# ON LONGITUDINAL SPECTRAL COHERENCE

L. KRISTENSEN
*Risø National Laboratory, DK-4000 Roskilde, Denmark.*

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**Abstract.** It is demonstrated that the longitudinal spectral coherence differs significantly from the transversal spectral coherence in its dependence on displacement and frequency. An expression for the longitudinal coherence is derived and it is shown how the scale of turbulence, the displacement between observation sites and the turbulence intensity influence the results. The limitations of the theory are discussed.

## 1. Introduction

Spectral coherence as defined in Lumley and Panofsky (1964) and discussed in detail by Pielke and Panofsky (1970), Ropelewski *et al.* (1973), Panofsky *et al.* (1974) and Panofsky and Mizuno (1975) is a commonly used statistical description of the time-space behaviour of turbulent velocity components. In the paper by Pielke and Panofsky it is suggested that the coherence has the form

$$\mathrm{coh}\,(n) = \exp\left(-a\frac{nD}{U}\right) \tag{1}$$

for transverse as well as longitudinal displacements with respect to the mean wind direction. In (1) $n$ is the frequency in Hz, $U$ the mean wind speed in m sec$^{-1}$, $D$ the displacement in m and $a$ is a dimensionless 'decay parameter' of the order 10 (Panofsky, 1973). The idea that the coherence should have this form was suggested by Davenport (1961) in an analysis of data from vertically displaced cup anemometers. It seems plausible to assume that the coherence has the same behaviour for lateral displacements.

The longitudinal coherence, however, should be expected to behave differently because of the finite travel time of the eddies from one anemometer to the other. In this connection, it is worth mentioning that a straightforward application of Taylor's hypothesis to the longitudinal space-time autocorrelation function implies that longitudinal coherence is unity. Ropelewski *et al.* (1973) introduced the concept 'eddy turnover time' $\tau$ to describe 'eddy decay' in a semiqualitative way. Panofsky and Mizuno (1975) suggested a correction for wind direction fluctuations that effectively decreased the value of $\tau$. This idea was modified by Perry *et al.* (1978) with the effect that the influence of wind direction fluctuations is enhanced. In all cases an expression of the form (1) was derived.

In the present paper, these ideas are extended in order to deduce an alternative expression based on a change in the assumptions about $\tau$ and also to demonstrate how the scale $l$ and the r.m.s. velocity $\sigma$ of the turbulence affect the expression for the longitudinal coherence.

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## 2. The Model

Consider an eddy of size $\lambda$ travelling with the mean wind speed $U$ from station 1 to station 2, displaced a distance $D$ along the mean wind. An illustration of the situation is given in Figure 1.

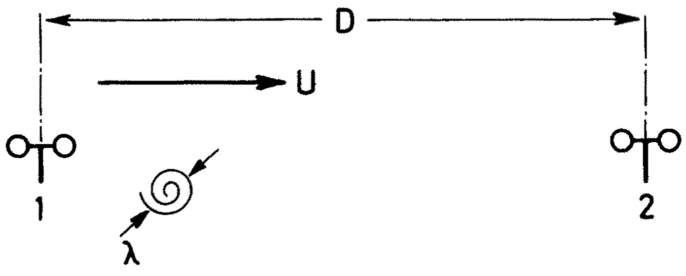


Fig. 1. Illustration of the model geometry. The two stations are symbolized by cup anemometers.

The frequency $n$ associated with eddies of size $\lambda$ is given by

$$n \sim U/\lambda\,, \tag{2}$$

since an eddy can be considered as a 'wave packet' with wavenumbers concentrated at $2\pi/\lambda$ (Tennekes and Lumley, 1972).

The present model considers an eddy as an entity with a certain mean lifetime $\tau_\lambda$. Since eddies of size $\lambda$ are most active in the eddy destruction (self destruction), we assume that

$$\tau_\lambda \sim (B(\lambda)/\lambda)^{-1/2}\,, \tag{3}$$

where $B(\lambda)$ is the turbulent kinetic energy per unit wavelength. This energy density is related to the energy spectrum as defined by Lumley and Panofsky (1964) by

$$B(\lambda) = \frac{2\pi}{\lambda^2} E\left(\frac{2\pi}{\lambda}\right). \tag{4}$$

The total turbulent kinetic energy $\sigma^2/2$ is given by

$$\frac{\sigma^2}{2} = \int_0^\infty E(k)\,\mathrm{d}k = \int_0^\infty B(\lambda)\,\mathrm{d}\lambda\,. \tag{5}$$

In the inertial subrange, $B(\lambda)$ is determined entirely by $\lambda$ and the rate of dissipation $\varepsilon$ of turbulent kinetic energy, so that we can write

$$B(\lambda) \sim \varepsilon^{2/3}\lambda^{-1/3}\,. \tag{6}$$

In this case $\tau_\lambda$ is given by

$$\tau_\lambda \sim \varepsilon^{-1/3}\lambda^{2/3}\,. \tag{7}$$

Expressing balance between the rate of production of turbulent kinetic energy and $\varepsilon$ by

$$\varepsilon \sim \sigma^3/l \tag{8}$$

where $l$ is the scale of the turbulence, we can rewrite (7) as

$$\tau_\lambda \sim l^{1/3}\lambda^{2/3}/\sigma. \tag{9}$$

Expression (6) is valid only in the inertial subrange, where

$$\lambda < l. \tag{10}$$

When $\lambda$ is larger than $l$, $B(\lambda)$ is directly dependent on $l$. Assuming that no other length scale enters, we can generalize (6) by writing

$$B(\lambda) \sim \varepsilon^{2/3}\lambda^{-1/3}g(l/\lambda) \tag{11}$$

where $g(l/\lambda)$ is a dimensionless function of the parameter $l/\lambda$.

Similarly, the expression for $\tau_\lambda$ becomes

$$\tau_\lambda \sim (l^{1/3}\lambda^{2/3}/\sigma)g^{-1/2}(l/\lambda). \tag{12}$$

Suppose that an eddy is observed at station 1. Whether it is observed at station 2 is determined by the probability that it does not decay en route and that it 'hits the target' at station 2. The square of the ratio of the number of eddies observed at both station 1 and station 2 to the total number observed at station 1 is identified as the coherence at frequency $n$.

Since the time $t$ of travel from station 1 to station 2 is $D/U$ the probability $P_1$ that an eddy does not decay during the transport from station 1 to station 2 is hypothesized to be

$$P_1 \sim \exp\,(-D/(U\tau_\lambda)). \tag{13}$$

The exponential form (13) is an obvious analogy to the decay of radioactive particles.

The probability that the eddy is observed at station 2 must depend on its size $\lambda$ and the amount of transversal turbulent diffusion during the travel between the two stations. The variance $\sigma_t^2$ of the lateral distance from station 2 to the center of the eddy, when it has travelled the distance $D$ from station 1, is

$$\sigma_t^2 = \int_0^{D/U} \mathrm{d}t' \int_0^{D/U} \mathrm{d}t'' \,\langle v_t(t')v_t(t'')\rangle, \tag{14}$$

where $v_t$ is the transversal Lagrangian velocity of the eddy and brackets denote ensemble averaging. We assume that $v_t$ is stationary and consequently (14) can be reduced to a single integral.

$$\sigma_t^2 = 2\sigma^2\frac{D}{U}\int_0^{D/U}\left(1-\frac{U\tau}{D}\right)\rho_L(\tau)\,\mathrm{d}\tau. \tag{15}$$

The function $\rho_L(\tau)$ is the Langrangian autocorrelation function. In order to reduce the number of parameters, we have assumed that the variance of $v_t$ is equal to $\sigma^2$. For values of $D/U$ large compared to the Lagrangian time scale $T_L$, the integral can be set equal to $T_L$. Since this time scale is also the 'eddy turnover time' $\tau_l$ for eddies of size $l$, we have according to (9)

$$\int_0^{D/U} \left(1-\frac{U\tau}{D}\right) \rho_L(\tau)\,\mathrm{d}\tau \approx \int_0^{\infty} \rho_L(\tau)\,\mathrm{d}\tau = \mathrm{T_L} \sim \tau_l \sim l/\sigma . \tag{16}$$

In the other limit, $D/U \ll \mathrm{T_L}$, the integral is approximately equal to $(D/U)/2$. Consequently we approximate $\sigma_t^2$ by

$$\sigma_t^2 = \begin{cases} \left(\dfrac{\sigma}{U}\right)^2 D^2 & \text{for} \quad \dfrac{\sigma}{U}\dfrac{D}{l} \leqslant 1 \\ \dfrac{\sigma}{U} Dl & \text{for} \quad \dfrac{\sigma}{U}\dfrac{D}{l} \geqslant 1. \end{cases} \tag{17}$$

The probability $P_2$ that an eddy of size $\lambda$ is observed at station 2 is the chance that it will pass station 2 within the distance $\lambda$; on the assumption of axisymmetric Gaussian transversal diffusion we find

$$\mathrm{P}_2 = \int_0^{\lambda} r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \frac{\exp(-r^2/2\sigma_t^2)}{2\pi\sigma_t^2} = \int_0^{\lambda} \exp(-r^2/2\sigma_t^2) r\,\mathrm{d}r/\sigma_t^2 = 1-\exp(-\lambda^2/2\sigma_t^2). \tag{18}$$

Combining (13) and (18) and keeping in mind that the coherence is the square of a relative frequency, we arrive at the expression

$$\mathrm{coh}\,(n) = P_1^2 P_2^2 = \exp(-2D/U\tau_\lambda) \times (1-\exp(-\lambda^2/2\sigma_t^2))^2 . \tag{19}$$

By application of (2), (12) and (17) to (19), the expression for the coherence can be written

$$\mathrm{coh}\,(n) = \exp(-2\alpha G(nl/U)) \times \begin{cases} (1-\exp(-(2\alpha^2(nl/U)^2)^{-1}))^2 & \text{for } \alpha \leqslant 1 \\ (1-\exp(-(2\alpha(nl/U)^2)^{-1}))^2 & \text{for } \alpha \geqslant 1 \end{cases} \tag{20}$$

where the function $G(\xi)$ is given by

$$G(\xi) \sim \xi^{2/3} g^{1/2}(\xi) \tag{21}$$

and

$$\alpha = \frac{\sigma}{U} \frac{D}{l} . \tag{22}$$

In the inertial subrange

$$g(\xi) = 1 \tag{23}$$

according to (6) and (11).

For $\lambda \gg l$, a model of homogeneous turbulence (Batchelor, 1953) predicts that

$$B(\lambda) \sim \sigma^2 l^5 \lambda^{-6} \tag{24}$$

in which case

$$g(\xi) = \xi^{17/3} \tag{25}$$

and

$$G(\xi) \sim \xi^{7/2} . \tag{26}$$

If $G(\xi)$ can be written generally in a power-law form as

$$G(\xi) \sim \xi^{\mu} , \tag{27}$$

where $\mu$ is a slowly varying function of $\xi$, then the coherence becomes

$$\mathrm{coh}\,(n) = \exp\,(-2\alpha\xi^{\mu}) \begin{cases} (1 - \exp\,(-(2\alpha^2\xi^2)^{-1}))^2 & \text{for } \alpha \leqslant 1 \\ (1 - \exp\,(-(2\alpha\xi^2)^{-1}))^2 & \text{for } \alpha \geqslant 1 \end{cases} \tag{28}$$

As seen above, we expect that $2/3 \leqslant \mu \leqslant 7/2$.

Kaimal *et al.* (1972) have given a semiempirical expression for the one-dimensional, longitudinal energy spectrum for neutral lapse rates. Assuming isotropy we can derive an expression for $B(\lambda)$ (Batchelor, 1953). The function $G(\xi)$ is in this case given by

$$G(\xi) = (33)^{-2/3} \frac{(33\xi)^2 (33\xi + 3/11)^{1/2}}{(33\xi + 1)^{11/6}} . \tag{29}$$

Figure 2 shows how $G(\xi)$ in (29) behaves as a function of $\xi = l/\lambda = nl/U$. The coherence is shown for three values of $\alpha$ in Figure 3.

## 3. Discussion and Concluding Remarks

It is useful to review the conditions and assumptions used to derive (20). First of all, it was assumed that the decay part is an exponential function of the time of travel $t$ divided by an eddy lifetime. Ropelewski *et al.* (1973) made the same assumption, but

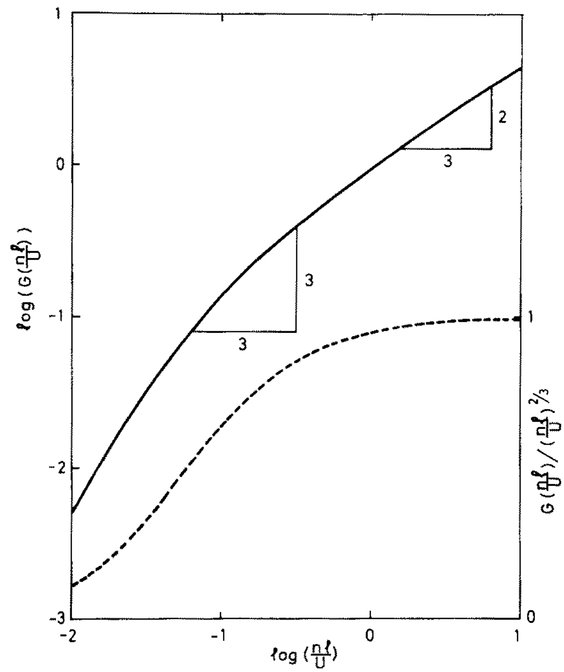


Fig. 2. The function $G$ in neutral conditions, assuming a semi-empirical longitudinal spectrum of Kaimal *et al.* (1972) and isotropy for $\lambda < 0.01\ l$. The full line shows $\log(G(l/\lambda))$ as a function of $\log(l/\lambda)$. The dashed line represents $G(l/\lambda)/(l/\lambda)^{2/3}$ as a function of $\log(l/\lambda)$. Note that the slope is approximately 1 for $nl/U = l/\lambda \sim 0.1$ and that it approaches its limiting value 2/3 as $l/\lambda$ approaches infinity. For practical purposes the slope is equal to 2/3 for $\lambda < l$.

they assumed that the lifetime is determined by $\lambda$ and the r.m.s. turbulent velocity. Instead of (3), they assumed that $\tau_\lambda \sim \lambda/\sigma \sim \lambda l^{-1/3} \varepsilon^{-1/3}$, which in the present model is true only for a limited range of $\lambda$. The frequency associated with the turbulent eddies is assumed to be $U/\lambda$ in their model also. The argument in the exponential function thus becomes

$$-t/\tau_\lambda \sim -\frac{\sigma}{U}\frac{D}{l}\frac{nl}{U} = -\frac{\sigma}{U}\frac{nD}{U} \tag{30}$$

and they obtain the same analytical form for the longitudinal coherence as for transversal coherence. This is consistent with the experimental findings of Perry *et al.* (1978), from which they concluded that the longitudinal coherence is given by (1) with $a$ of the order 4–6, depending somewhat on the intensity of the turbulence and its scale.

The model described in the preceding section is similar to the model of Perry *et al.* (1978) in the sense that the coherence decay is attributed to the effect of destruction and lateral diffusion of eddies. Here the similarity stops. The present model deviates from their model in several ways. The most significant difference probably lies in the ways the models account for coherence loss due to eddy destruction. Perry *et al.*

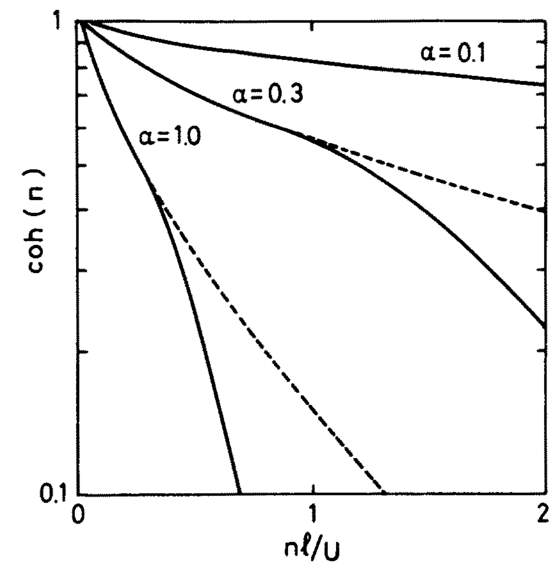


Fig. 3. The longitudinal coherence as function of the dimensionless frequency $nl/U$ for three values of $\alpha$. The full lines represent (20) with $G$ given by (29). The dashed lines show the coherence, if the effect of lateral diffusion is neglected.

(1978) assume that this loss is of the form $\exp(-c \cdot \sigma_w/U \cdot D/l \cdot nl/U)$, $\sigma_w$ being the r.m.s. vertical velocity component. Here a more general frequency dependence is derived, namely $\exp(-2 \cdot \sigma/U \cdot D/l \cdot G(nl/U))$. The function $G$ depends on the form of the turbulent energy spectrum. Unfortunately it is not possible on the basis of the data taken by Perry *et al.* (1978) to determine whether or not this generalization is justified. This is so because according to Figure 2 these data are taken in a range where we expect $G(nl/U) \sim nl/U$. For larger values of the dimensionless frequency $nl/U$, we should expect $G(nl/U) \sim (nl/U)^{2/3}$. Mizuno (1976) measured the longitudinal coherence at a height of 200–400 m by means of two tethered balloons. Presumably the condition $nl/U > 1$ was fulfilled, but he gives no information about $l$ (neither directly nor indirectly by reporting the depth of the boundary layer (see Kaimal *et al.*, 1976)). Consequently it becomes difficult to draw definite conclusions from this work.

The present model accounts for the coherence loss due to lateral diffusion by the square of the probability $P_2$ that the eddy does not miss target at station 2. In this picture the analytical form (18) for $P_2$ is derived. The parameter $\alpha = \sigma/U \cdot D/l$ determines how rapidly the coherence decreases with the dimensionless frequency $nl/U$. Perry *et al.* (1978) assume that the lateral wind direction fluctuations are responsible for a coherence loss of the form $\exp(-C \cdot \sigma_v/U \cdot D/L_y \cdot nD/U)$ $\cdot \exp(-C \cdot \sigma_v/U \cdot (D/L_y)^2 \cdot nL_y/U)$. Here $\sigma_v$ and $L_y$ are the r.m.s. lateral velocity and the scale of the lateral turbulence. We see that not only is their analytical expression for diffusion coherence loss different from the present one, but it also has another key parameter. Even if we assume that $l \sim L_y$, we see that their key parameter is $\sigma_v/U \cdot (D/l)^2$.

To decide whether the approach described in this paper is correct and fruitful, it seems necessary to perform carefully designed experiments and to initiate additional theoretical investigations. In particular, the assumption that eddies decay spontaneously in analogy to radioactive particles may not be a good model. Strictly speaking, the similarity considerations can only establish that $P_1$ is a universal function of $\sigma/U \cdot D/l \cdot G(nl/U)$. The specific functional behaviour should be verified on the basis of the formal definition of spectral coherence (Lumley and Panofsky, 1964). Relevant discussions by Comte-Bellot and Corrsin (1971) and Tennekes (1975) will probably prove helpful in further pursuit of these ideas.

It seems to be demonstrated – at least qualitatively – that longitudinal coherence behaves differently with the frequency than does the transversal coherence, and it is consequently difficult to develop a model for coherence that is neither transversal nor longitudinal. Before this is done it will probably be difficult to interpret experimental results, except when the mean wind direction is either perpendicular to or parallel with the line between measuring points.

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