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Asymptotic Analysis of Linearly Elastic Shells. I. Justification of Membrane Shell Equations

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Abstract

We consider a family of linearly elastic shells with thickness 2ε, clamped along their entire lateral face, all having the same middle surface $S = \boldsymbol{\varphi}(\overline{\omega}) \subset \mathbf{R}^3$, where $\omega \subset \mathbf{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary γ, and $\boldsymbol{\varphi} \in \mathscr{C}^3(\overline{\omega}; \mathbf{R}^3)$. We make an essential geometrical assumption on the middle surface S, which is satisfied if γ and $\boldsymbol{\varphi}$ are smooth enough and S is "uniformly elliptic", in the sense that the two principal radii of curvature are either both > 0 at all points of S, or both < 0 at all points of S.

We show that, if the applied body force density is $O(1)$ with respect to ε, the field $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$, where $u_i(\varepsilon)$ denote the three covariant components of the displacement of the points of the shell given by the equations of three-dimensional elasticity, once "scaled" so as to be defined over the fixed domain $\Omega = \omega \times]-1, 1[$, converges in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ as $\varepsilon \to 0$ to a limit $\boldsymbol{u}$, which is independent of the transverse variable. Furthermore, the average $\boldsymbol{\zeta} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u} dx_3$, which belongs to the space

$$\boldsymbol{V}_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

satisfies the (scaled) two-dimensional equations of a "membrane shell" viz.,

$$\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy = \int_\omega \left\{ \int_{-1}^{1} f^i dx_3 \right\} \eta_i \sqrt{a}\, dy$$

for all $\eta = (\eta_i) \in \boldsymbol{V}_M(\omega)$, where $a^{\alpha\beta\sigma\tau}$ are the components of the two-dimensional elasticity tensor of the surface S,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \tfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3$$

are the components of the linearized change of metric tensor of S, $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols of S, $b_{\alpha\beta}$ are the components of the curvature tensor of S,

and f^i are the scaled components of the applied body force. Under the above assumptions, the two-dimensional equations of a "membrane shell" are therefore justified.

Contents

Introduction

This is the first part of a three-part work, the second (Ciarlet, Lods & Miara [1996]) and third (Ciarlet & Lods [1996]) being henceforth simply referred to as "Part II" and "Part III".

Lower-dimensional plate, shell, and rod theories that rely on *a priori* assumptions of a mechanical or geometrical nature have been proposed by Cauchy, Sophie Germain, Kirchhoff, von Kármán, Love, Reissner, Jakob Bernoulli, Navier, Euler, Poisson, the Cosserats, Donnell, Flügge, Timoshenko, Novozhilov, Vekua, Green, Koiter, Simmonds, Naghdi, and others.

There are two reasons why these lower-dimensional theories are often preferred to the three-dimensional theory that they are supposed to "replace" when the thickness, or the diameter of the cross section, is "small enough".

One reason is their *simpler mathematical structure,* which in turn generates a richer variety of results. For instance, the existence, regularity, or bifurcation theories, and more generally "global analysis", are by now on firm mathematical grounds for nonlinearly elastic rods (see Antman [1995] for a scholarly and comprehensive exposition) or for nonlinearly elastic von Kármán plates (see Ciarlet & Rabier [1980]). By contrast, these theories are still partly in their infancies in nonlinear three-dimensional elasticity (see Marsden & Hughes [1978] and Ciarlet [1988] for comprehensive surveys): After the fundamental ideas set forth by Ball [1977], who was able to establish the existence of a minimizer of the energy for a wide class of realistic nonlinearly elastic materials, there indeed remain manifold challenging open problems; for instance, there is no known set of sufficient conditions guaranteeing that such a minimizer satisfies the equilibrium equations even in the weak sense of the principle of virtual work (another existence theory, based on the implicit function theorem, does not share this drawback, but it is restricted to problems with very smooth data and especially, to special boundary conditions, unrealistic in practice; see CIARLET [1988] and the comprehensive treatment of VALENT [1988]). The origin of this discrepancy is the *semi-linearity* of most lower-dimensional equations modeling nonlinearly elastic plates, shells, and rods, as opposed to the *quasi-linearity* of the equations of nonlinear three-dimensional elasticity.

Another virtue of lower-dimensional theories is their far better amenability to *numerical computations.* For instance, directly approximating the three-dimensional displacement field of a cooling tower seems out of reach at the present time, even in the linearly elastic realm: The existing codes use two-dimensional equations, such as those of KOITER; see BERNADOU [1994] for a comprehensive account. Likewise, although substantial progress has recently been achieved for directly approximating the "three-dimensional" displacement field of a linearly elastic rectangular plate (see BABUŠKA & LI [1992] and SCHWAB [1996]), current codes are almost invariably based on two-dimensional equations, such as those of the Kirchhoff-Love or Reissner-Mindlin models. Be that as it may, the *locking phenomenon* that arises in the numerical approximation of two-dimensional plate or shell equations still pose challenging problems; see in particular BREZZI, FORTIN & STENBERG [1991], PITKÄRANTA [1992], CHENAIS & ZERNER [1993], ARNOLD & BREZZI [1993, 1995], CHENAIS & PAUMIER [1994], ZERNER [1994], PAUMIER [1995].

Lower-dimensional models being thus widely used, two essential, and in fact intimately related, questions arise:

Given a "lower-dimensional" elastic body, together with specific loadings and boundary conditions, how to choose between the manifold lower-dimensional models that are available? For instance, given a linearly elastic shell, which model should be preferred, among those of KOITER, NAGHDI, NOVOZHILOV, BUDIANSKY & SANDERS? This question is of paramount *practical* importance, for *it makes no sense to devise accurate methods for approximating the solution of a "wrong" model!* Consequently, before approximating the exact solution of a given lower-dimensional model, we should first know whether it is "close enough" to the exact solution of the three-dimensional model it is intended to approximate. This observation leads to the second question:

How to mathematically justify in a rational fashion a lower-dimensional model from the three-dimensional model? This question has been answered through three different approaches.

The first approach consists *in directly estimating the difference between the three-dimensional solution and the solution of a given, i.e., "known in advance", lower-dimensional model* (this difference makes sense once the three-dimensional solution is properly averaged or the lower-dimensional one is extended in some fashion to a three-dimensional field). For *linearly elastic plates,* the first such estimate seems to be due to MORGENSTERN [1959], who cleverly used the dual variational principle of the linear theory; see also MORGENstern & Szabó [1961], Nordgren [1971], Simmonds [1971a], Shoiket [1976], Kohn & Vogelius [1985]. This approach was likewise successfully applied to *linearly elastic shells* by Koiter [1970] and Simmonds [1971b].

The *second approach* consists in using the *constraint method,* whose governing principle is an *a priori assumption that the admissible displacement fields are restricted to a specific form.* For a plate (to fix ideas), such "test functions" are finite sums of products of unspecified functions of the in-plane variables times given linearly independent functions of the "transverse" variable. The functions of the in-plane variables are then determined by inserting these test functions into the three-dimensional equations or into the three-dimensional energy, a process that leads to the solution of a finite number of two-dimensional boundary-value problems. Increasing the number of linearly independent functions of the transverse variable thus yields a "hierarchy" of models, which may be deemed two-dimensional, inasmuch as they are determined by solving two-dimensional problems.

References to this approach are numerous. For plates, we refer to Naghdi [1972], Podio-Guidugli [1989], Destuynder [1980, Ch. 5], Miara [1989], Schwab [1996]; for rods, to Miara & Trabucho [1992], Mascarenhas & Trabucho [1992], Figueiredo & Trabucho [1993], Antman [1972, 1995]; for shells, to Naghdi [1972], Figueiredo & Trabucho [1992], Podio-Guidugli [1990]; for a general analysis, to Antman [1976], Antman & Marlow [1991].

The two approaches described so far nevertheless rely on some *a priori* assumptions of a mechanical or geometrical nature, intended to account for the "smallness" of a geometrical parameter and intended to be more effective as this parameter approaches zero. Hence the need arises to *mathematically justify these* a priori *assumptions, together with the lower-dimensional theories they engender, directly from three-dimensional elasticity.* Otherwise, these assumptions and theories can be thought of as being "handed down by some higher power (a Hungarian wizard, say)", to quote Truesdell [1977, p. 601].

This direct justification is achieved by the *third approach,* which consists in applying an *asymptotic method.* It has recently received considerable attention, as exemplified by the books of Ciarlet [1990, 1997a] and Le Dret [1991] for plates; Le Dret [1991] and Trabucho & Viaño [1996] for beams (straight rods); Ciarlet [1997b] for shells.

In a *formal asymptotic method*, the three-dimensional solution (the displacement field and, in some cases, the stress field) is first "scaled" in an appropriate manner so as to be defined on a fixed domain, then expanded as a *formal series expansion* in terms of a "*small*" *parameter* ε, which is the "dimensionless" half-thickness of a plate or a shell, or the "dimensionless" half-diameter of the cross section of the rod. "Dimensionless" means that ε measures the *ratio* between the thickness or diameter and some "characteristic" dimension. For a cooling tower for instance, where common values for the average thickness and height are 0.3 m and 150 m, the ratio 2ε is thus equal to $\frac{1}{500}$. It is worthwhile to keep in mind this order of magnitude.

The formal series expansion of the scaled solution is then inserted into the three-dimensional boundary-value problem, and sufficiently many factors of the successive powers of ε found in this fashion are equated to zero until the leading term of the expansion can be computed and, hopefully, identified with the scaled solution of a known lower-dimensional problem. Such a method is "formal" in that the successive terms of the expansion, except the leading one, cannot usually "fully satisfy" the boundary conditions of the three-dimensional problem. This situation is typical of such *singular perturbations problems*; see, in this respect, the comprehensive treatments given in LIONS [1973] and ECKHAUS [1979] (there are however "exceptional" boundary conditions for which all the terms can be computed and the convergence of the series even established; PAUMIER [1991] has found such an occurrence for a rectangular plate).

The fundamental contributions of FRIEDRICHS & DRESSLER [1961] and GOLDENVEIZER [1962, 1964] for plates, RIGOLOT [1972, 1976] for rods, GOLDENVEIZER [1963, 1964] for shells, are among the first successful attempts to apply formal asymptotic methods in linearized elasticity. Some restrictions or *a priori* assumptions were however still needed. For instance, FRIEDRICHS & DRESSLER [1961, p.4] and GOLDENVEIZER [1962, eqs. (1.1) and (1.3)] assume that the three components of the body force and the in-plane components of the surface force vanish; GOLDENVEIZER [1962, p. 1001] *a priori* assumes that the required state of strain and stress is skew-symmetrical about the middle plane, etc. Another shortcoming is the *lack of convergence theorems*, essentially because the asymptotic method is applied in these works to the *partial differential equations* of the three-dimensional problem; in this case, convergence results usually rely on a *maximum principle* (see Eckhaus [1979]), which does not hold for the system of linearized three-dimensional elasticity.

CIARLET & DESTUYNDER [1979a,b] applied instead the formal asymptotic method to the *weak*, or *variational*, formulation of the boundary value problem of three-dimensional *linearly and nonlinearly elastic plates*. Without making any *a priori* assumption, they justified in this fashion the linear and nonlinear *Kirchhoff-Love plate theories*: only the magnitudes of the components of the applied loads and of the Lamé constants must be "scaled" as appropriate powers of the thickness, but, as shown in a systematic way by MIARA [1994a, 1994b], such scalings are unavoidable. The approach of CIARLET & DESTUYNDER was then extended to *von Kármán plates* by CIARLET [1980], to *Marguerre-von Kármán shallow shells* by CIARLET & PAUMIER [1986] and BUSSE [1996], to *general nonlinear constitutive equations* by DAVET [1986], to *nonlinear elastodynamics* by RAOULT [1988] and KARWOWSKI [1993], to plates with rapidly varying thickness by QUINTELA-ESTEVEZ [1989]. By allowing a larger class of scalings on the applied loads, FOX, RAOULT & SIMO [1993] were also able to justify in this fashion two-dimensional quasilinear plate equations that are valid for "large" deformations and "invariant", in that they share the same invariances as the three-dimensional theory (while MIARA assumed at the onset that the nonlinear two-dimensional models found by the formal asymptotic method had to reduce to the classical ones once linearized, this assumption is not made by FOX, RAOULT & SIMO, who were thus able to consider other families of scalings). The one-dimensional equations of a *nonlinearly elastic beam* (straight rod) were likewise justified by CIMETIÈRE, GEYMONAT, LE DRET, RAOULT & TUTEK [1988] and KARWOWSKI [1990]. *Nonlinear rod theory* has also been related to the three-dimensional theory by MIELKE [1988, 1990], who justified *St. Venant's principle* by a remarkable use of the *center-manifold theorem*.

The most noticeable virtue of the asymptotic method applied to the *weak* formulation of elasticity problems is its amenability to a rigourous *asymptotic analysis*, which shows that the three-dimensional scaled solution converges in some Hilbert spaces (H^1 or L^2) to the leading term of the formal asymptotic expansion. Such *convergence theorems* have been established by DESTUYNDER [1980, 1981], CAILLERIE [1980], CIARLET & KESAVAN [1981], KOHN & VOGELIUS [1984, 1985, 1986], RAOULT [1985], BLANCHARD & FRANCFORT [1987], CIORANESCU & SAINT JEAN PAULIN [1995], DESTUYNDER & GRUAIS [1995], DAUGE & GRUAIS [1996], AGANOVIĆ, MARUŠIĆ-PALOKA & TUTEK [1995] for *linearly elastic plates* (see also CIARLET [1990, 1997a] and the works cited therein), CIARLET & MIARA [1992], BUSSE, CIARLET & MIARA [1996] *for linearly elastic shallow shells*, BERMUDEZ & VIAÑO [1984], AGANOVIĆ & TUTEK [1986], GEYMONAT, KRASUCKI & MARIGO [1987], TRABUCHO & VIAÑO [1987], RAOULT [1988], VEIGA [1995], LE DRET [1995] for *linearly elastic beams* (see also the comprehensive survey of TRABUCHO & VIAÑO [1996] and the works cited therein). The proofs essentially rely on the ideas and methods described and developed in LIONS [1973] for analyzing "abstract" linear variational problems that contain a small parameter.

Convergence theorems can also be obtained from *Γ-convergence theory*, as in BOURQUIN, CIARLET, GEYMONAT & RAOULT [1992] and ANZELLOTTI, BALDO & PERCIVALE [1994] for *linearly elastic plates*, and also *linearly elastic beams* in the latter reference. *Nonlinear "membrane" models* that are "invariant" and valid for "large" deformations have also been obtained in this fashion by LE DRET & RAOULT [1993, 1995a, 1995b], who themselves based their approach on that of ACERBI, BUTTAZZO & PERCIVALE [1991] for *strings*. Special mention must also be made of the approach of MIELKE [1995], who keeps the thickness fixed, but lets the lateral boundary of the plate "go away to infinity".

Let us now turn to the central theme of the present work, *shell theory*.

After the earlier formal attempts of GOLDENVEIZER (cited *supra*) for *linearly elastic shells*, a first major step was achieved by DESTUYNDER [1980] in his thesis (see also DESTUYNDER [1985]), where a convergence theorem for *membrane shells* was "almost" proved (further comments are given in Sec. 7 of this paper). Another major step is due to SANCHEZ-PALENCIA [1990], who clearly delineated specific geometrical and kinematical assumptions that yield either the two-dimensional *membrane shell model* or the two-dimensional *flexural shell model*, when the formal asymptotic expansion method is applied to the variational equations of three-dimensional linearized elasticity (see also CAILLERIE & SANCHEZ-PALENCIA [1995]) and MIARA & SANCHEZ-PALENCIA [1996]. Of particular interest are also the convergence theorems obtained by ACERBI, BUTTAZZO & PERCIVALE [1988] by means of techniques of Γ-convergence (we again refer to Sec. 7 for more detailed comments).

For *nonlinearly elastic shells*, a first noteworthy achievement is due to JOHN [1965, 1971], who showed that, in the absence of surface loads and "away from the edge", the state of stress is "approximately planar", and that the stresses "parallel to the middle surface" vary "approximately linearly" across the thickness if the thickness is sufficiently small. These remarkable results laid the ground for the two-dimensional, linear and nonlinear, shell theories of KOITER [1966, 1970] and KOITER & SIMMONDS [1973]. However, in spite of their elegance and depth, JOHN'S results hold only for special cases of loadings; besides, they do not provide information "up to the boundary" (of the middle surface of the shell), let alone about the boundary conditions of the associated two-dimensional problem.

Again for *nonlinearly elastic shells*, the *formal asymptotic method* has been successfully applied by RAO [1994] to spherical shells, and to "general" shells by MIARA [1994c, 1995], LODS & MIARA [1995], who showed that the leading term of the formal asymptotic expansion can be identified with the solution of nonlinear two-dimensional membrane or flexural equations, according to specific geometrical or kinematical assumptions as in the linear case. A *convergence theorem* has also been obtained by LE DRET & RAOULT [1995c, 1996], who also used Γ-*convergence theory* to obtain *nonlinear "membrane" shell models* that are "invariant" and valid for "large" deformations.

In this three-part work, we analyze the asymptotic behavior of the scaled three-dimensional displacement field of a linearly elastic shell as the thickness approaches zero. Under two distinct sets of assumptions on the geometry of the middle surface, on the boundary conditions, and on the order of magnitude of the applied forces, *convergence theorems in H^1 or L^2 are established that justify either the linear two-dimensional equations of a "membrane shell"* (Part I), *or those of a "flexural shell"* (Part II). Combining these convergences with results of DESTUYNDER [1985] and SANCHEZ-PALENCIA [1989a,b, 1992], *we also justify the two-dimensional linear shell model of* KOITER, under similar sets of assumptions (Part III). Our results have been announced in CIARLET & LODS [1994b], CIARLET, LODS & MIARA [1994] and CIARLET & LODS [1995a].

We use the following conventions and notations throughout this work: *Greek* indices and exponents (except ε) belong to the set $\{1,2\}$, *Latin* indices and exponents (except when otherwise indicated, as e.g. when they are used to index sequences) belong to the set $\{1,2,3\}$, and the *summation convention* with respect to repeated indices and exponents is systematically used. The sign := indicates that the right-hand side defines the left-hand side. Symbols such as $\delta^\alpha_\beta, \delta^{ij}, \delta_{ij}$, etc., designate Kronecker's symbol. The Euclidean scalar product and the vector product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{R}^3$ are noted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$; the Euclidean norm is noted $|\cdot|$.

Let A be an open subset in a finite-dimensional Euclidean space. For each integer m, $H^m(A)$ and $||\cdot||_{m,A}$ denote the usual Sobolev spaces of real-valued functions ($H^0(A) = L^2(A)$). Boldface letters denote vector-valued or tensor-valued functions and their associated function spaces; for instance, $\boldsymbol{v} = (v_i) \in \mathbf{L^2}(\Omega)$ means that $v_i \in L^2(\Omega), i = 1, 2, 3; ||\boldsymbol{v}||_{0,\Omega} = \{\Sigma_i ||v_i||^2_{0,\Omega}\}^{1/2}$, etc. In order to avoid clumsy style and notations, we often deliberately perpetrate various *abuses of language*. In particular, we blithely ignore that families and sequences are not identical; likewise, we do not systematically mention that some equalities hold only almost everywhere (in some specific sense that should always be clear from the context).

1. The three-dimensional shell problem

All notions of differential geometry needed for shell theory may be found, e.g., in GREEN & ZERNA [1968], NIORDSON [1985], and CIARLET [1997b]. Let ω be a bounded, open, and connected subset of $\mathbf{R}^2$, with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ. Ley $y = (y_\alpha)$ denote a generic point in the set $\overline{\omega}$, and let $\partial_\alpha := \partial/\partial y_\alpha$. Let $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$ be an injective mapping of class $\mathscr{C}^3$ such that the two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\varphi}(y)$$

are linearly independent at all points $y \in \overline{\omega}$. They form the *covariant basis* of the tangent plane to the surface

$$S = \boldsymbol{\varphi}(\overline{\omega})$$

at the point $\boldsymbol{\varphi}(y)$; the two vectors $\boldsymbol{a}^\alpha(y)$ of the same tangent plane defined by the relations

$$\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$$

constitute its *contravariant basis*. We also define the unit vector

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)|},$$

which is normal to S at the point $\boldsymbol{\varphi}(y)$.

One then defines the *first fundamental form*, also known as the *metric tensor*, $(a_{\alpha\beta})$ or $(a^{\alpha\beta})$ (in covariant or contravariant components), the *second fundamental form*, also known as the *curvature tensor*, $(b_{\alpha\beta})$ or (b^β_α) (in covariant or mixed components), and the *Christoffel symbols* $\Gamma^\sigma_{\alpha\beta}$, *of the surface S* by letting (whenever no confusion should arise, we henceforth drop the explicit dependence on the variable $y \in \overline{\omega}$):

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta, \tag{1.1}$$

$$b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, \quad b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha}, \tag{1.2}$$

$$\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha. \tag{1.3}$$

Note the symmetries:

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad a_{\alpha\beta} = a^{\beta\alpha}, \quad b_{\alpha\beta} = b_{\beta\alpha}, \quad \Gamma^{\sigma}_{\alpha\beta} = \Gamma^{\sigma}_{\beta\alpha}.$$

The *area element* along S is $\sqrt{a}\,dy$, where

$$a := \det(a_{\alpha\beta}). \tag{1.4}$$

All the functions defined in (1.1)–(1.4) are at least continuous over the set $\overline{\omega}$. In particular, there exists a constant a_0 such that

$$0 < a_0 \leqq a(y) \quad \text{for all } y \in \overline{\omega}. \tag{1.5}$$

For each $\varepsilon > 0$, we define the sets

$$\Omega^{\varepsilon} = \omega \times]-\varepsilon, \varepsilon[, \quad \Gamma^{\varepsilon}_{+} = \omega \times \{\varepsilon\}, \quad \Gamma^{\varepsilon}_{-} = \omega \times \{-\varepsilon\}, \quad \Gamma^{\varepsilon}_{0} = \gamma \times [-\varepsilon, \varepsilon].$$

Note that $\Gamma^{\varepsilon}_{+} \cup \Gamma^{\varepsilon}_{-} \cup \Gamma^{\varepsilon}_{0}$ constitutes a partition of the boundary of the set Ω^{ε}. Let $x^{\varepsilon} = (x^{\varepsilon}_i)$ denote a generic point in the set $\overline{\Omega}^{\varepsilon}$, and let $\partial^{\varepsilon}_i := \partial/\partial x^{\varepsilon}_i$; hence $x^{\varepsilon}_{\alpha} = y_{\alpha}$ and $\partial^{\varepsilon}_{\alpha} = \partial_{\alpha}$.

We then define a mapping $\boldsymbol{\Phi} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ by letting

$$\boldsymbol{\Phi}(x^{\varepsilon}) := \boldsymbol{\varphi}(y) + x^{\varepsilon}_3 \boldsymbol{a}^3(y) \quad \text{for all } x^{\varepsilon} = (y, x^{\varepsilon}_3) \in \overline{\Omega}^{\varepsilon}. \tag{1.6}$$

One can then show (cf. Ciarlet & Paumier [1986, Prop. 3.2]) that there exists $\varepsilon_0 > 0$ such that the three vectors

$$\boldsymbol{g}^{\varepsilon}_i(x^{\varepsilon}) := \partial^{\varepsilon}_i \boldsymbol{\Phi}(x^{\varepsilon})$$

are linearly independent at all points $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$ and the mapping $\boldsymbol{\Phi} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ is injective for all $0 < \varepsilon \leqq \varepsilon_0$. The injectivity of the mapping $\boldsymbol{\Phi} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$, which itself relies on the assumed injectivity of the mapping $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$, ensures in particular that the physical problem described below is meaningful.

The three vectors $\boldsymbol{g}^{\varepsilon}_i(x^{\varepsilon})$ form the *covariant basis* (of the tangent space, here $\mathbf{R}^3$, to the manifold $\boldsymbol{\Phi}(\overline{\Omega}^{\varepsilon})$) at the point $\boldsymbol{\Phi}(x^{\varepsilon})$, and the three vectors $\boldsymbol{g}^{i,\varepsilon}(x^{\varepsilon})$ defined by

$$\boldsymbol{g}^{j,\varepsilon}(x^{\varepsilon}) \cdot \boldsymbol{g}^{\varepsilon}_i(x^{\varepsilon}) = \delta^j_i$$

form the *contravariant basis*. We then define the *metric tensor* (g^{ε}_{ij}) or $(g^{ij,\varepsilon})$ (in covariant or contravariant components) and the *Christoffel symbols* of the manifold $\boldsymbol{\Phi}(\overline{\Omega}^{\varepsilon})$ by letting (we omit explicit dependence on x^{ε})

$$g^{\varepsilon}_{ij} := \boldsymbol{g}^{\varepsilon}_i \cdot \boldsymbol{g}^{\varepsilon}_j, \quad g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}, \tag{1.7}$$

$$\Gamma^{p,\varepsilon}_{ij} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial^{\varepsilon}_i \boldsymbol{g}^{\varepsilon}_j. \tag{1.8}$$

Note the symmetries:

$$g_{ij}{}^{\varepsilon} = g_{ji}{}^{\varepsilon}, \quad g^{ij,\varepsilon} = g^{ji,\varepsilon}, \quad \Gamma^{p,\varepsilon}_{ij} = \Gamma^{p,\varepsilon}_{ji}. \tag{1.9}$$

The *volume element* in the set $\boldsymbol{\Phi}(\Omega^{\varepsilon})$ is $\sqrt{g^{\varepsilon}}dx^{\varepsilon}$, where

$$g^{\varepsilon} := \det(g_{ij}{}^{\varepsilon}). \tag{1.10}$$

For each $0 < \varepsilon \leqq \varepsilon_0$, the set $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is the reference configuration of an *elastic shell*, with *middle surface* $S = \boldsymbol{\varphi}(\overline{\omega})$ and *thickness* 2ε. We assume that the material constituting the shell is homogeneous and isotropic and that $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is a natural state, so that the material is characterized by its two *Lamé constants* $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. The *unknown* of the problem is the vector field $\boldsymbol{u}^\varepsilon = (u_i{}^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbf{R}^3$, where the three functions $u_i{}^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbf{R}$ are the covariant components of the displacement field $u_i{}^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the points of the shell; this means that $u_i{}^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ is the displacement of the point $\boldsymbol{\Phi}(x^\varepsilon)$. Finally, we assume that the shell is *clamped* along its whole "lateral" face $\boldsymbol{\Phi}(\Gamma_0{}^\varepsilon)$, i.e., that the displacement vanishes there (the subsequent analysis does not apply if the shell is only clamped over a portion of its lateral face, of the form $\boldsymbol{\Phi}(\gamma_0 \times [-\varepsilon, \varepsilon])$, with $\gamma_0 \subset \gamma$; cf. Sec. 7).

Then it is classical (cf., e.g., CIARLET [1997b]) that the variational formulation of the corresponding *three-dimensional problem of linearized elasticity* reads as follows, when it is expressed in terms of the *curvilinear coordinates* $x_i{}^\varepsilon$ of the reference configuration $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$: The unknown $\boldsymbol{u}^\varepsilon = (u_i{}^\varepsilon)$ satisfies

$$\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon); \boldsymbol{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0{}^\varepsilon\}, \tag{1.11}$$

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(\boldsymbol{u}^\varepsilon) e_{i\|j}{}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i{}^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \quad \text{for all } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon), \tag{1.12}$$

where

$$A^{ijkl,\varepsilon} := \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) \tag{1.13}$$

designate the contravariant components of the *three-dimensional elasticity tensor*,

$$e_{i\|j}{}^\varepsilon(\boldsymbol{v}^\varepsilon) := \tfrac{1}{2}(\partial_i{}^\varepsilon v_j{}^\varepsilon + \partial_j{}^\varepsilon v_i{}^\varepsilon) - \Gamma^{p,\varepsilon}_{ij} v_p{}^\varepsilon \tag{1.14}$$

designate the covariant components of the *linearized strain tensor* associated with an arbitrary displacement field $v_i{}^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the set $\boldsymbol{\Phi}(\bar{\Omega}^\varepsilon)$, $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ are the contravariant components of the *applied body force density*. Note the symmetries

$$A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon}. \tag{1.15}$$

Surface forces on $\boldsymbol{\Phi}(\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-)$ may be also taken into account, at the expense however of various additional technicalities in the ensuing asymptotic analysis. For this reason, they are treated separately, in Sec. 6.

The *three-dimensional shell problem* (1.11)–(1.12) has one and only one solution for each $\varepsilon > 0$. To see this, one may express it in Cartesian coordinates and then use the classical Korn inequality, as in, e.g., DUVAUT & LIONS [1972, p. 115]. The $\boldsymbol{V}(\Omega^\varepsilon)$-ellipticity of the bilinear form appearing in (1.12) may also be directly established in curvilinear coordinates, as in CIARLET [1997b].

Definitions (1.7)–(1.10) apply *verbatim* to a general manifold $\boldsymbol{\Phi}(\overline{\Omega}^{\varepsilon})$, where Ω^{ε} is any bounded, open, connected subset of $\mathbf{R}^3$ with a Lipschitz-continuous boundary, and $\boldsymbol{\Phi} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ is *any* injective mapping of class $\mathscr{C}^1$ such that the three vectors $\partial_i^{\varepsilon}\boldsymbol{\Phi}(x^{\varepsilon})$ are linearly independent at all points $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$; the elasticity tensor $(A^{ijkl,\varepsilon})$ of a linearly elastic material with Lamé constants λ^{ε} and μ^{ε}, occupying the set $\boldsymbol{\Phi}(\overline{\Omega}^{\varepsilon})$ in its reference configuration, is likewise always given by (1.13). Note however that, when the set Ω^{ε} is of the *special form* $\Omega^{\varepsilon} = \omega \times]-\varepsilon, \varepsilon[$ and the mapping $\boldsymbol{\Phi}$ is of the *special form* (1.6) as here, the following *additional relations* are satisfied:

(1.16) $$\Gamma^{3,\varepsilon}_{\alpha 3} = \Gamma^{p,\varepsilon}_{33} = 0 \quad \text{in } \overline{\Omega}^{\varepsilon},$$

(1.17) $$A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = 0 \quad \text{in } \overline{\Omega}^{\varepsilon}.$$

Remark. Shells whose middle surface has *no boundary,* such as an ellipsoid or a torus, are not covered by the present asymptotic analysis (nor by those of Parts II and III), which applies to surfaces S that can be described by a *single injective* mapping $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$. The needed corresponding two-dimensional existence theory is however available; cf. Ramos [1995].

2. The "scaled" three-dimensional shell problem over a domain independent of ε

Let

$$\Omega = \omega \times]-1, 1[, \quad \Gamma_+ = \omega \times \{1\}, \quad \Gamma_- = \omega \times \{-1\}, \quad \Gamma_0 = \gamma_0 \times [-1, 1],$$

let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$, and let $\partial_i = \partial/\partial x_i$. With $x^{\varepsilon} = (x_i^{\varepsilon}) \in \overline{\Omega}^{\varepsilon}$, we associate the point $x = (x_i) \in \overline{\Omega}$ defined by $x_\alpha = x_\alpha^{\varepsilon} (= y_\alpha)$ and $x_3 = (1/\varepsilon) x_3^{\varepsilon}$; we thus have $\partial_\alpha^{\varepsilon} = \partial_\alpha$ and $\partial_3^{\varepsilon} = (1/\varepsilon)\partial_3$.

With the unknown $\boldsymbol{u}^{\varepsilon} = (u_i^{\varepsilon}) : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ and the vector fields $\boldsymbol{v}^{\varepsilon} = (v_i^{\varepsilon}) : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$ appearing in the three-dimensional problem (1.11), (1.12), we associate the *scaled unknown* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbf{R}^3$ and the *scaled vector fields* $\boldsymbol{v} = (v_i)$ defined by

(2.1) $$u_i(\varepsilon)(x) = u_i^{\varepsilon}(x^{\varepsilon}) \quad \text{and} \quad v_i(x) = v_i^{\varepsilon}(x^{\varepsilon}) \qquad \text{for all } x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}.$$

Note that, in relations (2.1) and (2.3)–(2.6) *infra,* it is understood that x stands for the point of $\overline{\Omega}$ that is associated with the point $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$ as indicated above.

We next make the following *assumptions on the data,* i.e., on the *Lamé constants* and on the *forces:* There exist constants $\lambda > 0$ and $\mu > 0$ *independent of* ε, and there exist functions $f^i \in L^2(\Omega)$ independent of ε such that

(2.2) $$\lambda^{\varepsilon} = \lambda, \quad \mu^{\varepsilon} = \mu,$$

(2.3) $$f^{i,\varepsilon}(x^{\varepsilon}) = f^i(x) \quad \text{for all } x \in \Omega.$$

Remarks. (1) By contrast with (2.1) and (2.3), *different scalings* are made in the asymptotic analysis of *plates* on the "*horizontal*" components u_α^ε and "*vertical*" component u_3^ε of the unknown, and *different* assumptions are made on the "horizontal" components $f^{\alpha,\varepsilon}$, and "vertical" component $f^{3,\varepsilon}$, of the applied forces; cf. Ciarlet [1990, pp. 106–107] and Sec. 7 of the present article.
(2) Assumptions (2.3) could be replaced by the more general ones: "There exist functions $f^i(\varepsilon) \in L^2(\Omega)$ and $f^i \in L^2(\Omega)$ such that $f^{i,\varepsilon}(x^\varepsilon) = f^i(\varepsilon)(x)$ for all $x \in \Omega$ and $\varepsilon > 0$, and $f^i(\varepsilon) \to f^i$ in $L^2(\Omega)$ as $\varepsilon \to 0$ ".

A simple computation then shows that the scaled unknown $\boldsymbol{u}(\varepsilon)$ satisfies the *scaled three-dimensional shell problem* (2.10), (2.11), now posed over the set Ω, thus over a domain which is *independent of* ε :

Theorem 2.1. *Let the functions* $\Gamma_{ij}^{p,\varepsilon}$, $g^\varepsilon, A^{ijkl,\varepsilon} : \overline{\Omega}^\varepsilon \to \mathbf{R}$ *defined in* (1.8), (1.10), (1.13), *be associated with the functions* $\Gamma_{ij}^p(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon) : \overline{\Omega} \to \mathbf{R}$ *defined by*

$$\Gamma_{ij}^p(\varepsilon)(x) := \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon) \quad \textit{for all } x^\varepsilon \in \Omega^\varepsilon, \tag{2.4}$$

$$g(\varepsilon)(x) := g^\varepsilon(x^\varepsilon) \quad \textit{for all } x^\varepsilon \in \Omega^\varepsilon, \tag{2.5}$$

$$A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^\varepsilon) \quad \textit{for all } x^\varepsilon \in \Omega^\varepsilon. \tag{2.6}$$

With any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$, *let there be associated the symmetric tensor* $(e_{i\|j}(\varepsilon)(\boldsymbol{v})) \in L^2(\Omega)$ *defined by*

$$e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) := \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^p(\varepsilon) v_p, \tag{2.7}$$

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\right) - \Gamma_{\alpha 3}^\sigma(\varepsilon) v_\sigma, \tag{2.8}$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{2.9}$$

Then the scaled unknown $\boldsymbol{u}(\varepsilon)$ *defined in* (2.1) *satisfies*

$$\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \mathbf{0} \textit{ on } \Gamma_0\}, \tag{2.10}$$

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{i\|j}(\varepsilon)(\boldsymbol{v}) \sqrt{g(\varepsilon)}\, dx = \int_\Omega f^i v_i \sqrt{g(\varepsilon)}\, dx \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{2.11}$$

Remarks. (1) As already noted in (1.16), the Christoffel symbols $\Gamma_{\alpha 3}^{3,\varepsilon}$ and $\Gamma_{33}^{p,\varepsilon}$ vanish in Ω^ε for the special class (1.6) of mappings $\boldsymbol{\Phi}$ considered here.

Consequently, the functions $\Gamma^{3}_{\alpha 3}(\varepsilon)$ and $\Gamma^{p}_{33}(\varepsilon)$ likewise vanish in Ω, so that the functions $e_{i\|3}(\varepsilon)(\boldsymbol{v})$ of (2.8), (2.9) are equivalently defined as

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\right) - \Gamma^{p}_{\alpha 3}(\varepsilon)v_p,$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3 - \Gamma^{p}_{33}(\varepsilon)v_p,$$

i.e., in a form more reminiscent of (1.14).

(2) The functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$ are *not* defined for $\varepsilon = 0$. By contrast, the functions $\Gamma^{p}_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ converge in the space $\mathscr{C}^0(\overline{\Omega})$ as $\varepsilon \to 0$ (cf. Lemma 3.1).

3. Technical preliminaries

We henceforth assume without loss of generality that the number $\varepsilon_0 > 0$ (which is such that the "original" three-dimensional problem is well-defined for $0 < \varepsilon \leqq \varepsilon_0$; cf. Sec. 1) also satisfies $\varepsilon_0 \leqq 1$.

In this section and in Secs. 4 and 5, whenever a symbol such as C_1, C_2, etc., or c_1, c_2. etc., appears in an inequality, it means that *there exists a constant,* denoted by this symbol, *that is positive and independent of the various variables* (e.g., the parameter ε, functions in a specific space, etc.) *involved in this inequality.* For instance, inequality (3.10) in Lemma 3.1 means that *there exists a constant* $C_2 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$, of the point $x \in \overline{\Omega}$, and of the symmetric tensor (t_{ij}), such that this inequality holds.

Our first result gathers all the properties needed in the sequel concerning *the behavior of the functions* $\Gamma^{p}_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ as $\varepsilon \to 0$. A noteworthy conclusion in this respect is that, while these are functions of $x = (y, x_3) \in \overline{\Omega} = \overline{\omega} \times [-1, 1]$, *their limits for* $\varepsilon = 0$ *are functions of* $y \in \overline{\omega}$ *only, i.e., the limits are independent of the "transverse" variable* x_3 : see relations (3.2)–(3.5), where the functions $\Gamma^{\sigma}_{\alpha\beta}, b_{\alpha\beta}, b^{\sigma}_{\alpha}, a, A^{ijkl}(0)$ are identified with functions defined over the set $\overline{\Omega}$ by letting these be constant with respect to x_3. Observe that the notational distinction between the "three-dimensional" and "two-dimensional" Christoffel symbols $\Gamma^{\sigma}_{\alpha\beta}(\varepsilon)$ and $\Gamma^{\sigma}_{\alpha\beta}$ is automatic, as in relation (3.2) for instance, since the symbol ε appears only in the former. Also, note that the following symmetries hold (cf. (1.9) and (1.15))

$$\text{(3.1)} \qquad \Gamma^{p}_{ij}(\varepsilon) = \Gamma^{p}_{ji}(\varepsilon), \quad A^{ijkl}(\varepsilon) = A^{jikl}(\varepsilon) = A^{klij}(\varepsilon).$$

If $w \in \mathscr{C}^0(\overline{\Omega})$, we let

$$\| w \|_{0,\infty,\overline{\Omega}} = \sup\{|w(x)|; x \in \overline{\Omega}\}.$$

Lemma 3.1. *Let the functions* $\Gamma^{p}_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ *be defined for* $\varepsilon > 0$ *as in* (2.4)–(2.6); *let the functions* $a^{\alpha\beta}, b_{\alpha\beta}, b^{\sigma}_{\alpha}, \Gamma^{\sigma}_{\alpha\beta}, a \in \mathscr{C}^0(\overline{\omega})$ *be defined as in* (1.1)–(1.3) *and be identified with functions in* $\mathscr{C}^0(\overline{\Omega})$. *Then*

(3.2)
$$\| \Gamma^{\sigma}_{\alpha\beta}(\varepsilon) - \Gamma^{\sigma}_{\alpha\beta} \|_{0,\infty,\overline{\Omega}} + \| \Gamma^{3}_{\alpha\beta}(\varepsilon) - b_{\alpha\beta} \|_{0,\infty,\overline{\Omega}} + \| \Gamma^{\sigma}_{\alpha 3}(\varepsilon) + b^{\sigma}_{\alpha} \|_{0,\infty,\overline{\Omega}} \leqq C_1\varepsilon,$$

(3.3)
$$\Gamma^{3}_{\alpha 3}(\varepsilon) = \Gamma^{p}_{33}(\varepsilon) = 0,$$

(3.4)
$$\| g(\varepsilon) - a \|_{0,\infty,\overline{\Omega}} \leqq C_1\varepsilon,$$

(3.5)
$$\| A^{ijkl}(\varepsilon) - A^{ijkl}(0) \|_{0,\infty,\overline{\Omega}} \leqq C_1\varepsilon,$$

(3.6)
$$A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0,$$

for all $0 < \varepsilon \leqq \varepsilon_0$, *where*

(3.7)
$$A^{\alpha\beta\sigma\tau}(0) := \lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),$$

(3.8)
$$A^{\alpha\beta 33}(0) := \lambda a^{\alpha\beta}, \quad A^{\alpha 3\sigma 3}(0) := \mu a^{\alpha\sigma}, \quad A^{3333}(0) := \lambda + 2\mu,$$

(3.9)
$$A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) := 0,$$

and finally,

(3.10)
$$t_{ij}t_{ij} \leqq C_2 A^{ijkl}(\varepsilon)(x) t_{kl} t_{ij}$$

for all $0 < \varepsilon \leqq \varepsilon_0$, *all* $x \in \overline{\Omega}$, *and all symmetric tensors* (t_{ij}).

Proof. We only sketch the proof. First, it is clear that relations (3.3) and (3.6) are simply a re-writing of relations (1.16), (1.17). Next, let $\boldsymbol{g}_i(\varepsilon)(x) = \boldsymbol{g}_i^{\,\varepsilon}(x^{\varepsilon})$ and $\boldsymbol{g}^i(\varepsilon)(x) = \boldsymbol{g}^{i,\varepsilon}(x^{\varepsilon})$ for all $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$. Since $\boldsymbol{g}_{\alpha}(\varepsilon) = \boldsymbol{a}_{\alpha} + \varepsilon x_3 \partial_{\alpha}\boldsymbol{a}_3$ and $\boldsymbol{g}_3(\varepsilon) = \boldsymbol{a}_3$, it follows that $g(\varepsilon) = a + O(\varepsilon)$ in $\mathscr{C}^0(\overline{\Omega})$, whence inequality (3.4) is proved. *Because the mapping* $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$ *is assumed to be of class* $\mathscr{C}^3$ (third-order derivatives of $\boldsymbol{\varphi}$ appear in $\partial_{\beta}\boldsymbol{g}_{\alpha}(\varepsilon)$), we likewise have

$$\Gamma^{p}_{ij}(\varepsilon) = \boldsymbol{g}^{p}(\varepsilon) \cdot \partial_j \boldsymbol{g}_i(\varepsilon) = \Gamma^{p}_{ij}(0) + O(\varepsilon) \quad \text{in } \mathscr{C}^0(\overline{\Omega}),$$

with

$$\Gamma^{\sigma}_{\alpha\beta}(0) := \Gamma^{\sigma}_{\alpha\beta}, \quad \Gamma^{3}_{\alpha\beta}(0) := b_{\alpha\beta}, \quad \Gamma^{\sigma}_{\alpha 3}(0) := -b^{\sigma}_{\alpha},$$

whence inequality (3.2) is proved. Relations (3.5) and (3.7)–(3.9) are analogously proved.

For each $\varepsilon > 0$, the three-dimensional elasticity tensor defined in (1.13) is positive-definite, uniformly with respect to $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$ (see, e.g., Ciarlet [1997b]). This implies that there exists $c(\varepsilon) > 0$ such that (cf. (2.7))

$$A^{ijkl}(\varepsilon)(x) t_{kl} t_{ij} \geqq c(\varepsilon) t_{ij} t_{ij}$$

for all $x \in \overline{\Omega}$ and all $(t_{ij}) \in \boldsymbol{S}$, where $\boldsymbol{S}$ denotes the set of all symmetric matrices of order 3. Using definitions (3.7)–(3.9), we have

$$A^{ijkl}(0)t_{kl}t_{ij} = \lambda(a^{\alpha\beta}t_{\alpha\beta} + t_{33})^2 + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})t_{\sigma\tau}t_{\alpha\beta}$$
$$+ 4\mu a^{\alpha\beta}t_{\alpha 3}t_{\beta 3} + (\lambda + 2\mu)t_{33}t_{33},$$

on the other. Since there exists $c_1 > 0$ such that (cf., e.g., BERNADOU, CIARLET & MIARA [1994, Lemma 2.1])

$$(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})t_{\sigma\tau}t_{\alpha\beta} \geqq c_1 t_{\alpha\beta}t_{\alpha\beta},$$

it easily follows that there exists $c_2 > 0$ such that

$$A^{ijkl}(0)(x)t_{kl}t_{ij} \geqq c_2 t_{ij}t_{ij}$$

for all $x \in \overline{\Omega}$ and all $(t_{ij}) \in \boldsymbol{S}$. The continuity of the mapping

$$(\varepsilon, x, (t_{ij})) \in [0, \varepsilon_0] \times \overline{\Omega} \times \boldsymbol{S}_1 \to A^{ijkl}(\varepsilon)(x)t_{kl}t_{ij},$$

where $\boldsymbol{S}_1 := \{(t_{ij}) \in \boldsymbol{S}; t_{ij}t_{ij} = 1\}$, then yields the existence of a constant C_2 such that relation (3.10) holds for $0 \leqq \varepsilon \leqq \varepsilon_0$. □

Since *averages with respect to the "transverse" variable* x_3 play a fundamental rôle in the ensuing analysis, their relevant properties are gathered in the next lemma. If v and $\boldsymbol{v}$ are respectively real-valued and vector-valued functions defined almost everywhere over $\Omega = \omega \times]-1, 1[$, their *averages* $\overline{v}$ and $\overline{\boldsymbol{v}}$ are respectively the real-valued and vector-valued functions defined almost everywhere over ω by letting

$$(3.11) \qquad \overline{v}(y) := \frac{1}{2}\int_{-1}^{1} v(y, x_3)dx_3, \quad \overline{\boldsymbol{v}}(y) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{v}(y, x_3)dx_3$$

for almost all $y \in \omega$, whenever these definitions make sense (cf. Lemma 3.2(i) for such instances). The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ introduced in (3.17) are the covariant components of the *linearized change of metric,* or *strain, tensor,* associated with an arbitrary displacement field $\eta_i\boldsymbol{a}^i$ of the surface S.

Lemma 3.2. (i) *Let $v \in L^2(\Omega)$. Then $\overline{v}(y)$ as given in* (3.11) *is finite for almost all $y \in \omega$, the function $\overline{v}$ defined in this fashion belongs to $L^2(\omega)$, and*

$$(3.12) \qquad \| \overline{v} \|_{0,\omega} \leqq \frac{1}{\sqrt{2}} \| v \|_{0,\Omega} .$$

If $\partial_3 v = 0$ in the sense of distributions, i.e., if $\int_\Omega v\partial_3\varphi dx = 0$ for all $\varphi \in \mathscr{D}(\Omega)$, then v does not depend on x_3, and

$$(3.13) \qquad v(y, x_3) = \overline{v}(y) \quad \textit{for almost all } (y, x_3) \in \Omega.$$

(ii) *Let $v \in H^1(\Omega)$. Then $\overline{v} \in H^1(\omega), \partial_\alpha\overline{v} = \overline{\partial_\alpha v}$, and*

$$(3.14) \qquad \| \overline{v} \|_{1,\omega} \leqq \frac{1}{\sqrt{2}} \| v \|_{1,\Omega} .$$

Let γ_0 denote a measurable subset of γ. If $v = 0$ on $\gamma_0 \times [-1, 1]$, then $\overline{v} = 0$ on γ_0; in particular, $\overline{v} \in H_0^1(\omega)$ if $v = 0$ on $\Gamma_0 = \gamma \times [-1, 1]$.

(iii) *Let* $(v(\varepsilon))_{\varepsilon>0}$ *be a sequence of functions in* $H^1(\Omega)$ *and let* $\overline{v} \in L^2(\omega)$ *be such that*

$$\partial_3 v(\varepsilon) \to 0 \text{ in } L^2(\Omega), \quad \overline{v}(\varepsilon) \to \overline{v} \text{ in } L^2(\omega) \quad \text{as } \varepsilon \to 0. \tag{3.15}$$

Then

$$v(\varepsilon) \to \overline{v} \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \to 0, \tag{3.16}$$

where $\overline{v}$ *is identified in* (3.16) *with a function in* $L^2(\Omega)$, *by letting* $\overline{v}(y,x_3) := \overline{v}(y)$ *for all* $(y,x_3) \in \Omega$.

(iv) *Let* $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ *be a sequence of functions* $\boldsymbol{v}(\varepsilon) = (\boldsymbol{v}_i(\varepsilon)) \in \boldsymbol{H}^1(\Omega)$ *bounded in* $\boldsymbol{L}^2(\Omega)$, *let the functions* $e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}(\varepsilon))$ *be defined according to* (2.7), *and let*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \tfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3 \tag{3.17}$$

for any $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$. *Then*

$$\{\overline{e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})(\varepsilon)} - \gamma_{\alpha\beta}(\overline{\boldsymbol{v}(\varepsilon)})\} \to 0 \text{ in } L^2(\omega) \quad \text{as } \varepsilon \to 0. \tag{3.18}$$

Proof. (i) Let $v \in L^2(\Omega)$. For almost all $y \in \omega$, the function $v(y,\cdot)$ belongs to the space $L^2(]-1,1[)$ by Fubini's theorem. For such points y, the Cauchy-Schwarz inequality gives

$$\left| \int_{-1}^{1} v(y,x_3)dx_3 \right|^2 \leqq 2 \int_{-1}^{1} |v(y,x_3)|^2 dx_3 < +\infty;$$

hence

$$\int_\omega |\overline{v}(y)|^2 dy \leqq \frac{1}{2} \int_\omega \left\{ \int_{-1}^{1} |v(y,x_3)|^2 \, dx_3 \right\} dy = \frac{1}{2} \int_\Omega |v|^2 dx,$$

and inequality (3.12) is proved. If $\partial_3 v = 0$ in the sense of distributions, then there exists $\zeta \in L^2(\omega)$ such that $v(y,x_3) = \zeta(y)$ for almost all $(y,x_3) \in \Omega$ (cf., e.g., Le Dret [1991, Lemma 4.1, p. 74]). But $\overline{v} = \zeta$ in this case, and thus relation (3.13) is proved.

(ii) Let $v \in H^1(\Omega)$. Given an arbitrary function $\overline{\varphi} \in \mathscr{D}(\omega)$, let $\varphi : \Omega \to \mathbf{R}$ be defined by $\varphi(y,x_3) = \overline{\varphi}(y)$ for all $(y,x_3) \in \Omega$. Since φ vanishes on Γ_0 and the "horizontal" components of the unit outer normal vector vanish on $\Gamma_+ \cup \Gamma_-$, we have

$$\int_\Omega v \partial_\alpha \varphi \, dx = - \int_\Omega \partial_\alpha v \, \varphi \, dx$$

by Green's formula in Sobolev spaces. Since $v \in L^2(\Omega), \partial_\alpha v \in L^2(\Omega)$, and φ and $\partial_\alpha \varphi$ are independent of x_3, Fubini's theorem yields

$$\int_\omega \overline{v}\partial_\alpha\overline{\varphi}\,dy = -\int_\omega \overline{\partial_\alpha v}\overline{\varphi}\,dy;$$

hence $\overline{v} \in H^1(\omega)$, with $\partial_\alpha\overline{v} = \overline{\partial_\alpha v}$. These relations, combined with inequality (3.12), imply inequality (3.14).

Assume in addition that $v = 0$ on $\tilde{\Gamma}_0 := \gamma_0 \times [-1,1]$. There exist functions $\phi^k \in \mathscr{C}^\infty(\overline{\Omega})$, $k = 0,1,\ldots$, such that $\phi^k \to v$ in $H^1(\Omega)$ as $k \to \infty$. Hence, $\phi^k_{|\tilde{\Gamma}_0} \to 0 = v_{|\tilde{\Gamma}_0}$ in $L^2(\tilde{\Gamma}_0)$ as $k \to \infty$; consequently, $\overline{\phi}^k_{|\gamma_0} \to 0$ in $L^2(\gamma_0)$ since $\|\overline{\phi}^k\|_{L^2(\gamma_0)} \leqq \frac{1}{\sqrt{2}}\|\phi^k\|_{L^2(\tilde{\Gamma}_0)}$, and thus $\overline{\phi}^k_{|\gamma_0} \to 0 = \overline{v}_{|\gamma_0}$ in $L^2(\gamma_0)$.

(iii) Let $v \in L^2(\Omega)$ be such that its derivative $\partial_3 v$ in the sense of distributions belongs to $L^2(\Omega)$. Then for almost all $(y,s) \in \omega\times]-1,1[$, we may write (cf. Le Dret [1991, p. 9])

$$v(y,s) = v(y,-1) + \int_{-1}^{s} \partial_3 v(y,x_3)dx_3,$$

and thus

$$\overline{v}(y) = \frac{1}{2}\int_{-1}^{1} v(y,s)ds = v(y,-1) + \frac{1}{2}\int_{-1}^{1}\left(\int_{-1}^{t} \partial_3 v(y,x_3)dx_3\right)dt.$$

Hence the following identity holds:

$$v(y,s) = \overline{v}(y) + \int_{-1}^{s} \partial_3 v(y,x_3)dx_3 - \frac{1}{2}\int_{-1}^{1}\left(\int_{-1}^{t} \partial_3 v(y,x_3)dx_3\right)dt.$$

This identity, combined with the triangular inequality and the relations

$$\int_\omega \left\{\int_{-1}^{1} |\overline{v}(y)|^2 ds\right\}dy = 2\|\overline{v}\|^2_{0,\omega},$$

$$\int_\omega \left\{\int_{-1}^{1} \left|\int_{-1}^{s} \partial_3 v(y,x_3)dx_3\right|^2 ds\right\}dy \leqq 4\|\partial_3 v\|^2_{0,\Omega},$$

$$\int_\omega \left\{\int_{-1}^{1} \left|\int_{-1}^{1}\Big(\int_{-1}^{t} \partial_3 v(y,x_3)dx_3\Big)dt\right|^2 ds\right\}dy \leqq 8\|\partial_3 v\|^2_{0,\Omega},$$

shows that

$$\|v\|_{0,\Omega} \leqq \sqrt{2}\|\overline{v}\|_{0,\omega} + (2+\sqrt{2})\|\partial_3 v\|_{0,\Omega}. \tag{3.19}$$

The desired convergence (3.16) is then proved by letting $v = v(\varepsilon) - \overline{v}$ in inequality (3.19).

(iv) Since

$$\tfrac{1}{2}(\overline{\partial_\alpha v_\beta(\varepsilon) + \partial_\beta v_\alpha(\varepsilon)}) = \tfrac{1}{2}(\partial_\alpha \overline{v_\beta(\varepsilon)} + \partial_\beta \overline{v_\alpha(\varepsilon)})$$

by (ii), it suffices to establish that

$$\{\overline{\Gamma^p_{\alpha\beta}(\varepsilon)v_p(\varepsilon)} - \Gamma^p_{\alpha\beta}(0)\overline{v_p(\varepsilon)}\} \to 0 \quad \text{in}\, L^2(\omega) \;\text{ as } \varepsilon \to 0,$$

where the functions $\Gamma^\sigma_{\alpha\beta}(0) := \Gamma^\sigma_{\alpha\beta}$ and $\Gamma^3_{\alpha\beta}(0) := b_{\alpha\beta}$ are independent of x_3. By (3.2) and (3.12),

$$\|\overline{\Gamma^p_{\alpha\beta}(\varepsilon)v_p(\varepsilon)} - \Gamma^p_{\alpha\beta}(0)\overline{v_p(\varepsilon)}\|_{0,\omega} = \|\overline{\Gamma^p_{\alpha\beta}(\varepsilon)v_p(\varepsilon) - \Gamma^p_{\alpha\beta}(0)v_p(\varepsilon)}\|_{0,\omega}$$

$$\leqq \tfrac{1}{\sqrt{2}}\|\Gamma^p_{\alpha\beta}(\varepsilon)v_p(\varepsilon) - \Gamma^p_{\alpha\beta}(0)v_p(\varepsilon)\|_{0,\Omega} \leqq C_1\sqrt{\tfrac{3}{2}}\varepsilon\|\boldsymbol{v}(\varepsilon)\|_{0,\Omega},$$

and the convergence (3.18) follows from the boundedness (cf. (3.12)) of the sequence $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ in the space $L^2(\Omega)$. □

4. A generalized Korn inequality for an elliptic surface

If no specific assumption is made on the "geometry" of the surface S, it is shown in Part II that there exist constants $\varepsilon_1 > 0$ and $C > 0$ such that, for all $0 < \varepsilon \leqq \varepsilon_1$,

$$\Big\{\sum_i \|v_i\|^2_{1,\Omega}\Big\}^{1/2} \leqq \frac{C}{\varepsilon}\Big\{\sum_{i,j}\|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|^2_{0,\Omega}\Big\}^{1/2} \quad \text{for all } \boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega);$$

in fact, such an inequality also holds if the set $\Gamma_0 = \gamma \times [-1,1]$ where the functions in the space $\boldsymbol{V}(\Omega)$ vanish is replaced by the more general set $\gamma_0 \times [-1,1]$, where γ_0 is any subset of γ with *length* $\gamma_0 > 0$. This relation is a *generalized Korn inequality*, the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$ of (2.7)–(2.9) replacing the "traditional" functions

$$e_{ij}(\boldsymbol{v}) := \tfrac{1}{2}(\partial_j v_i + \partial_i v_j). \tag{4.1}$$

It is remarkable that, *in some cases*, the "constant" C/ε may be replaced by a constant that is *independent of* ε (at the expense, however, of replacing $\|v_3\|_{1,\Omega}$ by $\|v_3\|_{0,\Omega}$ in the left-hand side). More specifically, under the crucial assumption (4.3) (which is given a geometrical interpretation in Theorems 4.2, 4.3 *infra*), *another generalized Korn inequality* holds (cf. (4.4)), which plays a key rôle in the proof of Theorem 5.1; it is used there to establish the fundamental *a priori* bounds that the family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ satisfies.

Theorem 4.1. *Define the space*

$$\boldsymbol{V}_M(\omega) := \{\boldsymbol{\eta} = (\eta_i);\, \eta_\alpha \in H^1_0(\omega), \eta_3 \in L^2(\omega)\} = H^1_0(\omega)\times H^1_0(\omega)\times L^2(\omega), \tag{4.2}$$

and assume that there exists a constant $c > 0$ *such that*

$$(4.3)\qquad \left\{ \sum_\alpha \| \eta_\alpha \|^2_{1,\omega} + \| \eta_3 \|^2_{0,\omega} \right\}^{1/2} \leqq c \left\{ \sum_{\alpha,\beta} \| \gamma_{\alpha\beta}(\boldsymbol{\eta}) \|^2_{0,\omega} \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega)$,

where the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ *are defined as in* (3.17). *Then there exists a constant* ε_1 *satisfying* $0 < \varepsilon_1 \leqq \varepsilon_0$ *and a constant C such that, for all* $0 < \varepsilon \leqq \varepsilon_1$,

$$(4.4)\qquad \left\{ \sum_\alpha \| v_\alpha \|^2_{1,\Omega} + \| v_3 \|^2_{0,\Omega} \right\}^{1/2} \leqq C \left\{ \sum_{i,j} \| e_{i\|j}(\varepsilon)(\boldsymbol{v}) \|^2_{0,\Omega} \right\}^{1/2}$$

for all $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$,

where the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$ *and the space* $\boldsymbol{V}(\Omega)$ *are defined as in* (2.7)–(2.10).

Proof. (i) We first establish that, *for all* $0 < \varepsilon \leqq \varepsilon_0$,

$$(4.5)\qquad \left\{ \sum_\alpha \| v_\alpha \|^2_{1,\Omega} + \| \varepsilon v_3 \|^2_{1,\Omega} \right\}^{1/2} \leqq c_1 \left\{ \sum_{i,j} \| e_{i\|j}(\varepsilon)(\boldsymbol{v}) \|^2_{0,\Omega} + \sum_i \| v_i \|^2_{0,\Omega} \right\}^{1/2}$$

for all $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$. Given $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$, let $\boldsymbol{v}(\varepsilon) := (v_1, v_2, \varepsilon v_3) \in \boldsymbol{H}^1(\Omega)$ for $\varepsilon \leqq \varepsilon_0$. Then

$$e_{\alpha\beta}(\boldsymbol{v}(\varepsilon)) = e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) + \Gamma^p_{\alpha\beta}(\varepsilon) v_p,$$
$$e_{\alpha 3}(\boldsymbol{v}(\varepsilon)) = \varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) + \varepsilon \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma,$$
$$e_{33}(\boldsymbol{v}(\varepsilon)) = \varepsilon^2 e_{3\|3}(\varepsilon)(\boldsymbol{v}),$$

where the functions $e_{ij}(\cdot)$ are those of (4.1), and consequently, by (3.2),

$$(4.6)\qquad \left\{ \sum_{i,j} \| e_{ij}(\boldsymbol{v}(\varepsilon)) \|^2_{0,\Omega} \right\}^{1/2} \leqq c_2 \left\{ \sum_{i,j} \| e_{i\|j}(\varepsilon)(\boldsymbol{v}) \|^2_{0,\Omega} + \sum_i \| v_i \|^2_{0,\Omega} \right\}^{1/2}$$

since $\varepsilon_0 \leqq 1$ by assumption. By the "classical" Korn inequality (cf. the proof given in Duvaut & Lions [1972, p. 110] and its extension to domains with Lipschitz-continuous boundaries given, e.g., in Ciarlet [1997a]),

$$(4.7)\qquad \| \boldsymbol{v}(\varepsilon) \|_{1,\Omega} = \left\{ \sum_\alpha \| v_\alpha \|^2_{1,\Omega} + \| \varepsilon v_3 \|^2_{1,\Omega} \right\}^{1/2} \leqq c_3 \left\{ \sum_{i,j} \| e_{ij}(\boldsymbol{v}(\varepsilon)) \|^2_{0,\Omega} + \| \boldsymbol{v}(\varepsilon) \|^2_{0,\Omega} \right\}^{1/2},$$

and inequality (4.5) follows from inequalities (4.6) and (4.7).

(ii) In order to establish (4.4), it suffices to show that *there exists* $\varepsilon_1 \in]0, \varepsilon_0]$ *such that, for all* $0 < \varepsilon \leqq \varepsilon_1$,

$$\left\{ \sum_i \| v_i \|_{0,\Omega}^2 \right\}^{1/2} \leqq c_4 \left\{ \sum_{i,j} \| e_{i\|j}(\varepsilon)(\boldsymbol{v}) \|_{0,\Omega}^2 \right\}^{1/2} \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{4.8}$$

because inequalities (4.5) and (4.8) together imply

$$\min\left\{1, \frac{c_4^{-2}}{2}\right\} \left(c_1^{-2} \sum_\alpha \| v_\alpha \|_{1,\Omega}^2 + \| v_3 \|_{0,\Omega}^2 \right) \leqq 2 \sum_{i,j} \| e_{i\|j}(\varepsilon)(\boldsymbol{v}) \|_{0,\Omega}^2 .$$

Assume that inequality (4.8) is false. Then there exist $\varepsilon_m > 0$ and $\boldsymbol{v}^m = (v_i^m) \in \boldsymbol{V}(\Omega), m = 0, 1, \ldots$, such that (the Latin letters m and n are used here for indexing sequences)

$$\varepsilon_m \to 0 \quad \text{as } m \to \infty, \tag{4.9}$$

$$e_{i\|j}(\varepsilon_m)(\boldsymbol{v}^m) \to 0 \text{ in } L^2(\Omega) \quad \text{as } m \to \infty, \tag{4.10}$$

$$\sum_i \| v_i^m \|_{0,\Omega}^2 = 1 \quad \text{for all } m. \tag{4.11}$$

By (4.5), (4.10), (4.11), both sequences $(v_\alpha^m)_{m=0}^\infty$ are bounded in $H^1(\Omega)$. Hence there exist subsequences $(v_\alpha^n)_{n=0}^\infty$ and there exist functions $v_\alpha \in H^1(\Omega)$ satisfying $v_\alpha = 0$ on Γ_0 and a function $v_3 \in L^2(\Omega)$ such that

$$v_\alpha^n \rightharpoonup v_\alpha \text{ in } \boldsymbol{H}^1(\Omega), \; v_\alpha^n \to v_\alpha \text{ in } L^2(\Omega) \quad \text{and} \quad v_3^n \rightharpoonup v_3 \text{ in } L^2(\Omega) \tag{4.12}$$

as $n \to \infty$, where $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively (to ensure that $v_\alpha \in H^1(\Omega)$, consider subsequences $(v_\alpha^n)_{n=0}^\infty$ that weakly converge in $H^1(\Omega)$). The remainder of the proof consists in showing that the sequence $(v_3^n)_{n=0}^\infty$ converges strongly in $L^2(\Omega)$ and that the three limit functions v_i appearing in (4.12) vanish, in contradiction to (4.11). To these ends, we proceed in three steps.

(iii) We first show that

$$\overline{\boldsymbol{v}}^n \to \boldsymbol{0} \quad \text{in the space } \boldsymbol{V}_M(\omega) \text{ as } n \to \infty. \tag{4.13}$$

To see this, observe that, by (3.12),

$$e_{\alpha\|\beta}(\varepsilon_n)(\boldsymbol{v}^n) \to 0 \text{ in } L^2(\Omega) \Longrightarrow \overline{e_{\alpha\|\beta}(\varepsilon_n)(\boldsymbol{v}^n)} \to 0 \text{ in } L^2(\omega),$$

and that, by (3.18),

$$\overline{e_{\alpha\|\beta}(\varepsilon_n)(\boldsymbol{v}^n)} \to 0 \text{ in } L^2(\omega) \Longrightarrow \gamma_{\alpha\beta}(\overline{\boldsymbol{v}}^n) \to 0 \text{ in } L^2(\omega)$$

as $n \to \infty$. Hence the convergence (4.13) follows from assumption (4.3).

(iv) We next show that

$$v_\alpha^n \to 0 \quad \text{in } L^2(\Omega). \tag{4.14}$$

By (3.2), (4.9)–(4.11),

$$\partial_3 v_\alpha^n + \varepsilon_n \partial_\alpha v_3^n = 2\varepsilon_n e_{\alpha\|3}(\varepsilon_n)(\boldsymbol{v}^n) + 2\varepsilon_n \Gamma_{\alpha 3}^{\sigma}(\varepsilon_n) v_\sigma^n \to 0 \quad \text{in } L^2(\Omega).$$

Let $\varphi \in \mathscr{D}(\Omega)$; since the sequence $(v_3^n)_{n=0}^{\infty}$ is bounded in $L^2(\Omega)$, we have (recall that $v_\alpha := \lim_{n\to\infty} v_\alpha^n$ in $L^2(\Omega)$; cf. (4.12))

$$\int_\Omega \partial_3 v_\alpha \, \varphi \, dx = -\int_\Omega v_\alpha \partial_3 \varphi \, dx = -\lim_{n\to\infty}\left\{\int_\Omega v_\alpha^n \partial_3 \varphi \, dx + \varepsilon_n \int_\Omega v_3^n \partial_\alpha \varphi \, dx\right\}$$

$$= \lim_{n\to\infty}\left\{\int_\Omega (\partial_3 v_\alpha^n + \varepsilon_n \partial_\alpha v_3^n)\varphi \, dx\right\} = 0,$$

and thus $\partial_3 v_\alpha = 0$ in $L^2(\Omega)$. Hence v_α can be identified with $\overline{v}_\alpha$ by Lemma 3.2(i); but

$$v_\alpha^n \to v_\alpha \text{ in } L^2(\Omega) \Longrightarrow \overline{v}_\alpha^{\,n} \to \overline{v}_\alpha \text{ in } L^2(\omega)$$

on the one hand, and $\overline{v}_\alpha = 0$ by (4.13) on the other. Hence the convergence (4.14) is established.

(v) Finally, we show that

$$v_3^n \to 0 \quad \text{in } L^2(\Omega). \tag{4.15}$$

By (4.10),

$$\partial_3 v_3^n = \varepsilon_n e_{3\|3}(\varepsilon_n)(\boldsymbol{v}^n) \to 0 \quad \text{in } L^2(\Omega),$$

and by (4.13),

$$\overline{v}_3^n \to 0 \quad \text{in } L^2(\omega),$$

as $n \to \infty$. The convergence (4.15) is then a consequence of Lemma 3.2(iii). We have therefore reached a contradiction, and the proof is complete. □

We next show that assumption (4.3) is in fact an assumption "in disguise" about the allowed "geometries" of the surface S. To this end, we need a definition : A surface $S = \boldsymbol{\varphi}(\overline{\omega})$ with $\boldsymbol{\varphi} \in \mathscr{C}^2(\overline{\omega}; \mathbf{R}^3)$ is *elliptic* if there exists a constant $b > 0$ such that

$$| b_{\alpha\beta}(y)\xi^\alpha \xi^\beta | \geqq b \xi^\alpha \xi^\alpha \tag{4.16}$$

for all $y \in \overline{\omega}$ and $(\xi^\alpha) \in \mathbf{R}^2$; equivalently, the two principal radii of curvature are either > 0 at all points of S or < 0 at all points of S, and their moduli lie in a compact interval of $]0, +\infty[$.

The following sufficient conditions guaranteeing that the crucial assumption (4.3) holds were announced in Ciarlet & Sanchez-Palencia [1993] and Ciarlet & Lods [1994], and proved in Ciarlet & Sanchez-Palencia [1996] and Ciarlet & Lods [1996a], respectively:

Theorem 4.2. *Assume either that the boundary γ of ω is of class $\mathscr{C}^3$ and $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$ is the restriction to $\overline{\omega}$ of an analytic mapping or that γ is of class $\mathscr{C}^4$ and $\boldsymbol{\varphi} \in \mathscr{C}^5(\overline{\omega}; \mathbf{R}^3)$. Then relation* (4.3) *is satisfied if the surface $S = \boldsymbol{\varphi}(\overline{\omega})$ is elliptic.*

Remarkably, this condition is also necessary, as recently shown by Şlicaru [1996]:

Theorem 4.3. *Assume that γ is Lipschitz-continuous, $\boldsymbol{\varphi} \in \mathscr{C}^2(\overline{\omega}; \mathbf{R}^3)$, and relation* (4.3) *holds. Then the surface S is elliptic.*

Remark. Brezzi [1994] has shown that the rather stringent regularity conditions of Theorem 4.2 can be substantially relaxed, and parts of the proof significantly simplified, when the mapping $\boldsymbol{\varphi}$ takes the special form $\boldsymbol{\varphi}(y_1, y_2) = (y_1, y_2, \theta(y_1, y_2))$ for $(y_1, y_2) \in \overline{\omega}$.

5. Asymptotic analysis as $\varepsilon \to 0$

We now establish our main results, namely that *the scaled three-dimensional solutions $\boldsymbol{u}(\varepsilon)$ converge* (in a specific sense ; cf. (5.2)) *as $\varepsilon \to 0$ toward a limit $\boldsymbol{u}$*, and that *this limit, which is independent of the "transverse" variable x_3, can be identified with the solution $\overline{\boldsymbol{u}}$ of a two-dimensional problem* (cf. (5.5)), *posed over the set ω*. This limit problem will be identified in Sec. 7 as a *two-dimensional "membrane" shell problem.*

The functions $\gamma_{\alpha\beta}(\cdot)$ and $a^{\alpha\beta\sigma\tau}$ defined in the next theorem respectively represent the covariant components of the *change of metric tensor* of the surface S and the contravariant components of the *elasticity tensor* of S.

Theorem 5.1. *Let the space $\boldsymbol{V}_M(\omega)$ and the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ be defined by*

$$\begin{aligned} \boldsymbol{V}_M(\omega) &:= \{\boldsymbol{\eta} = (\eta_i); \eta_\alpha \in H_0^1(\omega), \eta_3 \in L^2(\omega)\} \\ &= H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \tfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - b_{\alpha\beta}\eta_3 \quad \text{for } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega), \end{aligned}$$

and assume that there exists a constant c such that

$$\Big\{\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{0,\omega}^2\Big\}^{1/2} \leqq c\Big\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\Big\}^{1/2} \tag{5.1}$$

$$\textit{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega).$$

For $0 < \varepsilon \leqq \varepsilon_0$, let $\boldsymbol{u}(\varepsilon)$ denote the solution of the scaled variational problem (2.11), (2.12). *Then there exist functions $u_\alpha \in H^1(\Omega)$ satisfying $u_\alpha = 0$ on Γ_0 and a function $u_3 \in L^2(\Omega)$ such that*

$$u_\alpha(\varepsilon) \to u_\alpha \text{ in } H^1(\Omega), \quad u_3(\varepsilon) \to u_3 \text{ in } L^2(\Omega) \quad \text{as } \varepsilon \to 0, \tag{5.2}$$

$$\boldsymbol{u} := (u_i) \text{ is independent of the "transverse" variable } x_3, \tag{5.3}$$

$$\overline{\boldsymbol{u}} := (\overline{u}_i) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3 \in \boldsymbol{V}_M(\omega), \tag{5.4}$$

and $\overline{\boldsymbol{u}}$ satisfies the two-dimensional variational equations

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\overline{\boldsymbol{u}})\gamma_{\alpha\beta}(\eta)\sqrt{a}\,dy = \int_\omega \Big\{\int_{-1}^{1} f^i dx_3\Big\}\eta_i\sqrt{a}\,dy \tag{5.5}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega)$,

where (*cf.* (1.1) *and* (1.4) *for the definitions of the functions* $a^{\alpha\beta}$ *and* a)

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{(\lambda+2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}). \tag{5.6}$$

Proof. For the sake of clarity, the proof is divided into eight steps, numbered (i) to (viii). For notational brevity, we let

$$e_{i\|j}(\varepsilon) := e_{i\|j}(\varepsilon)(\boldsymbol{u}(\varepsilon))$$

throughout the proof.

(i) *A priori bounds and extraction of weakly convergent sequences: The norms* $\| e_{i\|j}(\varepsilon) \|_{0,\Omega}$, $\| u_\alpha(\varepsilon) \|_{1,\Omega}$, $\| u_3(\varepsilon) \|_{0,\Omega}$ *are bounded independently of* $\varepsilon \in]0,\varepsilon_1]$. Consequently, *there exists a subsequence, still denoted* $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ *for convenience, and there exist functions* $e_{i\|j} \in L^2(\Omega)$, $u_\alpha \in H^1(\Omega)$ *satisfying* $u_\alpha = 0$ *on* Γ_0, *and* $u_3 \in L^2(\Omega)$ *such that*

$$e_{i\|j}(\varepsilon) \rightharpoonup e_{i\|j} \quad \text{in } L^2(\Omega), \tag{5.7}$$

$$u_\alpha(\varepsilon) \rightharpoonup u_\alpha \text{ in } H^1(\Omega), \quad u_\alpha(\varepsilon) \to u_\alpha \text{ in } L^2(\Omega), \tag{5.8}$$

$$u_3(\varepsilon) \rightharpoonup u_3 \quad \text{in } L^2(\Omega). \tag{5.9}$$

Recall that $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

From inequalities (1.5) and (3.4), we infer that there exist constants g_0 and g_1 such that

$$0 < g_0 \leqq g(\varepsilon)(x) \leqq g_1 \quad \text{for all } \varepsilon \in]0,\varepsilon_0] \text{ and all } x \in \overline{\Omega}. \tag{5.10}$$

From the variational equations (2.11), inequality (3.10), and the generalized Korn inequality (4.4), we infer that

$$\begin{aligned} C^{-2}\sum_i \|u_i(\varepsilon)\|^2_{0,\Omega} &\leqq C^{-2}\Big(\sum_\alpha \|u_\alpha(\varepsilon)\|^2_{1,\Omega} + \|u_3(\varepsilon)\|^2_{0,\Omega}\Big) \\ &\leqq \sum_{i,j}\|e_{i\|j}(\varepsilon)\|^2_{0,\Omega} \leqq C_2\, g_0^{-1/2}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}\,dx \\ &= C_2\, g_0^{-1/2}\int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}\,dx \\ &\leqq C_2\, g_0^{-1/2}g_1^{1/2}\Big\{\sum_i \|f^i\|^2_{0,\Omega}\Big\}^{1/2}\Big\{\sum_i \|u_i(\varepsilon)\|^2_{0,\Omega}\Big\}^{1/2}; \end{aligned}$$

hence the assertions follow.

(ii) *The limit functions u_i found in* (5.8), (5.9) *are independent of* x_3: By (3.2) and Step (i),

$$\partial_3 u_\alpha(\varepsilon) + \varepsilon \partial_\alpha u_3(\varepsilon) = 2\varepsilon\{e_{\alpha\|3}(\varepsilon) + \Gamma^\sigma_{\alpha 3}(\varepsilon) u_\sigma(\varepsilon)\} \to 0 \quad \text{in } L^2(\Omega).$$

Let $\varphi \in \mathscr{D}(\Omega)$; since $u_\alpha(\varepsilon) \rightharpoonup u_\alpha$ in $H^1(\Omega)$ and $(u_3(\varepsilon))_{\varepsilon>0}$ is bounded in $L^2(\Omega)$ by Step (i),

$$\int_\Omega \partial_3 u_\alpha \, \varphi \, dx = \lim_{\varepsilon\to 0} \int_\Omega \partial_3 u_\alpha(\varepsilon) \, \varphi \, dx,$$

$$\lim_{\varepsilon\to 0}\left\{ \int_\Omega \varepsilon \partial_\alpha u_3(\varepsilon) \, \varphi \, dx \right\} = -\lim_{\varepsilon\to 0}\left\{ \int_\Omega \varepsilon u_3(\varepsilon) \partial_\alpha \varphi \, dx \right\} = 0,$$

whence $\int_\Omega \partial_3 u_\alpha \, \varphi dx = 0$. Therefore $\partial_3 u_\alpha = 0$ in $L^2(\Omega)$. Likewise, by Step (i),

$$\partial_3 u_3(\varepsilon) = \varepsilon e_{3\|3}(\varepsilon) \to 0 \quad \text{in } L^2(\Omega).$$

Let $\varphi \in \mathscr{D}(\Omega)$; since $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$ by Step (i),

$$\int_\Omega u_3 \partial_3 \varphi \, dx = \lim_{\varepsilon\to 0} \int_\Omega u_3(\varepsilon) \partial_3 \varphi \, dx = -\lim_{\varepsilon\to 0} \int_\Omega \partial_3 u_3(\varepsilon) \, \varphi \, dx = 0,$$

whence $\partial_3 u_3 = 0$ in the sense of distributions; it then suffices to apply Lemma 3.2(i).

(iii) *The limit functions $e_{i\|j}$ found in* (5.7) *are independent of* x_3; *they are moreover related to the limit* $\boldsymbol{u} := (u_i)$ *by*

(5.11) $$e_{\alpha\|\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) := \tfrac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\sigma_{\alpha\beta} u_\sigma - b_{\alpha\beta} u_3,$$

(5.12) $$e_{\alpha\|3} = 0,$$

(5.13) $$e_{3\|3} = -\frac{\lambda}{\lambda + 2\mu} a^{\alpha\beta} e_{\alpha\|\beta}.$$

The convergences $e_{\alpha\|\beta}(\varepsilon) \rightharpoonup e_{\alpha\|\beta}$ in $L^2(\Omega)$, $u_\alpha(\varepsilon) \rightharpoonup u_\alpha$ in $H^1(\Omega)$, $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$, and $\Gamma^\sigma_{\alpha\beta}(\varepsilon) \to \Gamma^\sigma_{\alpha\beta}, \Gamma^3_{\alpha\beta}(\varepsilon) \to b_{\alpha\beta}$ in $\mathscr{C}^0(\overline{\Omega})$ (cf. Lemma 3.1), imply that

$$e_{\alpha\|\beta}(\varepsilon) = \tfrac{1}{2}(\partial_\alpha u_\beta(\varepsilon) + \partial_\beta u_\alpha(\varepsilon)) - \Gamma^p_{\alpha\beta}(\varepsilon) u_p(\varepsilon) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{u}) = e_{\alpha\|\beta} \quad \text{in} \quad L^2(\Omega),$$

which shows that the functions $e_{\alpha\|\beta}$ satisfy (5.11) and are independent of x_3 (the functions u_i are independent of x_3; cf. Step (ii)).

Let $\boldsymbol{v} = (v_i)$ be an arbitrary function in the space $\boldsymbol{V}(\Omega)$ of (2.11). The following relations are immediate consequences of definitions (2.7)–(2.9) of the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$:

(5.14) $$\varepsilon e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) \to 0 \quad \text{in } L^2(\Omega),$$

(5.15) $$\varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \to \tfrac{1}{2}\,\partial_3 v_\alpha \quad \text{in } L^2(\Omega),$$

(5.16) $$\varepsilon e_{3\|3}(\varepsilon)(\boldsymbol{v}) = \partial_3 v_3 \quad \text{for all } \varepsilon > 0.$$

Using the variational equations (2.11) of the scaled three-dimensional problem, and relations (3.6), we have

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon)\Big\{\varepsilon e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)(\boldsymbol{v})\Big\}\sqrt{g(\varepsilon)}\,dx \\
&= \int_\Omega \Big\{A^{\alpha\beta\sigma\tau}(\varepsilon)e_{\sigma\|\tau}(\varepsilon) + A^{\alpha\beta 33}(\varepsilon)e_{3\|3}(\varepsilon)\Big\}\Big\{\varepsilon e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})\Big\}\sqrt{g(\varepsilon)}\,dx \\
&\quad + \int_\Omega \Big\{4A^{\alpha 3\sigma 3}(\varepsilon)e_{\sigma\|3}(\varepsilon)\Big\}\Big\{\varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v})\Big\}\sqrt{g(\varepsilon)}\,dx \\
&\quad + \int_\Omega \Big\{A^{33\sigma\tau}(\varepsilon)e_{\sigma\|\tau}(\varepsilon) + A^{3333}(\varepsilon)e_{3\|3}(\varepsilon)\Big\}\Big\{\varepsilon e_{3\|3}(\varepsilon)(\boldsymbol{v})\Big\}\sqrt{g(\varepsilon)}\,dx \\
&= \varepsilon\int_\Omega f^i v_i\sqrt{g(\varepsilon)}\,dx.
\end{aligned}$$

Keep $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *fixed* and let $\varepsilon \to 0$. Using relations (3.4), (3.5), (3.7), (3.8), (5.14)–(5.16), and the weak convergences (5.7), we obtain

$$\int_\Omega \Big\{2\mu a^{\alpha\sigma} e_{\sigma\|3}\partial_3 v_\alpha + [\lambda a^{\sigma\tau} e_{\sigma\|\tau} + (\lambda + 2\mu)e_{3\|3}]\partial_3 v_3\Big\}\sqrt{a}\,dx = 0.$$

Letting $\boldsymbol{v}$ *vary* in $\boldsymbol{V}(\Omega)$ then yields relations (5.12), (5.13) (if $w \in L^2(\Omega)$ and $\int_\Omega w\partial_3 v dx = 0$ for all $v \in H^1(\Omega)$ that vanish on Γ_0, then $w = 0$; cf. Ciarlet [1990, p. 19]).

(iv) *The function* $\overline{\boldsymbol{u}} := (\overline{u}_i)$ *belongs to the space* $\boldsymbol{V}_M(\omega)$ *and satisfies the variational equations* (5.5). Consequently, since these equations have a *unique* solution, by the positive-definiteness of the fourth-order tensor $(a^{\alpha\beta\sigma\tau})$ defined in (5.6) (cf. e.g. Lemma 2.1 of Bernadou, Ciarlet & Miara [1994]) and by assumption (5.1), *the convergences* (5.7)–(5.9) *hold for the whole family* $\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ (if the functions $\overline{u}_i$ are unique, so are the functions u_i and $e_{i\|j}$ by Steps (ii) and (iii)).

That $\boldsymbol{u} \in \boldsymbol{V}_M(\omega)$ follows from Lemma 3.2. Let $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$ be *independent of the variable* x_3; then (cf. inequality (3.2))

(5.17) $$e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) \to \gamma_{\alpha\beta}(\boldsymbol{v}) := \Big\{\tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta}v_\sigma - b_{\alpha\beta}v_3\Big\} \quad \text{in } L^2(\Omega),$$

(5.18) $$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \to \Big\{\tfrac{1}{2}\partial_\alpha v_3 + b^\sigma_\alpha v_\sigma\Big\} \quad \text{in } L^2(\Omega),$$

(5.19) $$e_{3\|3}(\varepsilon)(\boldsymbol{v}) = 0,$$

as $\varepsilon \to 0$. Keep such a function $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *fixed* in the variational equations (2.11) and let $\varepsilon \to 0$. Relations (3.4)–(3.9), the strong convergences (5.17), (5.18), relation (5.19), and the weak convergences (5.7) to the limits $e_{i\|j}$ given by (5.11)–(5.13) together yield

$$\int_\Omega \left\{ \frac{2\lambda\mu}{\lambda+2\mu} a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}) \right\} e_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,dx = \int_\Omega f^i v_i \sqrt{a}\,dx, \tag{5.20}$$

which we may also write as (both functions $\boldsymbol{u}$ and $\boldsymbol{v}$ are independent of x_3)

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\overline{\boldsymbol{u}})\gamma_{\alpha\beta}(\overline{\boldsymbol{v}})\sqrt{a}\,dy = \int_\omega \left\{ \int_{-1}^{1} f^i dx_3 \right\} \overline{v}_i \sqrt{a}\,dy, \tag{5.21}$$

where the functions $a^{\alpha\beta\sigma\tau}$ are those defined in (5.6).

Given $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{H}_0^1(\omega)$, let $\boldsymbol{v} = (v_i)$ be defined by

$$\boldsymbol{v}(y, x_3) = \boldsymbol{\eta}(y) \quad \text{for } (y, x_3) \in \Omega.$$

Then $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, $\boldsymbol{v}$ is independent of x_3, and thus equations (5.21) are satisfied with $\overline{\boldsymbol{v}} = \boldsymbol{\eta}$ (Lemma 3.2). Since both sides of (5.21) are continuous linear forms with respect to $\overline{v}_3 = \eta_3 \in L^2(\omega)$ for fixed $\overline{v}_\alpha \in H_0^1(\omega)$, and since $H_0^1(\omega)$ is dense in $L^2(\omega)$, these equations are valid for all $\boldsymbol{\eta} \in \boldsymbol{V}_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$.

(v) *The weak convergences of* (5.7) *are strong, i.e.,*

$$e_{i\|j}(\varepsilon) \to e_{i\|j} \quad \textit{in } L^2(\Omega). \tag{5.22}$$

Combining inequalities (3.10) and (5.10) with the variational equations (2.11) where we let $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$, we first infer that

$$C_2^{-1} g_0^{1/2} \sum_{i,j} \|e_{i\|j}(\varepsilon) - e_{i\|j}\|_{0,\Omega}^2 \leqq \Lambda(\varepsilon), \tag{5.23}$$

where

$$\begin{aligned}\Lambda(\varepsilon) :&= \int_\Omega A^{ijkl}(\varepsilon)(e_{k\|l}(\varepsilon) - e_{k\|l})(e_{i\|j}(\varepsilon) - e_{i\|j})\sqrt{g(\varepsilon)}\,dx \\ &= \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_\Omega A^{ijkl}(\varepsilon)(e_{k\|l} - 2e_{k\|l}(\varepsilon))e_{i\|j}\sqrt{g(\varepsilon)}\,dx.\end{aligned}$$

Using the weak convergences (5.7)–(5.9) and the convergences (3.4), (3.5), we next have

$$\Lambda := \lim_{\varepsilon\to 0}\Lambda(\varepsilon) = \int_\Omega f^i u_i \sqrt{a}\,dx - \int_\Omega A^{ijkl}(0)e_{k\|l}e_{i\|j}\sqrt{a}\,dx. \tag{5.24}$$

Using (3.7)–(3.9), then (5.12), (5.13), we finally obtain

$$\int_\Omega A^{ijkl}(0)e_{k||l}e_{i||j}\sqrt{a}\,dx$$

$$= \int_\Omega \left\{ \left[\lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\right]e_{\sigma||\tau} + \lambda a^{\alpha\beta}e_{3||3}\right\}e_{\alpha||\beta}\sqrt{a}\,dx$$

$$+ 4\mu\int_\Omega a^{\alpha\sigma}e_{\sigma||3}e_{\alpha||3}\sqrt{a}\,dx + \int_\Omega \left\{\lambda a^{\sigma\tau}e_{\sigma||\tau} + (\lambda+2\mu)e_{3||3}\right\}e_{3||3}\sqrt{a}\,dx$$

$$= \int_\Omega \left\{\frac{2\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\right\}e_{\sigma||\tau}e_{\alpha||\beta}\sqrt{a}\,dx.$$

Hence

(5.25) $$\Lambda = 0$$

(let $\boldsymbol{v} = \boldsymbol{u}$ in (5.20) and use (5.11)), and the convergences (5.22) follow from (5.23)–(5.25).

(vi) *The family* $(\overline{\boldsymbol{u}}(\varepsilon))_{\varepsilon>0)}$ *converges strongly to* $\overline{\boldsymbol{u}}$ *in the space* $\boldsymbol{V}_M(\omega)$, *i.e.,*

(5.26) $$\overline{u}_\alpha(\varepsilon) \to \overline{u}_\alpha \text{ in } H^1(\omega), \quad \overline{u}_3(\varepsilon) \to \overline{u}_3 \text{ in } L^2(\omega).$$

By virtue of assumption (5.1), proving (5.26) is equivalent to proving

(5.27) $$\gamma_{\alpha\beta}(\overline{\boldsymbol{u}}(\varepsilon)) \to \gamma_{\alpha\beta}(\overline{\boldsymbol{u}}) = \overline{e_{\alpha||\beta}} \quad \text{in } L^2(\omega),$$

by (5.11). But, since $e_{\alpha||\beta}(\varepsilon) \to e_{\alpha||\beta}$ in $L^2(\Omega)$ by Step (v), we infer from Lemma 3.2 (i) that

$$\overline{e_{\alpha||\beta}(\varepsilon)} \to \overline{e_{\alpha||\beta}} \quad \text{in } L^2(\omega),$$

on the one hand, and we infer from Lemma 3.2 (iv) that

$$\left(\overline{e_{\alpha||\beta(\varepsilon)}} - \gamma_{\alpha\beta}(\overline{\boldsymbol{u}}(\varepsilon))\right) \to 0 \quad \text{in } L^2(\omega),$$

on the other hand (recall that $e_{\alpha||\beta}(\varepsilon) := e_{\alpha||\beta}(\varepsilon)(\boldsymbol{u}(\varepsilon))$). Hence the strong convergences (5.27) hold.

(vii) *The weak convergence of* (5.9) *is strong, i.e.,*

(5.28) $$u_3(\varepsilon) \to u_3 \quad \text{in } L^2(\Omega).$$

First, we have $\partial_3 u_3(\varepsilon) = \varepsilon e_{3||3}(\varepsilon) \to 0$ in $L^2(\Omega)$; secondly, we have already shown that $\overline{u}_3(\varepsilon) \to \overline{u}_3$ in $L^2(\omega)$ (cf. (5.26)). Hence the conclusion follows from Lemma 3.2(iii) and from the independence of the function u_3 with respect to the "transverse" variable x_3.

(viii) It remains to show that *the weak convergences of* (5.8) *are strong, i.e.,*

(5.29) $$u_\alpha(\varepsilon) \to u_\alpha \quad \text{in } H^1(\Omega).$$

To this end, we observe that proving (5.29) is equivalent to proving that

(5.30) $$e_{ij}(\boldsymbol{u}'(\varepsilon)) \to e_{ij}(\boldsymbol{u}') \quad \text{in } L^2(\Omega),$$

where the functions $e_{ij}(\cdot)$ are those of (3.20) and

$$\boldsymbol{u}'(\varepsilon) := (u_1(\varepsilon), u_2(\varepsilon), 0), \quad \boldsymbol{u}' := (u_1, u_2, 0).$$

(By Korn's inequality, $\left\{\sum_{i,j} \|e_{ij}(\cdot)\|_{0,\Omega}^2\right\}^{1/2}$ is equivalent to $\|\cdot\|_{1,\Omega}$ over the space $\boldsymbol{V}(\Omega)$). We have shown (cf. (5.11) and (5.22)) that

$$(5.31) \qquad e_{\alpha\|\beta}(\varepsilon) = \{e_{\alpha\beta}(\boldsymbol{u}'(\varepsilon)) - \Gamma^p_{\alpha\beta}(\varepsilon)u_p(\varepsilon)\} \\ \to \{e_{\alpha\beta}(\boldsymbol{u}') - \Gamma^\sigma_{\alpha\beta}u_\sigma - b_{\alpha\beta}u_3\} = e_{\alpha\|\beta} \quad \text{in } L^2(\Omega).$$

Combining inequality (3.2) with the strong convergences $u_i(\varepsilon) \to u_i$ in $L^2(\Omega)$ (cf. (5.8), (5.28), (5.31)), we therefore obtain

$$(5.32) \qquad e_{\alpha\beta}(\boldsymbol{u}'(\varepsilon)) \to e_{\alpha\beta}(\boldsymbol{u}') \quad \text{in } L^2(\Omega).$$

Notice in passing that, if we do not use the strong convergences $\overline{u}_\alpha(\varepsilon) \to \overline{u}_\alpha$ in $H^1(\omega)$, we definitely need here the strong convergence $\overline{u}_3(\varepsilon) \to \overline{u}_3$ (all these convergences are established in Step (vi)), which in turn implies the strong convergence (5.28) in Step (vii).

Since $e_{33}(\boldsymbol{u}'(\varepsilon)) = e_{33}(\boldsymbol{u}') = 0$, relations (5.30) will be proved if we show that ($\partial_3 u_\alpha = 0$ by Step (ii))

$$(5.33) \qquad \partial_3 u_\alpha(\varepsilon) = 2e_{\alpha 3}(\boldsymbol{u}'(\varepsilon)) \to 0 = 2e_{\alpha 3}(\boldsymbol{u}') \quad \text{in } L^2(\Omega),$$

or equivalently, that

$$(5.34) \qquad \partial_3 u_\alpha(\varepsilon) \to 0 \text{ in } H^{-1}(\Omega), \quad \partial_i \partial_3 u_\alpha(\varepsilon) \to 0 \quad \text{in } H^{-1}(\Omega).$$

The equivalence between (5.33) and (5.34) is a consequence of a lemma of J.-L. Lions (first mentioned in MAGENES & STAMPACCHIA [1958, p. 320, Note ([27])] and proved in DUVAUT & LIONS [1972, p. 111], then extended to Lipschitz-continuous boundaries in BORCHERS & SOHR [1990] and AMROUCHE & GIRAULT [1994]), which, together with the open mapping theorem, implies that *the mapping* $v \in L^2(\Omega) \to (v, \partial_1 v, \partial_2 v, \partial_3 v) \in \boldsymbol{H}^{-1}(\Omega)$ *is an isomorphism* (cf. also DAUTRAY & LIONS [1984, Lemma 2, p. 1261]).

Since $\partial_3 u_\alpha(\varepsilon) = 2\varepsilon(e_{\alpha\|3}(\varepsilon) + \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)) - \varepsilon\partial_\alpha u_3(\varepsilon)$, we first have, for all $\varphi \in \mathscr{D}(\Omega)$, that

$$\int_\Omega \partial_3 u_\alpha(\varepsilon)\varphi dx = \varepsilon \int_\Omega \{2(e_{\alpha\|3}(\varepsilon) + \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon))\varphi + u_3(\varepsilon)\partial_\alpha\varphi\} dx,$$

and consequently, by (3.2) and step (i),

$$\|\partial_3 u_\alpha(\varepsilon)\|_{-1,\Omega} \leqq c\varepsilon,$$

where, here and subsequently in this proof, c denotes constants that are independent of ε. Hence the first convergence in (5.34) is proved.

We next have the identity

$$(5.35) \qquad \partial_\beta \partial_3 u_\alpha(\varepsilon) = \partial_3 e_{\alpha\beta}(\boldsymbol{u}'(\varepsilon)) + \partial_\beta(\varepsilon e_{\alpha\|3}(\varepsilon) + \varepsilon\Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)) \\ -\partial_\alpha(\varepsilon e_{\beta\|3}(\varepsilon) + \varepsilon\Gamma^\tau_{\beta 3}(\varepsilon)u_\tau(\varepsilon))$$

in $\mathscr{D}'(\Omega)$. From (5.32), we infer that

$$\partial_3 e_{\alpha\beta}(\boldsymbol{u}'(\varepsilon)) \to \partial_3 e_{\alpha\beta}(\boldsymbol{u}') = 0 \quad \text{in } H^{-1}(\Omega),$$

since $\partial_3\partial_\beta u_\alpha = 0$ in $\mathscr{D}'(\Omega)$. Denoting by $\langle\cdot,\cdot\rangle$ the duality between $\mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega)$, we thus have, for all $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_\beta(\varepsilon e_{\alpha\|3}(\varepsilon) + \varepsilon \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)), \varphi\rangle = -\varepsilon \int_\Omega \{e_{\alpha\|3}(\varepsilon) + \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)\}\partial_\beta\varphi \, dx,$$

and consequently, by (3.2) and Step (i),

$$\|\partial_\beta(\varepsilon e_{\alpha\|3}(\varepsilon) + \varepsilon \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon))\|_{-1,\Omega} \leqq c\varepsilon.$$

The last term in (5.35) is treated in an analogous manner. Hence $\partial_\beta\partial_3 u_\alpha(\varepsilon) \to 0$ in $H^{-1}(\Omega)$.

Finally, we have, for all $\varphi \in \mathscr{D}(\Omega)$,

$$\begin{aligned}\langle \partial_3\partial_3 u_\alpha(\varepsilon), \varphi\rangle &= -\int_\Omega \partial_3 u_\alpha(\varepsilon)\partial_3\varphi \, dx \\ &= -2\varepsilon\int_\Omega \{e_{\alpha\|3}(\varepsilon) + \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)\}\partial_3\varphi\, dx + \varepsilon^2\int_\Omega e_{3\|3}(\varepsilon)\partial_\alpha\varphi\, dx,\end{aligned}$$

and consequently, by (3.2) and Step (i),

$$\|\partial_3\partial_3 u_\alpha(\varepsilon)\|_{-1,\Omega} \leqq c\varepsilon.$$

Hence $\partial_3\partial_3 u_\alpha(\varepsilon) \to 0$ in $H^{-1}(\Omega)$, and all the convergences in (5.34) are established. □

6. Consideration of surface forces

The notation is that of Secs. 1 and 2. The *area element* along the boundary of the set $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is

$$d\hat{\Gamma}^\varepsilon = (\det\nabla^\varepsilon\boldsymbol{\Phi})|\nabla^\varepsilon\boldsymbol{\Phi}^{-T}\boldsymbol{n}^\varepsilon|d\Gamma^\varepsilon,$$

where $\nabla^\varepsilon\boldsymbol{\Phi}$ is the matrix with $\boldsymbol{g}_1{}^\varepsilon, \boldsymbol{g}_2{}^\varepsilon, \boldsymbol{g}_3{}^\varepsilon$ as its column vectors, $\boldsymbol{n}^\varepsilon$ is the unit ($|\boldsymbol{n}^\varepsilon| = 1$) outer normal vector, and $d\Gamma^\varepsilon$ is the area element, along the boundary of the set Ω^ε. If *surface forces* are acting on the "upper" and "lower" faces $\boldsymbol{\Phi}(\Gamma^\varepsilon_+)$ and $\boldsymbol{\Phi}(\Gamma^\varepsilon_-)$ of the shell, the unknown $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) \in \boldsymbol{V}(\Omega^\varepsilon)$ satisfies

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\, dx^\varepsilon = \int_{\Gamma^\varepsilon_+\cup\Gamma^\varepsilon_-} h^{i,\varepsilon} v^\varepsilon_i \, d\hat{\Gamma}^\varepsilon \tag{6.1}$$

(compare with (1.12)) for all $\boldsymbol{v}^\varepsilon = (v^\varepsilon_i) \in \boldsymbol{V}(\Omega^\varepsilon)$, where $h^{i,\varepsilon} \in L^2(\Gamma^\varepsilon_+\cup\Gamma^\varepsilon_-)$ are the contravariant components of the applied surface force density. Without loss of generality, we assume in this section that the applied body force vanishes.

In addition to (2.2), we assume that there exist functions $h^i \in L^2(\Gamma_+ \cup \Gamma_-)$ *independent of* ε such that (the points $x^\varepsilon \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$ and $x \in \Gamma_+ \cup \Gamma_-$ are related as in Sec. 2):

(6.2) $$h^{i,\varepsilon}(x^\varepsilon) = \varepsilon h^i(x) \quad \text{for all } x \in \Gamma_+ \cup \Gamma_-.$$

Remark. Additional regularity will be assumed later on the function h^3; cf. Lemma 6.2.

The scaled unknown now satisfies *a scaled three-dimensional shell problem* with a "new" right-hand side (cf. (6.3), (6.4); note that the space $\boldsymbol{V}(\Omega)$ is the same as in Theorem 2.1):

Theorem 6.1. *The scaled unknown* $\boldsymbol{u}(\varepsilon)$ *defined in* (2.1) *satisfies*

(6.3) $$\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) = \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma_0\},$$

(6.4) $$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{i\|j}(\varepsilon)(\boldsymbol{v}) \sqrt{g(\varepsilon)}\, dx = \int_{\Gamma_+ \cup \Gamma_-} h^i v_i \sigma(\varepsilon) d\Gamma \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega),$$

where the function $\sigma(\varepsilon) : \Gamma_+ \cup \Gamma_- \to \mathbf{R}$ *is defined by*

(6.5) $$\sigma(\varepsilon)(x) := (\det \nabla^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)) |\nabla^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)^{-T} \boldsymbol{n}^\varepsilon(x^\varepsilon)| \quad \textit{for all } x^\varepsilon \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon,$$

$d\Gamma$ *is the area element along the boundary of the set* Ω, *and the other notations are as in Theorem 2.1.*

In order to carry out the asymptotic analysis as $\varepsilon \to 0$, two "technical preliminaries" (in addition to those of Sec. 3) are needed for properly handling the right-hand side of the variational equations (6.4). If $w \in \mathscr{C}^1(\overline{\omega})$, let

$$\|w\|_{1,\infty,\overline{\omega}} = \|w\|_{0,\infty,\overline{w}} + \sum_i \|\partial_i w\|_{0,\infty,\overline{\omega}}.$$

Lemma 6.1. *Let the functions* $\sigma(\varepsilon)^+ : \overline{\omega} \to \mathbf{R}$ *and* $\sigma(\varepsilon)^- : \overline{\omega} \to \mathbf{R}$ *be defined by*

(6.6) $$\sigma(\varepsilon)^+(y) := \sigma(\varepsilon)(y, 1),\ \sigma(\varepsilon)^-(y) := \sigma(\varepsilon)(y, -1) \quad \textit{for all } y \in \overline{\omega},$$

where the function $\sigma(\varepsilon) : \Gamma_+ \cup \Gamma_- \to \mathbf{R}$ *is defined in* (6.5) *and let the function* $a : \overline{\omega} \to \mathbf{R}$ *be defined as in* (1.4). *Then*

(6.7) $$\|\sigma(\varepsilon)^+ - \sqrt{a}\|_{1,\infty,\overline{\omega}} + \|\sigma(\varepsilon)^- - \sqrt{a}\|_{1,\infty,\overline{\omega}} \leqq C_3 \varepsilon.$$

Proof. The relations $\nabla^\varepsilon \boldsymbol{\Phi}^{-T} \boldsymbol{n}_\varepsilon = \pm \boldsymbol{a}_3$ for $x_3^\varepsilon = \pm\varepsilon$ and $\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \sqrt{a}\boldsymbol{a}_3$ imply that

$$\sigma(\varepsilon)^{\pm} = |\pm(\{(\boldsymbol{a}_1 \pm \varepsilon\partial_1\boldsymbol{a}_3) \times (\boldsymbol{a}_2 \pm \varepsilon\partial_2\boldsymbol{a}_3)\} \cdot \boldsymbol{a}_3)\boldsymbol{a}_3| = \sqrt{a} + O(\varepsilon)$$

in $\mathscr{C}^1(\overline{\omega})$. (The assumption that $\varphi \in \mathscr{C}^3(\overline{\omega})$ is needed there, since second-order partial derivatives of φ appear in $\partial_\alpha \boldsymbol{a}_3$.) $\square$

Lemma 6.2. *Assume that both functions $h^3_+ : \omega \to \mathbf{R}$ and $h^3_- : \omega \to \mathbf{R}$ defined by*

$$(6.8) \qquad h^3_+(y) := h^3(y,1), \quad h^3_-(y) := h^3(y,-1) \quad \text{for all } y \in \omega$$

belong to the space $H^1(\omega)$, where h^3 is the function appearing in (6.2). *Then the function $\tilde{h}(\varepsilon) : \overline{\Omega} \to \mathbf{R}$ defined by*

$$(6.9) \qquad \tilde{h}(\varepsilon)(y,x_3) = \tfrac{1}{2}(x_3+1)h^3_+(y)\sigma(\varepsilon)^+(y) + \tfrac{1}{2}(x_3-1)h^3_-(y)\sigma(\varepsilon)^-(y)$$

for all $(y,x_3) \in \overline{\Omega}$ belongs to the space $H^1(\Omega)$ for all $\varepsilon > 0$, and

$$(6.10) \qquad \tilde{h}(\varepsilon) \to \tilde{h}(0) \quad \text{in } H^1(\Omega) \text{ as } \varepsilon \to 0,$$

where

$$(6.11) \qquad \tilde{h}(0)(y,x_3) := \left\{\tfrac{1}{2}(x_3+1)h^3_+(y) + \tfrac{1}{2}(x_3-1)h^3_-(y)\right\}\sqrt{a(y)}$$

for all $(y,x_3) \in \overline{\Omega}$; futhermore,

$$(6.12) \qquad \int_{\Gamma_+\cup\Gamma_-} h^3 v_3 \sigma(\varepsilon) d\Gamma = \int_\Omega \tilde{h}(\varepsilon)\partial_i v_i dx + \int_\Omega \partial_i\tilde{h}(\varepsilon) v_i dx,$$

for all $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$, and there exists a constant C_4 such that

$$(6.13) \qquad \left| \int_{\Gamma_+\cup\Gamma_-} h^3 v_3 \sigma(\varepsilon) d\Gamma \right| \leqq C_4 \Big\{ \sum_{i,j} \|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|^2_{0,\Omega} \Big\}^{1/2},$$

for all $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$ and all $0 < \varepsilon \leqq \varepsilon_1$, where the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$ are those of (2.7)–(2.9), *and the space $\boldsymbol{V}(\Omega)$ is that of* (6.3). *The constant C_4 depends on the norms $\|h^3_+\|_{1,\omega}$ and $\|h^3_-\|_{1,\omega}$.*

Proof. Since both functions $\sigma(\varepsilon)^+$ and $\sigma(\varepsilon)^-$ belong to the space $\mathscr{C}^1(\overline{\omega})$, the function $\tilde{h}(\varepsilon)$ defined in (6.9) belongs to $H^1(\Omega)$ if the functions defined in (6.8) are in $H^1(\omega)$. By (6.7), both functions $\sigma(\varepsilon)^+$ and $\sigma(\varepsilon)^-$ converge to $\sqrt{a}$ in $\mathscr{C}^1(\overline{\omega})$; consequently, the convergence (6.10) holds, with $\tilde{h}(0)$ given in (6.11). Next, let $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$. Since $\boldsymbol{v} = \boldsymbol{0}$ on Γ_0 and $\tilde{h}(\varepsilon)v_3 = \pm h^3\sigma(\varepsilon)$ on $\Gamma_\pm$, we may write

$$(6.14) \qquad \int_{\Gamma_+\cup\Gamma_-} h^3 v_3 \sigma(\varepsilon) d\Gamma = \int_{\partial\Omega} \tilde{h}(\varepsilon) v_i n^i d\Gamma,$$

where (n^i) is the unit outer normal vector along the boundary $\partial\Omega$ of the set Ω. We thus obtain (6.12) by applying Green's formula to the right-hand side of (6.14). (Applying Green's formula is legitimate here since $\tilde{h}(\varepsilon) \in H^1(\Omega), \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, and $\partial\Omega$ is Lipschitz-continuous; see, e.g., Nečas [1967, p. 121].) From relation (6.12), we then deduce

$$(6.15) \qquad \left| \int_{\Gamma_+ \cup \Gamma_-} h^3 v_3 \sigma(\varepsilon) d\Gamma \right| \leqq \|\tilde{h}(\varepsilon)\|_{1,\Omega} \Big(\|\partial_1 v_1\|_{0,\Omega} + \|\partial_2 v_2\|_{0,\Omega} + \|\partial_3 v_3\|_{0,\Omega} + \sum_i \|v_i\|_{0,\Omega} \Big),$$

and inequality (6.13) follows from (6.15) combined with the boundedness of the norms $\|\tilde{h}(\varepsilon)\|_{1,\Omega}$ for $0 \leqq \varepsilon \leqq \varepsilon_0$, the inequality

$$\|\partial_1 v_1\|_{0,\Omega} + \|\partial_2 v_2\|_{0,\Omega} + \sum_i \|v_i\|_{0,\Omega} \leqq \sqrt{5} C \Big\{ \sum_{i,j} \|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|^2_{0,\Omega} \Big\}^{1/2}$$

for $0 < \varepsilon \leqq \varepsilon_1$ (which itself follows from the fundamental inequality (4.4)), and the inequality

$$\|\partial_3 v_3\|_{0,\Omega} \leqq \|e_{3\|3}(\varepsilon)(v)\|_{0,\Omega}$$

valid for $0 < \varepsilon \leqq 1$. □

The behavior of the solution of problem (6.3), (6.4) as $\varepsilon \to 0$ is described in the following analog of Theorem 5.1:

Theorem 6.2. *Define the functions* $h^i_+ : \omega \to \mathbf{R}$ *and* $h^i_- : \omega \to \mathbf{R}$ *by*

$$(6.16) \qquad h^i_+(y) := h^i(y, 1), \quad h^i_-(y) := h^i(y, -1) \quad \textit{for all } y \in \omega,$$

where the functions h^i *are those of* (6.2), *and assume that*

$$(6.17) \qquad h^\alpha_+, h^\alpha_- \in L^2(\omega), \quad h^3_+, h^3_- \in H^1(\omega).$$

Assume that there exists a constant c such that inequality (5.1) *holds. For* $0 < \varepsilon \leqq \varepsilon_0$, *let* $\boldsymbol{u}(\varepsilon)$ *denote the solution of the scaled variational problem* (6.3), (6.4). *Then there exist functions* $u_\alpha \in H^1(\Omega)$ *vanishing on* Γ_0 *and* $u_3 \in L^2(\Omega)$ *that satisfy relations* (5.2)–(5.4), *and the function* $\overline{\boldsymbol{u}} = (\overline{u}_i) \in \boldsymbol{V}_M(\omega)$ *satisfies the two-dimensional variational equations*

(6.18)

$$\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\overline{\boldsymbol{u}}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy = \int_\omega (h^i_+ + h^i_-) \eta_i \sqrt{a}\, dy \quad \textit{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega).$$

Proof. The proof involves the same eight steps as that of Theorem 5.1. We only indicate the modifications needed for handling the "new" right-hand side in equations (6.4).

In Step (i), the chain of inequalities that leads to the *a priori bounds* now reads, thanks to inequality (6.13),

$$C^{-2}\Big\{\sum_\alpha \|u_\alpha(\varepsilon)\|^2_{1,\Omega} + \|u_3(\varepsilon)\|^2_{0,\Omega}\Big\} \leqq \sum_{i,j}\|e_{i\|j}(\varepsilon)\|^2_{0,\Omega}$$
$$\leqq C_2 g_0^{-1/2}\Big(\int_{\Gamma_+\cup\Gamma_-} h^\alpha u_\alpha(\varepsilon)\sigma(\varepsilon)\, d\Gamma + \int_{\Gamma_+\cup\Gamma_-} h^3 u_3(\varepsilon)\sigma(\varepsilon)\, d\Gamma\Big)$$
$$\leqq C_2 g_0^{-1/2}\Big(C_5\Big\{\sum_\alpha \|u_\alpha(\varepsilon)\|^2_{1,\Omega}\Big\}^{1/2} + C_4\Big\{\sum_{i,j}\|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|^2_{0,\Omega}\Big\}^{1/2}\Big)$$
$$\leqq C_2 g_0^{-1/2}(CC_5 + C_6)\Big\{\sum_{i,j}\|e_{i\|j}(\varepsilon)\|^2_{0,\Omega}\Big\}^{1/2},$$

and thus the conclusions are the same. Note that the constant C_5 depends on the norms $\|h^\alpha\|_{L^2(\Gamma_+\cup\Gamma_-)}$ and on the norm of the trace operator acting from $H^1(\Omega)$ into $L^2(\Gamma_+\cup\Gamma_-)$.

Step (ii) is the same. In Step (iii), the right-hand side of the relation

$$\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)\Big\{\varepsilon e_{i\|j}(\varepsilon)(\boldsymbol{v})\Big\}\sqrt{g(\varepsilon)}\, dx = \varepsilon\int_{\Gamma_+\cup\Gamma_-} h^i v_i\sigma(\varepsilon)\, d\Gamma$$

again converges to 0 as $\varepsilon\to 0$ for $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ fixed, thanks to (6.7). In Step (iv), again let $\boldsymbol{v}=(v_i)\in\boldsymbol{V}(\Omega)$ be independent of x_3; then, again by (6.7) and by Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\int_{\Gamma_+\cup\Gamma_-} h^i v_i\sigma(\varepsilon)\, d\Gamma = \int_{\Gamma_+\cup\Gamma_-} h^i v_i\sqrt{a}\, dy = \int_\omega (h^i_+ + h^i_-)\overline{v}_i\sqrt{a}\, dy.$$

The same denseness argument then shows that $\overline{\boldsymbol{u}}$ satisfies equations (6.18).

In Step (v), we now have

$$\Lambda(\varepsilon) = \int_{\Gamma_+\cup\Gamma_-} h^i u_i(\varepsilon)\sigma(\varepsilon)\, d\Gamma + \int_\Omega A^{ijkl}(\varepsilon)(e_{k\|l} - e_{k\|l}(\varepsilon))e_{i\|j}\sqrt{g(\varepsilon)}\, dx.$$

Since $u_\alpha(\varepsilon)\rightharpoonup u_\alpha$ in $L^2(\Gamma_+\cup\Gamma_-)$ whenever $u_\alpha(\varepsilon)\rightharpoonup u_\alpha$ in $H^1(\Omega)$, it follows from (6.7) that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_+\cup\Gamma_-} h^\alpha u_\alpha(\varepsilon)\sigma(\varepsilon)\, d\Gamma = \int_{\Gamma_+\cup\Gamma_-} h^\alpha u_\alpha\sqrt{a}\, d\Gamma = \int_\omega (h^\alpha_+ + h^\alpha_-)\overline{u}_\alpha\sqrt{a}\, dy$$

as $\varepsilon\to 0$, on the one hand. Identity (6.12) allows us to write

$$\int_{\Gamma_+\cup\Gamma_-} h^3 u_3(\varepsilon)\sigma(\varepsilon)\, d\Gamma = \int_\Omega \tilde{h}(\varepsilon)\big(\partial_\alpha u_\alpha(\varepsilon) + \partial_3 u_3(\varepsilon)\big)\, dx + \int_\Omega \partial_i\tilde{h}(\varepsilon)u_i(\varepsilon)\, dx,$$

on the other hand. Since $\tilde{h}(\varepsilon)\to\tilde{h}(0)$ in $H^1(\Omega)$, $u_\alpha(\varepsilon)\rightharpoonup u_\alpha$ in $H^1(\Omega)$, $\partial_3 u_3(\varepsilon)\to 0$ in $L^2(\Omega)$ (cf. Step (ii)), and $u_3(\varepsilon)\rightharpoonup u_3$ in $L^2(\Omega)$, we conclude that

$$\lim_{\varepsilon\to 0} \int_{\Gamma_+\cup\Gamma_-} h^3 u_3(\varepsilon)\sigma(\varepsilon)\, d\Gamma = \int_\Omega \tilde{h}(0)\partial_\alpha u_\alpha dx + \int_\Omega \partial_i \tilde{h}0 u_i\, dx.$$

Using Green's formula as in the proof of Lemma 6.2, we obtain (note that $u_\alpha = 0$ on $\Gamma_{\mathbf{0}}$)

$$\int_\Omega \tilde{h}(0)\partial_\alpha u_\alpha dx + \int_\Omega \partial_i \tilde{h}(0) u_i dx = \int_\Omega \partial_3 \tilde{h}(0) u_3 dx$$

$$= \frac{1}{2}\int_\Omega (h_+^3 + h_-^3) u_3 \sqrt{a}\, dx = \int_\omega (h_+^3 + h_-^3)\overline{u}_3 \sqrt{a}\, dy,$$

and thus we again conclude that $\Lambda = 0$, as in (5.25). The remaining Steps (vi), (viii) are unaltered. □

7. Conclusions and comments

7.1. Assume that both body and surface forces satisfying assumptions (2.3) and (6.2) respectively, are acting on the shell, and let $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) \in \boldsymbol{V}(\Omega)$ denote the scaled unknown (cf. (2.1) and (2.10)) that satisfies the corresponding three-dimensional shell problem. If there exists a constant c such that inequality (5.1) holds, Theorems 5.1 and 6.2 together imply that there exist functions $u_\alpha \in H^1(\Omega)$ vanishing on $\Gamma_0 = \gamma \times [-1, 1]$ and $u_3 \in L^2(\Omega)$ such that

(7.1) $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(\Omega)$, $u_3(\varepsilon) \to u_3$ in $L^2(\Omega)$ as $\varepsilon \to 0$,

(7.2) $\boldsymbol{u} = (u_i)$ is independent of the transverse variable x_3,

(7.3) $$\boldsymbol{\zeta} = (\zeta_i) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3 \in \boldsymbol{V}_M(\omega),$$

(7.4) $$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\, dy = \int_\omega p^{i,\varepsilon}\eta_i \sqrt{a}\, dy \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega),$$

where

(7.5) $$\boldsymbol{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

the tensors $(a^{\alpha\beta\sigma\tau})$, $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$, and the function a are defined in (5.6), (3.17) and (1.4), and

(7.6) $$p^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} dx_3^\varepsilon + (h_+^{i,\varepsilon} + h_-^{i,\varepsilon}),$$

where $h_+^{i,\varepsilon} := \varepsilon h_+^i, h_-^{i,\varepsilon} := \varepsilon h_-^i$ and the functions h_+^i, h_-^i are defined in (6.16).

Under the essential assumption that the surface S is elliptic (cf. Theorems 5.2 and 5.3), *we have therefore justified by a convergence result (cf.* (7.1)) *two-dimensional variational equations* (7.4) *that are classically those of a linearly* *elastic "membrane" shell* (cf., e.g., DIKMEN [1982, eqs. (7.10)], GREEN & ZERNA [1968, Sec. 11.1], or NIORDSON [1985, eq. (10.3)]). In so doing, *we have also justified the formal asymptotic approach of* SANCHEZ-PALENCIA [1990] (see also MIARA & SANCHEZ-PALENCIA [1996] and CAILLERIE & SANCHEZ-PALENCIA [1995]) *in the "well-inhibited"* case, according to the terminology of SANCHEZ-PALENCIA.

7.2. The *existence* and *uniqueness* of a solution to the two-dimensional membrane shell equations (7.4) is a corollary of Theorem 4.2 and of the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$ (cf., e.g., BERNADOU, CIARLET & MIARA [1994, Lemma 2.1]). The *regularity* of the solution has been established by GENEVEY [1995]; her proof relies on the theory of elliptic systems of AGMON, DOUGLIS & NIRENBERG [1964] (in the same vein, see also GEYMONAT & SANCHEZ-PALENCIA [1995]). Note in passing that the variational problem (7.4) is atypical, in that one of the unknowns "only" lies in the space $L^2(\omega)$.

7.3. The convergences (7.1), the scalings (2.1), and inequalities (3.12), (3.14) together imply the following convergences of the *averages* across the thickness of the shell of the covariant components of the "original" three-dimensional displacement:

$$(7.7) \qquad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon dx_3^\varepsilon \to \zeta_\alpha \text{ in } H^1(\omega), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon dx_3^\varepsilon \to \zeta_3 \text{ in } L^2(\omega).$$

These convergences can be further improved and given a more "intrinsic" character by considering instead the averages of the *tangential* component $u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon}$ and *normal* component $u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon}$ of the three-dimensional displacement *vector* itself (note that, along a given normal direction to the surface S, the vectors $\boldsymbol{g}^{\alpha,\varepsilon}$ and $\boldsymbol{g}^{3,\varepsilon}$ remain respectively parallel to the tangent plane and normal to S, since $\boldsymbol{g}_\alpha^\varepsilon = \boldsymbol{a}_\alpha - x_3^\varepsilon b_\alpha^\sigma \boldsymbol{a}_\sigma, \boldsymbol{g}^{3,\varepsilon} = \boldsymbol{a}_3$, and $\boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}^{j,\varepsilon} = \delta_i^j$). More specifically, the convergences (7.7) combined with the behavior as $\varepsilon \to 0$ of the vectors $\boldsymbol{g}^{i,\varepsilon}$ (once "scaled" for convenience as vectors defined over the fixed set Ω) imply that

$$(7.8) \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon} dx_3^\varepsilon \to \zeta_\alpha \boldsymbol{a}^\alpha \text{ in } H^1(\omega), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon} dx_3^\varepsilon \to \zeta_3 \boldsymbol{a}^3 \text{ in } L^2(\omega).$$

7.4. The first *convergence results* for "membrane" shells have been obtained by DESTUYNDER [1980] in his doctoral dissertation. In particular, the convergences established there in Theorem 7.9 (p. 305), under the assumption that the surface S is elliptic, are almost identical to those established in Theorem 5.1 for the components $u_\alpha(\varepsilon)$, but "weaker" for the component $u_3(\varepsilon)$, since DESTUYNDER only established that $\varepsilon u_3(\varepsilon) \to 0$ in $L^2(\Omega)$. Besides, the justification of the membrane shell equations remained partially formal in that it still relied on an assumed asymptotic expansion of $u_3(\varepsilon)$.

Using Γ-*convergence theory*, ACERBI, BUTTAZZO & PERCIVALE [1988] were able to obtain convergence theorems for shells viewed as "thin inclusions" in a larger, surrounding elastic body. As a consequence, the distinction between "membrane shells" and "flexural shells" (cf. Part II) is no longer related to the geometry of the middle surface and the boundary condition as here, but instead to the ratio (as a power of ε) between the Lamé constants of the two elastic materials in presence. This asymptotic analysis is thus reminiscent of that of CIARLET, LE DRET & NZENGWA [1989], who considered a plate partly inserted in an elastic body; if the shell were a plate, the approach of ACERBI et al. would only apply to the inserted portion, however.

7.5. Our asymptotic analysis covers two essentially distinct situations regarding the "geometry" of the surface S and boundary conditions: Either the shell is clamped on its entire lateral surface and assumption (5.1) holds (this is the situation considered here), or (cf. Part II) the space of "inextensional displacements"

$$\mathbf{V}_F(\omega) := \Big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0,\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\Big\}$$

does not reduce to $\{\mathbf{0}\}$, where $\boldsymbol{\Phi}(\gamma_0 \times [-\varepsilon, \varepsilon])$, with $\gamma_0 \subset \gamma$, denotes the portion of the lateral face where the "original" three-dimensional shell is clamped (here, $\gamma_0 = \gamma$). In addition to these cases, which were respectively labeled "well-inhibited" and "non-inhibited" by SANCHEZ-PALENCIA [1989a, 1989b], there remain the "badly-inhibited cases" (following again the terminology of SANCHEZ-PALENCIA), occurring when the space $\mathbf{V}_F(\omega)$ reduces to $\{\mathbf{0}\}$, but relation (5.1) does not hold. This happens for instance if the surface S is elliptic but γ_0 is only a portion of γ (ŞLICARU [1996]), or if the surface S is a hyperboloid of revolution (MARDARE [1996]).

For such *generalized membrane shells,* a *formal* asymptotic analysis of the *three-dimensional* shell equations can still be carried out (cf. CAILLERIE & SANCHEZ-PALENCIA [1995]), and *a convergence theorem* has been established by CIARLET & LODS [1995b,c]. The limit problems found in this fashion possess two unusual features: Their solutions are not necessarily distributions, and they are "extremely sensitive" to arbitrary small perturbations of the data. Examples of such problems have been recently studied by SANCHEZ-PALENCIA [1993] and LIONS & SANCHEZ-PALENCIA [1994, 1996].

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