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REMARKS ON THE METHOD OF PAIRED COMPARISONS: I. THE LEAST SQUARES SOLUTION ASSUMING EQUAL STANDARD DEVIATIONS AND EQUAL CORRELATIONS*

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Thurstone's Case V of the method of paired comparisons assumes equal standard deviations of sensations corresponding to stimuli and zero correlations between pairs of stimuli sensations. It is shown that the assumption of zero correlations can be relaxed to an assumption of equal correlations between pairs with no change in method. Further the usual approach to the method of paired comparisons Case V is shown to lead to a least squares estimate of the stimulus positions on the sensation scale.

1. *Introduction.* The fundamental notions underlying Thurstone's method of paired comparisons (4) are these:

(1) There is a set of stimuli which can be located on a subjective continuum (a sensation scale, usually not having a measurable physical characteristic).

(2) Each stimulus when presented to an individual gives rise to a sensation in the individual.

(3) The distribution of sensations from a particular stimulus for a population of individuals is normal.

(4) Stimuli are presented in pairs to an individual, thus giving rise to a sensation for each stimulus. The individual compares these sensations and reports which is greater.

(5) It is possible for these paired sensations to be correlated.

(6) Our task is to space the stimuli (the sensation means), except for a linear transformation.

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There are numerous variations of the basic materials used in the analysis—for example, we may not have n different individuals, but only one individual who makes all comparisons several times; or several individuals may make all comparisons several times; the individuals need not be people.

Furthermore, there are "cases" to be discussed—for example, shall we assume all the intercorrelations equal, or shall we assume them zero? Shall we assume the standard deviations of the sensation distributions equal or not?

The case which has been discussed most fully is known as Thurstone's Case V. Thurstone has assumed in this case that the standard deviations of the sensation distributions are equal and that the correlations between pairs of stimulus sensations are zero. We shall discuss a standard method of ordering the stimuli for this Case V. Case V has been employed quite frequently and seems to fit empirical data rather well in the sense of reproducing the original proportions of the paired comparison table. The assumption of equal standard deviations is a reasonable first approximation. We will not stick to the assumption of zero correlations, because this does not seem to be essential for Case V.

2. *Ordering Stimuli with Error-Free Data.* We assume there are a number of objects or stimuli, $O_1, O_2, \cdots, O_n$. These stimuli give rise to sensations which lie on a single sensation continuum S. If X_i and X_j are single sensations evoked in an individual I by the ith and jth stimuli, then we assume X_i and X_j to be jointly normally distributed for the population of individuals with

$$\begin{aligned} &\text{mean of } X_i = S_i && (i = 1, 2, \cdots, n) \\ &\text{variance of } X_i = \sigma^2(X_i) = \sigma^2 && (i = 1, 2, \cdots, n) \qquad (1) \\ &\text{correlation of } X_i \text{ and } X_j = \rho_{ij} = \rho && (i,j = 1, 2, \cdots, n). \end{aligned}$$

The marginal distributions of the X_i's appear as in Figure 1.

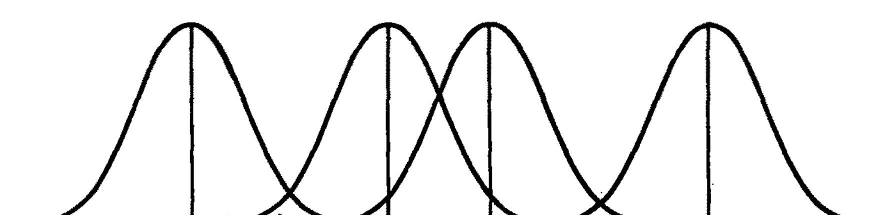

FIGURE 1

The Marginal Distributions of the Sensations Produced by the Separate Stimuli in Thurstone's Case V of the Method of Paired Comparisons.

The figure indicates the possibility that $X_2 < X_1$, even though $S_1 < S_2$. In fact this has to happen part of the time if we are to build anything more than a rank-order scale.

An individual I compares O_i and O_j and reports whether $X_i \gtrless X_j$ (no ties are allowed).

We can best see the tenor of the method for ordering the stimuli if we first work through the problem in the case of nonfallible data. For the case of nonfallible data we assume we know the true proportion of the time X_i exceeds X_j, and that the conditions given above (1) are exactly fulfilled.

Our problem is to find the spacing of the stimuli (or the spacing of the mean sensations produced by them, the $S_1 \cdots S_n$ points in Figure 1). Clearly we cannot hope to do this except within a linear transformation, for the data reported are merely the percentages of times X_i exceeds X_j, say p_{ij}.

$$p_{ij} = P(X_i > X_j) = \frac{1}{\sqrt{2\pi}\,\sigma(d_{ij})} \int_0^\infty e^{\frac{-[d_{ij} - (S_i - S_j)]^2}{2\,\sigma^2(d_{ij})}}\, dd_{ij} \quad (2)$$

where $d_{ij} = X_i - X_j$, and $\sigma^2(d_{ij}) = 2\sigma^2(1 - \rho)$. There will be no loss in generality in assigning the scale factor so that

$$2\sigma^2(1 - \rho) = 1. \quad (3)$$

It is at this point that we depart slightly from Thurstone, who characterized Case V as having equal variances and zero correlations. However, his derivations only assume the correlations are zero explicitly (and artificially), but are carried through implicitly with equal correlations (not necessarily zero). Actually this is a great easing of conditions. We can readily imagine a set of attitudinal items on the same continuum correlated .34, .38, .42, i.e., nearly equal. But it is difficult to imagine them all correlated zero with one another. Past uses of this method have all benefited from the fact that items were not *really* assumed to be uncorrelated. It was only *stated* that the model assumed the items were uncorrelated, but the model was unable to take cognizance of the statement. Guttman (2) has noticed this independently.

With the scale factor chosen in equation (3), we can rewrite equation (2)

$$p_{ij} = \frac{1}{\sqrt{2\pi}} \int_{-(S_i - S_j)}^{\infty} e^{-\frac{1}{2}y^2} \, dy. \tag{4}$$

From (4), given any p_{ij} we can solve for $-(S_i - S_j)$ by use of a normal table of areas. Then if we arbitrarily assign as a location parameter $S_1 = 0$, we can compute all other S_i. Thus given the p_{ij} matrix we can find the S_i. The problem with fallible data is more complicated.

3. *Paired Comparison Scaling with Fallible Data.* When we have fallible data, we have p'_{ij} which are estimates of the true p_{ij}. Analogous to equation (4) we have

$$p'_{ij} = \frac{1}{\sqrt{2\pi}} \int_{-D'_{ij}}^{\infty} e^{-\frac{1}{2}y^2} \, dy, \tag{5}$$

where the D'_{ij} are estimates of $D_{ij} = S_i - S_j$. We merely look up the normal deviate corresponding to p'_{ij} to get the matrix of D'_{ij}. We notice further that the D'_{ij} need not be consistent in the sense that the D_{ij} were; i.e.,

$$D_{ij} + D_{jk} = S_i - S_j + S_j - S_k = D_{ik},$$

does not hold for the D'_{ij}.

We conceive the problem as follows: from the D'_{ij} to construct a set of estimates of the S_i's called S'_i, such that

$$\Sigma = \sum_{i,j} [D'_{ij} - (S'_i - S'_j)]^2 \quad \text{is to be a minimum.} \tag{6}$$

It will help to indicate another form of solution for nonfallible data. One can set up the $S_i - S_j$ matrix:

MATRIX OF $S_i - S_j$

	1	2	3	.....	n
1	$S_1 - S_1$	$S_1 - S_2$	$S_1 - S_3$		$S_1 - S_n$
2	$S_2 - S_1$	$S_2 - S_2$	$S_2 - S_3$		$S_2 - S_n$
3	$S_3 - S_1$	$S_3 - S_2$	$S_3 - S_3$		$S_3 - S_n$
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n	$S_n - S_1$	$S_n - S_2$	$S_n - S_3$		$S_n - S_n$
Totals	$\Sigma S_i - nS_1$	$\Sigma S_i - nS_2$	$\Sigma S_i - nS_3$		$\Sigma S_i - nS_n$
Means	$\bar{S} - S_1$	$\bar{S} - S_2$	$\bar{S} - S_3$		$\bar{S} - S_n$

Now by setting $S_1 = 0$, we get $S_2 = (\bar{S} - S_1) - (\bar{S} - S_2)$, $S_3 = (\bar{S} - S_1) - (\bar{S} - S_3)$, and so on. We will use this plan shortly for the S'_i.

If we wish to minimize expression (6) we take the partial derivative with respect to S'_i. Since $D'_{ij} = -D'_{ji}$ and $S'_i - S'_j = -(S'_j - S'_i)$ and $D'_{ii} = S'_i - S'_i = 0$, we need only concern ourselves with the sum of squares from above the main diagonal in the $D'_{ij} - (S'_i - S'_j)$ matrix, i.e., terms for which $i < j$. Differentiating with respect to S'_i we get:

$$\frac{\partial(\Sigma/2)}{\partial S'_i} = 2\left[\sum_{j=1}^{i-1}(D'_{ji} - S'_j + S'_i) - \sum_{j=i+1}^{n}(D'_{ij} - S'_i + S'_j)\right] \quad (i = 1, 2, \cdots, n). \tag{7}$$

Setting this partial derivative equal to zero we have

$$\begin{aligned} &+S'_1 + S'_2 \cdots + S'_{i-1} - (n-1)S'_i + S'_{i+1} + \cdots + S'_n \\ &= \sum_{j=1}^{i-1} D'_{ji} - \sum_{j=i+1}^{n} D'_{ij} \qquad (i = 1, 2, \cdots, n), \end{aligned} \tag{8}$$

but $D'_{ij} = -D'_{ji}$, and $D'_{ii} = 0$; this makes the right side of (8)

$$\sum_{j=1}^{i-1} D'_{ji} + D'_{ii} + \sum_{j=i+1}^{n} D'_{ji} = \sum_{j=1}^{n} D'_{ji}.$$

Thus (8) can be written

$$\sum_{j=1}^{n} S'_j - nS'_i = \sum_{j=1}^{n} D'_{ji} \qquad (i = 1, 2, \cdots, n). \tag{9}$$

The determinant of the coefficients of the left side of (9) vanishes. This is to be expected because we have only chosen our scale and have not assigned a location parameter. There are various ways to assign this location parameter, for example, by setting $\bar{S}' = 0$ or by setting $S'_1 = 0$. We choose to set $S'_1 = 0$. This means we will measure distances from S'_1. Then we try the solution (10) which is suggested by the similarity of the left side of (9) to the total column in the matrix of $S_i - S_j$.

$$S'_i = \sum_{j=1}^{n} D'_{j1}/n - \sum_{j=1}^{n} D'_{ji}/n. \tag{10}$$

Notice that when $i=1$, $S'_i = 0$ and that

$$\sum_{i=1}^{n} S'_i = \sum_{i=1}^{n} D'_{j1}$$

because

$$\sum_i \sum_j D'_{ji} = 0 ,$$

which happens because every term and its negative appear in this double sum. Therefore, substituting (10) in the left side of (9) we have

$$\sum_{i=1}^{n} D'_{j1} - n \left[\sum_{j=1}^{n} D'_{j1}/n - \sum_{j=1}^{n} D'_{ji}/n \right] = \sum_{j=1}^{n} D'_{ji} , \qquad (11)$$

which is an identity, and the equations are solved. Of course, any linear transformation of the solutions is equally satisfactory.

The point of this presentation is to provide a background for the theory of paired comparisons, to indicate that the assumption of zero correlations is unnecessary, and to show that the customary solution to paired comparisons is a least squares solution in the sense of condition (6). That this is a least squares solution seems not to be mentioned in the literature although it may have been known to Horst (3), since he worked closely along these lines.

This least squares solution is not entirely satisfactory because the p'_{ij} tend to zero and unity when extreme stimuli are compared. This introduces unsatisfactorily large numbers in the D'_{ij} table. This difficulty is usually met by excluding all numbers beyond, say, 2.0 from the table. After a preliminary arrangement of columns so that the S'_i will be in approximately proper order, the quantity

$$\sum (D'_{ij} - D'_{i,j+1})/k$$

is computed where the summation is over the k values of i for which entries appear in both column j and $j+1$. Then differences between such means are taken as the scale separations (see for example Guilford's discussion (1) of the method of paired comparisons). This method seems to give reasonable results. The computations for methods which take account of the differing variabilities of the p'_{ij} and therefore of the D'_{ij} seem to be unmercifully extensive.

It should also be remarked that this solution is not entirely a reasonable one because we really want to check our results against the original p'_{ij} . In other words, a more reasonable solution might be one such that once the S'_i are computed we can estimate the p'_{ij} by p''_{ij}, and minimize, say,

$$\sum (p'_{ij} - p''_{ij})^2$$

or perhaps

$$\sum (\text{arc sin} \sqrt{p'_{ij}} - \text{arc sin} \sqrt{p''_{ij}})^2.$$

Such a thing can no doubt be done, but the results of the author's attempts do not seem to differ enough from the results of the present method to be worth pursuing.

REFERENCES

1. Guilford, J. P. Psychometric Methods. New York: McGraw-Hill Book Co., 1936, 227-8.
2. Guttman, L. An approach for quantifying paired comparisons and rank order. *Annals of math. Stat.*, 1946, 17, 144-163.
3. Horst, P. A method for determining the absolute affective values of a series of stimulus situations. *J. educ. Psychol.*, 1932, 23, 418-440.
4. Thurstone, L. L. Psychophysical analysis. *Amer. J. Psychol.*, 1927, 38, 368-389.

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