Combining Photoelasticity and Finite-element Methods for Stress Analysis Using Least Squares

by Donald G. Berghaus

ABSTRACT—Photoelastic data are combined with the finite-element method for stress solutions over regions partially bounded by free surfaces and axes of symmetry. Least-squares solutions are obtained without presumed values of applied forces at element nodes and without isoclinic data. Varied example problems are used to compare the results to independent photoelastic and finite-element solutions and to theoretical stress values.

Introduction

Numerical methods can provide valuable assistance for obtaining photoelastic solutions. For plane problems, to determine the state of stress at a point, three independent data values are needed; with two values available in the photoelastic data. One general method for solving for the stress values consists of inclusion of stress-equilibrium and static-equilibrium equations.[1] When placed in finite-difference and summation form, these equations contribute to an overdetermined problem which may be solved, using least-squares, for stresses over the region.

Finite-element methods have also been used to assist in solving the problem. Lukas[2] has used a finite-element method to solve the strain-compatibility equation (in terms of stress) for the sum of the principal stresses over the region of interest, providing a third data value at each element. Chambless, Swinson, Suhling and Turner[3] have used a weighting procedure to include selected photoelastic data as constraints in the finite-element solution.

The approach used in this paper provides simplifications and advantages when compared to either of the methods used separately. A solution region will be bounded on most sides by free surfaces and lines of symmetry. Photoelastic data from these boundaries will be used to obtain additional displacement equations which eliminate the need to specify applied node forces. Static-force equilibrium is also used. Isoclinic data are not needed for the solution.

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The method is applied to three problems to compare with: (1) theoretical stress values, (2) finite-element results, and (3) least-squares photoelastic results.

Finite-element Relations

The relationships between load, displacement, stress and strain in the finite-element construction are briefly given to aid in understanding the inclusion of the photoelastic information.

The region for stress analysis is subdivided into triangular elements[4] as seen in Fig. 1. (Subdivided regions are shown in each of the example problems.) Each element is presumed to experience a state of stress and strain which is constant over the element. The elements are interconnected at their nodes, which transmit forces in two orthogonal directions; the same directions as the operational node displacements. The node displacements are related to the element strains using the simple strain-displacement relations together with an assumed element displacement field which assures the constant state of strain. For the displacement field for the triangular element:

$$\begin{aligned} u(x,y) &= a_0 + a_1 x + a_2 y \\ v(x,y) &= b_0 + b_1 x + b_2 y \end{aligned} \tag{1}$$

The strain-displacement equations give

$$\epsilon_x = \frac{\partial u}{\partial x} = a_1 \qquad \epsilon_y = \frac{\partial v}{\partial y} = b_2$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + b_1 \tag{2}$$

For an element of given geometry, the coefficients a_j, b_j may be solved for in terms of the unknown displacements using eqs (1) at each of the three nodes. These expressions for a_j, b_j are then used in eqs (2) to express the state of strain in terms of the node displacements u_i, v_i. Or, succinctly stated,

$$\{\epsilon\} = [B]\begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{or} \qquad \{\epsilon\}^T = \begin{Bmatrix} u \\ v \end{Bmatrix} [B]^T \tag{3}$$

Linear-elastic material properties are used to obtain the element state of stress:

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \qquad (4)$$
$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

or $\{\sigma\} = [D]\{\epsilon\}$. Thus for each element, the state of stress is given in terms of the node displacements which are yet unknown:

$$\{\sigma\} = [D]\,[B]\begin{Bmatrix} u \\ v \end{Bmatrix} \qquad (5)$$

Solution for the displacements proceeds from a consideration of the virtual work produced by the forces at the element nodes and the strain-energy experienced by the element. For the strain energy: $\delta U = \int_v \epsilon^T \sigma dV$ where V is the volume of the element: $V = At$ with the plane area and the thickness equal to A and t. For the virtual work,

$$\delta W = \sum_{i=1}^{3}(F_{xi}u_i + F_{yi}v_i) = \begin{Bmatrix} u \\ v \end{Bmatrix}^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

Equating the strain energy to the virtual work and remembering the constant state of strain,

$$\begin{Bmatrix} u \\ v \end{Bmatrix}^T [B]^T[D]\,[B]\begin{Bmatrix} u \\ v \end{Bmatrix}(V) = \begin{Bmatrix} u \\ v \end{Bmatrix}^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \qquad (6)$$

which may be reduced to

$$[K]\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \qquad (7)$$

where K is the element stiffness matrix: $K = [B]^T[D][B](V)$. Using eqs (7) it is possible to specify a consistent combination of six-node displacements and/or forces, and solve for the remaining forces and displacements.

For the stress analysis over the region, force equilibrium is used at all nodes. At each node, two orthogonal equilibrium equations are written. They include force contributions for each element connected at the node, but they are written in terms of the node displacements using eq (7). The loading system of applied forces is placed at certain of the nodes. Displacement constraints are placed at certain of the nodes; usually boundary nodes. Thus a system of equations is obtained over the entire region in terms of the unknown node displacements. This linear system of equations is solved for the displacements. For each element the node displacements are then used with eqs (5) to obtain the state of stress.

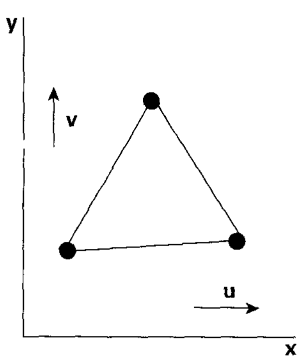


Fig. 1—Triangular element

Photoelastic Information for Finite-element Analysis

At a point in a plane model, photoelastic data are used in two commonly recognized stress-optical equations,

$$\tau_{xy} = \frac{1}{2}\left(\frac{Nf}{t}\right)\sin 2\theta$$
$$\sigma_x - \sigma_y = \frac{Nf}{t}\cos 2\theta \qquad (8)$$

where f is the stress-optical material constant, θ is the isoclinic angle (or principal-stress direction) and N is the compensated isochromatic fringe order. The information given in these two linear equations can be used to supplement the finite-element equilibrium equations. When placed into displacement form, eqs (8) can be added to the equations of force equilibrium at the nodes to obtain an overdetermined system of equations. A least-squares solution is then used.

Photoelastic data are straightforward to obtain along solution region boundaries which are free surfaces or lines of symmetry. For such locations, the principal directions are known and it is not necessary to determine isoclinic angles. There is an attendant simplicity in obtaining compensated fringe orders at these locations. The overdetermined system is produced when data from these boundary locations are placed into the appropriate displacement equations.

Due to the excess amount of information in hand, it is not necessary to use applied node forces. Presumed values of these loads are ordinarily placed in the selected node force-equilibrium equations, but with the present method equilibrium equations may simply be omitted at these nodes. It is possible to introduce the overall loadings using static-equilibrium equations.

The displacement photoelastic equations are developed next, followed by a static-equilibrium equation.

Displacement Equations for Photoelastic Data

Free Boundary of Region

Along a free boundary the stress component normal to the boundary and the surface shear stress are both zero. The tangential stress may be written in terms of the isochromatic fringe order,

$$\sigma_T = \frac{Nf}{t}$$

For the plane model the tangential boundary strain (Fig. 2) may be written

$$\epsilon_T = \frac{\Delta s}{s} = \frac{\sigma_T}{E} = \frac{Nf}{Et} \qquad (9)$$

Kinematic considerations for small changes of a short-line length, s, with end points (x_1, y_1) and (x_2, y_2) give

$$\Delta s = (\Delta u)\cos\phi + (\Delta v)\sin\phi \quad (10)$$

where $\Delta u = u_2 - u_1$ and $\Delta v = v_2 - v_1$, $\cos\phi = (x_2 - x_1)/s$, $\sin\phi = (y_2 - y_1)/s$ and $s = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}$. Subscripts (1) and (2) refer to the nodes in Fig. 2 only. It is assumed that the rotation of s is negligible. Equations (9) and (10) may be combined to give

$$(u_{2i} - u_{1i})\cos\phi + (v_{2i} - n_{1i})\sin\phi = \frac{s_i}{E}\left(\frac{N_i f}{t}\right) \quad (11)$$

Line length, s_i, is one side of free boundary element i. In eq (11), the displacements are unknown. All other parameters are given experimentally or in the element geometry.

Line of Symmetry

Along a horizontal line of symmetry, the principal stresses are the coordinate normal stresses, σ_x and σ_y. Equations (8) reduce simply to

$$\sigma_x - \sigma_y = \frac{Nf}{t}$$

For the strains, from eq (4),

$$\epsilon_x - \epsilon_y = \frac{1+\nu}{E}(\sigma_x - \sigma_y) = \left(\frac{1+\nu}{E}\right)\left(\frac{Nf}{t}\right) \quad (12)$$

An isosceles triangle is taken for the element, i, with base, ℓ, along the axis and height, h, to the vertex (Fig. 3). For the horizontal direction,

$$\epsilon_x = (\Delta\ell)/\ell = (u_{2i} - u_{1i})/(x_{2i} - x_{1i})$$

For the vertical direction, considering an identical symmetric element below the element shown,

$$\epsilon_y = [v_{3i} - (-v_{3i})]/2h_i = v_{3i}/h_i$$

Subscripts 1, 2 and 3 refer to the nodes given in Fig. 3. Equation (12) may be rewritten

$$u_{2i} - u_{1i} = \frac{\ell_i}{h_i}v_{3i} = \frac{\ell_i(1+\nu)}{E}\left(\frac{N_i f}{t}\right) \quad (13)$$

A similar equation may be written for an element along a vertical line of symmetry.

Photoelastic equations (11) or (13) are written for every appropriate element.

Static-equilibrium Equation

Static-equilibrium equations relate the total applied loads in the horizontal and/or vertical directions to the loads produced by stresses along appropriate solution lines. These may be the lines of symmetry. For equilibrium considerations the free-body diagram consists of the solution region with two of the region boundaries being the lines of symmetry. Static-equilibrium equations are useful in least-squares photoelastic solutions, and one of them can often be used in the problem at hand. Because the applied loads at the nodes are not present, these equations are not redundant.

For force equilibrium in the vertical direction, the static-equilibrium equation may be written

$$\Sigma F_y = \sum_{i=1}^{n_l} \sigma_{yi}\,\ell_i\,t$$

where n_l is the number of elements along the line of symmetry. The side of the triangle along the line of symmetry is ℓ_i. The external vertical loads total ΣF_y. Using eqs (4) to obtain a displacement equation, as previously done for the photoelastic equations,

$$\left(\frac{1-\nu^2}{Et}\right)\Sigma F_y = \sum_{i=1}^{n_l}\left[\frac{v_{3i}}{h_i} + \nu\frac{(u_{2i} - u_{1i})}{\ell_i}\right] \quad (14)$$

In this equation the subscripts 1, 2 and 3 refer to the nodes in Fig. 3 for element, i, along the line of symmetry.

Solution Procedure

A solution region for the model is chosen which is bounded mostly by lines of symmetry and free boundaries. For the example problems to be described, the regions are essentially four-sided. Two edges of the region are lines of symmetry, one each vertical and horizontal. Various free surfaces are used, one of these being a third edge of the region. The fourth edge of the region closes it within the model and is opposite the line of symmetry which may be called the 'baseline'. The region is subdivided into triangular elements with elements of smaller size clustered at stress concentrations in the usual finite-element manner. Isosceles triangles are used along the lines of symmetry.

Photoelastic data-displacement equations [eqs (11) or (13)] are constructed using nodes of elements along the free boundaries and lines of symmetry. A static-equilibrium equation is constructed along the baseline. Force-displacement equilibrium equations are constructed at all nodes, except those along the region boundary which is opposite the baseline. Neither photoelastic nor equilibrium equations are written along this boundary. The system of

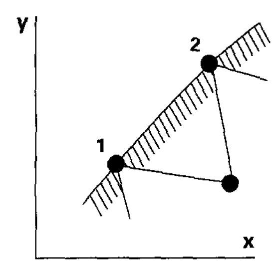


Fig. 2—Boundary element

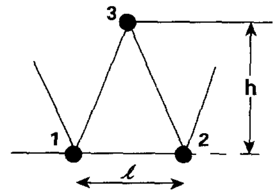


Fig. 3—Element on line of symmetry

equations produces an overdetermined solution for all node displacements.

Precautions

There are two important considerations for including the photoelastic and static-equilibrium equations with the finite-element equations. These are concerned with ordering of the principal stresses along the free boundary and line of symmetry and with weighting of the equations.

Equation 11 presumes a tensile tangential stress and eq (13) presumes that $\sigma_x > \sigma_y$. If these situations are not present the signs of the right sides of these equations should be changed.

The signs of the stresses, in either case, may be determined readily using a supplementary retarder.[5] In many problems the signs are intuitively apparent, especially on free boundaries.

Least-squares solutions require that attention be given to equation weights to assure the intended influence for a given equation in the solution and to avoid inadvertent weighting.[6] Increased weighting is produced by multiplying both sides of an equation by a coefficient greater than unity, and weighting is reduced by using a low weighting coefficient. A high magnitude weighting coefficient produces a relatively close fit of the solution to the equation where this coefficient is used, at the expense of a poorer solution fit to equations with low weights. In the limit, a weight of zero removes an equation. For relatively equal influence, the relative magnitudes of coefficients should be adjusted to be about the same in the various equations.

As written in this paper, the photoelastic equations have coefficients of order unity. In comparison, the static equilibrium-equation coefficients are divided by element length, and the finite-element equations have coefficients of order E. The coefficients in the static-equilibrium equation should be multiplied by an average element length and those in the finite-element equations should be divided by E to prevent inadvertent weighting.

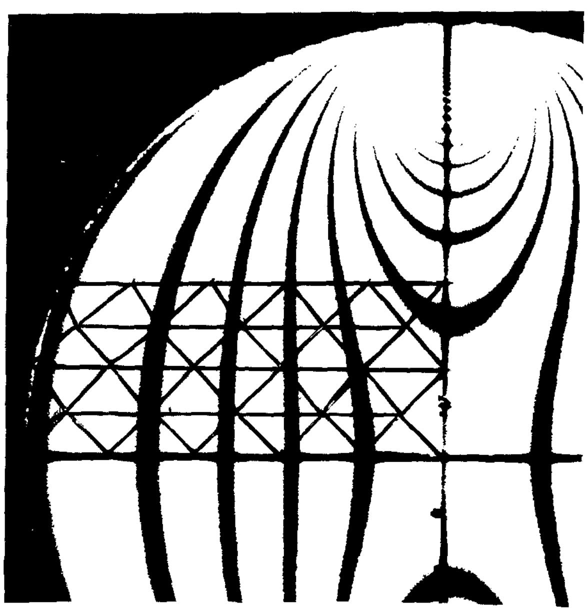


Fig. 4—Isochromatics and finite-element grid for quadrant of disk in diametral compression. Disk has diameter of 5.08 cm

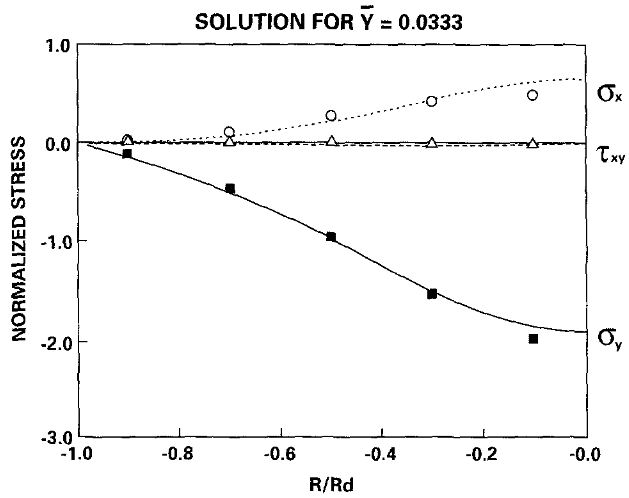
The subject of equation weights is discussed more completely in Ref. 6, which gives a procedure for determining them and shows the effects of adjusting weights in photoelastic solutions.

Application

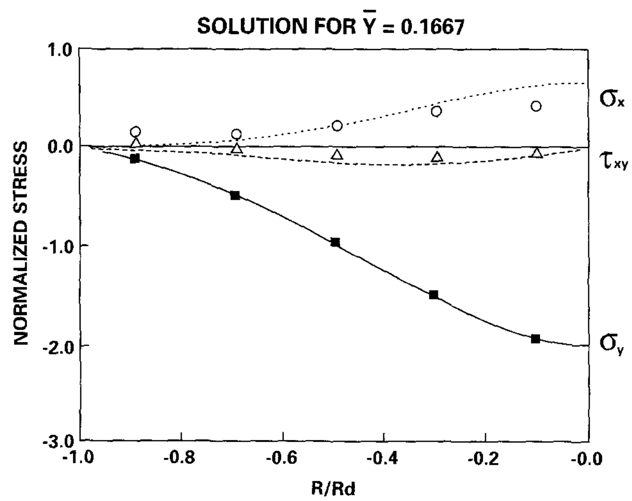
The combined method is demonstrated in application to three problems. The problems enable comparison to closed form theoretical stress values, to straightforward photoelastic stress results and to finite-element results.

The problems contain relatively small numbers of elements. They were solved using a personal computer and are intended to demonstrate the method and to compare it with other methods of stress analysis. The method could also be used in larger problems containing more and smaller elements, with improved accuracy.

The first problem consists of the disk in diametral compression. The photoelastic dark-field fringe pattern

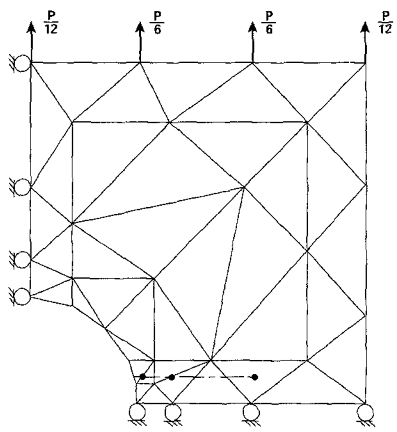


(a)

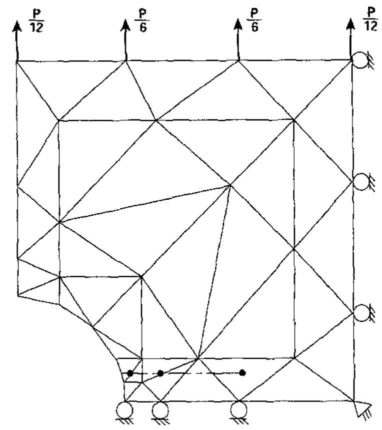


(b)

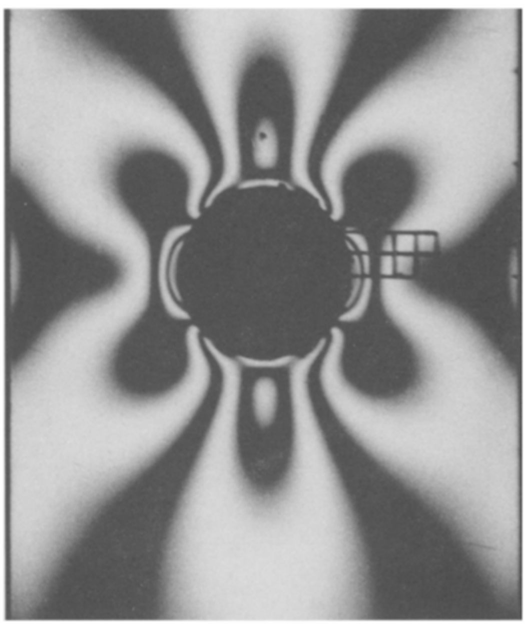
Fig. 5—Comparison of combined method results with theory for disk in diametral compression; $\bar{Y} = y/Rd$, normalized stress $= \sigma/(2t\,Rd)$. Rd is the disk radius. The thickness is t. R denotes the horizontal location



6(a)



7(a)



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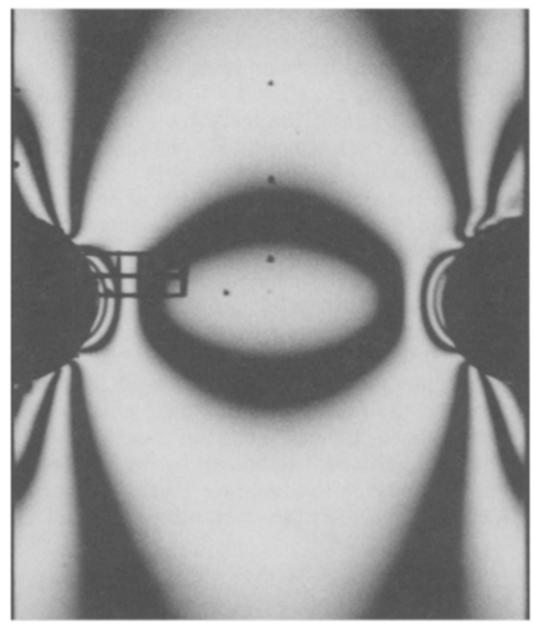

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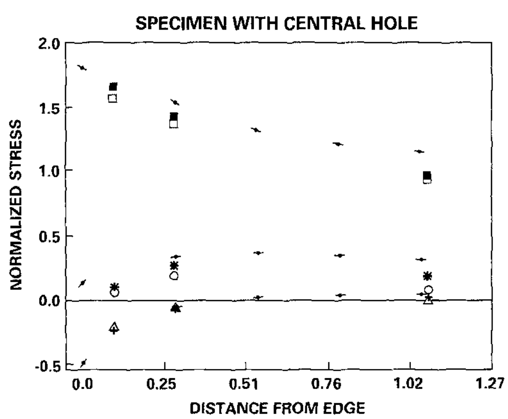
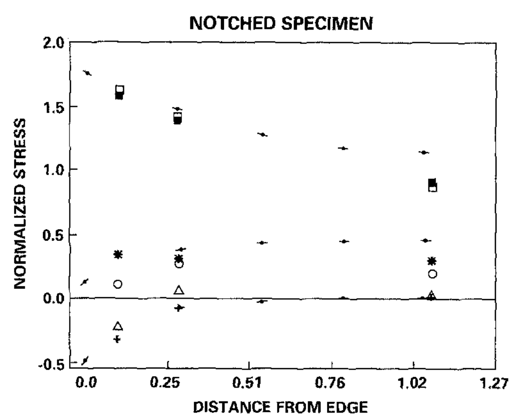


6(c)



7(c)

Figs. 6 and 7—Finite-element grids (a) and isochromatic fringe patterns (b) for tensile specimen with central hole (Fig. 6) and notched specimen (Fig. 7), each under the influence of total load, *P*. Stress results (c) include least-squares photoelastic results (marked tangentially), combined method (marked as in Fig. 5) and finite-element results. Results are given for $y = 0.25$ cm. Locations of combined method and finite-element results are shown in (a). Overall width of each specimen is 6 cm. Along line of symmetry width, $w = 4.1$ cm. Distances are given in centimeters. Normalized stress $= \sigma/(P/wt)$

is shown with the subdivided finite-element region superposed on it in Fig. 4 for the region of the disk being studied.

The discretization produces 40 elements from 28 nodes. There are no steep stress gradients in this problem, and it is possible to subdivide the region into elements of approximately the same size. For convenient theoretical comparison they are mostly aligned in eight rows parallel to the horizontal diameter. Nodes at the bottom of the region or baseline, along the horizontal diameter, are constrained in the vertical direction. Nodes on the vertical diameter (right edge of region) are constrained horizontally. Photoelastic data are taken for the seven elements on these diameters and for the two elements along the free boundary (left side of the region).

Results are compared with theoretical stress values[7] in Fig. 5 for the first and fourth rows. For the most part the results are good in this region, especially for the largest of the three stress components, σ_y. At the sixth row (not shown), values of σ_x and τ_{xy} deteriorate.

Two additional problems are used to compare the combined method with finite-element results alone. Both are stress-concentration problems. They consist of tensile bars, one containing a central hole (Fig. 6) and the other containing two notches at the edges (Fig. 7). The subdivided finite-element regions are shown in Figs. 6(a) and 7(a). These figures also show the applied loads for the finite-element method. For each problem there are 39 elements from 29 nodes.

Figures 6(b) and 7(b) show the dark-field isochromatic patterns with a small solution station grid used for least-squares photoelastic results. The results shown in Figs. 6(c) and 7(c) compare the combined and finite-element methods with the least-squares photoelasticity values.

The values are compared at three locations, which are shown on Figs. 6(a) and 7(a). Two locations (to the left) are each midway between two adjacent element centroids. Average stress values for the two adjacent elements are used for these locations. The third location is at an element centroid. The locations were chosen to be aligned with the central grid line of the photoelastic solution. The comparison is favorable for the largest stress values, and in the subregion (to the left) containing small elements. Values become less accurate with large elements, as expected.

Both problems show the essential similarity of the combined method and the traditional finite-element method. The advantage of the combined method lies in the ability to concentrate on small solution regions where assignment of loads at nodes may be difficult. Such a region is used in the disk problem.

Discussion of Method

The purpose of this paper is to show how photoelastic data may be combined with the finite-element method for stress analysis. The procedure has been applied using a personal computer. The program was adapted from that given by Brown.[8] The least-squares reduction is described in Ref. 1.

Variations in the method are possible which will be briefly discussed. Each of them necessarily increases the effective size of the problem.

Improved strain (and stress) gradients may be obtained using a linear-strain triangle for the element.[9] Such an element uses quadratic displacement functions for u and v instead of the linear functions of eqs (1). These functions each contain six unknown coefficients which are solved for in terms of displacements at six nodes. These are located at the midpoints of triangle sides and at the vertices. Isosceles triangles used along the lines of symmetry will produce two photoelastic data locations. The photoelastic equation [eq (13)] for these locations presumes statements for strain which are identical to those produced by the quadratic functions, indicating that the linear-strain triangle can be employed directly. This is not possible for elements which use cubic or higher order functions as there are differences in the strain expressions between those used in the photoelastic equation and those produced by these functions.

The second variation is to include photoelastic data at additional locations. This would improve the results, especially away from the lines of symmetry. To do this it is necessary to obtain photoelastic displacement equations written about element centroids using the general state of plane stress. Photoelastic data, including principal directions (isoclinic data), are then taken at each location. Such method must be compared with the least-squares photoelastic method[1,6] which may offer advantages in organization and execution.

It is possible to use different solution region boundaries for the combined method. Regions may be bounded entirely with external surfaces, and may include surface loads. It is possible to use internal boundaries which are not lines of symmetry. For this case it is necessary to use the general photoelastic equations and isoclinic data discussed in the preceding paragraph. It is important to establish a properly constrained finite-element problem. (Displacement constraints are provided along the lines of symmetry in problems containing them.) The static equilibrium equation(s) for such problems will include shear and normal stresses and likely will include contributions from more than one boundary line.

Conclusion

A method has been presented and demonstrated which combines the finite-element method with photoelastic data for stress analysis using least-squares. The results obtained compare favorably with photoelastic results, finite-element results and theory. The method offers the advantage of simplified data measurement (no isoclinics) and local application to regions of high stress. The method was applied using a personal computer. Variations of the method are discussed.

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