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On the Range of Eigenvalues of an Interval Matrix

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Abstract — Zusammenfassung

On the Range of Eigenvalues of an Interval Matrix. We describe a method for enclosing the set of real eigenvalues of an interval matrix pertaining to eigenvectors of a given sign pattern.

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Zur Einschliessung der Eigenwerte einer Intervallmatrix. Wir beschreiben eine Methode zur Einschliessung der Menge aller reellen Eigenwerte einer Intervallmatrix die zu Eigenvektoren von einer vorgeschriebenen Zeichenstruktur gehören.

1. Introduction

Eigenvalues of interval matrices were recently studied by Deif [1] and Rohn [2]. The problem considered there was the following: given a square interval matrix A^I, determine the exact range of eigenvalues of all the matrices contained in A^I. Formulae for exact bounds on eigenvalues were given for the complex case in [1] and for the case of real eigenvalues in [2]. A common assumption in both papers was a constancy of the sign patterns of the real or imaginary parts of the eigenvectors.

In the present paper we pursue a slightly different approach: we investigate the set of real eigenvalues (of matrices in A^I) pertaining to eigenvectors of a given sign pattern. This approach enables us to circumvent introducing additional assumptions (made in [1] and [2]) which are generally difficult to verify. In the main Theorem 1 we describe a method for enclosing this set of eigenvalues by means of two nonnegative nonzero vectors satisfying certain rather weak constraints. In Theorem 2 we formulate conditions under which exact bounds can be obtained by an appropriate choice of the two vectors. The quality of enclosures for different choices of auxiliary vectors is illustrated on a small size example. Finally, in Theorem 3 we give an application of the main result for solving a tolerance problem for eigenvalues.

2. The Results

Consider an interval matrix

$$A^I = \{A; A_c - \Delta \le A \le A_c + \Delta\}$$

where A_c and Δ are two real $n \times n$ matrices, $\Delta \ge 0$ (this form of expressing the bounds is most appropriate for our purposes; inequalities are to be understood componentwise). We shall study the set

$$L_S = \{\lambda \in R^1; Ax = \lambda x \text{ for some } A \in A^I \text{ and an } x \text{ with } Sx > 0\}$$

where S is a given $n \times n$ signature matrix, i.e. a diagonal matrix whose each diagonal entry is equal to 1 or -1. Notice that the condition $Sx > 0$ means that $x_j > 0$ if $S_{jj} = 1$ and $x_j < 0$ if $S_{jj} = -1$, hence L_S is the set of real eigenvalues of matrices in A^I pertaining to eigenvectors with a constant sign pattern prescribed by the diagonal vector of the matrix S. In Theorem 1 to follow we describe a method for computing lower and upper bounds on L_S. We shall formulate the result in terms of the matrices

$$\underline{A}_S = SA_cS - \Delta$$

and

$$\bar{A}_S = SA_cS + \Delta .$$

A^T denotes the transpose of a matrix A.

Theorem 1. *Let q and p be nonnegative nonzero vectors satisfying*

$$(\forall j)(q_j = 0 \Rightarrow (\underline{A}_S^T q)_j \ge 0) \tag{1}$$

and

$$(\forall j)(p_j = 0 \Rightarrow (\bar{A}_S^T p)_j \le 0) . \tag{2}$$

Then we have

$$L_S \subset [\underset{\sim}{\lambda}_S, \tilde{\lambda}_S] \tag{3}$$

where

$$\underset{\sim}{\lambda}_S = \min\left\{\frac{(\underline{A}_S^T q)_j}{q_j}; q_j > 0\right\} \tag{4}$$

and

$$\tilde{\lambda}_S = \max\left\{\frac{(\bar{A}_S^T p)_j}{p_j}; p_j > 0\right\} . \tag{5}$$

Comment. Note that the conditions (1) and (2) imposed upon q and p are very weak. They are satisfied e.g. if either (a) or (b) holds:

(a) $q > 0$ and $p > 0$,
(b) $\underline{A}_S^T q \ge 0$ and $\bar{A}_S^T p \le 0$.

Proof. First, from (4) we can see that

$$(\underline{A}_S^T q)_j \geq \underset{\sim}{\lambda}_S q_j$$

holds for each j with $q_j > 0$, but this inequality is also satisfied if $q_j = 0$ in view of (1). Hence we conclude that

$$\underline{A}_S^T q \geq \underset{\sim}{\lambda}_S q \tag{6}$$

holds. In a similar way, from (5) and (2) we can derive the inequality

$$\overline{A}_S^T p \leq \tilde{\lambda}_S p. \tag{7}$$

Now, let $\lambda \in L_S$, so that $Ax = \lambda x$ for some $A \in A^I$ and an x with $Sx > 0$. Denote $y = Sx$, then $y > 0$ and we have

$$|\lambda y - SA_c Sy| = |S(\lambda x - A_c x)| = |(A - A_c)x| \leq \Delta |x| = \Delta y,$$

which gives

$$\underline{A}_S y \leq \lambda y \leq \overline{A}_S y. \tag{8}$$

We shall prove that there exists a j with

$$(\underline{A}_S y)_j \geq \underset{\sim}{\lambda}_S y_j. \tag{9}$$

Assume to the contrary that this is not so, so that

$$\underline{A}_S y < \underset{\sim}{\lambda}_S y$$

holds. Premultiplying this inequality by the nonnegative nonzero vector q, we obtain

$$q^T \underline{A}_S y < \underset{\sim}{\lambda}_S q^T y,$$

but premultiplying (6) by the positive vector y yields

$$q^T \underline{A}_S y \geq \underset{\sim}{\lambda}_S q^T y,$$

which is a contradiction. Hence (9) holds for some j and from (8), (9) we obtain

$$\lambda \geq \frac{(\underline{A}_S y)_j}{y_j} \geq \underset{\sim}{\lambda}_S.$$

In a similar way we can prove from (7) that

$$(\overline{A}_S y)_k \leq \tilde{\lambda}_S y_k$$

holds for some k, which in conjunction with (8) gives

$$\lambda \leq \frac{(\overline{A}_S y)_k}{y_k} \leq \tilde{\lambda}_S.$$

Hence $L_S \subset [\underset{\sim}{\lambda}_S, \tilde{\lambda}_S]$, which concludes the proof. ∎

In this way, any pair of nonnegative nonzero vectors satisfying (1), (2) (e.g. an arbitrary pair of positive vectors) provides us via (4), (5) with some bounds on L_S. We shall show that under certain conditions exact bounds can be achieved in (4),

(5) by an appropriate choice of q and p. Similar results were given in [1] and [2] under assumption that *each* matrix in A^I has an eigenvalue in L_S, while here we require this property to hold for *two* matrices only:

Theorem 2.

(i) *If the matrix* $A_c - S\Delta S$ *has an eigenvalue* $\underline{\lambda}_S \in L_S$ *pertaining to a left eigenvector* y *satisfying* $Sy > 0$, *then*

$$\underline{\lambda}_S = \min L_S . \tag{10}$$

(ii) *If the matrix* $A_c + S\Delta S$ *has an eigenvalue* $\overline{\lambda}_S \in L_S$ *pertaining to a left eigenvector* y' *satisfying* $Sy' > 0$, *then*

$$\overline{\lambda}_S = \max L_S . \tag{11}$$

Proof. We shall prove (i) only; the proof of (ii) is quite analogous. Let $q = Sy$, then q is a positive vector and from

$$(A_c - S\Delta S)^T y = \underline{\lambda}_S y$$

we obtain

$$\underline{A}_S^T q = \underline{\lambda}_S q ,$$

hence Theorem 1 implies that for each $\lambda \in L_S$ we have

$$\lambda \geq \min\left\{\frac{(\underline{A}_S^T q)_j}{q_j}; q_j > 0\right\} = \underline{\lambda}_S$$

and since $\underline{\lambda}_S \in L_S$ by assumption, (10) follows. ∎

Note that if both the matrices A_c and Δ are symmetric, then $A_c - S\Delta S$ and $A_c + S\Delta S$ are also symmetric, hence in this case a left eigenvector can be replaced by an eigenvector in the formulation of (i) and (ii).

Example. Consider the interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$, where

$$A_c = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and

$$\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let $S = I$ = unit matrix. Let us take an arbitrary pair of positive vectors q, p. Then from (4), (5) we obtain the estimates

$$\underset{\sim}{\lambda}_S = \min\left\{\frac{q_1 + q_2}{q_1}, \frac{q_1 + q_2}{q_2}\right\} = 1 + \min\left\{\frac{q_2}{q_1}, \frac{q_1}{q_2}\right\} \tag{12}$$

and

$$\tilde{\lambda}_S = \max\left\{\frac{p_1 + 3p_2}{p_1}, \frac{3p_1 + p_2}{p_2}\right\} = 1 + 3\max\left\{\frac{p_2}{p_1}, \frac{p_1}{p_2}\right\}. \tag{13}$$

This shows that $\underset{\sim}{\lambda}_S \in (1, 2]$ and $\tilde{\lambda}_S \geq 4$, but $\tilde{\lambda}_S$ may get arbitrarily large for an inappropriate choice of p. Since the matrices $A_c - S\Delta S = A_c - \Delta$ and $A_c + S\Delta S = A_c + \Delta$ are symmetric and positive, they have eigenvalues $\underline{\lambda}_S = 2$ and $\overline{\lambda}_S = 4$, respectively, corresponding to positive Perron vectors, hence $\underline{\lambda}_S \in L_S$ and $\overline{\lambda}_S \in L_S$ and from Theorem 2 we conclude that $\min L_S = 2$ and $\max L_S = 4$. These extremal eigenvalues are achieved in (12), (13) e.g. for $q = p = (1, 1)^T$.

Theorem 1 can be also applied to solve a tolerance problem for eigenvalues: given a real interval $[\lambda_1, \lambda_2]$, check whether $L_S \subset [\lambda_1, \lambda_2]$ holds. We have this sufficient condition:

Theorem 3. *For a given real interval* $[\lambda_1, \lambda_2]$, *let the systems of linear inequalities*

$$(\underline{A}_S^T - \lambda_1 I)q \geq 0 \tag{14}$$

and

$$(\overline{A}_S^T - \lambda_2 I)p \leq 0 \tag{15}$$

have nonnegative nonzero solutions. Then

$$L_S \subset [\lambda_1, \lambda_2]$$

holds.

Proof. Obviously, the solutions q, p to (14), (15) satisfy (1) and (2) and from (4), (5), (14), (15) we obtain $\lambda_1 \leq \underset{\sim}{\lambda}_S$ and $\tilde{\lambda}_S \leq \lambda_2$, hence Theorem 1 gives $L_S \subset [\underset{\sim}{\lambda}_S, \tilde{\lambda}_S] \subset [\lambda_1, \lambda_2]$. ■

3. Concluding Remark

We have described a method for enclosing the set L_S for a particular S. A repeated use of this procedure may yield an enclosure of the whole real part of the spectrum of an interval matrix provided the sign pattern structure of the eigenvectors is known.

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