Optimization of Geometric Discontinuities in Stress Fields

In this paper, the problem of the optimization of holes and fillets is reviewed, the concept of efficiency factor is introduced, and attempts are made at optimizing complete boundaries, even those subjected to stresses of opposite signs

by A.J. Durelli, K. Brown and P. Yee

ABSTRACT—The ideal boundary of a discontinuity is defined as that boundary along which there is no stress concentration. Photoelastically an isochromatic coincides with the ideal boundary. This property is used to develop experimentally ideal boundaries for some cases of technological interest. The concept of 'coefficient of efficiency' is introduced to evaluate the degree of optimization. The procedure to idealize boundaries is illustrated for the two cases of the circular tube and of the perforated rectangular plate, with prescribed functional restraints and a particular criterion for failure. An ideal design of the inside boundary of the tube is developed which decreases its maximum stress by 25 percent, at the time it also decreases its weight by 10 percent. The efficiency coefficient is increased from 0.59 to 0.95. Tests with a brittle material show an increase in strength of 20 percent. An ideal design of the boundary of the hole in the plate reduces the maximum stresses by 26 percent and increases the coefficient of efficiency from 0.54 to 0.90.

Introduction

Some fifty years ago, the subject of stress concentrations deserved a great deal of interest from scientists and engineers. Changes in the uniform shape of a component disturb the stress distribution, and most of the time increase the maximum stress. This fact was likely to have an influence on failure and excited the imagination of theoreticians first, and experimentalists later, to find means to determine the value of the increase in stress. Kirsch[1] was probably the first one who obtained a meaningful answer to the problem when he presented the equations giving the stress distribution around an empty circular hole. Today, several handbooks summarize the findings on stress concentrations and make them available to engineers in an easy-to-use form. Among the most popular, the books by Peterson[2] and Roark[3] can be mentioned.

Scientists and engineers, after worrying about the increase in stress associated with changes in shape, are beginning to consider now the possibility of controlling those changes to minimize the stress and optimize the shape. It seems logical in the development of these studies that the optimization of shapes has to follow the knowledge of the stress concentrations. The subject is of particular importance today when mankind will be making a strong effort to save energy and materials. The methods and criteria to be presented in this paper are general, but the new examples of application have been selected less for the technological importance than as illustrations and means of exciting the imagination of designers and students.

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Previous Contributions

Optimization of the shape of fillets and holes in stress fields has interested few people so far. One of the first references, by implication, can be found in a discussion by Richmond[4] of a paper by Mindlin. It is pointed out that, if a square tunnel with rounded corners is present in a semi-space, there is a particular value of the radius of the fillets at the corners of the square that will optimize the stress distribution. Whether the value of the radius is smaller or larger than $\frac{D}{6}$, D being the side of the square, the stress concentration will increase.

An important contribution was made in an early paper by Berkey[5] who studied systematically the stress concentration associated with elliptical fillets with the purpose of reducing the concentration at a shoulder. The attempts by Baud[6] and by Lansard[7] should be mentioned in spite of the unfortunate reference to a nonexisting analogy. Some further reference to the problem is implied in a section in Peterson's handbook[8] when the study of concentrations associated with noncircular fillets is introduced.

Kuske[9] refers to the problem but it is in Heywood's[10,11] books that the subject has been dealt with more extensively, and in a more practical way.

Sometime ago, the first author used the concept of an ideal fillet, defined it as a fillet without stress concentration and related it photoelastically to the coincidence of the boundary with an isochromatic fringe. Some references can be found in a book,[12] reports and early papers.[13-15] Recently, Francavilla *et al.*[16] attempted the optimization of fillets using finite-element methods. The geometries they obtained, however, show some stress concentration. In this paper, besides reviewing the problem of the optimization of holes and fillets, the concept of efficiency factor will be introduced, and attempts will be made at optimizing complete boundaries, even those subjected to stresses of opposite signs.

For the purpose of completeness, it should be mentioned that another approach has been followed sometimes, with the same objective of increasing strength by decreasing weight. It consists in adding new discontinuities to the original one. So it can be shown that a row of holes, or fillets may produce a smaller stress concentration than a

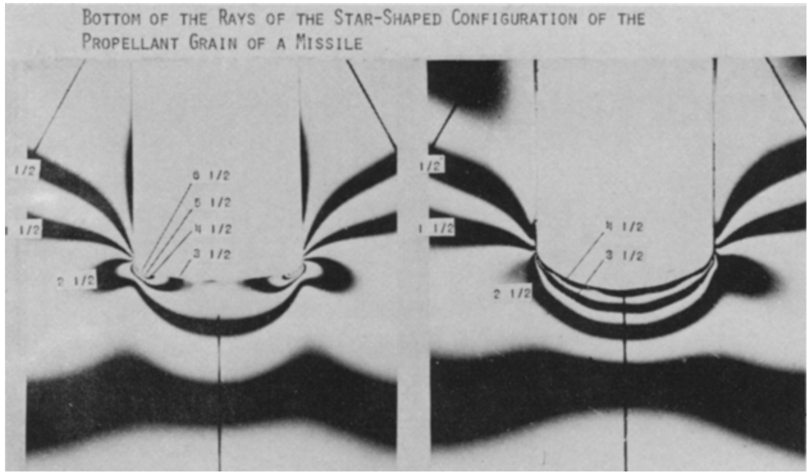

Fig. 1—Isochromatics obtained for the original and optimized designs of a fillet contour. (When an isochromatic lies along the length of a fillet, the fillet geometry is optimum)

single hole, or fillet. One of the first contributions to this method was made by Thum and Svenson.[17] Further studies can be found in other papers by the first author[18,19] and more recently by Erickson and Riley.[20] The scope of this paper is limited to the optimization by changing the contour of the discontinuity.

Approach to the Solution

It is possible to use a computer and an appropriate program to develop a contour that will minimize the stress as was done in Ref. 16. It seems more efficient, however, to use photoelasticity. Two-dimensional photoelasticity is very well developed by now, and the machining of models and the photographing of records can be done as routine operations in well-organized laboratories. The optimization can be accomplished by manual filing of boundaries as suggested in Ref. 12.

The proposed method has already been applied to the solution of two problems of technological interest: (1) the tip of the several rays of stars in perforated solid-propellant grains used for rocket propulsion, and (2) the transition between the blade and the dove-tail joint in a turbine.

The tip of the star in solid-propellant grains was originally designed either with one circular fillet tangent to the two sides of each ray or with a flat bottom connected through two small circular fillets to the sides of each ray (Fig. 1). In the first case, the maximum stress, given in a photoelastic model by the maximum order of the isochromatic fringe, is at the axis of the ray. In the second case, left side of Fig. 1, the maximum stress takes place at the corners. It can be observed that, at these points, both the fringe order and the density of the fringes are higher. A very small amount of material at the boundary of the fillet is subjected to a very high stress.

The second example refers to the transition between the blade and the dove tail that joins it to the rotor of the turbine. It was originally designed using circular fillets, of relatively large radii in this case, as shown on the left of Fig. 2. The isochromatic pattern in the photoelastic model indicates dense fringes with a high order at the bottom of the fillet.

The appearance of isochromatic fringes at the boundary of ideal fillets is shown on the right side of Figs. 1 and 2. The deciding characteristic is that a fringe coincides with appreciable length of the boundary of the fillet. When the boundary intersects another fringe, the latter is of a lower

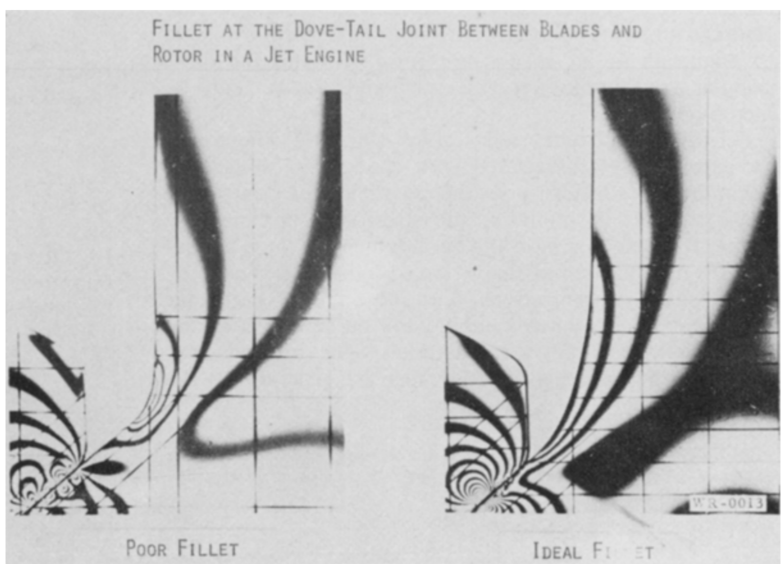

Fig. 2—Isochromatics about a poorly designed fillet and a nearly ideal fillet

order. Strain and energy, therefore, are not concentrated on a small portion of the boundary but distributed on a long part of it. An even more striking example is shown in Fig. 3 which represents the tip of a star in a solid-propellant grain.

The transformation of the shape from the original design to the optimum design shown on the right side of the figures can be done in a relatively short time with a hand file. The operator starts removing material by filing off zones of low stress. This decreases the order of the fringe at the zone of concentration and increases it at the zone of low stress. If the operation is conducted in a large-field diffused-light polariscope, the operator can watch the transferring of fringes as he files and, in a short while, reaches the moment when one single fringe coincides with the boundary of the model. The manual operation of filing may introduce some local irregularities as shown in Fig. 1. If more precision is desired, a second model should be machined as shown in Fig. 2, with a fillet shape corresponding to the one obtained by filing and, if necessary, further refinement can be obtained by applying the same operation of filing to the second model. In the two cases reported above, the optimum shape is obtained after only slight changes in geometry.

From a practical point of view, a further consideration should be made. It has been found that, frequently, the optimum fillet shape can be fitted with two or more circles so that neighboring circles have common tangents.

Criteria

The definition of the problem requires the specification of the constraints imposed by the design. In the two cases mentioned above, the optimization was obtained with very little change in geometry. That was all that was permitted by the functional requirements. Of course, the optimization problem may have several answers if the functional requirements permit appreciable changes in design.

An improved design, obtained following the procedure outlined above, always brings the stress-concentration value down. However, it may not always be clear whether the design is optimum. It is proposed here that the 'degree of optimization' be evaluated quantitatively as a coefficient of efficiency, k_{eff}. For the case where the tangential stress σ_t is of the same sign all along the boundary, k_{eff} can be defined as

$$k_{eff} = \int_{S_0}^{S_1} \frac{\sigma_t ds}{(S_1 - S_0)\sigma^+_{all}}$$

where σ_{all} represents the maximum allowable stress and S_1 and S_0 are the limiting points along the boundary. For the case of both tensile and compressive stresses, k_{eff} is computed as a weighted average of the efficiency factors along the tensile and compressive portions of the boundary. Taking the weighting factor in terms of boundary lengths yields

$$k_{eff} = \frac{\int_{S_0}^{S_1} \sigma_t^+ ds}{(S_1 - S_0)\sigma^+_{all}} \frac{S_1 - S_0}{S_2 - S_0} + \frac{\int_{S_1}^{S_2} \sigma_t^- ds}{(S_2 - S_1)\sigma^-_{all}} \frac{S_2 - S_1}{S_2 - S_0}$$

$$k_{eff} = \frac{1}{S_2 - S_0} \left\{ \frac{\int_{S_0}^{S_1} \sigma_t^+ ds}{\sigma^+_{all}} + \frac{\int_{S_1}^{S_2} \sigma_t^- ds}{\sigma^-_{all}} \right\}$$

where the positive and negative superscripts refer to tensile and compressive stresses, respectively.

A coefficient of efficiency equal to one is a limiting case and corresponds to a boundary without stress concentration, subjected everywhere to the same stress. The circular hole in a hydrostatic field is an example. The closer k_{eff} is to one, the more efficient the design.

The criteria for optimization will depend on the criterion for failure. If the boundary to be optimized is subjected to both positive and negative stresses, the integration along the boundary should be conducted using absolute values for the stresses. If the component is designed for a material that has the same allowable maximum stress under tension as under compression, the ideal shape would have equal values for both peak stresses, the tensile and the compressive. If, as is the case for brittle materials, the maximum allowable tensile stress is only a fraction of the maximum allowable compressive stress, the ratio between the two peak stresses in the optimum design would be the same as the ratio of the two allowable maximum stresses.

The redesign of a circular tube or ring, to optimize the inside boundary, will be used as example of the application of the criteria and the procedure mentioned above. The problem has application in the field of tunnel and pipe design, but it will be presented mainly as an academic problem to illustrate the method. It will be assumed that the material to be used in the manufacture of the tube has the same maximum allowable tensile and compressive stresses. Another example to be shown will be the case of a thin straight bar of rectangular cross section. The bar has a transverse circular hole and is subjected to axial loading. The optimization will be conducted for a different allowable stress in tension and in compression.

The Ring Under Diametral Compression

The circular ring subjected to diametral compression has been the object of many experimental analyses (see,

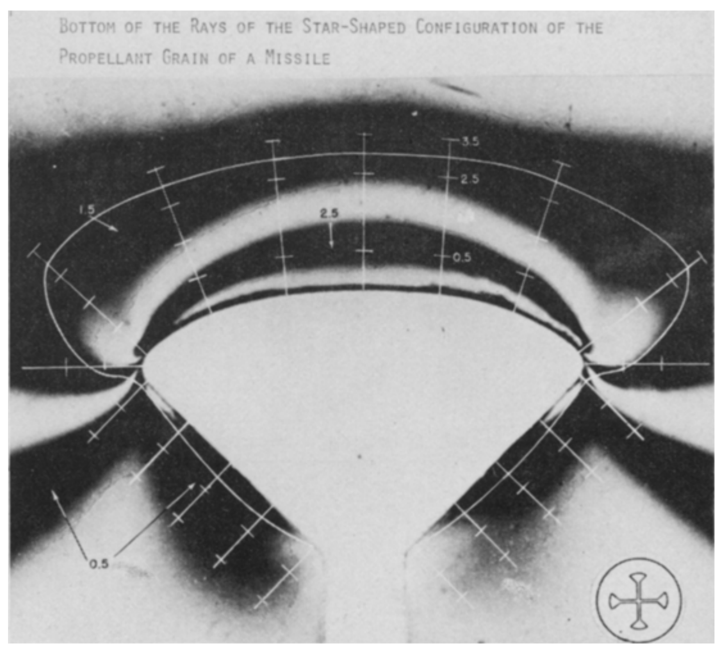

Fig. 3—Example of a fillet of optimum shape. The magnitude of the boundary stress is proportional to the distance between the boundary and the white line

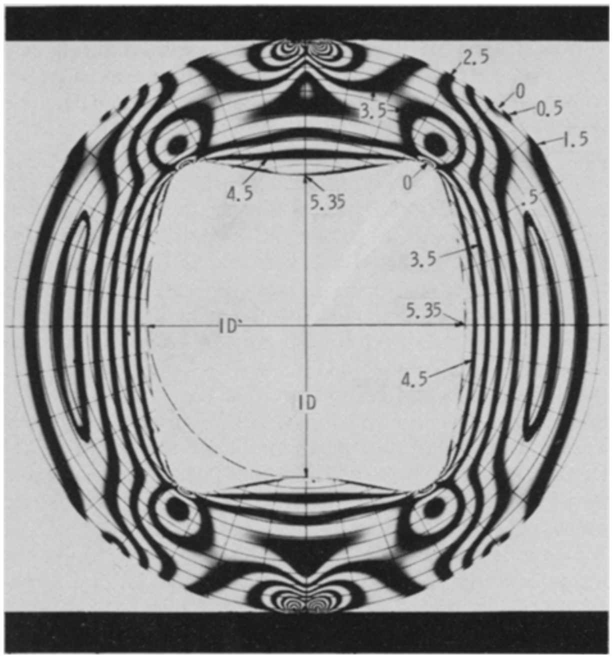

Fig. 4—Optimization of the inside boundary of a circular ring subjected to diametral compression

among others, Ref. 21). In a future paper, it is planned to study parametrically the properties of the ring as the ratio between the outside and inside diameters varies and to attempt optimization of the inside boundary for the whole range of thickness. In this paper, the procedure will be illustrated for the case $\frac{ID}{OD} = 0.53$. The constraints of the problem are: (a) the outside boundary has to be kept circular; (b) the inside boundary has to clear the circle of diameter 0.53 *OD*; (c) the allowable maximum stress for tension is the same as for compression.

Optimization of the Ring

Two stress-concentration factors (taking the average σ_y over the horizontal section of symmetry as reference) are of particular interest. The one of compression takes place at the intersection of the inside boundary with the horizontal axis and the one of tension at the intersection of the vertical axis with the same boundary. For the $\frac{ID}{OD} = 0.53$ ring, these factors are 6.0 and 6.6 respectively. The efficiency coefficient is 0.587.

Following the procedure of removing material from low-stress regions, the shape shown in Fig. 4 was developed. Photoelastic analysis of the pattern indicates that both stress-concentration factors have decreased to 5 and the efficiency coefficient has increased to 0.952. The tensile stress concentration which is the governing one in many designs has been decreased by nearly 25 percent. The saving in the weight of the material used is 10 percent.

The stress distribution over the inside boundary for the circular ring and for the optimized geometry is shown in Fig. 5.

The empirically developed inside geometry has been fitted with a combination of circles of different diameters and common tangents at the points of intersections. The geometry of the optimized shape is shown in Fig. 6.

The improvement obtained in the strength of rings designed using optimized inside boundaries has been determined by breaking three plain circular rings and three optimized rings, made of 0.5-in.-thick Homelite-100 plates. The increase in strength was 20.6 percent. The range of values of each set of measurements was limited by a variation of ± 6 percent of the average.

The Perforated Plate Under Axial Loading

In industrial applications, a plate may have to be perforated for different reasons: to permit the passage of another component (a bar for instance) or to make the plate lighter, and still sufficiently rigid. In the case of walls, or tunnels, the perforation is a passage. Frequently, the geometry given to the perforation is circular but, for

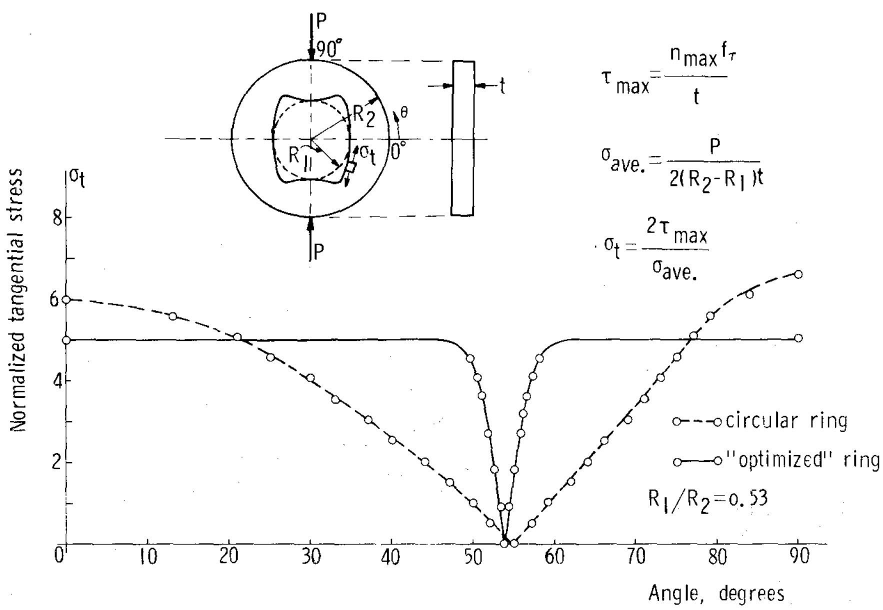

Fig. 5—Stress tangential to the boundary of an optimized ring subjected to diametral compression

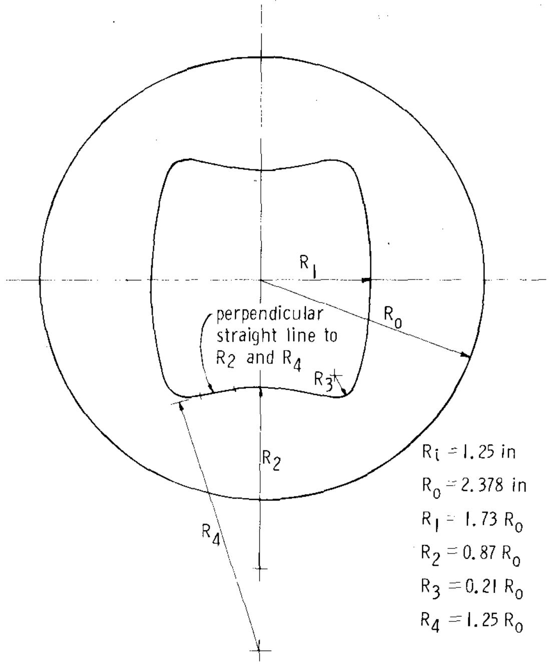

Fig. 6—Nondimensionalized geometry of the optimized ring for $\frac{ID}{OD} = 0.53$

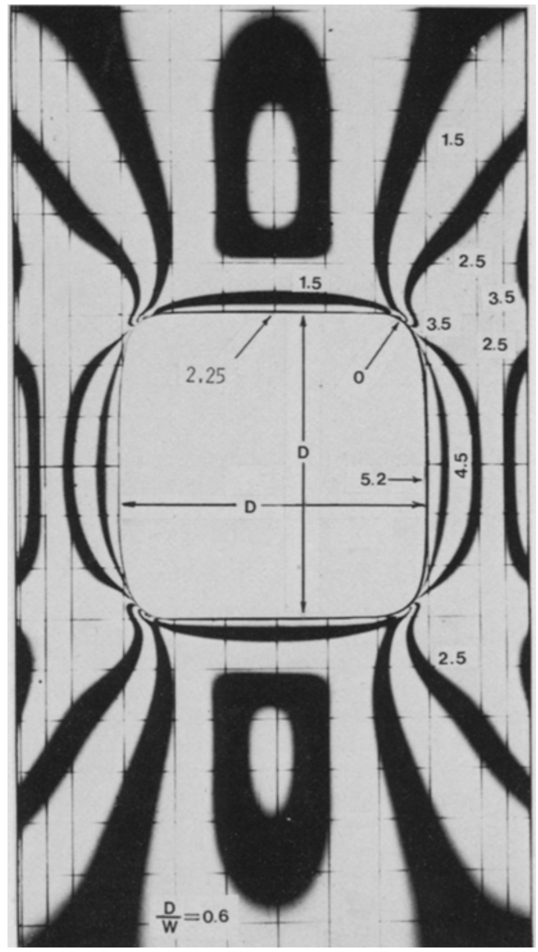

Fig. 7—Optimization of the boundary of a hole in a rectangular plate of finite width subjected to axial load

functional requirements, the perforation may be square or rectangular.

The maximum stress on the edge of the circular hole takes place at the transverse cross section and, for the very wide plate, its order is 3 when the stress on the gross area is taken as reference. A stress of opposite sign, of order 1, on the edge of the hole takes place at the longitudinal axis. As the width of the plate decreases

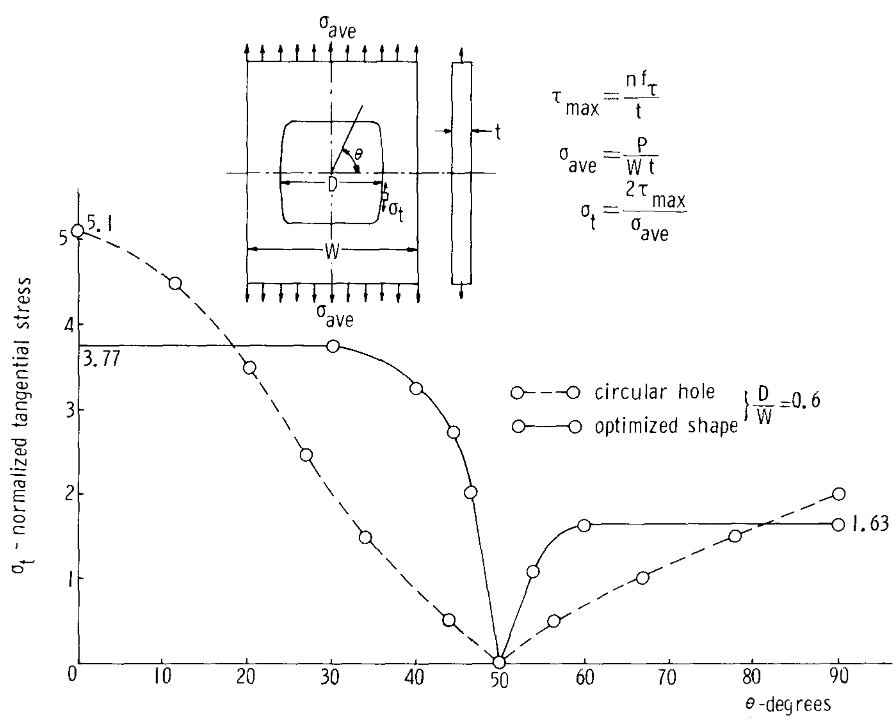

Fig. 8—Stress tangential to the boundary of an optimized hole in a plate subjected to axial loading ($\frac{D}{W} = 0.6$)

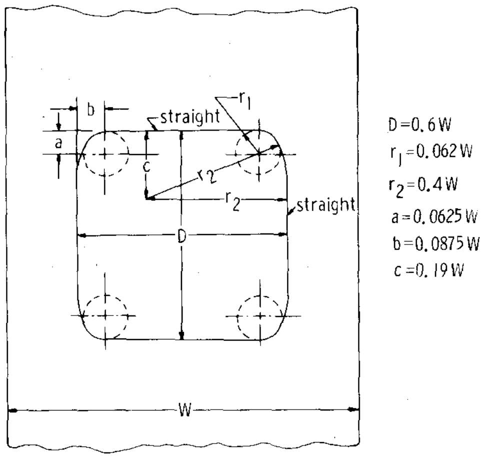

Fig. 9—Nondimensionalized geometry of the optimized hole in a plate subjected to axial loading ($\frac{D}{W}$ = 0.6)

in relation to the diameter of the hole, those values of stresses increase. If material used for the plate is metal, and the plate is subjected to axial tensile loading, points on the edge of the boundary near the transverse cross section are much closer to failure than those near the longitudinal cross section (unless buckling is involved). If the problem is a tunnel under compressive load, and the material used is brittle, the points at the longitudinal axis may fail under tension, much before those at the transverse axis would fail under compression. Similar considerations can be applied to the square hole, the situation being more complicated because of the possible appearance of concentrations at the corners.

The optimization of this type of discontinuity depends on the relation between the width W of the plate and the diameter D of the hole, and on the relative allowable stress of the material under tension and compression. The optimization procedure will be illustrated for the case $\frac{D}{W}$ = 0.6. The constraints of the problem are: (a) the inside boundary has to lie in between the circle of diameter D and the square of side D; (b) the allowable maximum stress for compression is 2.3 times the allowable stress for tension (case of some brittle materials).

Optimization of the Hole in the Plate

The stress-concentration factors (taking the average σ_y over the transverse gross area as reference) are of particular interest. The one at the transverse axis is 5.1 for the circular hole, and 3.77 for the optimized hole; the one at the longitudinal axis is 2.2 for the circular hole and 1.63 for the optimized hole. The maximum stresses have been reduced by 26 percent. The size of the hole has been increased by 22.8 percent.

The efficiency coefficient of the circular hole is 54 percent. The efficiency coefficient of the optimized hole is 90 percent.

Conclusion

It has been shown that two-dimensional photoelasticity can be used effectively to optimize the boundaries of plates loaded in their plane. The concept of 'coefficient of efficiency' has been introduced to evaluate the degree of the optimization. Two illustrative problems have been solved: a circular tube (or ring) under diametral compression and a perforated plate loaded axially. The efficiency coefficient of the tube has been increased from 0.587 to 0.952, and the one of the plate from 0.54 to 0.90. In both cases, the maximum stress has been decreased by about 25 percent. The weight of the ring has been reduced by 10 percent and the size of the hole of the plate has been increased by about 23 percent. The increase in the strength of the ring made of a brittle material was 20.6 percent.

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