_{k=1}^{n} \frac{z-z_k}{1-\bar{z}_k z}$. Then there is constant C such that*

$$\int_0^1 (1-x)^{\gamma} |B_n(x)|\, dx \geqq C\sqrt{n} \exp\left(-\pi \sqrt{n(\gamma+1)}\right)$$

for $\gamma>-1$ *and $n=1, 2, \ldots$.*

Proof. This lemma is essentially due to NEWMAN [13] we only have to make some modifications. For $r \in (0.5, 1)$ we let

$$w(x) := C_r \left[\sqrt{x}(1-x)\right]^{-1} \quad \text{for} \quad x \in (0, r)$$

where C_r is choosen so that $\int_0^r w(x)dx=1$. Then using Jensen's inequality we get

$$\int_0^1 [1-x)^\gamma |B_n(x)|\,dx \geqq \int_0^r (1-x)^\gamma |B_n(x)|\big(w(x)\big)^{-1} w(x)\,dx \geqq$$

$$C_r^{-1}\exp\int_0^r \big(\log|B_n(x)|+(\gamma+1)\log(1-x)+0.5\log x\big)w(x)\,dx. \tag{13}$$

Here we observe that

$$\int_0^r \log(1-x)w(x)\,dx = C_r\cdot 2\int_0^{\sqrt{r}} \frac{\log(1-x^2)}{1-x^2}\,dx \geqq$$

$$\geqq C_r\int_0^{\sqrt{r}} \log(1-x)\cdot\left[\frac{1}{1+x}+\frac{1}{1-x}\right]dx \geqq -C_r(1+0.5\log^2(1-r)).$$

To take care of the rest of the integral we use Newman's observation that

$$\int_{-1}^1 \log\left|\frac{t-w}{1-\bar{w}t}\right|\cdot\frac{dt}{1-t^2} \geqq -\frac{\pi^2}{4}$$

for all $w\in U$ and hence

$$\int_0^1 \log\left|\frac{x-z}{1-\bar{z}x}\right|\cdot\frac{x^{-1/2}}{1-x}\,dx \geqq -\frac{\pi^2}{2}$$

for all $z\in U$. The integral in (13) is therefore not smaller than

$$-C_r[(\gamma+1)\log^2(1-r)+(n+1)\pi^2].$$

The definition of C_r gives the estimate $C_r\leqq|\ln(1-r)|^{-1}$. We now pick r so that $\log(1-r)=-\pi[n(\gamma+1)^{-1}]^{1/2}$. Then by our estimates for (13) the lemma follows.

§ 4. Upper estimates for $\varrho_{n,p}(\hat{\mu})$

In order to include the case $p<1$, p^{-1} non-integer, we make estimates for the approximation with weights which could be compared to similar results in [15].

Lemma 4. *Suppose that $p\in(1,\infty)$ and $\varepsilon\in(0,0.5)$. Let $\beta:=N+\gamma$ where N is a non-negative integer and γ a real number such that $\gamma+1/p\in(\varepsilon,1-\varepsilon)$. Furthermore we assume that $d\mu$ is a positive measure on $(0,1)$ such that $d\mu\leqq w_\alpha dx$ where $\alpha+\gamma+$ $+1/p\geqq\varepsilon$. Then there is a constant $C=C(\alpha,p,\varepsilon,N)$ such that*

$$\varrho_{n,p}(\hat{\mu}) \leqq C\varepsilon(n,\alpha+\beta,p)$$

for $n=1,2,\ldots$.

Proof. It is enough to consider $n>N$. Let $x_1, \ldots, x_{n-N}$ be the optimal nodes for quadrature of type (10) at $n-N$ points for $H^q, p^{-1}+q^{-1}=1$, and with respect to the measure $w_\beta d\mu$. After letting $x_k=1$ for $k=n-N+1, \ldots, n$ we can define $\mathbf{z}\in\mathbf{C}^n$ by $\mathbf{z}:=(x_1, \ldots, x_n)$.

We now return to the observations in section 2 and especially to (2) and (6)—(8). The condition $\Phi(r)=0$ for $r\in\mathscr{R}(\mathbf{z})$ gives $G_\beta(x_k)=0$ for $k=1, \ldots, n-N$. Applying the optimal quadrature formula for

$$\Phi(w_\beta \hat{\mu}) = \int_0^1 w_{-\beta}(x) G_\beta(x) w_\beta(x)\, d\mu(x)$$

we therefore obtain

$$|\Phi(w_\beta \hat{\mu}) \leqq \|w_{-\beta} G_\beta\|_q \cdot e_{n-N,q}. \tag{14}$$

However, it is not immediate that $w_{-\beta}G_\beta\in H^q$ since the conditions on β leading to (8) are fulfilled only if $N=0$. With our $\mathbf{z}$ we find that the function g fulfils

$$\int_T (1-\zeta)^{-k} w_\beta(\zeta) g(\zeta)\, d\zeta = 0 \quad \text{for} \quad k = 1, \ldots, N.$$

Since

$$(1-z\zeta)^{-1} = \sum_{k=1}^N (-z)^{-k}(1-z)^{k-1}(1-\zeta)^{-k} - (-z)^{-N}(1-z)^N(1-\zeta)^{-N}(1-z\zeta)^{-1}$$

we see in (7) that

$$G_\beta(z) = -(-z)^{-N} w_N(z) G_{\beta-N}(z)$$

and hence $\|w_{-\beta}G_\beta\|_q=\|w_{-\gamma}G_\gamma\|_q$. The conditions on γ, however, guarantee that $\|w_{-\gamma}G_\gamma\|_q\leqq C(\gamma, q)\cdot\|g\|_q=C(\gamma, q)$. In fact it is possible to replace the constant $C(\gamma, q)$ by a constant depending only on ε and p.

The lemma then follows from (14) and lemma 2 (for the measure $w_\beta d\mu$).

We can now proceed with the proof of part i) of theorem 2. The case $1<p<\infty$ follows at once from lemma 4 by letting $N=\gamma=0$. So let us assume that $0<p<1$ and as before $p^{-1}=N+\delta$. Since p^{-1} is not an integer we have $0<\delta<1$.

For $n=1, 2, \ldots,$ we define

$$\beta := p^{-1} - 0.5 - n^{-1/2} =: N+\gamma$$

where $\gamma=\delta-0.5-n^{-1/2}$. Let s be the conjugate exponent of $2/p$, i.e. $s=(1-0.5p)^{-1}$. For each n we pick an $r\in\mathscr{R}_n$ so that $\|(\hat{\mu}-r)w_\beta\|_2\leqq 2\varrho_{n,2}(\hat{\mu}, \beta)$. With the notation $h=\hat{\mu}-r$ we find by Hölder's inequality and lemma 4 that

$$\|h\|_p^p \leqq \|w_{-\beta p}\|_s \|hw_\beta\|_2^p \leqq Cn^{1/2s}(2\varrho_{n,2}(\hat{\mu}, \beta))^p \leqq Cn^{1/2} \exp\left(-\pi p \sqrt{2n(\alpha+\beta+1/2)}\right).$$

Hence we get

$$\varrho_{n,p}(\hat{\mu}) \leqq C\varepsilon(n, \alpha, p)$$

with $C=C(\alpha, p)$. This concludes the proof of part i) of theorem 2.

§ 5. Lower estimates for $\varrho_{n,p}(\hat{\mu})$

We start with part ii) of theorem 2, i.e. estimates for $1 \leqq p \leqq \infty$. Let $\mathbf{z}=(z_1, \dots, z_n) \in U^n$ be given. We denote by B_n the Blaschke product

$$B_n(z) := \prod_{k=1}^{n} \frac{z-z_k}{1-\bar{z}_k z}$$

and by $\bar{B}_n$ the product $\bar{B}_n(z) := \overline{B_n(\bar{z})}$. In what follows we let q be the conjugate exponent of p. For each n we define a function g by

$$g(z) := C_s \cdot z(1-z)^{-s} B_n(z) \bar{B}_n(z)$$

where $s=q^{-1}-n^{-1/2}$. The constant C_s is choosen so that $\|g\|_q=1$. Since

$$\|g\|_q \leqq C_s \|w_{-s}\|_q \leqq C_s \cdot C n^{1/2q}$$

with $C=C(q)$ we find that $C_s \geqq C^{-1} n^{-1/2q}$.

We now define a linear functional Φ on H^p by

$$\Phi(f) := \frac{1}{2\pi i} \int_T f(\zeta) g(\bar{\zeta}) \bar{\zeta} \, d\zeta.$$

Then $\|\Phi\| \leqq \|g\|_q = 1$ and for $z \in U$ it holds that

$$\hat{\Phi}(z) := \Phi\big((1-(\cdot)z)^{-1}\big) = g(z).$$

Consequently $\Phi(r)=0$ for each $r \in \mathscr{R}(\mathbf{z})$. We also see that

$$\Phi(\hat{\mu}) = \int_0^1 g(x) \, d\mu(x).$$

By Lemma 3 we get

$$\Phi(\hat{\mu}) \geqq C_s \cdot C \sqrt{n} \exp\big(-\pi \sqrt{(2n+1)(\alpha-s+1)}\big) \geqq C n^{1/2p} \exp\big(-\pi \sqrt{2n(\alpha+1/p)}\big)$$

with $C=C(\alpha, p)$. Hence $\varrho_p(\hat{\mu}, \mathbf{z}) \geqq C(\alpha, p) \varepsilon(n, \alpha, p)$ for all $\mathbf{z} \in U^n$ and and consequently part ii) of the theorem follows.

When $0<p<1$ it is possible for the approximating rational functions to have poles also on T. But for each n we can pick $\mathbf{z} \in U^n$ so that $\varrho_p(\hat{\mu}, \mathbf{z}) \leqq 2\varrho_{n,p}(\hat{\mu})$. Therefore we can assume $\mathbf{z} \in U^n$ when we prove the estimate from below. The proof is somewhat more complicated than for $1 \leqq p \leqq \infty$ and we need an auxiliary function. Let for $\gamma \in (0.5, 1)$ the function $\varphi = \varphi_\gamma$ be defined for $z \notin [1, \infty)$ by

$$\varphi(z) := [1-(1-z)^\gamma]/[1+(1-z)^\gamma].$$

This φ defines a conformal mapping $w=\varphi(z)$ of the sector $-\pi(2\gamma)^{-1} < \arg(1-z) < \pi(2\gamma)^{-1}$ onto $|w|<1$.

Let $\mathbf{z}$ be as above. We define a point $\mathbf{w}\in U^n$ by $w_k=\varphi(z_k)$, $k=1, \ldots, n$, and Blaschke products B_n, $\bar{B}_n$ by

$$B_n(w) := \prod_{k=1}^{n} \frac{w-w_k}{1-\bar{w}_k w} \quad \text{and} \quad \bar{B}_n(w) := \overline{B_n(\bar{w})}.$$

Moreover, for each n we define g by

$$g(z) := C_s \varphi(z)(1-\varphi(z))^s B_n(\varphi(z)) \bar{B}_n(\varphi(z))$$

where $\gamma^{-1}:=1+2(\pi\sqrt{n})^{-1}$, $s:=(p^{-1}-1)\gamma^{-1}$ and $C_s:=n^{(1-p^{-1})/2}$. With this C_s we shall show that there is a constant $C=C(p)$ such that

$$(15) \qquad |g(z)|+|g^{(N+1)}(z)|(1-|z|)^{2-\delta} \leqq C$$

for all $z\in U$. From section 2 we know that this means that

$$\Phi(f) = \lim_{r\to 1-} \int_T f(r\zeta)\zeta g(\bar{\zeta})\, d\zeta$$

defines a linear functional on B^p with $\|\Phi\| \leqq C(p)$. As above we find that $\Phi(r)=0$ for $r\in\mathscr{R}(\mathbf{z})$ and with $\psi:=\varphi^{-1}$

$$\Phi(\hat{\mu}) = \int_0^1 g(x)\, d\mu(x) = \int_0^1 g(\psi(t))\, d\mu\circ\psi(t) \geqq$$

$$\geqq C_s \int_0^1 (1-t)^s(1-\psi(t))^\alpha \psi'(t)|B_n(t)|^2 t\, dt.$$

For ψ we have the estimates $C \leqq (1-\psi(t))(1-t)^{-1/\gamma} \leqq C^{-1}$ and $\psi'(t) \geqq C(1-t)^{-1+1/\gamma}$ for all $t\in(0, 1)$ and some constant $C>0$. Consequently

$$\Phi(\hat{\mu}) \geqq C_s \cdot C \int_0^1 (1-t)^{s+\alpha/\gamma-1+1/\gamma} |B_n(t)|^2 t\, dt$$

and by lemma 3 and the definition of C_s

$$\Phi(\hat{\mu}) \geqq Cn^{(1/2)-(1/2p)} \cdot n^{1/2} \exp\left(-\pi\sqrt{2n(s+(\alpha+1)/\gamma)}\right) \geqq$$

$$\geqq Cn^{1-(1/2p)} \exp\left(-\pi\sqrt{2n(\alpha+1/p)}\right).$$

As in the previous case this gives the estimate for $\varrho_{n,p}(\hat{\mu})$.

In order to finish the proof it remains to show the estimate (15). It is obvious that $\|g\|_\infty \leqq 2^{1/p}$ so we can concentrate on $g^{(N+1)}$.

Let Γ be the boundary of the intersection of the disc $|z|<2$ and the sector $\varphi^{-1}(U)$. It consists of a part of the circle $|z|=2$ that we denote by Γ_1 and the union Γ_2 of two segments. Observe that $|\varphi(\zeta)| \leqq 1$ for ζ on and inside Γ. Hence for $|z|<1$

we get

$$|g^{(N+1)}(z)| \leqq \frac{C_s N!}{2\pi} \int_{\Gamma} \frac{|1-\varphi(\zeta)|^s}{|\zeta - z|^{N+2}} |d\zeta|.$$

We first observe that

$$\int_{\Gamma_1} \frac{|1-\varphi(\zeta)|^s}{|\zeta - z|^{N+2}} |d\zeta| \leqq 2^{s+1}\pi.$$

For $\zeta \in \Gamma_2$ and $z \in U$ the inequality $|\zeta - z| \geqq |1-\zeta| \sin \varepsilon$ is valid with $\varepsilon := := \pi(\gamma^{-1}-1)/2 = n^{-1/2}$. Moreover $|1-\varphi(\zeta)| \leqq 2|1-\zeta|^\gamma$ for $\zeta \in \Gamma_2$. These estimates yield

$$\int_{\Gamma_2} \frac{|1-\varphi(\zeta)|^s}{|\zeta - z|^{N+2}} |d\zeta| \leqq 2^s \int_{\Gamma_2} \frac{|1-\zeta|^{\gamma s}}{|\zeta - z|^{N+2}} |d\zeta| \leqq 2^s (\sin \varepsilon)^{-\gamma s} \int_{\Gamma_2} |\zeta - z|^{\gamma s - N - 2} |d\zeta|.$$

Note that $\gamma s - N - 2 = \delta - 3$ so that the whole expression can be estimated by $Cn^{(1/p-1)/2}(1-|z|)^{\delta-2}$. Our choice of C_s therefore gives $|g^{(N+1)}(z)| \leqq C(1-|z|)^{\delta-2}$ where $C = C(p)$ and consequently also (15).

Thereby theorem 2 is proved.

§ 6. Approximation of $(1-z)^\alpha$

To be able to apply the preceding results to the problem of approximating $(1-z)^\alpha$ we have to represent it as a Markov—Stieltjes function. A standard use of Cauchy's integral formula gives for $z \in U$

$$(1-z)^\alpha = -\frac{\sin \pi\alpha}{\pi} \int_1^2 \frac{(t-1)^\alpha}{t-z} dt + \frac{1}{2\pi i} \int_{\Gamma} \frac{(1-\zeta)^\alpha}{\zeta - z} d\zeta$$

where Γ is the circle $|\zeta| = 2$.

We define a measure $d\mu$ by $d\mu(x) := w(x)\,dx$ with

$$w(x) := \begin{cases} (1-x)^\alpha x^{-\alpha-1} & \text{for} \quad x \in [0.5, 1] \\ 2 & \text{for} \quad x \in [0, 0.5], \end{cases}$$

Then we have for $z \in U$ that

$$\pi(1-z)^\alpha = \sin \pi\alpha \cdot \hat{\mu}(z) + g(z) \tag{16}$$

where g is analytic in $|z| < 2$. We can get an estimate for $\varrho_{n,p}((1-z)^\alpha)$ from theorem 2 if we can show that the function g is not significant for the order of approximation of $(1-z)^\alpha$. This will be a consequence of the following lemma.

Lemma 5. *Suppose that for* $f, g \in H^p$ *there are positive constants* a, b, c, A, B *such that*

i) $A^{-1} < \varrho_k(f) \cdot k^{-a} e^{b\sqrt{k}} < A$ *for* $k = 1, 2, \ldots$.

ii) $\varrho_m(g) < Be^{-cm}$ *for* $m = 1, 2, \ldots$.

If $\varrho_n(f+g) > 0$ *for every* $n = 1, 2, \ldots$, *then there is a constant* C *such that*

$$C^{-1} \leqq \varrho_n(f+g)/\varrho_n(f) \leqq C$$

for $n = 1, 2, \ldots$.

Proof. Because of the condition $\varrho_n(f+g) > 0$ it is enough to show that lim sup and lim inf of $\varrho_n(f+g)/\varrho_n(f)$ are positive numbers.

For all natural numbers k and m it holds that

$$\varrho_{k+2m}(f) - \varrho_m(g) \leqq \varrho_{k+m}(f+g) \leqq \varrho_k(f) + \varrho_m(g). \tag{17}$$

To each n we pick m so that $\varrho_m(g) < \varrho_{2n}(f)$. Because of i) and ii) this can be done with m such $m \leqq d\sqrt{n}$ for some d independent of n. Then we let $k := n - m$. We may assume that $k \geqq 1$ and we observe that $-d \leqq \sqrt{n \pm m} - \sqrt{n} \leqq d$. From i) it follows that $\limsup \varrho_{n-m}(f)/\varrho_n(f)$ is finite and that $\liminf \varrho_{n+m}(f)/\varrho_n(f)$ is positive. Since moreover $\limsup \varrho_m(g)/\varrho_n(f) = 0$ the desired properties for lim sup and lim inf of $\varrho_n(f+g)/\varrho_n(f)$ follows from (17).

With $f := \sin \pi\alpha \cdot \hat{\mu}$ and g as in (16) the conditions of the lemma are satisfied if α is not an integer. Hence we have the following consequence of theorem 2.

Theorem 3. *For* $1 < p < \infty$ *and* $\alpha > -1/p$ *there are constants* $B = B(\alpha, p) > 0$, $C = C(\alpha, p)$ *such that*

$$B\,|\sin \alpha\pi|\, \varepsilon(n, \alpha, p) \leqq \varrho_{n,p}\big((1-z)^\alpha\big) \leqq C\varepsilon(n, \alpha, p)$$

for $n = 1, 2, \ldots$.

7. Approximation of x^α on [0, 1]

Except for the introduction we have only considered approximation in H^p. We shall, hovever, finish by giving a proof of the upper estimate in theorem 1. This estimate will follow in the same way as for $\varrho_{n,p}\big((1-z)^\alpha\big)$ as soon as we have proved the following counterpart of theorem 2. For $f \in L^p = L^p[0, 1]$ we here use the notation

$$R_{n,p}(f) := \inf \{\|f - r\|_p \colon r \in \mathscr{R}_n\}.$$

Theorem 4. *Let* $1 < p < \infty$ *and* $\alpha > -1/p$. *Suppose the measure* $d\mu$ *on* (0, 1) *fulfils* $A^{-1} w_\alpha dx \leqq d\mu \leqq A w_\alpha dx$ *for some constant* A. *Then there are constants* $B =$

$=B(\alpha, p)>0$ *and* $C=C(\alpha, p)$ *such that*

$$B \leqq R_{n,p}(\hat{\mu}) \cdot n^{-1/2p} \exp\left(2\pi \sqrt{n(\alpha+1/p)}\right) \leqq C$$

for $n=1, 2, \ldots$.

Proof. In analogy with the H^p-case we let

$$R_p(\hat{\mu}, \mathbf{z}) := \inf\{\|f-r\|:\ r \in \mathscr{R}(\mathbf{z})\}$$

for each $\mathbf{z} \in \mathbf{C}^n$ such that $\mathscr{R}(\mathbf{z}) \subseteqq L^p$. As in section 2 we have

$$R_p(\hat{\mu}, \mathbf{z}) = \sup \left| \int_0^1 \hat{\mu}(t)\, g(t)\, dt \right|$$

with the sup over all $g \in L^q$ (where $p^{-1}+q^{-1}=1$) such that $\|g\|_q=1$ and $\int_0^1 r(t) g(t)\, dt=0$ for all $r \in \mathscr{R}(\mathbf{z})$. We observe that

$$\int_0^1 \hat{\mu}(t)\, g(t)\, dt = \int_0^1 G(x)\, d\mu(x)$$

where

$$G(z) := \int_0^1 \frac{g(t)}{1-zt}\, dt, \quad z \notin [1, \infty).$$

The function G is analytic in particular in the domain D bounded by $[1, 2]$ and the circle $|z|=2$. Let φ be the conformal mapping of $|w|<1$ onto D normalized by $\varphi(0)=0$ and $\varphi'(0)>0$. It follows by the inequalities of M. Riesz and Fejér—Riesz that $H=G\circ\varphi \cdot (\varphi')^{1/q}$ belongs to H^q and that there is a constant $C(q)$ such that

$$\|H\|_q \leqq C(q) \|g\|_q.$$

The symmetry gives that φ maps $[0, 1)$ onto $[0, 1)$ so that

$$\int_0^1 G(x)\, d\mu(x) = \int_0^1 H(t)\, d\nu(t) \tag{18}$$

where $d\nu$ is the positive measure defined by $d\nu(t) := (\varphi'(t))^{-1/q} d\mu \circ \varphi(t)$. On $[0, 1)$ the mapping φ has the properties

a) $C^{-1}|1-t|^2 \leqq 1-\varphi(t) \leqq C|1-t|^2$,

b) $C^{-1}|1-t| \leqq \varphi'(t) \leqq C|1-t|$

for some constant C. Our conditions on $d\mu$ therefore imply that

$$d\nu(t) \leqq C(1-t)^{2\alpha+1/p}\, dt$$

for some C.

We now proceed as in section 2. Let $\mathbf{w}=(w_1, \ldots, w_n)$ be the optimal nodes for quadrature with respect to dv in H^q and let $\mathbf{z}:=(z_1, \ldots, z_n)$ with $z_k:=\varphi(w_k)$, $k=1, \ldots, n$. Observe that if $\int_0^1 r(t)g(t)\,dt=0$ for all $r\in\mathscr{R}(\mathbf{z})$ then $H(w_k)=0$, $k=1, \ldots, n$. When we apply the optimal quadrature formula to (18) we therefore get by lemma 2 that

$$R_p(\hat{\mu}, \mathbf{z}) \leqq C\varepsilon(n, 2\alpha+1/p, p).$$

This gives the estimate from above in the theorem.

To obtain the estimate from below we could proceed as in section 5 but instead we have choosen to use the results of that section. Unfortunately we need to introduce another conformal map. Let Ψ be the mapping of $|w|>1$ onto the complement of $[0, 1]$ such that $\Psi(\infty)=\infty$ and $\Psi'(\infty)>0$. Then we define a linear transformation $S\colon L^p[0, 1]\to H^p$ by

$$Sf(w) := \frac{1}{2\pi i}\int_T \frac{f\circ\Psi(u)\cdot\Psi'(u)^{1/p}}{u-w}\,du, \quad \text{for} \quad |w|<1.$$

By M. Riesz' inequality there is a $C=C(p)$ such that $\|Sf\|_p\leqq C\|f\|_p$ if $1<p<\infty$. Moreover a simple calculus of residues gives $S(\mathscr{R}_n\cap L^p)=\mathscr{R}_n\cap H^p$. Consequently

$$\varrho_{n,p}(Sf) \leqq CR_{n,p}(f)$$

for each $f\in L^p[0, 1]$. With $f=\hat{\mu}$ we find that

$$S\hat{\mu}(w) = \int_0^1 \frac{dv(u)}{1-uw}$$

where this time $dv(u)=(\Psi'(1/u))^{-1/q}u\Psi(1/u)\,d\mu(1/\Psi(1/u))$. The function Ψ has properties of the same type as φ above so for dv we have

$$dv(u) \geqq C(1-u)^{2\alpha+1/p}\,du$$

for some C. But this means that

$$\varrho_{n,p}(S\hat{\mu}) \geqq B\varepsilon(n, 2\alpha+1/p, p)$$

by theorem 2. Hence the wanted estimate follows.

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Рациональная аппроксимация функций типа x^α в интегральных нормах

ЯН-ЭРИК АНДЕРССОН

Основной результат работы касается порядков рациональных приближений в H^p функций типа Маркова—Стилтьеса с некоторыми специальными условиями на меры. Как следствие, получено распространение на случай иррациональных показателей α тех оценок порядка рациональных приближений функций x^α в $L^p[0, 1]$, $1<p<\infty$, которые ранее были известны для рациональных α.

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