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ON DISK-HOMOGENEOUS SYMMETRIC SPACES

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We prove a classification theorem for disk-homogeneous locally symmetric spaces.

1. INTRODUCTION

The major problem in the theory of harmonic spaces (see [1],[6]) is the classification problem. To the present time, the following particular results are known :

(A) Each harmonic space of dimension $n \leq 4$ is either locally Euclidean or locally isometric to a rank one symmetric space (A. Lichnerowicz and A.G. Walker).

(B) Each locally symmetric harmonic space is either locally Euclidean or locally isometric to a rank one symmetric space (A.J. Ledger).

In our previous joint article [5] we proved an analogous theorem to (A) for the ball- and disk-homogeneous Riemannian manifolds, or for the strongly disk-homogeneous Riemannian manifolds respectively.

In the present paper we consider the analogue of (B) for the disk-homogeneous locally symmetric spaces. In particular we prove

THEOREM 1. Let M be a connected locally symmetric space of dimension ≥ 3. Then M is disk-homogeneous up to order 6, i.e. the volume $V_m^x(r)$ of a disk satisfies

$$V_m^x(r) = V_0^{n-1}(r)\{1 + ar^2 + br^4 + cr^6 + O(r^8)\}$$

with constant a,b and c, if and only if one of the following cases occurs :

i) M is locally flat;

ii) M is locally isometric to a rank one symmetric space;

iii) M is locally isometric to the Lie group E_8, the symmetric space E_8/D_8, the symmetric space $E_{8(-24)}$, or to some of their noncompact duals.

Further, we state the following

CONJECTURE. If a connected locally symmetric space is disk-homogeneous up to order 8, then M is locally flat or locally isometric to a rank one symmetric space.

This would prove the full analogue of (B). But such a program would necessarily involve very long calculations.

Our theorem as well as the conjecture is based on a algebraic classification theorem by P. Carpenter, A. Gray and T.J. Willmore [2], in which the authors analyse in detail the meaning of the harmonicity conditions of small orders in the class of all symmetric spaces. This theorem is in fact a deep extension of Ledger's theorem (B). (See the very end of this paper.)

2. DISK-HOMOGENEOUS MANIFOLDS

First, let us recall some basic concepts and formulas from [5]. Let (M,g) be a connected analytic Riemannian manifold of dimension $n \geq 3$. A *geodesic disk* with center m and radius r, perpendicular to a unit tangent vector $x \in T_mM$, is defined as the set

$$D_m^x(r) = \{m' \in M | d(m',m) < r\} \cap \exp_m(\{x\}^\perp)$$

where $\exp_m : T_mM \to M$ is the exponential map at m. We always suppose $r < i(m)$ where i(m) denotes the injectivity radius at m. Further, $V_m^x(r)$ will denote the (n-1)-dimensional volume of $D_m^x(r)$.

In [5] we derived a power series expansion up to order 6 for $V_m^x(r)$ in the form

$$V_m^x(r) = V_0^{n-1}(r)\{1 + A(x)r^2 + B(x)r^4 + C(x)r^6 + O(r^8)\}(m)$$

where $V_0^{n-1}(r)$ denotes the volume of a Euclidean ball of dimension n-1 and with radius r. Here

(1) $$A(x) = -\frac{1}{6(n+1)}(\tau - 2\rho(x,x))$$

where ρ denotes the Ricci tensor and τ the scalar curvature. The expression for B(x) was also given in full, and the expression for C(x) was determined in the case of an Einstein manifold. For our purposes, we shall need only special cases of these complicated formulas, namely the formula for B(x) in an Einstein manifold and the formula for C(x) in a locally symmetric Einstein space.

From theorem 2.1 in [5] we see easily that for an Einstein manifold we have

(2) $$B(x) = \frac{1}{360(n+1)(n+3)}\{-3\|R\|^2 + 24\|R^x\|^2 + 20\|R^{xx}\|^2\} + \text{constant}$$

where

(3) $$\|R\|^2 = \sum_{i,j,k,\ell=1}^{n} R^2_{ijk\ell},$$

(4) $$\begin{cases} \|R^x\|^2 = \sum_{i,j,k=2}^{n} R^2_{1ijk}, \\ \|R^{xx}\|^2 = \sum_{i,j=2}^{n} R^2_{1i1j}. \end{cases}$$

The summation in (3) is considered with respect to any orthonormal basis, and the summations in (4) are considered with respect to any *adapted* orthonormal basis $\{e_1 = x, e_2, \ldots, e_n\}$.

The formula for C(x) will be given later.

The manifold (M,g) is said to be *disk-homogeneous* (or *strongly disk-homogeneous*, respectively) if the volume $V_m^x(r)$ depends only on the point m and radius r (or only on the radius r, respectively).

We see immediately that in a disk-homogeneous manifold the functions A(x), B(x) and C(x) are constant on each unit sphere $S_mM \subset T_mM$, $m \in M$, and in a strongly disk-homogeneous manifold the functions A(x), B(x), C(x) are constant on the whole unit sphere bundle $SM \subset TM$. For a homogeneous space (in particular, for a symmetric space) "disk-homogeneous" is equivalent to "strongly disk-homogeneous".

Finally, let us recall that in a homogeneous space all the (scalar) Riemannian invariants are constants.

3. FOURTH ORDER GEOMETRY

The following is immediate from formulas (1) and (2).

PROPOSITION 1. Let (M,g) be a Riemannian manifold of dimension ≥ 3. We have :

a) A(x) is constant on the unit sphere bundle SM if and only if (M,g) is Einsteinian.

b) If (M,g) is Einsteinian, then B(x) is constant on SM if and only if $24\|R^x\|^2 + 20\|R^{xx}\|^2 - 3\|R\|^2$ is constant on SM.

We shall now prove the following

LEMMA 1. Let (M,g) be an n-dimensional Einstein space and let a,b,c be real numbers such that $(n-2)a + 3b \neq 0$ and $2n(n+2)c + 3b + 2(n-1)a \neq 0$. If the relation

$$a\|R^x\|^2 + b\|R^{xx}\|^2 + c\|R\|^2 = \text{constant}$$

holds on the unit sphere bundle $SM \subset TM$, then the symmetric bilinear form

$$\dot{R}(x,y) = \sum_{i,j,k=1}^{n} R_{xijk}R_{yijk}$$

on $\otimes^2 TM$ satisfies

$$\dot{R} = \frac{1}{n} \|R\|^2 g.$$

Moreover, $\|R\|^2$ is constant and $\|R^x\|^2 + 2\|R^{xx}\|^2$ is constant.

REMARK. This generalizes proposition 6.57 from [1] which appears here as a special case for $a = 0$, $b = 1$, $c = 0$.

Proof. Suppose that

$$a\|R^x\|^2 + b\|R^{xx}\|^2 + c\|R\|^2 = d \ , \ d \text{ constant},$$

holds on SM. We can rewrite this as the identity

$$a \sum_{i,j,k=2}^{n} R^2_{xijk}\, g(x,x) + b \sum_{i,j=2}^{n} R^2_{xixj} = \gamma g(x,x)^2,$$

valid for all $x \in T_m M$, $m \in M$, where $\gamma = d - c\|R\|^2$. Using once more an adapted orthonormal frame $\{e_1 = \frac{x}{\|x\|}, e_2, \ldots, e_n\}$, we obtain a tensorial identity :

$$a \sum_{i,j,k=1}^{n} R^2_{xijk}\, g(x,x) + (b-2a) \sum_{i,j=1}^{n} R^2_{xixj} = \gamma g(x,x)^2.$$

Now we use the linearization procedure. We get

$$a \sum_{i,j,k=1}^{n} R^2_{xijk}\, g(y,y) + 4a \sum_{i,j,k=1}^{n} R_{xijk}R_{yijk}\, g(x,y) + a \sum_{i,j,k=1}^{n} R^2_{yijk}\, g(x,x)$$

$$+ (b-2a) \left\{ 2 \sum_{i,j=1}^{n} R_{xixj}R_{yiyj} + 2 \sum_{i,j=1}^{n} R^2_{xiyj} + 2 \sum_{i,j=1}^{n} R_{xiyj}R_{yixj} \right\} =$$

$$= 2\gamma g(x,x)g(y,y) + 4\gamma g(x,y)^2.$$

Next we put $y = e_i$, $i=1,\dots,n$, and sum up. Hence

$$na \sum_{i,j,k=1}^{n} R_{xijk}^2 + 4a \sum_{i,j,k=1}^{n} R_{xijk}^2 + a\|R\|^2 g(x,x) + (b-2a)\left\{2\frac{\tau^2}{n^2} g(x,x)\right.$$

$$\left. + 2\sum_{i,j,k=1}^{n} R_{xijk}^2 + \sum_{i,j,k=1}^{n} R_{xijk}^2\right\} = 2(n+2)\gamma\, g(x,x)$$

and

$$\left\{(n-2)a + 3b\right\} \sum_{i,j,k=1}^{n} R_{xijk}^2 = \left\{2(n+2)\gamma - a\|R\|^2 - 2(b-2a)\frac{\tau^2}{n^2}\right\} g(x,x).$$

This can be rewritten in the form

$$\{(n-2)a + 3b\}\dot{R}(x,x) = \{\beta - [a + 2(n+2)c]\|R\|^2\}g(x,x)$$

where β is a constant. Hence $\dot{R}(x,x) = \lambda(m)g(x,x)$ at each point $m \in M$, and by a new linearization we get

$$\dot{R}(x,y) = \lambda(m)g(x,y).$$

On the other hand, putting above $x = e_\ell$, $\ell = 1,\dots,n$, and summing up we get

$$\{(n-2)a + 3b\}\|R\|^2 = n\{\beta - [a + 2(n+2)c]\|R\|^2\}$$

and

$$\|R\|^2 = n\lambda(m).$$

Hence

$$\{2n(n+2)c + 3b + 2(n-1)a\}\|R\|^2 = n\beta,\ \lambda(m) = \frac{1}{n}\|R\|^2.$$

Thus $\|R\|^2$ is constant and $\dot{R} = \frac{1}{n}\|R\|^2 g$.

Finally, we see easily that

$$\dot{R}(x,x) = |R^x|^2 + 2|R^{xx}|^2$$

on SM, and hence $|R^x|^2 + 2|R^{xx}|^2 = \frac{1}{n}|R|^2 = \text{constant}$.

PROPOSITION 2. Let (M,g) be an Einstein space and $n > 3$. Then B(x) is constant on SM if and only if $|R^{xx}|^2$ (or $|R^x|^2$, respectively) is constant on SM.

Proof. (a) Let B(x) be a constant on SM. Putting a = 24, b = 20 and c = -3 in lemma 1. we obtain that $|R|^2$ and $|R^x|^2 + 2|R^{xx}|^2$ are constant functions. Because $24|R^x|^2 + 20|R^{xx}|^2 - 3|R|^2$ is also constant, we have proved that $|R^x|^2$ and $|R^{xx}|^2$ are constants.

(b) Suppose $|R^{xx}|^2 = \text{constant}$, or $|R^x|^2 = \text{constant}$, respectively. Using lemma 1 for the case a = 0, b = 1, c = 0, or for the case a = 1, b = 0, c = 0 respectively, we always deduce that the second quantity is constant, and also $|R|^2$ is constant. Hence B(x) is constant according to proposition 1. (In the second case we used the inequality $n \neq 2$.)

PROPOSITION 3. Let (M,g) be an Einstein manifold with $n > 3$ and such that $|R^{xx}|^2 = \text{constant}$, or B(x) = constant on SM. Then (M,g) is either irreducible or locally Euclidean.

Proof. For the case $|R^{xx}|^2 = \text{constant}$, an easy proof is given in [3,theorem 6.22] for a *general* Riemannian manifold. In the Einstein case, B(x) = constant yields $|R^{xx}|^2 = \text{constant}$, and hence the result follows.

In accordance with [2], an analytic Riemannian manifold (M,g) will be called a *2-stein space* if (M,g) is Einsteinian and $|R^{xx}|^2 = \text{constant}$ on SM. We then have the following

THEOREM 2. An analytic Riemannian manifold (M,g) of dimension $n > 3$ is strongly disk-homogeneous up to order 4 if and only if it is a 2-stein space. If this is the case, then (M,g) is irreducible or locally Euclidean.

4. SIXTH ORDER GEOMETRY

We continue with the "sixth order geometry" of the geodesic disks.

PROPOSITION 4. Let (M,g) be a locally symmetric Einstein space of dimension $n > 3$. Then

$$C(x) = \sum_{i=1}^{7} a_i C_i(x) + \mu \| R^x \|^2 + \nu \| R^{xx} \|^2 + \varepsilon,$$

where $a_1, \ldots, a_7, \mu, \nu, \varepsilon$ are absolute constants such that $(a_1 + 2a_3 + 4a_4 - 2a_5) \neq 0$, $15(a_1 + 2a_3 + 4a_4 - 2a_5) - (8 + n)(a_2 + 4a_3 + 8a_4 - 2a_5 - a_6) \neq 0$, and the quantities $C_1(x), \ldots, C_7(x)$ are given with respect to any *adapted* orthonormal basis $\{e_1 = x, e_2, \ldots, e_n\}$ by the following formulas (with the summations ranging over the index set $\{2,3,\ldots,n\}$) :

$$C_1(x) = \sum R_{1i1j} R_{1j1k} R_{1k1i},$$

$$C_2(x) = \sum R_{1i1j} R_{1k1\ell} R_{ikj\ell},$$

$$C_3(x) = \sum R_{1i1j} R_{ipqr} R_{jpqr},$$

$$C_4(x) = \sum R_{1jk\ell} R_{k\ell pq} R_{1jpq},$$

$$C_5(x) = \sum R_{1j1\ell} R_{1\ell pq} R_{pq1j},$$

$$C_6(x) = \sum R_{1kj\ell} R_{kp\ell q} R_{p1qj},$$

$$C_7(x) = \sum R_{1kj1} R_{kp1q} R_{p1qj}.$$

Proof. This is a special case of theorem 2.2 in [5]. (In the original theorem, the coefficients $a_1, a_2, \ldots, a_6$ have not been specified numerically, but the necessary calculation is only a tedious routine.)

PROPOSITION 5. Let $\bar{C}_i(x)$, $i=1,\ldots,7$ denote the functions defined by the same formulas as $C_i(x)$ but with the summations extended over the set $\{1,2,\ldots,n\}$. Then

$$\bar{C}_1(x) = C_1(x),$$

$$\bar{C}_2(x) = C_2(x),$$

$$\bar{C}_3(x) = C_3(x) + 2C_1(x) + 2C_5(x) - 2C_7(x),$$

$$\bar{C}_4(x) = C_4(x) + 4C_1(x) + 4C_5(x),$$

$$\bar{C}_5(x) = C_5(x) + 2C_1(x),$$

$$\bar{C}_6(x) = C_6(x) + C_2(x) + 2C_7(x),$$

$$\bar{C}_7(x) = C_7(x).$$

Proof. The relations are obtained by direct calculation, using the first Bianchi identity in the case of $\bar{C}_3(x)$.

PROPOSITION 6. Let M be an irreducible locally symmetric manifold. Then the following identities hold :

$$\bar{C}_3(x) = \text{constant},$$

$$\bar{C}_4(x) = \text{constant},$$

$$\bar{C}_6(x) = \text{constant},$$

$$\bar{C}_7(x) = 0,$$

$$\bar{C}_2(x) + \bar{C}_7(x) + \frac{1}{2}\bar{C}_5(x) - \frac{\tau}{n}\|R^{xx}\|^2 = 0.$$

Proof. Because M is irreducible, any covariantly constant and symmetric bilinear form $B(x,y)$ on TM is a multiple of the metric tensor g at each point, and hence a constant multiple of g on M. In particular, we see that the *quadratic* forms $\bar{C}_3(x)$, $\bar{C}_4(x)$, $\bar{C}_6(x)$ are constant functions on SM.

Because (M,g) is locally symmetric, we have the following general identity as a consequence of the Ricci identity :

$$(5) \quad \sum_{c=1}^{n} \{R_{abrc}R_{csuv} + R_{absc}R_{rcuv} + R_{abuc}R_{rscv} + R_{abvc}R_{rsuc}\} = 0,$$

$a,b,r,s,u,v = 1,2,\dots,n$. Making the substitutions $a \to j$, $b \to 1$, $c \to k$, $r \to 1$, $s \to p$, $u \to 1$, $v \to q$ and contracting (to the right) with the tensor R_{1pjq} from 1 to n, we obtain (using also the first Bianchi identity) the relation $\bar{C}_7(x) = 0$. If we make in (5) the substitutions $a \to 1$, $b \to \ell$, $c \to k$, $r \to 1$, $s \to i$, $u \to \ell$, $v \to j$ and contract the new expression (to the left) with the term R_{1i1j}, we obtain the last formula.

PROPOSITION 7. If M is a locally symmetric 2-stein space with $n > 3$, then the following holds on SM :

$$C_3(x) = 2C_1(x) + 4C_2(x) + \text{constant},$$

$$C_4(x) = 4C_1(x) + 8C_2(x) + \text{constant},$$

$$C_5(x) = -2C_1(x) - 2C_2(x) + \text{constant},$$

$$C_6(x) = -C_2(x) + \text{constant},$$

$$C_7(x) = 0.$$

Proof. M is irreducible or locally flat according to theorem 2, and $\|R^{xx}\|^2 =$ constant on SM. The assertion now follows from propositions 5 and 6.

We shall now prove a sixth order analogue of lemma 1.

LEMMA 2. Let M be an n-dimensional locally symmetric 2-stein space ($n > 3$), and let α,β be real numbers such that $15\alpha - (8+n)\beta \neq 0$. If $\alpha C_1(x) + \beta C_2(x)$ is constant on SM, then $C_2(x)$ is also constant on SM.

Proof. Suppose M to be irreducible (the flat case is trivial). For any vectors $u,v,x,y,w,z \in T_mM$, $m \in M$, put

$$P_{uvxywz} = \sum_1^n R_{uavb}R_{xbyc}R_{wcza},$$

$$T_{uvxy} = \sum_1^n R_{ijk\ell}R_{iukv}R_{jx\ell y},$$

where the summation is taken with respect to any orthonormal basis. Then, for any $x \in SM$,

$$C_1(x) = \bar{C}_1(x) = P_{xxxxxx},$$

$$C_2(x) = \bar{C}_2(x) = T_{xxxx}.$$

Further, put

$$S_{xxxx} = \sum_1^n R_{xikj}R_{xi\ell j}R_{xkx\ell},$$

$$U_{xxxx} = -\sum_1^n R_{xijk}R_{xji\ell}R_{xkx\ell}.$$

Then we have on SM (taking into account the Bianchi identity) :

$$(6)\qquad \begin{cases} \bar{C}_7(x) = U_{xxxx}, \\ \bar{C}_5(x) = 2(S_{xxxx} + U_{xxxx}). \end{cases}$$

By proposition 6 we obtain

$$(7)\qquad U_{xxxx} = 0,\quad T_{xxxx} + S_{xxxx} = \frac{\tau}{n}\|R^{xx}\|^2.$$

Now, we can write our assumption in the form

$$\alpha P_{xxxxxx} + \beta T_{xxxx} = \gamma \quad \text{(on SM)},$$

or, equivalently, in the form

$$\alpha P_{xxxxxx} + \beta T_{xxxx} g(x,x) = \gamma g(x,x)^3 \quad \text{(on TM)}.$$

Next, we apply the linearization procedure and contraction as in the proof of lemma 1. We obtain

$$\alpha \sum_{i=1}^{n} (P_{iixxxx} + P_{ixixxx} + \dots + P_{xxxxii}) + \beta(8 + n)T_{xxxx}$$
$$+ \beta \sum_{i=1}^{n} (T_{iixx} + T_{ixix} + \dots + T_{xxii})g(x,x) = \gamma(12 + 3n)g(x,x)^2.$$

Here $T(x,x) = \sum (T_{iixx} + \dots + T_{xxii})$ is a covariantly constant quadratic form on TM, and since M is irreducible, $T(x,x)$ is a constant multiple of $g(x,x)$. By a lengthy but routine calculation we get that

$$\alpha \left\{ \frac{3\tau}{n} \| R^{xx} \|^2 + 9S_{xxxx} + 6U_{xxxx} + 3\bar{C}_5(x) \right\} + \beta(8 + n)T_{xxxx} = \text{constant}$$

on SM. M is 2-stein and thus $\| R^{xx} \|^2$ is constant. From (6), (7) it follows that $[-15\alpha + (8+n)\beta]T_{xxxx}$ is constant on SM, which completes the proof.

<u>PROPOSITION 8</u>. Let (M,g) be an irreducible locally symmetric space with $n > 3$ and satisfying the condition $B(x) = \text{constant}$. Then $C(x) = \text{constant}$ if and only if $C_1(x) = \text{constant}$.

<u>Proof</u>. According to propositions 2,4 and 7, $C(x)$ is constant on SM if and only if $\alpha C_1(x) + \beta C_2(x)$ is constant on SM, where

$$\alpha = a_1 + 2a_3 + 4a_4 - 2a_5,$$

$$\beta = a_2 + 4a_3 + 8a_4 - 2a_5 - a_6.$$

Also, we have $\alpha \neq 0$, $15\alpha - (8+n)\beta \neq 0$. Due to lemma 2, this implies $C_2(x) = \text{constant}$, and hence $C_1(x) = \text{constant}$.

Conversely, using lemma 2 once more, we see that $C_1(x)$ = constant implies $C_2(x)$ constant, and hence $C(x)$ = constant.

5. PROOF OF THEOREM 1

Let us denote by R_x the endomorphism $u \mapsto R(x,u)x$, $x \in S_mM$, $u \in T_mM$. In this denotation we have

$$\operatorname{trace} R_x = \rho(x,x), \quad \operatorname{trace} R_x^2 = \|R^{xx}\|^2, \quad x \in SM.$$

Following [2] again, a locally symmetric space is called a k-*stein space* if $\operatorname{trace} R_x^{\ell}$ = constant on SM for $\ell = 1,\ldots,k$. (Let us recall that the condition $\operatorname{trace} R_x^{\ell}$ = constant is nothing but Ledger's harmonicity condition of order 2ℓ for a locally symmetric space [6].) In particular we have $\operatorname{trace} R_x^3 = C_1(x)$.

We have the following

PROPOSITION 9. For a locally symmetric space (M,g) of dimension ≥ 3, the following two conditions are equivalent :

i) All functions $A(x)$, $B(x)$, $C(x)$ are constant on SM ;

ii) (M,g) is a 3-stein space.

Proof. The result follows immediately from theorem 2 and proposition 8.

Now, our theorem 1 as well as the justification of our conjecture will follow from the following result by Carpenter, Gray and Willmore, which is a part of theorem 1.1 in [2] : Let M be a nonflat locally symmetric space. If M is 4-stein, then it is locally isometric to a rank one symmetric space. Further, M is 3-stein but not 4-stein if and only if it is locally isometric to one of the following symmetric spaces : the Lie group E_8, the symmetric space E_8/D_8, the symmetric space $E_{8(-24)}$, or the noncompact dual of one of these spaces.

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