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A GENERAL ROTATION CRITERION AND ITS USE IN ORTHOGONAL ROTATION*

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Measures of test parsimony and factor parsimony are defined. Minimizing their weighted sum produces a general rotation criterion for either oblique or orthogonal rotation. The quartimax, varimax and equamax criteria are special cases of the expression. Two new criteria are developed. One of these, the parsimax criterion, apparently gives excellent results. It is argued that one of the most important factors bearing on the choice of a rotation criterion for a particular problem is the amount of information available on the number of factors that should be rotated.

Test Parsimony and Rotation

Consider the following expression

$$\sum_i^n \left(\sum_p^m a_{ip}^2\right)^2 = \sum_i^n \sum_p^m a_{ip}^4 + \sum_i^n \sum_p^m \sum_{\substack{q \\ p \neq q}}^m a_{ip}^2 a_{iq}^2 \tag{1}$$

where i refers to tests and p and q to factors. For convenience, (1) may be written

$$H = Q + T \tag{2}$$

where

H = the sum of squares of the communalities and is constant under orthogonal transformation

Q = the quartimax criterion for orthogonal rotation [Ferguson, 1954; Neuhaus and Wrigley, 1954; Saunders, 1953], and

T = the quartimin criterion for oblique rotation [Carroll, 1953].

Criteria based on T or Q are here referred to as test parsimony criteria because they can be developed by defining the parsimony of test i as

$$T_i = \sum_p^m \sum_{\substack{q \\ p \neq q}}^m a_{ip}^2 a_{iq}^2 \tag{3}$$

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or

$$Q_i = \sum_{p}^{m} a_{ip}^2 \tag{4}$$

and summing over all n tests. Maximum test parsimony occurs when T_i is small or when Q_i is large. T_i is not a measure of test parsimony in the oblique case since H is not a constant under oblique rotation.

In order to evaluate rotation criteria consider Thurstone's five principles of simple structure [Harman, 1960]:

"1. Each row of the factor matrix should have at least one zero.
2. If there are m common factors, each column of the factor matrix should have at least m zeros.
3. For every pair of columns of the factor matrix there should be several variables whose entries vanish in one column but not in the other.
4. For every pair of columns of the factor matrix, a large proportion of the variables should have vanishing entries in both columns when there are four or more factors.
5. For every pair of columns of the factor matrix there should be only a small number of variables with non-vanishing entries in both columns."

Although these principles do not provide a definitive standard for rotation they do provide a useful framework for discussing analytic rotation criteria.

Because they are defined for rows, criteria based on test parsimony tend to produce large and small values in the rows of the factor loading matrix and usually produce rotated factor matrices which satisfy the first and third principles of simple structure. However, the value of these criteria would by unchanged by a reordering of the values in the rows of the factor loading matrix, and therefore, during convergence, they can allow many high loadings to accumulate on the same factor and simple structure condition 2 is not necessarily satisfied. This condition is very important in determining an adequate rotation; it locates and overdetermines the hyperplanes and, unless it is satisfied, conditions 4 and 5 are unlikely to be satisfied.

Criteria, such as the quartimax and quartimin criteria, which are based on test parsimony tend to produce a general factor. The maximum number of zero loadings occurs when each test is loaded on only one factor. In practice when there are several factors and when the tests are not all univocal this situation is most nearly approximated when the factor matrix contains a general factor. The value of this type of criterion would not be changed by a rearrangement of the values in the rows of the factor matrix and therefore during convergence it can allow many high loadings to accumulate on the same factor. Thus, in practice, criteria based on test parsimony usually reach their maximum or minimum value only when they produce a general factor.

Factor Parsimony and Rotation

The last four principles of simple structure are defined with respect to the columns of the factor loading matrix. With this fact in mind consider the following expression

$$\sum_{p}^{m} \left(\sum_{i}^{n} a_{ip}^{2}\right)^{2} = \sum_{p}^{m} \sum_{i}^{n} a_{ip}^{4} + \sum_{p}^{m} \sum_{i}^{n} \sum_{\substack{j \\ i \neq j}}^{n} a_{ip}^{2} a_{jp}^{2} \tag{5}$$

where i and j refer to tests and p refers to factors. For convenience, (5) may be written

$$V = Q + F \tag{6}$$

where

V = the sum of the squares of the factor variances
F = a measure of factor parsimony and
Q = the familiar quartimax criterion.

Since F gets smaller as the loadings on a single factor approach 1 and 0 it can be used as a criterion for rotation. It will be referred to as the factor parsimony criterion. Since V is not a constant, Q is not a measure of factor parsimony.

Since F is defined for columns its value would be changed by a reordering of the values in the rows of the factor matrix and therefore it does not allow a large number of high loadings to accumulate on the same factor during convergence. A matrix rotated by this criterion should have approximately the same number of zeros in each column and should have approximately equal factor variances. If sufficient factors have been rotated, a considerable number of near zero loadings should appear in each column and simple structure requirements 2, 4 and 5 should be satisfied. However, since F is defined for columns, it does not necessarily produce a zero in every row of the factor matrix and at convergence its results do not necessarily satisfy simple structure requirements 1 and 3. Apparently neither test nor factor parsimony, by itself, provides an adequate basis for defining a rotation criterion.

A General Criterion for Analytic Rotation

A criterion based on some combination of test and factor parsimony would apparently embody all the conditions for simple structure. Consider the minimum value of the function

$$G(\phi) = K_1 T + K_2 F \tag{7}$$

where K_1 and K_2 are weights that determine the relative importance of test and factor parsimony in the rotation criterion, $G(\phi)$. Equation (7) is a general expression for describing the parsimony of a factor loading matrix and may be used to define both oblique and orthogonal criteria. By varying the weights an infinite number of particular criteria can be generated. Several of the existing orthogonal criteria are special cases of $G(\phi)$.

The relative size of K_1 and K_2 is very important since it determines the size of the general factor as well as how much the number of factors rotated affects the final result. If K_1 is large relative to K_2, variance tends to concentrate in the first few factors and a pronounced general factor can occur. If K_2 is large relative to K_1, variance is spread over all factors rotated and factors with approximately equal variances are produced, making the results dependent on the number of factors rotated. A rational method of determining the relative importance of K_1 and K_2 must be found.

The Parsimax Criterion

The weights can be chosen so that test and factor parsimony always have the same weight regardless of the number of factors rotated. Since there are $nm(m - 1)$ terms in T and $nm(n - 1)$ terms in F, setting $K_1 = nm(m - 1)$ and $K_2 = nm(n - 1)$, substituting into (7) and simplifying produces

$$P(\phi) = (n - 1)T + (m - 1)F, \tag{8}$$

a criterion which weights test and factor parsimony equally for any number of factors. It is referred to as the parsimax criterion for analytic rotation and may be used for oblique or orthogonal rotation.

The Orthomax Criterion

If attention is restricted to the orthogonal case then from (2) $T = H - Q$ and from (6) $F = V - Q$. Substituting these expressions into (7) produces

$$O'(\phi) = K_1(H - Q) + K_2(V - Q), \tag{9}$$

rearranging terms and multiplying through by $n/(K_1 + K_2)$ produces

$$O(\phi) = \frac{nK_1H}{K_1 + K_2} - \left[nQ - \frac{nK_2}{K_1 + K_2} V\right]. \tag{10}$$

For convenience this expression may be written as

$$HC_1 - [nQ - C_2V] \tag{11}$$

where

$$C_1 = nK_1/(K_1 + K_2) \tag{12}$$

and

$$C_2 = nK_2/(K_1 + K_2). \tag{13}$$

Since HC_1 is a constant, minimizing (11) is the same as maximizing

$$O(\phi) = nQ - C_2V \tag{14}$$

which is Saunders' [1962] trans-varimax criterion for analytic rotation. Carroll [Harman, 1960] has more appropriately called it the orthomax criterion. Equation (14) is a general expression for orthogonal rotation; C_2 determines the relative importance of test and factor parsimony in the criterion.

If $K_2 = 0$ then from (13) $C_2 = 0$ and $O(\phi)$ is independent of factor parsimony. If $K_1 = 0$, then from (13 and 6) $C_2 = n$ and $O(\phi)$ is independent of test parsimony. If C_2 is unequal to either 0 or n, then $O(\phi)$ is a function of both test and factor parsimony. Although C_2 can be given any value, these facts suggest that

$$0 \leq C_2 \leq n$$

is a meaningful range for C_2.

Oblique Analogues of Some Orthogonal Criteria

C_2 can be chosen so that $O(\phi)$ becomes the quartimax, varimax, equamax [Saunders, 1962], parsimax or orthogonal factor parsimony criteria and K_1 and K_2 can be chosen to produce oblique analogues of these criteria. If $K_2 = 0$, then from (13) $C_2 = 0$, and both $G(\phi)$ and $O(\phi)$ are independent of factor parsimony. In this case $G(\phi)$ becomes the quartimin criterion [Carroll, 1953] and $O(\phi)$ becomes the quartimax criterion [Ferguson, 1954; Neuhaus and Wrigley, 1954; Saunders, 1953]. For the varimax criterion $C_2 = nK_2/(K_1 + K_2) = 1$ and $C_2 = 1$ if K_1 and K_2 are proportional to 1 and $1/(n - 1)$. Hence without loss of generality, K_1 and K_2 can be set to 1 and $1/(n - 1)$ and (7) becomes

$$G(\phi) = (n - 1)\ T + F, \tag{15}$$

an oblique analogue of the varimax criterion. This analogue, however, is not equivalent to the covarimin criterion, Kaiser's [1958] oblique version of the varimax criterion. For the equamax criterion $C_2 = nK_2/(K_1 + K_2) = m/2$ and by reasoning similar to the above (7) becomes

$$G(\phi) = (2n - m)\ T + mF, \tag{16}$$

an oblique analogue of the equamax criterion. For the oblique parsimax criterion $K_1 = (n - 1)$ and $K_2 = (m - 1)$. Substituting these values into (13) produces $C_2 = n(m - 1)/(m + n - 2)$, which can be used to provide an orthogonal analogue of the parsimax criterion. And finally if $K_2 = 0$, $G(\phi)$ becomes the oblique factor parsimony criterion and if $C_2 = n$, $O(\phi)$ becomes the orthogonal factor parsimony criterion. Both the oblique and orthogonal versions of the varimax, equamax and parsimax criteria are

Table 1

Value of C_2, K_1 and K_2 for Orthogonal and Oblique Versions of Five Rotation Criteria

CRITERION	ORTHOGONAL	OBLIQUE*
Quartimax	$C_2 = 0$	$K_1 = 1$, or any positive real number; $K_2 = 0$
Varimax	$C_2 = 1$	$K_1 = n - 1$; $K_2 = 1$
Equamax	$C_2 = m/2$	$K_1 = 2n - m$; $K_2 = m$
Parsimax	$C_2 = n(m - 1)/(m + n - 2)$	$K_1 = m - 1$; $K_2 = n - 1$
Factor Parsimony	$C_2 = n$	$K_1 = 0$; $K_2 = 1$, or any positive real number

* If both K_1 and K_2 are multiplied by a positive constant the criterion is unchanged

functions of both test and factor parsimony. The equamax and parsimax criteria are also explicitly functions of the number of factors rotated.

An Example

In this paper an empirical example of the orthomax criterion, the orthogonal case of the general rotation criterion, is reported. An investigation of the oblique case will be reported in a future paper. Five special cases of the orthomax criterion were investigated. They were the quartimax criterion, which depends solely on test parsimony, the varimax, equamax and parsimax criteria, which depend on both test and factor parsimony, and the factor parsimony criterion which depends solely on factor parsimony.

Method and Results

A computer program [Cooley and Lohnes, 1962] was modified to give: (a) quartimax rotations if $C_2 = 0$; (b) varimax rotations if $C_2 = 1$; (c) equamax rotations if $C_2 = m/2$; (d) parsimax rotations if $C_2 = n(m - 1)/(m + n - 2)$; and (e) factor parsimony rotations if $C_2 = n$ where n and m refer to tests and factors and C_2 is used as defined in equation (14). The results of using the parsimax and the factor parsimony criteria to rotate the four row normalized centroid factors from Harman's (1960) study of twenty-four psychological tests are shown in Table 2. The factor variances for the six different rotations of this problem are shown in Table 3. The last column of this table shows the standard deviations of these variances. The number of loadings less than .20 and .10 for each factor of all rotations is shown in

Table 4. The correlations (Kaiser, 1960) between the factors of the subjective rotation and those of five analytic rotations are shown in Table 5.

Many investigators desire a criterion which produces rotated factor matrices with approximately equal factor variances [Kaiser, 1964]. The standard deviation of the factor variances is one measure of the similarity of these variances. The standard deviation for the subjective rotation is smaller then the value for any of the analytic rotations. The values for the equamax, parsimax, varimax and factor parsimony rotations are close together and only a little larger than the value for the subjective rotation. Although the standard deviation for the varimax and factor parsimony rotations are a little larger than those of the equamax and parsimax rotations they are much smaller than the comparatively large value for the quartimax rotation. On the basis of this criterion the equamax rotation is the best of the analytic rotations, but is only a little better than the parsimax, factor parsimony and varimax rotations.

TABLE 2

Rotations of Holzinger and Harman's (1960) Twenty-four Psychological Variables*

I	II	III	IV	I	II	III	IV
Parsimax				Factor Parsimony			
15	23	64	18	16	26	63	19
11	09	42	11	12	12	41	11
17	05	53	09	17	08	52	10
21	12	52	08	21	15	51	09
75	24	17	13	74	26	14	13
75	12	18	21	75	15	16	22
82	19	16	08	82	21	14	08
54	29	33	12	54	31	31	12
80	04	17	26	80	06	15	26
13	70	-11	21	11	70	-14	19
16	62	02	34	15	63	-00	32
01	70	18	08	00	71	15	07
18	62	37	05	17	63	34	04
21	18	00	49	20	19	-01	49
11	09	11	50	11	11	09	50
09	13	38	44	09	15	37	45
13	20	02	63	12	22	00	63
00	29	28	53	-00	32	26	53
13	17	20	39	13	19	19	39
36	14	44	26	36	17	42	27
16	41	38	25	15	43	35	25
37	08	38	37	37	11	36	37
36	25	53	22	36	28	51	23
34	46	17	33	33	48	14	32
3.58	2.75	2.68	2.36	3.57	2.99	2.45	2.36

*Decimal points have been omitted in the body of the table.

Table 3

Contribution of Factors to Variance
for Several Methods of Rotation

Type of Rotation	Factors I	II	III	IV	Standard Deviation of Factor Variances
Subjective	3.43	2.92	2.68	2.36	.451
Quartimax	6.25	2.06	1.76	1.32	2.289
Varimax	3.50	3.08	2.44	2.36	.543
Equamax	3.58	2.78	2.67	2.36	.520
Parsimax	3.59	2.75	2.68	2.36	.524
Factor Parsimony	3.57	2.99	2.45	2.37	.556

Most investigators agree that in a good rotation there should be a large number of near zero loadings evenly distributed over the rows and columns of the factor loading matrix. The results bearing on this criterion are difficult to interpret since an orthogonal factor matrix can have more near zero loadings if it has a general factor, but if it has a general factor the near zero loadings cannot be evenly distributed over all factors. The results shown in Table 4 indicate that although the quartimax criterion produces a large number of near zero loadings it does not distribute them evenly over all factors. Although the subjective rotation does not have as many near zero loadings as some of the analytic rotations, its near zero loadings are evenly distributed over all factors. On the basis of the second criterion it is difficult to determine which of the varimax, equamax, parsimax or factor parsimony

TABLE 4

Number of Near Zero Loadings on
each Factor of the Six Rotations

Type of Rotation	Number of Loadings Less Than .20*					Number of Loadings Less Than .10*				
	I	II	III	IV	Total	I	II	III	IV	Total
Subjective	11	11	9	10	41	5	3	5	2	15
Quartimax	0	18	18	15	51	0	12	12	10	34
Varimax	14	5	14	9	42	3	4	8	5	20
Equamax	13	11	12	9	45	3	3	5	5	16
Parsimax	13	12	11	9	45	3	5	3	5	16
Factor Parsimony	13	11	12	10	46	3	2	4	4	13

*This number is an absolute value.

rotations is best. The varimax criterion is particularly good at producing a large number of near zero loadings while some of the other criteria distribute them more evenly.

The fact that many investigators believe that the results of analytic rotations must be improved through further subjective rotations suggests that one way of judging the quality of an analytic rotation is to compare it with a well known subjective rotation. Harman's [1960] subjective rotation provides a good standard for judging the results of an analytic rotation. The results shown in Table 5 indicate that with the exception of the quartimax rotation all analytic rotations are very similar to the subjective rotation. The varixmax and equamax rotations are not quite as similar to the subjective rotation as are the parsimax and factor parsimony rotations; the chief difference being that when the factors are ordered by their sums of squares the varimax factors are not in the same order sa the subjective factors. By the third criterion, factor parsimony, parsimax, equamax and varimax rotations, in that order, are the best approximations to Harman's [1960] subjective rotations.

On the basis of the results presented here the following conclusions seem warranted: (a) the quartimax criterion is less useful than the other criteria for most problems because it tends to produce a general factor, (b) the varimax, parsimax, equamax and factor parsimony criteria all give good results and (c) it is difficult to determine empirically which of these four criteria is most satisfactory. Further empirical research could, perhaps, determine which criterion is best, but it would be expensive and time consuming and would not necessarily lead to an unambiguous conclusion since there is probably no one criterion that is ideal for all problems.

Discussion

On the basis of what is known about orthogonal analytic rotation criteria, several criteria can be tentatively eliminated and provisional conclusions drawn about the kinds of situations for which the others are most appropriate. The two most important factors bearing on the usefulness of a rotation criterion for a particular problem are the relative weights of test and factor parsimony in the criterion and whether these weights are a function of the number of factors rotated. The test parsimony term of (7) tends to concentrate the variance in the first few factors rotated while the factor parsimony term tends to spread the variance over all factors that have been rotated. If a criterion weights factor parsimony heavily or makes the relative weights of test and factor parsimony a function of the number of factors rotated, its results will be greatly affected by the number of factors rotated. Since the quartimax criterion depends only on test parsimony it tends to produce a general factor and should not be used unless such a factor is desired. Since the factor parsimony criterion depends only on factor parsimony it

TABLE 5

Correlations (Kaiser, 1960) Between the Factors of the Subjective Rotation and those of the Analytic Rotations*

	Quartimax				Varimax				Equamax				Parsimax				Factor Parsimony			
	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV
I	26	37	81	37	98	-15	13	04	97	-16	16	05	97	-17	17	05	96	-18	19	05
II	95	-26	-15	-06	-10	21	97	09	-14	17	97	07	20	97	-12	-10	21	97	-08	-08
III	17	89	-38	-18	19	96	-17	-14	19	97	-14	-11	-15	16	97	06	-17	12	98	04
IV	-01	00	-41	91	-01	12	-11	99	-02	10	-09	99	-02	10	-08	99	-03	08	-05	1.00

*Decimal points omitted

tends to spread the variance over all factors that have been rotated and should not be used unless the investigator is certain about the number of factors that should be rotated.

A criterion which depends on both test and factor parsimony should be appropriate for most problems. Since the test and factor parsimony weights for the parsimax criterion are rationally chosen, and are very similar to those for the equamax criterion which are empirically chosen (Saunders, 1962), and since parsimax rotations are as good as equamax rotations, some investigators might prefer the parsimax criterion. The varimax or parsimax criterion should be used for most problems.

The most important factor bearing on which of these two criteria should be used for a particular problem is the amount of information that is available on the number of factors that should be rotated. The parsimax criterion is explicitly a function of the number of factors and is therefore very sensitive to the number of factors rotated. The varimax criterion is not explicitly a function of the number of factors and does not weight factor parsimony heavily; therefore it is not very sensitive to the number of factors rotated. If the wrong number of factors is rotated by a criterion which is very sensitive to the number of factors rotated, the relationships between the tests and factors may be distorted and the results difficult to interpret. If, however, sufficient information is available to make an accurate estimate of the number of factors, this information should be used. The parsimax criterion should usually be used when the investigator is confident that he has sufficient information to make an accurate estimate of the number of factors. The varimax criterion should usually be used whenever an accurate estimate of the number of factors is not available.

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