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Number of Lattice Points in the Hyperbolic Cross

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ABSTRACT. An asymptotic formula for the number of points of an arbitrary lattice in the hyperbolic cross is obtained.

KEY WORDS: complete lattice, polar lattice, asymptotic formulas for the number of lattice points, hyperbolic cross.

For $s \geq 2$, let $\Lambda \subset \mathbb{R}^s$ be a complete lattice. The domain $K(T) = \{\vec{x} \,|\, \overline{x}_1 \cdots \overline{x}_s \leq T\}$, where $\overline{x} = \max(1, |x|)$ for any real x, is called a *hyperbolic cross*. Let $D(T\,|\,\Lambda)$ be the number of nonzero points of the lattice Λ in the hyperbolic cross $K(T)$. The goal of this paper is to prove the following theorem.

Theorem. *For any lattice Λ and $T \geq 3$ the asymptotic equality*

$$D(T\,|\,\Lambda) = \frac{2^s T \ln^{s-1} T}{(s-1)! \det \Lambda} + \Theta C(\Lambda) \frac{T \ln^{s-2} T}{\det \Lambda},$$

where $C(\Lambda) = 2^s e(a_0 + 2)^s$ and $|\Theta| \leq 1$, holds.

§1. Notation

Let $\vec{\lambda}_j = (\lambda_{j1}, \dots, \lambda_{js})$ $(j = 1, \dots, s)$ be an arbitrary fixed basis of the lattice Λ, and let

$$A = A(\vec{\lambda}_1, \dots, \vec{\lambda}_s) = \max_{1 \leq j \leq s} \frac{1}{2} \sum_{\nu=1}^{s} |\lambda_{\nu j}|. \tag{1}$$

We denote by $\vec{\lambda}^*_j = (\lambda^*_{j1}, \dots, \lambda^*_{js})$ $(j = 1, \dots, s)$ the reciprocal basis of the polar lattice of Λ^* (as is known [1], the reciprocal basis is defined by the relations

$$(\vec{\lambda}_i, \vec{\lambda}^*_j) = \sum_{\nu=1}^{s} \lambda_{i\nu} \lambda^*_{j\nu} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

and the polar lattice Λ^* is uniquely defined by the lattice Λ).

We introduce the following domains:

$$\Pi(T\,|\,\Lambda) = \left\{ \vec{t} : \prod_{j=1}^{s} \overline{\sum_{\nu=1}^{s} \lambda_{\nu j} t_\nu + \sum_{\nu=1}^{s} \lambda_{\nu j} \Big(\frac{1}{2} - \Big\{ t_\nu + \frac{1}{2} \Big\} \Big)} \leq T \right\};$$

for an integer vector $\vec{m}$,

$$\Pi(\vec{m}) = \left\{ \vec{t} : \Big[t_\nu + \frac{1}{2} \Big] = m_\nu, \ \nu = 1, \dots, s \right\};$$

$$\Pi^*(T\,|\,\Lambda) = \left\{ \vec{y} : \prod_{j=1}^{s} \overline{y_j + \sum_{\nu=1}^{s} \lambda_{\nu j} \Big(\frac{1}{2} - \Big\{ \frac{1}{2} + \sum_{k=1}^{s} y_k \lambda^*_{\nu k} \Big\} \Big)} \leq T \right\};$$

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for $a \geq 0$,

$$\Pi_1(T,a) = \left\{ \vec{y} : \prod_{j=1}^{s} \overline{|y_j| + a} \leq T \right\}, \quad \Pi_2(T,a) = \left\{ \vec{y} : \prod_{j=1}^{s} u(y_j, a) \leq T \right\},$$

where

$$u(y,a) = \begin{cases} 1 & \text{for } |y| \leq a+1, \\ |y| - a & \text{for } |y| \geq a+1. \end{cases}$$

Notice that $\Pi_1(T,0) = \Pi_2(T,0) = K(T)$. For $a \geq 0$ and $T \geq 0$, let

$$I_s(a,T) = \int_{\substack{\prod_{j=1}^{s}(y_j + a) \leq T \\ y_1, \ldots, y_s \geq 0}} d\vec{y}, \qquad J_s(a,T) = \int_{\substack{\prod_{j=1}^{s} u(y_j, a) \leq T \\ y_1, \ldots, y_s \geq 0}} d\vec{y}.$$

§2. Auxiliary lemmas

Lemma 1.

$$\sum_{\prod_{j=1}^{s} \overline{\lambda_{1j} m_1 + \cdots + \lambda_{sj} m_s} \leq T} \int_{\Pi(\vec{m})} d\vec{t} = \int_{\Pi(T \mid \Lambda)} d\vec{t}.$$

Proof. For any two distinct integer vectors $\vec{n}$ and $\vec{m}$,the domains $\Pi(\vec{n})$ and $\Pi(\vec{m})$ are disjoint. On the other hand, if $m_\nu = [t_\nu + 1/2]$ ($\nu = 1, \ldots, s$), then

$$\sum_{\nu=1}^{s} \lambda_{\nu j} t_\nu + \sum_{\nu=1}^{s} \lambda_{\nu j} \Big(\frac{1}{2} - \Big\{ t_\nu + \frac{1}{2} \Big\} \Big) = \sum_{\nu=1}^{s} \lambda_{\nu j} m_\nu,$$

and so

$$\Pi(T \mid \Lambda) = \bigcup_{\prod_{j=1}^{s} \overline{\lambda_{1j} m_1 + \cdots + \lambda_{sj} m_s} \leq T} \Pi(\vec{m}).$$

Now the statement of the lemma follows from the additivity of the integral. □

Lemma 2.

$$\int_{\Pi(T \mid \Lambda)} d\vec{t} = \frac{1}{\det \Lambda} \int_{\Pi^*(T \mid \Lambda)} d\vec{y}, \tag{2}$$

$$\int_{\Pi_1(T,a)} d\vec{y} = 2^s I_s(a,T) \qquad \textit{for} \quad a \geq 1, \tag{3}$$

$$\int_{\Pi_2(T,a)} d\vec{y} = 2^s J_s(a,T) \qquad \textit{for} \quad a \geq 0. \tag{4}$$

Proof. Set

$$\vec{y} = (y_1, \ldots, y_s) = t_1 \vec{\lambda}_1 + \cdots + t_s \vec{\lambda}_s. \tag{5}$$

Then

$$y_\nu = \sum_{j=1}^{s} t_j \lambda_{j\nu} \quad (\nu = 1, \ldots, s), \qquad t_j = (\vec{y}, \vec{\lambda}_j^*) = \sum_{\nu=1}^{s} y_\nu \lambda_{j\nu}^*.$$

After the change of variables (5), we get $d\vec{t} = (\det \Lambda)^{-1} d\vec{y}$ and

$$\int_{\Pi(T \mid \Lambda)} d\vec{t} = \frac{1}{\det \Lambda} \int_{\prod_{j=1}^{s} \overline{y_j + \sum_{\nu=1}^{s} \lambda_{\nu j}(1/2 - \{1/2 + \sum_{k=1}^{s} y_k \lambda_{\nu k}^*\})} \leq T} d\vec{y} = \frac{1}{\det \Lambda} \int_{\Pi^*(T \mid \Lambda)} d\vec{y},$$

completing the proof of equation (2).

We have $|y| + a = \overline{|y| + a}$ for any $a \geq 1$; hence, by the symmetry of the domain $\Pi_1(T,a)$, the relation

$$\int_{\Pi_1(T,a)} d\vec{y} = 2^s \int_{\substack{\prod_{j=1}^{s}(y_j + a) \leq T \\ y_1, \ldots, y_s \geq 0}} d\vec{y} = 2^s I_s(a,T)$$

follows, completing the proof of (3).

Finally, (4) follows from the symmetry of the domain $\Pi_2(T,a)$ for any $a \geq 0$ □

Lemma 3. *For any $a > 0$ and $T \geq a^s$,*

$$I_s(a,T) = (-1)^{s+1}(T - a^s) + T\sum_{n=1}^{s-1}\frac{(\ln T - s\ln a)^n(-1)^{s-1-n}}{n!}. \tag{6}$$

Proof. First of all, we establish a recurrence relation for $I_s(a,T)$. We have

$$\begin{aligned} I_s(a,T) &= \int_0^{T/a^{s-1}-a} dy_s \int_{\substack{\prod_{j=1}^{s-1}(y_j+a)\leq \frac{T}{y_s+a} \\ y_1,\dots,y_{s-1}\geq 0}} dy_1\cdots dy_{s-1} \\ &= \int_0^{T/a^{s-1}-a} I_{s-1}\left(a, \frac{T}{y+a}\right) dy = \int_a^{T/a^{s-1}} I_{s-1}\left(a, \frac{T}{y}\right) dy. \end{aligned} \tag{7}$$

We shall use (7) and proceed by induction on s. For $s = 1$ we have

$$I_1(a,T) = \int_0^{T-a} dy = T - a$$

and (6) is valid. Eq. (7) and the induction conjecture for $s-1$ yield

$$\begin{aligned} I_s(a,T) &= \int_a^{T/a^{s-1}} \Bigg((-1)^s\left(\frac{T}{y} - a^{s-1}\right) \\ &\quad + \frac{T}{y}\sum_{n=1}^{s-2}\frac{(\ln(T/y) - (s-1)\ln a)^n(-1)^{s-2-n}}{n!}\Bigg) dy \\ &= (-1)^s T(\ln T - (s-1)\ln a - \ln a) + (-1)^{s+1}a^{s-1}\left(\frac{T}{a^{s-1}} - a\right) \\ &\quad + T\int_{\ln a}^{\ln T-(s-1)\ln a}\sum_{n=1}^{s-2}\frac{(\ln T - u - (s-1)\ln a)^n(-1)^{s-2-n}}{n!} du \\ &= (-1)^{s+1}(T - a^s) + (-1)^s T(\ln T - s\ln a) + T\sum_{n=1}^{s-2}\frac{(-1)^{s-2-n}}{n!}\int_0^{\ln T - s\ln a} t^n\, dt \\ &= (-1)^{s+1}(T - a^s) + T\sum_{n=1}^{s-2}\frac{(-1)^{s-2-n}}{n!}\frac{(\ln T - s\ln a)^{n+1}}{n+1} + (-1)^s T(\ln T - s\ln a) \\ &= (-1)^{s+1}(T - a^s) + T\sum_{n=1}^{s-1}\frac{(-1)^{s-1-n}(\ln T - s\ln a)^n}{n!}, \end{aligned}$$

which completes the proof of the lemma. □

Lemma 4. *For any $a \geq 0$, $T \geq 1$, and $s \geq 1$,*

$$J_s(a,T) = a^s + \sum_{n=0}^{s-1}\frac{T\ln^n T}{n!}\sum_{k=0}^{s-1-n} C_s^k C_{s-k-1}^n a^k. \tag{8}$$

Proof. By the definition of $J_s(a,T)$, we have

$$\begin{aligned} J_s(a.T) &= \int_0^{a+1} dy_s \int_{\substack{y_1,\dots,y_{s-1}\geq 0 \\ u(y_1,a)\cdots u(y_{s-1},a)\leq T}} dy_1\cdots dy_{s-1} \\ &\quad + \int_{a+1}^{T+a} dy_s \int_{\substack{y_1,\dots,y_{s-1}\geq 0 \\ u(y_1,a)\cdots u(y_{s-1},a)\leq T/(y-a)}} dy_1\cdots dy_{s-1} \\ &= (a+1)J_{s-1}(a,T) + \int_1^T J_{s-1}\left(a, \frac{T}{y}\right) dy. \end{aligned} \tag{9}$$

We use the recurrence relation (9) and proceed by induction on s.
For $s = 1$ we have

$$J_1(a,T) = \int_{\substack{y\geq 0\\ u(y,a)\leq T}} dy = \int_0^{a+1} dy + \int_{a+1}^{a+T} dy = a+1+T-1 = a+T,$$

which agrees with equality (8).
Set

$$Q_{s,n}(a) = \sum_{k=0}^{s-1-n} C_s^k C_{s-k-1}^n a^k, \qquad J_s(a,T) = a^s + \sum_{n=0}^{s-1} \frac{T\ln^n T}{n!} Q_{s,n}(a).$$

Then

$$\begin{aligned}
J_{s+1}(a,T) &= (a+1)J_s(a,T) + \int_1^T J_s\left(a, \frac{T}{y}\right) dy \\
&= (a+1)a^s + \sum_{n=0}^{s-1} \frac{T\ln^n T}{n!} Q_{s,n}(a)(a+1) + \int_1^T \left(a^s + \sum_{n=0}^{s-1} \frac{T/y\cdot \ln^n(T/y)}{n!} Q_{s,n}(a)\right) dy \\
&= a^{s+1} + a^s + Ta^s - a^s + \sum_{n=0}^{s-1} \frac{T\ln^n T}{n!} Q_{s,n}(a)(a+1) + \sum_{n=0}^{s-1} \frac{T}{n!} Q_{s,n}(a) \frac{\ln^{n+1} T}{n+1} \\
&= a^{s+1} + T(a^s + Q_{s,0}(a)(a+1)) + \sum_{n=1}^{s-1} \frac{T\ln^n T}{n!}(Q_{s,n}(a)(a+1) + Q_{s,n-1}(a)) + \frac{T\ln^s T}{s!} Q_{s,s-1}(a) \\
&= a^{s+1} + \sum_{n=0}^{s} \frac{T\ln^n T}{n!} Q_{s+1,n}(a),
\end{aligned}$$

where

$$\begin{aligned}
Q_{s+1,0}(a) &= a^s + (a+1)\sum_{k=0}^{s-1} C_s^k a^k = \sum_{k=1}^{s} C_s^{k-1} a^k + \sum_{k=0}^{s-1} C_s^k a^k + a^s \\
&= sa^s + \sum_{k=1}^{s-1}(C_s^{k-1} + C_s^k)a^k + 1 = \sum_{k=0}^{s} C_{s+1}^k a^k = \sum_{k=0}^{s-1-0} C_{s+1}^k C_{s+1-k-1}^0 a^k, \\
Q_{s+1,s}(a) &= Q_{s,s-1}(a) = 1 = \sum_{k=0}^{(s+1)-1-s} C_{s+1}^k C_{s+1-k-1}^s a^k,
\end{aligned}$$

and for $1 \leq n \leq s-1$,

$$\begin{aligned}
Q_{s+1,n}(a) &= Q_{s,n}(a)(a+1) + Q_{s,n-1}(a) \\
&= (a+1)\sum_{k=0}^{s-1-n} C_s^k C_{s-k-1}^n a^k + \sum_{k=0}^{s-1-(n-1)} C_s^k C_{s-k-1}^{n-1} a^k \\
&= \sum_{k=1}^{s-n} C_s^{k-1} C_{s-(k-1)-1}^n a^k + \sum_{k=0}^{s-1-n} C_s^k (C_{s-k-1}^n + C_{s-k-1}^{n-1}) a^k + C_s^{s-n} C_{n-1}^{n-1} a^{s-n} \\
&= \sum_{k=1}^{s-n} C_s^{k-1} C_{s-k}^n a^k + \sum_{k=0}^{s-n} C_s^k C_{s-k}^n a^k = \sum_{k=1}^{s-n} (C_s^{k-1} + C_s^k) C_{s-k}^n a^k + C_s^0 C_{s-0}^n a^0 \\
&= \sum_{k=0}^{s-n} (C_{s+1}^k C_{(s+1)-1-k}^n) a^k.
\end{aligned}$$

This means that $J_{s+1}(a,T)$ obeys (8), completing the proof of the lemma. □

Corollary 1. *For any $a > 1$ and $T \geq 3$, we have*

$$I_s(a,T) \geq \frac{T\ln^{s-1}T}{(s-1)!} - ea^sT\ln^{s-2}T - a^s. \tag{10}$$

Proof. Indeed, we have

$$\begin{aligned}
I_s(a,T) &= (-1)^{s+1}(T-a^s) + T\sum_{n=1}^{s-1}\frac{(\ln T - s\ln a)^n(-1)^{s-1-n}}{n!} \\
&= T\sum_{n=1}^{s-1}\frac{(-1)^{s-1-n}}{n!}\sum_{k=0}^{n}C_n^k\ln^k T(-1)^{n-k}(s\ln a)^{n-k} + (-1)^{s+1}T + (-1)^s a^s \\
&= (-1)^s a^s + T\sum_{k=1}^{s-1}\ln^k T\sum_{n=k}^{s-1}\frac{(-1)^{s-1-n}(-1)^{n-k}}{n!}C_n^k(s\ln a)^{n-k} \\
&\quad + T\left((-1)^{s+1} + \sum_{n=1}^{s-1}\frac{(-1)^{s-1-n}}{n!}(-1)^n(s\ln a)^n\right) \\
&= (-1)^s a^s + (-1)^{s+1}T\sum_{n=0}^{s-1}\frac{(s\ln a)^n}{n!} + T\sum_{k=1}^{s-1}\ln^k T(-1)^{s-1-k}\sum_{n=0}^{s-k-1}\frac{(s\ln a)^n}{k!n!} \\
&= \frac{T\ln^{s-1}T}{(s-1)!} + T\sum_{k=0}^{s-2}\frac{\ln^k T(-1)^{s-1-k}}{k!}\sum_{n=0}^{s-k-1}\frac{(s\ln a)^n}{n!} + (-1)^s a^s \\
&\geq \frac{T\ln^{s-1}T}{(s-1)!} - T\ln^{s-2}T\sum_{k=0}^{s-2}\frac{1}{k!}\sum_{n=0}^{s-k-1}\frac{(s\ln a)^n}{n!} - a^s \\
&\geq \frac{T\ln^{s-1}T}{(s-1)!} - T\ln^{s-2}T\sum_{k=0}^{s-1}\frac{1}{k!}a^s - a^s \geq \frac{T\ln^{s-1}T}{(s-1)!} - ea^sT\ln^{s-2}T - a^s,
\end{aligned}$$

and the proof is complete. □

Corollary 2.

$$J_s(a,T) \leq \frac{T\ln^{s-1}T}{(s-1)!} + (a+2)^sT\ln^{s-2}T + a^s. \tag{11}$$

Proof. Indeed, we have

$$\begin{aligned}
J_s(a,T) &= \frac{T\ln^{s-1}T}{(s-1)!} + \sum_{n=0}^{s-2}\frac{T\ln^n T}{n!}\sum_{k=0}^{s-1-n}\frac{s!(s-k-1)!}{k!(s-k)!n!(s-k-n-1)!}a^k + a^s \\
&= \frac{T\ln^{s-1}T}{(s-1)!} + T\sum_{n=0}^{s-2}\frac{\ln^n T}{n!}C_s^n\sum_{k=0}^{s-1-n}\frac{s-n}{s-k}C_{s-n-1}^k a^k + a^s \\
&\leq \frac{T\ln^{s-1}T}{(s-1)!} + T\sum_{n=0}^{s-2}\frac{\ln^n T}{n!}C_s^n\frac{s-n}{n+1}(a+1)^{s-n-1} + a^s \\
&\leq \frac{T\ln^{s-1}T}{(s-1)!} + (a+2)^sT\ln^{s-2}T + a^s,
\end{aligned}$$

and the proof is complete. □

Proof of the theorem. From the definition of $D(T\mid\Lambda)$ and Lemma 1, it follows that

$$D(T\mid\Lambda)+1=\sum_{\substack{\vec{x}\in\Lambda\\ \overline{x_1\cdots x_s}\le T}}1=\sum_{\prod_{j=1}^{s}\overline{\lambda_{1j}m_1+\cdots+\lambda_{sj}m_s}\le T}1=\int_{\Pi(T\mid\Lambda)}d\vec{t}.$$

Applying Lemma 2, we obtain

$$D(T\mid\Lambda)+1=\frac{1}{\det\Lambda}\int_{\Pi^*(T\mid\Lambda)}d\vec{t}.$$

Let $a=\overline{A}$, where A is defined by (1). Then the inclusions $\Pi_1(T,a)\subseteq\Pi^*(T\mid\Lambda)\subseteq\Pi_2(T,a)$ hold, because

$$\prod_{j=1}^{s}u(|y_j|,a)\le\prod_{j=1}^{s}\overline{y_j+\sum_{\nu=1}^{s}\lambda_{\nu j}\left(\frac{1}{2}-\left\{\frac{1}{2}+\sum_{k=1}^{s}y_k\lambda^*_{\nu k}\right\}\right)}\le\prod_{j=1}^{s}(|y_j|+a).$$

These inclusions and Lemma 2 imply the inequalities

$$\frac{2^sI_s(a,T)}{\det\Lambda}\le D(T\mid\Lambda)+1\le\frac{2^sJ_s(a,T)}{\det\Lambda}.$$

Inequalities (10) and (11) yield

$$\frac{2^s}{\det\Lambda}\left(\frac{T\ln^{s-1}T}{(s-1)!}-ea^sT\ln^{s-2}T-a^s\right)\le D(T\mid\Lambda)+1$$
$$\le\frac{2^s}{\det\Lambda}\left(\frac{T\ln^{s-1}T}{(s-1)!}+T\ln^{s-2}T(a+2)^s+a^s\right).$$

It follows that

$$D(T\mid\Lambda)=\frac{2^sT\ln^{s-1}T}{(s-1)!\det\Lambda}+C(\Lambda)\Theta\frac{T\ln^{s-2}T}{\det\Lambda}$$

with $C(\Lambda)=2^se(a_0+2)^s$ and $a_0=\min\overline{A}(\vec{\lambda}_1,\dots,\vec{\lambda}_s)$, where the minimum is taken over all the bases of lattice Λ.

This completes the proof of the theorem. □

The main result of this work was announced in the summary [2].

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