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On the Solvability of a Nonstationary Problem Describing the Dynamics of an Incompressible Viscoelastic Fluid

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ABSTRACT. We study the local solvability of the Cauchy–Dirichlet problem for the system

$$\begin{gathered}(1-\varkappa\nabla^2)\mathbf{v}_t=\nu\nabla^2\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{v}-\nabla p+\mathbf{f}(t),\\ 0=-\nabla(\nabla\cdot\mathbf{v}),\end{gathered}$$

which describes the dynamics of an incompressible viscoelastic Kelvin–Voigt fluid. The configuration space of the problem is described.

KEY WORDS: Kelvin–Voigt fluid, hydrodynamic equations, Cauchy–Dirichlet problem, local solvability.

Oskolkov's system [1]

$$\begin{gathered}(1-\varkappa\nabla^2)\mathbf{v}_t=\nu\nabla^2\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{v}-\nabla p+\mathbf{f}(t),\\ 0=\nabla\cdot\mathbf{v}\end{gathered}\tag{1}$$

provides a model of the dynamics of an incompressible viscoelastic Kelvin–Voigt fluid. Let $\Omega\subset\mathbb{R}^n$, $n=2,3,4$, be a bounded domain with C^∞ boundary $\partial\Omega$. Previously [2, 3], we have considered the Cauchy–Dirichlet problem

$$\begin{aligned}\mathbf{v}(x,0)&=\mathbf{v}_0(x), && x\in\Omega,\\ \mathbf{v}(x,t)&=0, && (x,t)\in\partial\Omega\times\mathbb{R},\end{aligned}\tag{2}$$

for system (1) under the assumption that the right-hand side $\mathbf{f}=(f_1,\dots,f_n)$ is independent of time, that is, $f_k=f_k(x)$, $x\in\Omega$, $k=1,\dots,n$. This restriction depreciates the heuristic power of the model and reduces generality (cf. [4], where problem (1), (2) is considered for the case in which $f_k=f_k(x,t)$ and $\varkappa^{-1}>-\lambda_1$; here λ_1 is the least eigenvalue of the Dirichlet problem for the Laplace operator in Ω). On the other hand, the approach suggested in [5] permits one to consider problem (1), (2) in its full generality, that is, for a nonstationary right-hand side $\mathbf{f}=\mathbf{f}(x,t)$ and arbitrary values of the parameter $\varkappa\in\mathbb{R}$, which characterizes the elastic properties of the fluid. (Note that negative values of $\varkappa$ have been observed experimentally [6, 7].)

To avoid the question of how can an incompressible fluid be elastic, one treats system (1) as the limit case (as $\varepsilon\downarrow 0$) of the system

$$\begin{gathered}(1-\varkappa\nabla^2)\mathbf{v}_t=\nu\nabla^2\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{v}-\nabla p+\mathbf{f},\\ \varepsilon p_t=-\nabla\cdot\mathbf{v},\end{gathered}\tag{3}$$

which models the dynamics of a weakly compressible viscoelastic Kelvin–Voigt fluid. In (3) we perform the substitution $\mathbf{p}=\nabla p$ and set $\varepsilon=0$; then we arrive at the system

$$\begin{gathered}(1-\varkappa\nabla^2)\mathbf{v}_t=\nu\nabla^2\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{p}+\mathbf{f},\\ 0=-\nabla(\nabla\cdot\mathbf{v}).\end{gathered}\tag{4}$$

Although the kernel of ∇ is nontrivial (it consists of constants), problem (4), (2) has no new solutions as compared with problem (1), (2); this is due to the boundary condition in (2). The substitution $\mathbf{p}=\nabla p$

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is motivated by the fact that in most hydrodynamic problems [8] it is preferable to consider the pressure gradient instead of pressure itself.

We are interested in the local unique solvability of problem (2), (4). It is convenient to consider this problem in the framework of the theory of Sobolev type equations. Therefore, in the first part of the paper we develop some formalism, the main result being that a semilinear nonstationary Sobolev type equation possesses quasistationary trajectories [5]. In the second part, this formalism is applied to problem (2), (4).

§1. The formal scheme

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, and let the operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in C^\infty(\mathfrak{U};\mathfrak{F})$ and a function $f\colon \mathbb{R} \to \mathfrak{F}$ be given. We consider the Cauchy problem

$$u(0) = u_0 \tag{5}$$

for the semilinear nonstationary Sobolev type equation

$$L\dot{u} = M(u) + f. \tag{6}$$

A linear operator $L\colon \mathfrak{U} \to \mathfrak{F}$ is said to be ***bisplitting*** [9] if the kernel $\ker L$ and the range $\operatorname{im} L$ are complemented in the spaces $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Suppose that L is bisplitting. By $M'_{u_0} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ we denote the Fréchet derivative of M at a point $u_0 \in \mathfrak{U}$. We consider chains of ***M'_{u_0}-associated vectors of the operator L*** [10]; these vectors will be chosen from some complement $\operatorname{coim} L = \mathfrak{U} \ominus \ker L$ of $\ker L$. We introduce the following condition:

A1) Regardless of the choice of $\operatorname{coim} L$, the chain of M'_{u_0}-associated vectors of every vector $\varphi \in \ker L \setminus \{0\}$ contains exactly p elements.

Let $\widetilde{L}$ be the restriction of L to $\operatorname{coim} L$. By Banach's closed graph theorem, the operator $\widetilde{L}\colon \operatorname{coim} L \to \operatorname{im} L$ is a linear topological isomorphism. We set $\mathfrak{U}_0^0 = \ker L$ and consider the sets $\mathfrak{U}_q^0 = \widetilde{A}^q[\mathfrak{U}_0^0]$, $q = 1,\dots,p$, where $\widetilde{A} = \widetilde{L}^{-1}M'_{u_0}$. Obviously, the sets $\mathfrak{U}_q^0 \subset \operatorname{coim} L$ are linear spaces. Consequently, the image $\mathfrak{F}_p^0 = M'_{u_0}[\mathfrak{U}_p^0]$ is also a linear space, and moreover, $\mathfrak{F}_p^0 \cap \operatorname{im} L = \{0\}$ (provided that condition A1) is satisfied). Let us introduce yet another condition:

A2) $\mathfrak{F}_p^0 \oplus \operatorname{im} L = \mathfrak{F}$.

We denote by $Q_p\colon \mathfrak{F} \to \mathfrak{F}_p^0$ the projection along $\operatorname{im} L$ and consider the operator $A = \widetilde{L}^{-1}(I - Q_p)M'_{u_0}$. Note that $A[\mathfrak{U}_q^0] = \mathfrak{U}_{q+1}^0$, $q = 0,1,\dots,p-1$, and $A[\mathfrak{U}_p^0] = \{0\}$. It follows that

$$A^q[\mathfrak{U}_r^0] = \begin{cases} \{0\}, & q + r > p, \\ \mathfrak{U}_{q+r}^0, & q + r \le p. \end{cases} \tag{7}$$

Let D be the restriction of the operator $Q_p M'_{u_0} A^p\colon \mathfrak{U} \to \mathfrak{F}_p^0$ to $\mathfrak{U}_0^0$. By construction, $D[\mathfrak{U}_0^0] = \mathfrak{F}_p^0$ and $D \in \mathcal{L}(\mathfrak{U}_0^0;\mathfrak{F}_p^0)$. Moreover, $\ker D = \{0\}$, since otherwise the vector $\varphi \in \ker D \setminus \{0\} \subset \ker L \setminus \{0\}$ would have an infinite chain $\{\varphi_1, \varphi_2, \dots, \varphi_p, 0, \dots\}$ of M'_{u_0}-associated vectors. Again by Banach's theorem, the operator $D\colon \mathfrak{U}_0^0 \to \mathfrak{F}_p^0$ is a linear topological isomorphism.

Let $P_0\colon \mathfrak{U} \to \mathfrak{U}_0^0$ be the projection along $\operatorname{coim} L$. We consider the operators $P_q = A^q D^{-1} Q_p M'_{u_0} A^{p-q}$, $q = 1,\dots,p$. The operators $P_q\colon \mathfrak{U} \to \mathfrak{U}_q^0$ are projections. Indeed, $\operatorname{im} P_q = \mathfrak{U}_q^0$, $P_q \in \mathcal{L}(\mathfrak{U})$ and

$$P_q^2 = A^q\big(D^{-1}(Q_p M'_{u_0} A^p)\big) D^{-1} Q_p M'_{u_0} A^{p-q} = P_q$$

by the definition of D. Moreover, by virtue of (7) and the definition of P_0, we have

$$P_q P_r = P_r P_q = 0, \qquad q, r = 0,1,\dots,p, \quad q \neq r.$$

Set

$$\mathfrak{U}^0 = \bigoplus_{q=0}^{p} \mathfrak{U}_q^0, \qquad P = \sum_{q=0}^{p} P_q.$$

The operator $P \in \mathcal{L}(\mathfrak{U})$ is a projection with $\operatorname{im} P = \mathfrak{U}^0$. Let $\mathfrak{U}^1 = \ker P$. Then $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$.

We introduce the linear manifolds $\mathfrak{F}_q^0 = M'_{u_0}[\mathfrak{U}_q^0]$, $q = 0, 1, \dots, p-1$, and the operator $B = M'_{u_0}\widetilde{L}^{-1}(I - Q_p)$. Since $B[\mathfrak{F}_q^0] = \mathfrak{F}_{q+1}^0$, $q = 0, 1, \dots, p-1$, and $B[\mathfrak{F}_p^0] = \{0\}$, we have

$$B^q[\mathfrak{F}_r^0] = \begin{cases} \{0\}, & q + r > p, \\ \mathfrak{F}_{q+r}^0, & q + r \le p. \end{cases} \tag{8}$$

By analogy with the preceding, it follows from (8) that the operators $Q_q = B^q M'_{u_0} D^{-1} Q_p B^{p-q}$, $q = 0, 1, \dots, p-1$, are projections on $\mathfrak{F}_q^0$, and moreover,

$$Q_q Q_r = Q_r Q_q = 0, \qquad q, r = 0, 1, \dots, p, \quad q \ne r.$$

We set

$$\mathfrak{F}^0 = \bigoplus_{q=0}^{p} \mathfrak{F}_q^0, \qquad Q = \sum_{q=0}^{p} Q_q.$$

The operator $Q \in \mathcal{L}(\mathfrak{F})$ is a projection, and hence $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$, where $\mathfrak{F}^0 = \operatorname{im} Q$ and $\mathfrak{F}^1 = \ker Q$.

Note that

$$LA^q D^{-1} Q_p = B^{q-1} M'_{u_0} D^{-1} Q_p, \qquad q = 1, \dots, p, \tag{9}$$

by construction. Further,

$$BL = M'_{u_0}(I - P_0). \tag{10}$$

From (9) and (10) with $q = 1, \dots, p$, we obtain

$$LP_q = LP_q(I - P_0) = LA^q D^{-1} Q_p M'_{u_0} A^{p-q}(I - P_0) = B^{q-1} M'_{u_0} D^{-1} Q_p B^{p-q} M'_{u_0}(I - P_0) = Q_{q-1} L. \tag{11}$$

Let us rewrite Eq. (7) in the form

$$L\dot{u} = M'_{u_0} u + F(u) + f, \tag{12}$$

where $F = M - M'_{u_0} \in C^\infty(\mathfrak{U}; \mathfrak{F})$ by definition. By successively applying the projections Q_q, $q = 0, 1, \dots, p$, and $I - Q$ to (12), we obtain, by virtue of (11), the equivalent system

$$\begin{aligned} L\dot{u}_1^0 &= M'_{u_0} u_0^0 + F_0(u) + f_0^0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ L\dot{u}_p^0 &= M'_{u_0} u_{p-1}^0 + F_{p-1}(u) + f_{p-1}^0, \\ 0 &= M'_{u_0} u_p^0 + F_p(u) + f_p^0, \\ L\dot{u}^1 &= (I - Q)M(u) + f^1, \end{aligned} \tag{13}$$

where $u_q^0 \in \mathfrak{U}_q^0$, $f_q^0 \in \mathfrak{F}_q^0$, $F_q(u) = Q_q F(u) + Q_q M'_{u_0} u^1$, $q = 0, 1, \dots, p$, $u^1 \in \mathfrak{U}^1$, and $f^1 \in \mathfrak{F}^1$. Thus we have proved the following assertion.

Lemma 1. *Suppose that $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $M \in C^\infty(\mathfrak{U}; \mathfrak{F})$, L is a bisplitting operator, and conditions* A1) *and* A2) *are satisfied. Then Eq.* (6) *is equivalent to system* (13).

Remark 1. Under the assumptions of Lemma 1, the operator M is L-bounded at the point u_0 [5], and the point at infinity is a pole of order p of the operator function $(\mu L - M'_{u_0})^{-1}$.

Let us now study problem (6), (7). A *solution* of this problem is a vector function $u \in C^\infty((-t_0, t_0); \mathfrak{U})$, $t_0 = t_0(u_0) > 0$, satisfying Eq. (6) and condition (5). Here we encounter two difficulties. First, it is well known [2, 3, 11] that problem (5), (6) is not solvable for some $u_0 \in \mathfrak{U}$. Second, even if problem (5), (6) has a solution, it need not be unique [5]. To overcome the first (and partly the second) difficulty, we introduce the following notion.

Definition 1. A set $\mathfrak{B}^t \subset \mathfrak{U} \times \mathbb{R}$ will be called a *configuration space* of Eq. (6) if for any point $u_0 \in \mathfrak{U}$ such that $(u_0, 0) \in \mathfrak{B}^0$ there exists a unique solution of problem (5), (6), and moreover, $(u(t), t) \in \mathfrak{B}^t$.

Remark 2. If $\mathfrak{B}^t = \mathfrak{B} \times \mathbb{R}$, where $\mathfrak{B} \subset \mathfrak{U}$, then the set $\mathfrak{B}$ is called the *phase space* of Eq. (6) (see [2, 3, 5]).

To remove the second difficulty completely, we restrict our considerations to quasistationary trajectories of Eq. (6), that is, solutions $u = u(t)$ of problem (5), (6) such that $L\dot{u}^0 \equiv 0$ for all $t \in (-t_0, t_0)$, where $u^0 = Pu$ [5]. To single out the quasistationary trajectories from the set of all possible solutions of problem (5), (6), we impose several conditions.

A3) $f_q^0(t) \equiv 0$ for all $t \in \mathbb{R}$ and $q = 1, \dots, p$.

Let us consider the set $\widetilde{\mathfrak{U}} = \{u \in \mathfrak{U} : u_q^0 = \text{const}, \ q = 1, \dots, p\}$. It is easily seen that $\widetilde{\mathfrak{U}}$ is a complete affine manifold modeled by the subspace $\mathfrak{U}_0^0 \oplus \mathfrak{U}^1$. Let $u_0 \in \widetilde{\mathfrak{U}}$; by $\mathfrak{O}_{u_0}$ we denote some neighborhood $\mathfrak{O}_{u_0} \subset \widetilde{\mathfrak{U}}$ of the point u_0.

A4) $F_q(u) \equiv 0$ for all $u \in \mathfrak{O}_{u_0}$ and $q = 1, \dots, p$.

Theorem 1. *Suppose that*

1) *the assumptions of Lemma* 1 *are satisfied*;
2) $(u_0, 0) \in \mathfrak{B}^0$, *where* $\mathfrak{B}^t = \{(u, t) \in \widetilde{\mathfrak{U}} \times \mathbb{R} : Q_0(M(u) + f(t)) = 0\}$;
3) $f \in C^\infty(\mathbb{R}; \mathfrak{F})$;
4) *conditions* A3) *and* A4) *hold.*

Then problem (5), (6) *has a unique solution, which is a quasistationary trajectory, and moreover,* $u(t) \in \mathfrak{B}^t$ *for every* $t \in (-t_0, t_0)$.

Proof. Suppose that we have found a solution of problem (5), (6). Then by virtue of conditions A3) and A4) it follows from (13) that $L\dot{u}^0 \equiv 0$, that is, the solution is a quasistationary trajectory. Let us establish the existence and uniqueness of the solution.

Lemma 1 and conditions A3) and A4) imply that system (13) can be reduced in the neighborhood $\mathfrak{O}_{u_0}$ to the form

$$\begin{aligned} 0 &= M'_{u_0} u_0^0 + F_0(u) + f_0^0, \\ L\dot{u}^1 &= (I - Q)M(u) + f^1. \end{aligned} \tag{14}$$

Note that, by construction, the operator $M'_{u_0} : \mathfrak{U}_0^0 \to \mathfrak{F}_0^0$ is nondegenerate and

$$F'_{0u}|_{u=u_0} \equiv 0,$$

where F'_{0u} is the Fréchet derivative of F_0 at u. By the implicit function theorem, it follows that there exists a neighborhood $\mathfrak{O}^1_{u_0} \subset (I - P)[\mathfrak{O}_{u_0}]$ and a vector function $\delta \in C^\infty(\mathfrak{O}^1_{u_0} \times \mathbb{R}; \mathfrak{O}^0_{u_0})$, where $\mathfrak{O}^0_{u_0} = P[\mathfrak{O}_{u_0}]$, such that

$$u(t) = u_0^0(t) + \sum_{q=1}^p u_q^0 + u^1 \in \mathfrak{B}^t \qquad \forall t \in \mathbb{R}.$$

Here $u_0^0(t) = \delta(u^1, t)$ for any $u^1 \in \mathfrak{O}^1_{u_0}$, and $u_q^0 = P_q u_0 = \text{const}$ for $q = 1, \dots, p$.

Next, it follows from (11) that $QL = LP$. This means that the operator L acts in the spaces $L : \mathfrak{U}^1 \to \mathfrak{F}^1$. Let L_1 be the restriction of L to $\mathfrak{U}^1$. The operator $L_1 \in \mathcal{L}(\mathfrak{U}^1; \mathfrak{F}^1)$ is injective by construction. Let us establish that it is surjective. Let $f^1 \in \mathfrak{F}^1$. Then the element $\widetilde{u} = \widetilde{L}^{-1} f^1 \in \operatorname{coim} L$ is well defined. Suppose that $P\widetilde{u} \neq 0$, that is,

$$P\widetilde{u} = \sum_{q=1}^p P_q \widetilde{u} = \sum_{q=1}^p \widetilde{u}_q^0 \neq 0.$$

Then

$$L\widetilde{u} = LP\widetilde{u} + L(I - P)\widetilde{u} = \sum_{q=1}^p L\widetilde{u}_q^0 + L_1(I - P)\widetilde{u} = f^1 \notin \mathfrak{F}^1.$$

This is a contradiction. Hence $L_1\colon \mathfrak{U}^1 \to \mathfrak{F}^1$ is a continuous bijection, and furthermore, L^{-1} is the restriction of $\widetilde{L}^{-1}$ to $\mathfrak{F}^1$.

It follows from the preceding that system (14) on $\mathfrak{D}^1_{u_0}$ can be reduced to the form

$$\dot{u}^1 = L^{-1}(I-Q)M\big(\delta(u^1,t)+(P-P_0)u_0+u^1\big)+g(t)\equiv \Phi(u^1,t), \tag{15}$$

where $\Phi \in C^\infty(\mathfrak{D}^1_{u_0}\times\mathbb{R};\mathfrak{U}^1)$ and $g(t)=L^{-1}f^1(t)$. The unique local solvability of the Cauchy problem $u^1(0)=(I-P)u_0$ for Eq. (15) is a classical result [12]. The desired quasistationary trajectory has the form $u(t)=\delta(u^1(t),t)+u^1(t)$, where $u^1\in C^\infty\big((-t_0,t_0);\mathfrak{D}^1_{u_0}\big)$ is the solution of the Cauchy problem for Eq. (15). □

Remark 3. By analyzing the proof of Theorem 1, one can see that the initial value can be taken arbitrarily from some neighborhood of u_0 in $\mathfrak{B}^0$. Thus $\mathfrak{B}^t$ is locally a configuration space.

§2. Interpretation of the formal scheme

Following [5], we reduce problem (2), (4) to problem (5), (6). To this end, we set

$$\mathfrak{U}=\mathbf{H}^2_\sigma\times\mathbf{H}^2_\pi\times\mathbf{H}_p,\qquad \mathfrak{F}=\mathbf{H}_\sigma\times\mathbf{H}_\pi\times\mathbf{H}_p, \tag{16}$$

where $\mathbf{H}_\sigma$ is the closure with respect to the norm of $\mathbf{L}^2(\Omega)=\big(L^2(\Omega)\big)^n$ of the linear manifold $\{\mathbf{v}\in\big(C_0^\infty(\Omega)\big)^n : \nabla\cdot\mathbf{v}=0\}$ of solenoidal vectors, $\mathbf{H}_\pi=\mathbf{H}^1_\sigma$, and $\mathbf{H}_p=\mathbf{H}_\pi$. Let $\Sigma\colon \mathbf{L}^2(\Omega)\to\mathbf{H}_\sigma$ be the orthogonal projection. Then $\Sigma\in\mathcal{L}\big(\big(W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega)\big)^n\big)$. We write $\operatorname{im}\Sigma=\mathbf{H}^2_\sigma$ and $\ker\Sigma=\mathbf{H}^2_\pi$ and define operators $L,M\colon\mathfrak{U}\to\mathfrak{F}$ by the formulas

$$L:=\begin{pmatrix}\Sigma A_\varkappa\Sigma & \Sigma A_\varkappa\Pi & 0\\ \Pi A_\varkappa\Sigma & \Pi A_\varkappa\Pi & 0\\ 0&0&0\end{pmatrix}, \tag{17}$$

where $\Pi=I-\Sigma$, $A_\varkappa=1-\varkappa\nabla^2$;

$$M(u):=\begin{pmatrix}\Sigma B(u_\sigma+u_\pi)\\ \Pi B(u_\sigma+u_\pi)-u_p\\ C(u_\sigma+u_\pi)\end{pmatrix}, \tag{18}$$

where $B(u_\sigma+u_\pi):=\nu\nabla^2(u_\sigma+u_\pi)-\big((u_\sigma+u_\pi)\cdot\nabla\big)(u_\sigma+u_\pi)$, $C(u_\sigma+u_\pi):=-\nabla\big(\nabla\cdot(u_\sigma+u_\pi)\big)$, $u=(u_\sigma,u_\pi,u_p)$.

Lemma 2. *Let spaces $\mathfrak{U}$ and $\mathfrak{F}$ be defined by formulas* (16), *where $n=2,3,4$, and let operators $L,M\colon\mathfrak{U}\to\mathfrak{F}$ be defined by formulas* (17) *and* (18). *Then*

1) $L\in\mathcal{L}(\mathfrak{U};\mathfrak{F})$; *moreover, if $\varkappa^{-1}\notin\sigma(-\nabla^2)$, then $\ker L=\{0\}\times\{0\}\times\mathbf{H}_p$ and $\operatorname{im}L=\mathbf{H}_\sigma\times\mathbf{H}_\pi\times\{0\}$*;
2) $M\in C^\infty(\mathfrak{U};\mathfrak{F})$.

Proof. Assertion 1) is obvious, and Assertion 2) can be verified in a straightforward manner. Let us only point out that

$$M'_u=\begin{pmatrix}\Sigma B_\sigma & \Sigma B_\pi & 0\\ \Pi B_\sigma & \Pi B_\pi & -I\\ 0 & C & 0\end{pmatrix}, \tag{19}$$

where B_σ (B_π) is the partial Fréchet derivative of the operator B at the point $u_\sigma+u_\pi$ with respect to u_σ (u_π). □

We set $f=(f_\sigma,f_\pi,0)$; thus completing the reduction of problem (2), (4) to problem (5), (6). Next, let us verify conditions A1)–A4). To verify the validity of A1), we consider the restriction $A_{\varkappa\sigma}$ of the operator $\Sigma A_\varkappa\Sigma$ to $\mathbf{H}^2_\sigma$.

Lemma 3. *Suppose that the assumptions of Lemma 2 are satisfied, and moreover,* $\ker A_{\varkappa\sigma} = \{0\}$. *Then each vector* $\varphi \in \ker L \setminus \{0\}$ *has exactly one* M'_u*-associated vector regardless of the point* $u \in \mathfrak{U}$.

Proof. Suppose that $\varphi \in (0, 0, \varphi_p) \in \ker L$, $\varphi_p \notin 0$. Let us find a vector $\psi \in \mathfrak{U}$ such that $L\psi = M'_u\varphi$. It follows from (17) and (19) that

$$A_{\varkappa\sigma}\psi_\sigma + \Sigma A_\varkappa \psi_\pi = 0, \qquad \Pi A_\varkappa \psi_\sigma + \Pi A_\varkappa \psi_\pi = -\varphi_p. \tag{20}$$

By the Solonnikov–Vorovich–Yudovich theorem (see [3]), the inverse operator $A_{\varkappa\sigma}^{-1} \in \mathcal{L}(\mathbf{H}_\sigma, \mathbf{H}_\sigma^2)$ exists. Hence from (20) we obtain $\psi_\sigma = -A_{\varkappa\sigma}\Sigma A_\varkappa \psi_\pi$. It follows that if $\psi_\pi = 0$, then $\psi_\sigma = 0$ and $\varphi_p = 0$. Thus $\psi_\pi \neq 0$. □

Let

$$\widetilde{L}^{-1} = \begin{pmatrix} \Sigma A_\varkappa^{-1}\Sigma & \Sigma A_\varkappa^{-1}\Pi & 0 \\ \Pi A_\varkappa^{-1}\Sigma & \Pi A_\varkappa^{-1}\Pi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Since

$$\widetilde{L}^{-1}L = \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathfrak{U}), \qquad L\widetilde{L}^{-1} = \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathfrak{F}),$$

we see that $\psi_\sigma = -\Sigma A_\varkappa^{-1}\varphi_p$, $\psi_\pi = -\Pi A_\varkappa^{-1}\varphi_p$, and the component ψ_p of the vector ψ may be arbitrary.

Next,

$$M'_u\psi = \begin{pmatrix} \Sigma(B_\sigma\psi_\sigma + B_\pi\psi_\pi) \\ \Pi(B_\sigma\psi_\sigma + B_\pi\psi_\pi) - \varphi_p \\ C\psi_\pi \end{pmatrix}.$$

Since $\psi_\pi \neq 0$, it follows that $C\psi_\pi \neq 0$ [13]. We conclude that $M'_u\psi \notin \operatorname{im} L$ regardless of $u \in \mathfrak{U}$.

Thus condition A1) is satisfied, and moreover, $p = 1$. Let us verify A2). By $A_{\varkappa\pi}$ we denote the restriction of the operator $\Pi A_{\varkappa\pi}^{-1}\Pi$ to $\mathbf{H}_\pi$.

Lemma 4. *Under the assumptions of Lemma 3, the operator* $A_{\varkappa\pi}\colon \mathbf{H}_\pi \to \mathbf{H}_\pi^2$ *is a linear topological isomorphism.*

Proof. By construction, $A_{\varkappa\pi} \in \mathcal{L}(\mathbf{H}_\pi, \mathbf{H}_\pi^2)$. Let us verify that $A_{\varkappa\pi}$ is bijective. Suppose that $f_\pi \in \ker A_{\varkappa\pi}$. Then $A_\varkappa^{-1} f_\pi = u_\sigma \in \mathbf{H}_\sigma^2$, that is, $f_\pi = A_\varkappa u_\sigma$. It follows that $A_{\varkappa\sigma}u_\sigma = 0$; hence $u_\sigma = 0$, and so $f_\pi = 0$. Thus $A_{\varkappa\pi}$ is injective.

Let us prove that this operator is surjective. Suppose that $u_\pi \in \mathbf{H}_\pi^2$. We set $u_\sigma = -A_{\varkappa\sigma}^{-1}\Sigma A_\varkappa u_\pi \in \mathbf{H}_\sigma^2$. Then

$$\Sigma A_\varkappa u_\sigma + \Sigma A_\varkappa u_\pi = 0, \qquad \Pi A_\varkappa u_\sigma + \Pi A_\varkappa u_\pi = f_\pi.$$

It follows that $A_\varkappa(u_\sigma + u_\pi) = f_\pi$, that is, $u_\sigma + u_\pi = A^{-1} f_\pi$, $u_\pi = A_{\varkappa\pi} f_\pi$. The proof is complete. □

By Lemma 2, the operator L in (17) is bisplitting. We set $\mathfrak{U}_0^0 = \ker L$ and $\operatorname{coim} L = \mathbf{H}_\sigma^2 \times \mathbf{H}_\pi^2 \times \{0\}$ and construct the linear manifolds

$$\mathfrak{F}_0^0 = M'_{u_0}[\mathfrak{U}_0^0] = \{0\} \times \mathbf{H}_p \times \{0\} = \{0\} \times \mathbf{H}_\pi \times \{0\} \subset \operatorname{im} L,$$
$$\mathfrak{U}_1^0 = \widetilde{L}^{-1}[\mathfrak{F}_0^0] = \Sigma A_\varkappa^{-1}[\mathbf{H}_p] \times A_{\varkappa\pi}[\mathbf{H}_p] \times \{0\} = \Sigma A_\varkappa^{-1} A_{\varkappa\pi}^{-1}[\mathbf{H}_\pi^2] \times \mathbf{H}_\pi^2 \times \{0\}.$$

By Lemma 4,

$$\mathfrak{F}_1^0 = M'_{u_0}[\mathfrak{U}_1^0] = \Sigma B_0 A_\varkappa^{-1}[\mathbf{H}_p] \times \Pi B_0 A_\varkappa^{-1}[\mathbf{H}_p] \times C A_\varkappa^{-1}[\mathbf{H}_p].$$

Let $\widetilde{C}$ be the restriction of C to $\mathbf{H}_\pi^2$. Since the inverse $\widetilde{C}^{-1}$ exists [13], it follows from Lemma 4 that

$$\mathfrak{F}_1^0 = \Sigma B_0 A_\varkappa^{-1} A_{\varkappa\pi}^{-1}\widetilde{C}^{-1}[\mathbf{H}_p] \times \Pi B_0 A_\varkappa^{-1} A_{\varkappa\pi}^{-1}\widetilde{C}^{-1}[\mathbf{H}_p] \times \mathbf{H}_p.$$

Here and in the preceding B_0 stands for the Fréchet derivative of B at $u_\sigma^0 + u_\pi^0$, and the operator $\widetilde{L}^{-1}$ is defined in (21).

Let us construct the operators

$$P_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Pi \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & P_1^{12} & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

where $P_1^{12} = \Sigma A_\varkappa^{-1} A_{\varkappa\pi}^{-1}\Pi$, and the operators

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ Q_0^{21} & \Pi & Q_0^{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0 & 0 & Q_1^{13} \\ 0 & 0 & Q_1^{23} \\ 0 & 0 & \Pi \end{pmatrix}, \tag{23}$$

where

$$Q_1^{13} = \Sigma B_0 A_\varkappa^{-1} A_{\varkappa\pi}^{-1}\widetilde{C}^{-1}\Pi,\ Q_1^{23} = \Pi B_0 A_\varkappa^{-1} A_{\varkappa\pi}^{-1}\widetilde{C}^{-1}\Pi,\ Q_0^{21} = -\Pi A_\varkappa A_{\varkappa\sigma}^{-1}\Sigma,\ Q_0^{23} = -Q_0^{21}Q_1^{13} - Q_1^{23}.$$

One can readily see that the operators $P_k \in \mathcal{L}(\mathfrak{U})$ and $Q_k \in \mathcal{L}(\mathfrak{F})$, $k = 0, 1$, are projections; moreover, $\operatorname{im} P_k = \mathfrak{U}_k^0$, $\operatorname{im} Q_k = \mathfrak{F}_k^0$, $k = 0, 1$, and $P_0P_1 = P_1P_0 = \mathbf{0}$, $Q_0Q_1 = Q_1Q_0 = \mathbf{0}$. Since $\ker Q_1 = \operatorname{im}(I - Q_1) = \operatorname{im} L$, it follows that $\mathfrak{F}_1^0 \oplus \operatorname{im} L = \mathfrak{F}$, that is, condition A2) is satisfied.

Condition A3) is satisfied, since $Q_1 f = Q_1(f_\sigma, f_\pi, 0) = (0, 0, 0)$.

To verify A4), let us consider the set $\widetilde{\mathfrak{U}} = \{u \in \mathfrak{U} : P_1 u = \text{const}\} = \{u \in \mathfrak{U} : u_\pi = \text{const}\}$. In our case, condition A4) consists of the single equation

$$Q_1 M(u) = \begin{pmatrix} Q_1^{13} C(u_\sigma + u_\pi) \\ Q_1^{23} C(u_\sigma + u_\pi) \\ C(u_\sigma + u_\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is satisfied identically [13] provided that $u_\pi = 0$. Thus if we set $\widetilde{\mathfrak{U}} = \{u \in \mathfrak{U} : u_\pi = 0\}$, then A4) is satisfied.

Let us construct the set $\mathfrak{B}^t$. By Theorem 1,

$$\mathfrak{B}^t = \{(u, t) \in \widetilde{\mathfrak{U}} \times \mathbb{R} : Q_0(M(u) + f(t)) = 0\}.$$

Since

$$Q_0 \left(M \begin{pmatrix} u_\sigma \\ 0 \\ u_p \end{pmatrix} + \begin{pmatrix} f_\sigma(t) \\ f_\pi(t) \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff (Q_0^{21}\Sigma + \Pi)B(u_\sigma) - u_p + Q_1^{21} f_\sigma(t) + f_\pi(t) = 0$$

and

$$Q_0^{21}\Sigma + \Pi = A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1}\Sigma + A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1}\Pi = A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1}, \tag{24}$$

we have

$$\mathfrak{B}^t = \{(u, t) \in \mathfrak{U} \times \mathbb{R} : A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1}(B(u_\sigma) + f_\sigma(t)) + f_\pi(t) = u_p,\ u_\pi = 0\}. \tag{25}$$

To prove (24), note that

$$\Pi A_\varkappa^{-1} A_{\varkappa\sigma}\Sigma + \Pi A_\varkappa^{-1}\Pi A_\varkappa\Sigma = \Pi A_\varkappa^{-1}(\Sigma A_\varkappa + \Pi A_\varkappa)\Sigma = \mathbf{0}.$$

Hence

$$\Pi A_\varkappa^{-1} A_{\varkappa\sigma}\Sigma = -A_{\varkappa\pi}\Pi A_\varkappa\Sigma, \qquad A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1} A_{\varkappa\sigma}\Sigma = -\Pi A_\varkappa\Sigma, \qquad A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1}\Sigma = -\Pi A_\varkappa A_{\varkappa\sigma}^{-1}\Sigma = Q_0^{21}\Sigma.$$

We have proved the following theorem.

Theorem 2. *Suppose that the assumptions of Lemma 3 are satisfied. Let $f \in C^\infty(\mathbb{R}; L^2(\Omega))$, and let $(\mathbf{v}_0, 0) \in \mathfrak{B}^0$ (see (25)). Then for some $t_0 = t_0(\mathbf{v}_0)$ there exists a unique solution $(\mathbf{v}, \mathbf{p})$ of problem (2), (4) such that $\mathbf{v} \in C^\infty((-t_0, t_0); \mathbf{H}_\sigma^2)$, $\mathbf{v}_\pi = 0$, and $\mathbf{p} = A_{\varkappa\pi}^{-1}\Pi A_\varkappa^{-1}(B(\mathbf{v}_\sigma) + f_\sigma(t)) + f_\pi(t)$.*

Remark 4. Our formalism for problem (2), (4) is different from the one considered in [2, 3, 14].

Remark 5. If we do not intend to construct the set $\mathfrak{B}^t$, then the proof of Theorem 2 can be simplified [6, 7].

References

1. A. P. Oskolkov, "Initial-boundary value problems for the equations of motion of Kelvin–Voigt and Oldroyd fluids," *Trudy Mat. Inst. Steklov* [*Proc. Steklov Inst. Math.*], **179**, 126–164 (1988).
2. G. A. Sviridyuk and T. G. Sukacheva, "Phase spaces for a class of operator equations," *Differentsial'nye Uravneniya* [*Differential Equations*], **26**, No. 2, 250–258 (1990).
3. G. A. Sviridyuk and T. G. Sukacheva, "The Cauchy problem for a class of semilinear equations of the Sobolev type," *Sibirsk. Mat. Zh.* [*Siberian Math. J.*], **31**, No. 5, 109–119 (1990).
4. A. P. Oskolkov, "On the theory of Voigt fluids," *Zap. Nauchn. Sem. LOMI*, **96**, 233–236 (1980).
5. G. A. Sviridyuk, "Semilinear equations of the Sobolev type with relatively bounded operators," *Dokl. Akad. Nauk SSSR* [*Soviet Math. Dokl.*], **318**, No. 4, 828–831 (1991).
6. G. A. Sviridyuk, "A dynamic model of an incompressible viscoelastic fluid," *Izv. Vyssh. Uchebn. Zaved. Mat.* [*Soviet Math. (Iz. VUZ)*], No. 1, 74–79 (1988).
7. G. A. Sviridyuk, "The solution manifold of a problem for an incompressible viscoelastic fluid," *Differentsial'nye Uravneniya* [*Differential Equations*], **24**, No. 10, 1846–1848 (1988).
8. L. D. Landau and E. M. Lifshits, *Hydrodynamics* [in Russian], 3d ed., Nauka, Moscow (1986).
9. Yu. G. Borisovich, V. G. Zvyagin, and Yu. I. Sapronov, "Nonlinear Fredholm maps and the Leray–Schauder theory," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **32**, No. 4, 3–54 (1977).
10. M. M. Vainberg and V. A. Trenogin, *Bifurcation Theory of Solutions to Nonlinear Equations* [in Russian], Nauka, Moscow (1969).
11. H. A. Levine, "Some nonexistence and instability theorems for solutions of formally parabolic equation of the form $Pu_t = -Au + F(u)$," *Arch. Rational Mech. Anal.*, **51**, No. 5, 371–386 (1973).
12. S. Lang, *Introduction to Differentiable Manifolds*, Wiley, New York (1962).
13. L. V. Kapitanskii and K. N. Piletskas, "Some problems of vector analysis," *Zap. Nauchn. Sem. LOMI*, **138**, 65–85 (1984).
14. A. P. Oskolkov, "Nonlocal problems for a class of nonlinear operator equations arising in the theory of equations of the Sobolev type," *Zap. Nauchn. Sem. LOMI*, **198**, 31–48 (1991).

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