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L^p-norm convergence of series in compact, totally disconnected groups

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The notations that we use in this paper are similar to those in the books of HEWITT–ROSS [6] and SCHIPP–WADE–SIMON [10]. Let σ be an equivalence class of continuous, irreducible, unitary representations of a compact group G. Denote by Σ the set of all such σ. Σ is called the dual object of G. The dimension of a representation $U^{(\sigma)}$, $\sigma \in \Sigma$, is denoted by d_σ, and let

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle \quad i,j \in \{1, \dots, d_\sigma\}$$

be the coordinate functions for $U^{(\sigma)}$, where $\xi_1, \dots, \xi_{d_\sigma}$ is an orthonormal basis in the representation space of $U^{(\sigma)}$. According to the Weyl–Peter theorem, the system of functions

$$\sqrt{d_\sigma} u_{i,j}^{(\sigma)}, \quad \sigma \in \Sigma, \ i,j \in \{1, \dots, d_\sigma\},$$

is an orthonormal basis for $L^2(G)$. If G is a finite group, then Σ is also finite. If $\Sigma := \{\sigma_1, \dots, \sigma_s\}$, then

$$|G| = d_{\sigma_1}^2 + \dots + d_{\sigma_s}^2.$$

A topological space X is connected if it is not the disjoint union of any two nonvoid sets that are both open and closed. A component of a topological space is a connected subset which is properly contained in no other connected subset. A topological space is totally disconnected if all of its components are points. We now restrict our attention to infinite, compact, totally disconnected groups. It is known that these groups have a countable neighborhood base $G = G_0 \supset G_1 \supset \cdots$ at the identity e consisting of open and closed normal subgroups which satisfy the property that for every $n \in \mathbf{N}$, the factor structure G_n/G_{n+1} is finite [9]. Moreover,

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G is a complete direct product of these factor structures. Hence an infinite, compact, totally disconnected group can be constructed in the following way.

Denote by $m := (m_k : k \in \mathbf{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbf{N}$, and by G_{m_k} a finite group with order m_k, $k \in \mathbf{N}$. For simplicity, we will use the same notation for the operation of G_{m_k}, $k \in \mathbf{N}$, and denote by e the identity of these groups. Suppose that each group has a discrete topology, and a right and left Haar measure μ_k with $\mu_k(G_{m_k}) = 1$. Thus, each group has a similar measure such that the measure of every singleton of G_{m_k} equals $1/m_k$, $k \in \mathbf{N}$. Let G_m be the compact group formed by the complete direct product of G_{m_k} with the product of the topologies, operations, and measures (μ). Thus, each $x \in G_m$ consists of sequences $x := (x_0, x_1, \ldots)$, where $x_k \in G_{m_k}$, $k \in \mathbf{N}$. Define G_0 by means of the set of finite sequences of G_m, $I_0(x) := G_m$,

$$I_n(x) := \{y \in G_m : y_k = x_k, \text{ for } 0 \leq k < n\} \quad (x \in G_m,\ n \in \mathbf{N}),$$

$I_n := I_n(e)$. The sets I_n form a countable neighborhood base at the identity in the product topology on G_m.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad 0 \leq n_k < m_k,\ n_k \in \mathbf{N}.$$

This allows us to say that the $(n_0, n_1, \ldots)$ sequence is the expansion of n with respect to m. We often use the following notations:

$$|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}, \quad n_{(k)} := \sum_{j=0}^{k-1} n_k M_k, \quad n^{(k)} = \sum_{j=k}^{\infty} n_k M_k.$$

Now we denote by Σ_k the dual object of G_{m_k}. Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be the set of all normalized coordinate functions of the group G_{m_k}, and suppose that $\varphi_k^0 \equiv 1$. Thus, for every $0 \leq s < m_k$ there exist $\sigma \in \Sigma_k$, $i, j \in \{1, \ldots, d_\sigma\}$ such that

$$\varphi_k^s = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \quad (x \in G_{m_k}).$$

Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G_m),$$

where

$$n = \sum_{k=0}^{\infty} n_k M_k \quad \text{and} \quad x = (x_0, x_1, \ldots).$$

The Weyl–Peter theorem and Theorem (27.43) in [6] ensure that the system ψ is orthonormal and complete in $L^2(G_m)$. Since the set $\Im \in L^1(G_m)$ of all finite complex linear combinations of the coordinate functions, called polynomials, is dense in $L^1(G_m)$, we can state the following

Theorem 1. *The system ψ is orthonormal and complete in $L^1(G_m)$.*

We remark that if G_{m_k} is the discrete cyclic group of order m_k, $k \in \mathbf{N}$, then G_m coincides with the Vilenkin group, and ψ is the Vilenkin system [10], [17].

For $f \in L^1(G_m)$, we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f\overline{\psi}_k \, d\mu \quad (k \in \mathbf{N}), \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbf{P}, \; S_0 f := 0),$$

where $\mathbf{P}$ is the set of positive integers.

The Dirichlet kernel is defined by

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{P}, \; D_0 := 0).$$

It is clear that

$$S_n f(x) = \int_{G_m} f(y) D_n(x,y) \, d\mu(y).$$

The Dirichlet kernel plays a prominent role in the convergence of Fourier series. The following formulas will be useful in this regard. The case of Abelian groups G_{m_k}, $k \in \mathbf{N}$, is discussed by VILENKIN [1, 17]. We follow his method.

Lemma 1. *If $n \in \mathbf{N}$, $x, y \in G_m$, then*

(a) $$D_n(x,y) = \sum_{k=0}^{\infty} D_{M_k}(x,y) \Big(\sum_{s=0}^{n_k - 1} \varphi_k^s(x_k) \overline{\varphi}_k^s(y_k) \Big) \psi_{n(k+1)}(x) \overline{\psi}_{n(k+1)}(y),$$

(b) $$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y), \end{cases}$$

where $(n_0, n_1, \ldots)$ is the expansion of n and $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$.

Proof. For each $n \in \mathbf{N}$, $x, y \in G_m$, it is easy to see that

$$D_n(x,y) = D_{M_{|n|}}(x,y) \Big(\sum_{s=0}^{n_{|n|}-1} \varphi_{|n|}^s(x_k) \overline{\varphi}_{|n|}^s(y_k) \Big) +$$
$$+ \varphi_{|n|}^{n_{|n|}}(x_{|n|}) \overline{\varphi}_{|n|}^{n_{|n|}}(y_{|n|}) D_{n_{(|n|)}}(x,y).$$

Formula (a) can be proved by induction, while taking into account that

$$\varphi_k^{n_k} \equiv 1 \quad \text{for} \quad k > |n|.$$

To prove (b), note that

$$D_{M_n}(x,y) = \prod_{k=0}^{n-1} \sum_{s=0}^{m_k-1} \varphi_k^s(x_k)\overline{\varphi}_k^s(y_k), \;\; D_{M_0} \equiv 1 \;\; (n \in \mathbf{N},\, n > 0,\, x,y \in G_m).$$

Then it is sufficient to prove that

$$\sum_{s=0}^{m_k-1} \varphi_k^s(x_k)\overline{\varphi}_k^s(y_k) = \begin{cases} m_k & \text{for } x_k = y_k, \\ 0 & \text{for } x_k \neq y_k, \end{cases} \tag{1}$$

for each $k \in \mathbf{N}$. In other words, it is sufficient to demonstrate that for every finite and compact group G of order m, we have

$$\sum_{\sigma\in\Sigma} \sum_{i,j=1}^{d_\sigma} d_\sigma u_{ij}^{(\sigma)}(x)\overline{u}_{ij}^{(\sigma)}(y) = \begin{cases} m & \text{for } x = y, \\ 0 & \text{for } x \neq y, \end{cases}$$

where $(x,y) \in G$. Using the equalities

$$\overline{u}_{i,j}^{(\sigma)}(x) = u_{ij}^{(\sigma)}(x^{-1}), \quad u_{i,j}^{(\sigma)}(xy) = \sum_{r=1}^{d_\sigma} u_{ir}^{(\sigma)}(x)u_{rj}^{(\sigma)}(y),$$

for $x,y \in G$, $i,j \in \{1,\dots,d_\sigma\}$, $\sigma \in \Sigma$, which are well known in the representation theory (see 27.5 in [6]), we can state the following:

$$\sum_{\sigma\in\Sigma} \sum_{i,j=1}^{d_\sigma} d_\sigma u_{ij}^{(\sigma)}(x)\overline{u}_{ij}^{(\sigma)}(y) = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} u_{ij}^{(\sigma)}(x)u_{ji}^{(\sigma)}(y^{-1}) =$$

$$= \sum_{\sigma\in\Sigma} d_\sigma \sum_{i=1}^{d_\sigma} d_\sigma u_{ii}^{(\sigma)}(xy^{-1}) =: \sum_{\sigma\in\Sigma} d_\sigma \chi_\sigma(xy^{-1}),$$

where χ_σ is called the character of the representation $U^{(\sigma)}$. Since the above sum is the identity element for the convolution (see Theorem 27.41 in [6]), we have

$$\sum_{\sigma\in\Sigma} d_\sigma \chi_\sigma(xy^{-1}) = \begin{cases} m & \text{for } x = y, \\ 0 & \text{for } x \neq y, \end{cases}$$

where $x \in G$. This completes the proof of Lemma 1.

Formula (b) is used to prove that the partial sums $S_{M_n}f$ of the Fourier series of a function $f \in L^p(G_m)$, $p \geq 1$, converge to f in L^p-norm and almost everywhere (a.e.). Indeed, the operator

$$S_{M_n}f(x) = \int_{G_m} f(y)D_{M_n}(x,y)\,d\mu(y) = \frac{1}{\mu(I_n(x))}\int_{I_n(x)} f\,d\mu$$

is the conditional expectation with respect to the σ- algebra $\mathcal{A}_n$ generated by the sets $I_n(x)$, $x \in G$, that is,

$$S_{M_n} f = E(f|\mathcal{A}_n), \quad n \in \mathbf{N}.$$

Thus, the following statement is a corollary of the martingale convergence theorem [8].

Corollary. *For each $f \in L^p(G_m)$, $p \geq 1$, and $n \in \mathbf{N}$, the partial sums $S_{M_n} f$ converge to f in L^p-norm and a.e.*

Now we study the whole sequence of the partial sums S_n. According to the Banach–Steinhaus theorem,

$$S_n f \to f \quad \text{in } L^p\text{-norm as } n \to \infty$$

for $f \in L^p(G_m)$ if and only if there exists a constant $C_p > 0$ such that

$$\|S_n f\|_p \leq C_p \|f\|_p, \quad f \in L^p(G_m).$$

Thus, the operators S_n are of type (p,p). Since the system ψ is an orthonormal basis in the Hilbert space $L^2(G_m)$, it is obvious that S_n is of type $(2,2)$.

For bounded Vilenkin systems the Paley theorem implies that the nth partial sum operators are bounded, uniformly in n, from $L^p(G_m)$ into itself for $1 < p < \infty$, that is, the S_n are uniformly of type (p,p) for $1 < p < \infty$. The Paley theorem on unbounded Vilenkin groups does not hold (see WATARI [18]). However, the partial sums of the Vilenkin–Fourier series are uniformly bounded in $L^p(G_m)$ $(1 < p < \infty)$ (see YOUNG [16], SCHIPP [12], SIMON [14]), thus they converge in $L^p(G_m)$ $(1 < p < \infty)$.

We cannot generalize this statement for every non-Abelian group. To illustrate the situation, we consider $G_{m_k} \equiv S_3$ for each $k \in \mathbf{N}$, where S_3 is the symmetric group on 3 elements. It is known that in a certain basis $\{\xi_1, \xi_2\}$ the representation operators of S_3 have the following matrices:

$$e \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (12) \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(13) \to \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \quad (23) \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix},$$

$$(123) \to \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \quad (132) \to \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}.$$

Put

$$\varphi_k^s(x) = \sqrt{2} u_{11}(x),$$

where $0 < s < 6$ is a fixed integer, u_{11} is the coordinate function of the above representation corresponding to the first row and the first column. Thus,

$$\varphi_k^s(e) = \sqrt{2}, \quad \|\varphi_k^s\|_1 = \frac{2\sqrt{2}}{3}.$$

Define $f_k \in L^1(S_3)$ by

$$f_k(x) = \begin{cases} 1 & \text{for } x = e, \\ 0 & \text{for } x \neq e, \end{cases}$$

where $x \in S_3$. Thus,

$$\int_{S_3} f_k \overline{\varphi}_k^s \, d\mu_k \|\varphi_k^s\|_1 = \frac{\sqrt{2}}{6} \cdot \frac{2\sqrt{2}}{3} = \frac{4}{3}\|f_k\|_1.$$

Since the norm $\|f\|_p$ is a continuous function of p for each $f \in L^p(G_m)$, there are constants $C > 1$ and $1 < p < 2$ such that

$$\Big(\int_{S_3} f_k \overline{\varphi}_k^s \, d\mu_k\Big)\|\varphi_k^s\|_p > C\|f_k\|_p.$$

Now we suppose that $j \in \mathbf{N}$, $j > 0$, and let

$$n = \sum_{k=0}^{j-1} s6^k.$$

Define $F_j \in L^1(G_m)$ by

$$F_j(x) := \prod_{k=0}^{j-1} f_k(x_k) \qquad (x \in G_m),$$

where $x = (x_0, x_1, \ldots)$. Since

$$\|F_j\|_p = \prod_{k=0}^{j-1} \|f_k\|_p,$$

it follows that

$$\|S_{n+1}F_j - S_nF_j\|_p = \Big|\int_{G_m} F_j \overline{\psi}_n \, d\mu\Big| \|\psi_n\|_p =$$

$$= \prod_{k=0}^{j-1} \int_{S_3} f_k \overline{\varphi}_k^s \, d\mu_k \|\varphi_k^s\|_p > C^j \|F_j\|_p.$$

Since S_n is of type (p,p), there is a constant $C_p > 0$ so that

$$\|S_{n+1}F_j - S_nF_j\|_p \leq \|S_{n+1}F_j\|_p + \|S_nF_j\|_p \leq 2C_p\|F_j\|_p$$

for each $j > 0$. For this reason the operators S_n are not uniformly of type (p,p).

Remark. In the same way, we can prove that the Paley theorem is not valid for groups for which there exists a constant $C > 0$ so that

$$\|\varphi_k^s\|_1 > \frac{1}{\sqrt{d_k^{(s)}}} + C$$

for infinite number of functions, where $d_k^{(s)}$ is the dimension of the representation corresponding to φ_k^s.

Finally, we prove the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case. The method of the proof is similar to that in [4]. As for the Fejér kernel in the case of Abelian groups G_{m_k}, $k \in \mathbf{N}$, see also [1]. In this regard, we introduce the following concepts. The compact, totally disconnected group G_m is called bounded if the sequence m is bounded. Denote by

$$\sigma_n f = \frac{1}{n}\sum_{k=0}^{n-1} S_k f \quad (n \in \mathbf{P},\ \sigma_0 f := 0)$$

the Fejér mean of the Fourier series and by

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k \quad (n \in \mathbf{P},\ K_0 := 0)$$

the Fejér kernel. Then

$$\sigma_n f(x) = \int_{G_m} f(y) K_n(x,y)\, d\mu(y) \quad (x \in G_m,\ n \in \mathbf{P}).$$

Lemma 2. *If G_m is a bounded group, then there is a constant $C > 0$ such that*

$$\sup_{x \in G_m} \int_{G_m} |K_n(x,y)|\, d\mu(y) \leq C.$$

Proof. Throughout this proof $C > 0$ will denote an absolute constant which will not necessarily be the same at different occurrences. Let r be a fixed natural number. To estimate $|K_n|$, we prove that for every $r \in \mathbf{P}$

$$\sum_{j=0}^{r} M_j d_{n^{(j)}} \leq \frac{\sqrt{2}}{\sqrt{2}-1} M_r d_{n^{(r)}}, \quad \text{where} \quad d_n = \prod_{k=0}^{\infty} d_k^{(n_k)} \tag{2}$$

and $d_k^{(n_k)}$ is the dimension of the representation corresponding to $\varphi_k^{n_k}$. Set

$$b_j := M_j d_{n^{(j)}} \quad (0 \leq s \leq r),$$

thus

$$b_{j+1} := M_{j+1} d_{n^{(j+1)}} = M_j d_{n^{(j)}} \frac{m_j}{d_j^{(n_j)}} \geq b_j \sqrt{2}$$

for $0 \le s < r$, since $(d_j^{(n_j)})^2 < m_j$. Then

$$\sum_{j=0}^{r} b_j \ge b_0 + \sqrt{2}\sum_{j=0}^{r} b_j - \sqrt{2} b_r.$$

Consequently,

$$\sum_{j=0}^{r} b_j \le \frac{\sqrt{2}}{\sqrt{2}-1} b_r.$$

This proves inequality (2).

First we will estimate the absolute value of the kernel

$$K_{n^{(s)},M_s} := \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} D_a \qquad (s \in \mathbf{N})$$

for $x \in G_m$, $y \in I_r(x) \setminus I_{r+1}(x)$ and use the identity

$$(3) \qquad nK_n = \sum_{s=0}^{|n|} \sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s, M_s} \qquad (n \in \mathbf{P}).$$

Let $s \le r$. Then, by Lemma 1,

$$K_{n^{(s)},M_s}(x,y) =$$

$$= \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} \sum_{k=0}^{r} M_k \Big(\sum_{j=0}^{a_k-1} \varphi_k^j(x_k)\overline{\varphi}_k^j(y_k)\Big) \psi_{a^{(k+1)}}(x)\overline{\psi}_{a^{(k+1)}}(y),$$

where $x \in G_m$, $y \in I_r(x) \setminus I_{r+1}(x)$. Since G_m is a bounded group, by (2) we have

$$|K_{n^{(s)},M_s}(x,y)| \le cM_s M_r d_{n^{(s)}}.$$

Then

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_{n^{(s)},M_s}(x,y)| \, d\mu(y) \le cM_s d_{n^{(s)}}.$$

Next we turn to the case $s > r$. In this case we have to find a better estimate of $|K_{n^{(s)},M_s}(x,y)|$. Namely,

$$K_{n^{(s)},M_s}(x,y) = \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} \sum_{k=0}^{r-1} M_k \Big(\sum_{j=0}^{a_k-1} |\varphi_k^j(x_k)|^2 \Big) \psi_{a^{(k+1)}}(x)\overline{\psi}_{a^{(k+1)}}(y) +$$

$$+ \sum_{a=n^{(s)}}^{n^{(s)}+M_s-1} M_r \Big(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r)\Big) \psi_{a^{(r+1)}}(x)\overline{\psi}_{a^{(r+1)}}(y) =: J_1 + J_2,$$

where $x \in G_m$, $y \in I_r(x) \setminus I_{r+1}(x)$. It is easy to see that

$$J_1 = \sum_{a_0=0}^{m_0-1} \cdots \sum_{a_{r-1}=0}^{m_{r-1}-1} \sum_{a_{r+1}=0}^{m_{r+1}-1} \cdots \sum_{a_{s-1}=0}^{m_{s-1}-1} \Big(\sum_{a_r=0}^{m_r-1} \varphi_r^{a_r}(x_r)\overline{\varphi}_r^{a_r}(y_r)\phi(x,y)\Big),$$

where $\phi(x,y)$ does not depend on a_r. By (1) in the proof of Lemma 1, we have $J_1 = 0$.

Next we estimate J_2 as follows:

$$J_2 = M_r \sum_{a_0=0}^{m_0-1} \cdots \sum_{a_{s-1}=0}^{m_{s-1}-1} \Big(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r)\Big)\psi_{a^{(r+1)}}(x)\overline{\psi}_{a^{(r+1)}}(y).$$

It is clear that J_2 does not depend on $a_1, a_2, \dots, a_{r-1}$. Therefore,

$$J_2 = M_r^2 \sum_{a_r=0}^{m_r-1} \Big(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r)\Big) \sum_{a_{r+1}=0}^{m_{r+1}-1} \cdots \sum_{a_{s-1}=0}^{m_{s-1}-1} \psi_{a^{(l)}}(x)\overline{\psi}_{a^{(l)}}(y) =$$

$$= M_r^2 \sum_{a_r=0}^{m_r-1} \Big(\sum_{j=0}^{a_r-1} \varphi_r^j(x_r)\overline{\varphi}_r^j(y_r)\Big) \prod_{l=r+1}^{s-1} \Big(\sum_{a_l=0}^{m_l-1} \varphi_l^{a_l}(x_l)\overline{\varphi}_l^{a_l}(y_l)\Big)\psi_{n^{(s)}}\overline{\psi}_{n^{(s)}}(y).$$

By (1), we have $J_2 = 0$ if $x_l \neq y_l$ for any $r < l < s$. Since G_m is a bounded group, if $x_l = y_l$ for some $r < l < s$, then we have

$$|K_{n^{(s)},M_s}(x,y)| \le cM_rM_sd_{n^{(s)}}.$$

Then denoting by

$$A := \Big\{ y \in G_m : y_0 = x_0, \dots, y_{r-1} = x_{r-1}, y_r \neq x_r, \\ y_{r+1} = x_{r+1}, \dots, y_{s-1} = x_{s-1} \Big\},$$

we may write that

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_{n^{(s)},M_s}(x,y)|\, d\mu(y) \le \int_A cM_rM_sd_{n^{(s)}}\, d\mu(y) =$$

$$= cM_rM_sd_{n^{(s)}} \frac{m_r-1}{M_s} \le cM_rd_{n^{(s)}}.$$

Since $n \ge M_{|n|}$, by (2) and (3),

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|\, d\mu(y) < \frac{c}{M_{|n|}} \sum_{s=0}^{r} M_sd_{n^{(s)}} + \frac{c}{M_{|n|}} \sum_{s=r+1}^{|n|} M_rd_{n^{(s)}} \le$$

$$\le \frac{c}{M_{|n|}} M_rd_{n^{(r)}} + \frac{c}{M_{|n|}} M_rd_{n^{(r)}}(|n|-r).$$

Then

$$\int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|\, d\mu(y) < \frac{c}{M_{|n|}} M_rd_{n^{(r)}}(|n|-r+1).$$

The set G_m is a disjoint union of the sets $I_r(x)\setminus I_{r+1}(x)$, $r \in \mathbf{N}$, where x is a fix element of G_m. If $r > |n|$, the modulus $|K_n(x,y)|$ depends only on x whenever $y \in I_r(x)$. For this reason if $x \in G_m$, then we get that

$$|K_n(x,x)| \le \frac{1}{n} \sum_{l=0}^{n-1} \sum_{k=0}^{|n|} M_k c d_{n^{(k)}} < c M_{|n|},$$

and hence

$$\sum_{r=|n|+1}^{\infty} \int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|\, d\mu(y) =$$

$$= \int_{I_{|n|+1}(x)} |K_n(x,y)|\, d\mu(y) = \frac{|K_n(x,x)|}{M_{|n|}} < C.$$

Since

$$\frac{d_j^{(n_j)}}{m_j} < \frac{1}{\sqrt{2}},$$

it follows that

$$\int_{G_m} |K_n(x,y)|\, d\mu(y) = \sum_{r=0}^{\infty} \int_{I_r(x)\setminus I_{r+1}(x)} |K_n(x,y)|\, d\mu(y) <$$

$$< \sum_{r=0}^{|n|} \frac{C}{M_{|n|}} M_r d_{n^{(r)}} (|n| - r + 1) + C <$$

$$< C d_{|n|}^{(n_{|n|})} \sum_{r=0}^{|n|} \frac{d_r^{(n_r)}}{m_r} \cdots \frac{d_{|n|-1}^{(n_{|n|-1})}}{m_{|n|-1}} (|n| - r + 1) + C < C \sum_{k=0}^{\infty} \frac{k+1}{(\sqrt{2})^k} + C$$

for each $x \in G_m$, where $m_r \cdots m_{|n|-1} = 1$ and $r = |n|$. Since the above series is convergent, for each $x \in G_m$ there exists a positive constant C such that

$$\int_{G_m} |K_n(x,y)|\, d\mu(y) \le C.$$

This completes the proof of Lemma 2.

Remark. In an analogous way, we can prove that there is a constant C such that

$$\sup_{y \in G_m} \int_{G_m} |K_n(x,y)|\, d\mu(x) \le C.$$

From Lemma 2 we can get the following

Theorem 2. *If G_m is a bounded, compact, totally disconnected group and $f \in L^p(G_m)$, $1 \le p \le \infty$, then $\sigma_n f \to f$ in L^p-norm.*

Proof. It is sufficient to prove that the operators σ_n are uniformly of type (p,p) when $1 \leq p \leq \infty$, since the convergence $\sigma_n f \to f$ is valid for each $f \in \Im$ trigonometric polynomial and then we can apply the Banach–Steinhaus theorem. By the interpolation theorem of Marcinkiewicz [10], it is sufficient to prove that the operators σ_n are uniformly of type $(1,1)$ and (∞,∞). From Lemma 2, using the Fubini theorem, for $f \in L^1(G_m)$, we have

$$\|\sigma_n f\|_1 \leq \int_{G_m}\int_{G_m} |f(y)||K_n(x,y)|\, d\mu(y)d\mu(x) =$$
$$= \int_{G_m} |f(y)| \int_{G_m} |K_n(x,y)|\, d\mu(x)d\mu(y) \leq C\|f\|_1.$$

Thus, the operators σ_n are uniformly of type $(1,1)$. For $f \in L^\infty(G_m)$,

$$\|\sigma_n f\|_\infty \leq \|f\|_\infty \int_{G_m} |K_n(\cdot,y)|\, d\mu(y)\|_\infty \leq C\|f\|_\infty.$$

Thus, the operators σ_n are uniformly of type (∞,∞). This completes the proof of Theorem 2.

Finally, we remark the one of the authors proved the pointwise convergence $\sigma_n f \to f$ a.e. ($f \in L^1(G)$) (see [5]). If $m_k = 2$ for each $k \in \mathbf{N}$ (the Walsh case), this was proved by FINE [2], and for bounded (Abelian) Vilenkin groups this was proved by SIMON and PÁL [15]. The two-dimensional (Walsh) case $\sigma_{m,n} f \to f$ a.e. is discussed by MÓRICZ, SCHIPP and WADE [7] (as $\min(m,n) \to \infty$ and $|f| \in H^\#$ (which is a certain "hybrid" Hardy space)), and by GÁT [3] and by WEISZ [19] (as $m,n \to \infty$ in such a manner that the integral lattice points (m,n) remain in some positive cone, $f \in L^1$). The two-dimensional "non-Abelian" case is open as to both norm and pointwise convergence.

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Сходимость в L^p-норме рядов на компактных вполне несвязных группах

Г. ГАТ и Р. ТОЛЕДО

Хорошо известно, что частные суммы рядов Фурье–Виленкина для каждой функции $f \in L^p$, $1 < p < \infty$, сходятся к f по норме. Для любого $1 \leq p \leq \infty$ операторы S_{M_n} также сходятся по норме к f для каждой $f \in L^p$.

В настоящей работе мы изучаем подобные свойства на вполне несвязных группах, не обязательных абелевых, и для систем, состоящих из произведений нормированных координатных функций для непрерывных неприводимых унитарных представлений координатных групп. Наконец, мы установливаем сходимость для $1 \leq p \leq \infty$ средних Фейера в случае ограниченных групп.

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