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ON THE UNIVERSALITY OF THE RIEMANN ZETA-FUNCTION

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Let $\zeta(s)$, $s = \sigma + it$, denote the Riemann zeta-function. In 1975 S. M. Voronin [1] discovered the following famous property of the function $\zeta(s)$.

THEOREM A (Voronin). *Let $0 < r < 1/4$. Let $f(s)$ be any continuous function on the disk $|s| \leqslant r$, which is analytic and nonvanishing in the interior of this disk. Then for every $\varepsilon > 0$ there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon. \tag{1}$$

A property of the function $\zeta(s)$, indicated in Theorem A usually is called the universality. Roughly speaking, this property asserts that any analytic function can be approximated uniformly on the disks of the strip $D = \{s \in \mathbf{C},\ 1/2 < \sigma < 1\}$, where $\mathbf{C}$ stands for the complex plane, by translations of the function $\zeta(s)$. A stronger concept of the universality was proposed by A. Reich [2]. Let $\underline{d}(A) = \underline{\lim}_{T\to\infty} 1/T \operatorname{meas}\{t \in [0, T] \cap A\}$ denote the lower density of the set A. Here $\operatorname{meas}\{A\}$ stands for the Lebesgue measure of the set A. This stronger version of the universality asserts that a set of numbers τ, satisfying the inequality (1) has a positive lower density. B. Bagchi in [3] applied the limit theorems in the space of analytic functions to prove the universality theorems, and he obtained that any analytic function can be approximated uniformly on some compact subsets of D by translations of $\zeta(s)$. A final version of the universality theorem for $\zeta(s)$ is presented in [4, 5] as the following theorem.

THEOREM B. *Let K be a compact subset of the strip D with conected complement. Let $f(s)$ be a nonvanishing continuous function on K which is analytic in the interior of K. Then for every $\varepsilon > 0$*

$$\underline{d}\left(\tau \in \mathbb{R},\ \sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right)>0.$$

Here, as usual, $\mathbb{R}$ denotes the set of all real numbers. In [6] a limit theorem for the Riemann zeta-function with weight in the space of analytic functions has been obtained. Moreover, in [7] the explicit form of the limit measure in the latter theorem was indicated. Thus a possibility turned up to prove the universality theorem with weight for $\zeta(s)$.

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, and such that its variation $V_a^b w$ on $[a, b]$ satisfies the inequality $V_a^b w \leqslant cw(a)$ with some $c > 0$ for all $b \geqslant a \geqslant T_0$. We define a function

$$U = U(T, w) = \int_{T_0}^{T} w(t)\,dt$$

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and suppose that $\lim_{T\to\infty} U(T, w) = \infty$. Moreover, we require some additional property of $w(t)$, related to ergodic theory. Let EX denote the mean of a random variable X. Let $X(\tau, \omega)$ be an ergodic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\tau \in \mathbb{R}$, $E|X(\tau, \omega)| < \infty$, such that the sample paths are integrable almost surely in the Riemann sense over every finite interval. Suppose that

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)X(t+\tau, \omega)\,d\tau = EX(t, \omega) + o(1+|t|)^{\alpha}$$

almost surely for all $t \in \mathbb{R}$ with some $\alpha > 0$ as $T \to \infty$. The latter relation is some generalization of the classical Birkhoff–Khintchine theorem which asserts that

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T} X(\tau, \omega)\,d\tau = EX(0, \omega)$$

almost surely.

Let I_A stand for the indicator function of the set A. Our aim is to prove the following assertion.

THEOREM. *Suppose that the function $w(t)$ satisfies all conditions indicated above. Let K be a compact subset of the strip D with connected complement. Let $f(s)$ be a nonvanishing continuous function on K which is analytic in the interior of K. Then for every $\varepsilon > 0$*

$$\varliminf_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)I_{\{\tau:\ \sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\}}\,d\tau > 0.$$

The proof of the theorem is based on a limit theorem with weight in the space of analytic functions. Let $\mathcal{B}(S)$ stand for the class of Borel sets of a space S. Denote by γ the unit circle on the complex plane, that is, $\gamma = \{s \in \mathbf{C},\ |s| = 1\}$, and take

$$\Omega = \prod_{p}\gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological group. Therefore there exists the probability Haar measure m on $(\Omega, \mathcal{B}(\Omega))$. Thus, one gets a probability space $(\Omega, \mathcal{B}(\Omega), m)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p. Let G be a simply connected region on the complex plane. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $D_1 = \{s \in \mathbf{C},\ \sigma > 1/2\}$. On the probability space $(\Omega, \mathcal{B}(\Omega), m)$ define an $H(D_1)$-valued random element $\zeta(s, \omega)$ by the formula

$$\zeta(s, \omega) = \prod_{p}\left(1 - \frac{\omega(p)}{p^s}\right)^{-1}$$

and denote by P_ζ the distribution of $\zeta(s, \omega)$. Let $P_{1,\zeta}$ be the restriction of P_ζ to $(H(D), \mathcal{B}(H(D)))$. Define on $(H(D), \mathcal{B}(H(D)))$ the probability measure

$$P_{T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau)I_{\{\tau:\ \zeta(s+i\tau)\in A\}}\,d\tau.$$

LEMMA 1. *The probability measure $P_{T,w}$, converges weakly to $P_{1,\zeta}$ as $T \to \infty$.*

Proof. The lemma is a part of a theorem from [7].

Now we state one lemma on the support of the measure $P_{1,\zeta}$. Let S be a separable metric space and let P be a probability measure on $(S, \mathcal{B}(S))$. We recall that a minimal closed set $S_P \subseteq S$, such that $P(S_P) = 1$, is called the support of P. Note that S_P consists of all $x \in S$ such that for every neighbourhood G of x the inequality $P(G) > 0$ is satisfied. Let

$$\mathcal{S} = \{f \in H(D),\ f(s) \neq 0 \text{ or } f(s) \equiv 0\}.$$

LEMMA 2. *The support of the measure $P_{1,\zeta}$ is the set $\mathcal{S}$.*

Proof of the lemma is given in [1], see also [5].

For the proof of theorem we will need one property of the weak convergence of probability measures.

LEMMA 3. *Let P_n and P be the probability measures on $(S, \mathcal{B}(S))$. Then two following assertions are equivalent*:

1^0 *P_n, converges weakly to P as $n \to \infty$;*

2^0 *$\underline{\lim}_{n\to\infty} P_n(G) \geqslant P(G)$ for all open sets G of S.*

Proof. The lemma is a part of Theorem 2.1 from [8] where its proof can be found also.

The following assertion is called the Mergelyan theorem.

LEMMA 4. *Let K be a compact subset of $\mathbf{C}$ whose complement is connected. Then any continuous function $f(s)$ on K which is analytic in the interior of K is approximable uniformly on K by the polynomials of s.*

Proof is given, for example, in [9].

Proof of theorem. First suppose that the function $f(s)$ has nonvanishing analytic continuation to D. Denote by G the set of functions $g \in H(D)$ such that

$$\sup_{s\in K} |g(s) - f(s)| < \varepsilon.$$

By Lemma 2 the function $f(s) \in \mathcal{S}$, i.e., it is contained in the support of the random element $\zeta(s, \omega)$. Clearly, the set G is open. Since by Lemma 1 the probability measure $P_{T,w}$ converges weakly to $P_{1,\zeta}$ as $T \to \infty$, it follows from Lemma 3 as well as from the properties of the support that

$$\underline{\lim}_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\ \sup_{s\in K} |\zeta(s+i\tau)-f(s)|<\varepsilon\}}\, d\tau \geqslant P_{1,\zeta}(G) > 0. \tag{2}$$

Now let $f(s)$ be as in the statement of the theorem. Then by Lemma 4 there exists a sequence of polynomials $\{p_n(s)\}$ such that $p_n(s) \underset{n\to\infty}{\to} f(s)$ uniformly on K. Since $f(s) \neq 0$ on K, we have that $p_{n_0}(s) \neq 0$ on K for a sufficiently large n_0 and

$$\sup_{s\in K} |f(s) - p_{n_0}(s)| < \varepsilon/4. \tag{3}$$

Obviously, the polynomial $p_{n_0}(s)$ has only finitely many zeros. Thus we may choose the region G_1 with connected complement such that $K \subset G_1$ and $p_{n_0}(s) \neq 0$ on G_1. Consequently, there exists a continuous version of $\log p_{n_0}(s)$ on G_1, and $\log p_{n_0}(s)$ is an analytic function in the interior of G_1. By Lemma 4 again we can find a sequence of polynomials $\{q_n(s)\}$ such that $q_n(s) \underset{n\to\infty}{\to} \log p_{n_0}(s)$ uniformly on K. Consequently, for sufficiently large n_1

$$\sup_{s\in K} |p_{n_0}(s) - \exp\{q_{n_1}(s)\}| < \varepsilon/4.$$

The latter inequality together with (3) gives

$$\sup_{s\in K} |f(s) - \exp\{q_{n_1}(s)\}| < \varepsilon/2. \tag{4}$$

Obviously, $\exp\{q_1(s)\} \neq 0$ for all s. Therefore by (2) from the first part of this proof we have

$$\underline{\lim}_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\ \sup_{s\in K} |\zeta(s+i\tau)-\exp\{q_{n_1}(s)\}|<\varepsilon/2\}}\, d\tau > 0.$$

Hence, taking into account the inequality (4), we find that

$$\underline{\lim}_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\ \sup_{s\in K} |\zeta(s+i\tau)-f(s)|<\varepsilon\}}\, d\tau > 0.$$

The theorem is proved.

REFERENCES

1. S. M. Voronin, Theorem on the "universality" of the Riemann zeta-function, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **39**(3), 475–486 (1975).
2. A. Reich, Universelle Werteverteilung von Eulerprodukten, *Nach. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, **1**, 1–17 (1977).
3. B. Bagchi, The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series, Ph. Thesis, Calcutta, Indian Statistical Institute (1981).
4. A. Laurinčikas, On limit theorems for the Riemann zeta-function in some spaces, in: *Proc. Sixth Vilnius Conf. on Prob. Theory and Math. Stat.*, VSP/TEV (1994), pp. 457–483.
5. A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer AP, Dordrecht (to appear).
6. A. Laurinčikas and G. Misevičius, A limit theorem with weight for the Riemann zeta-function in the space of analytic functions, *Lith. Math. J.*, **34**, 171–182 (1994).
7. A. Laurinčikas and G. Misevičius, On limit distribution of the Riemann zeta-function, submitted to *Acta Arith.*
8. P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York–London–Sydney–Toronto (1968).
9. J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Math. Soc. Coll. Publ., Vol. 20 (1960).

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