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The Final Version of the Mean Value Theorem for Harmonic Functions

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ABSTRACT. We construct examples of nonharmonic functions satisfying the mean value equation for some set of spheres. These results permit us to obtain the two-circle theorem in its definitive form.

§1 Introduction

Suppose that $n \geq 2$ is a positive integer, f is a locally integrable function on real Euclidean space $\mathbb{R}^n$ (we write $f \in L_{\text{loc}}(\mathbb{R}^n)$), and E is a given set of positive numbers. Suppose that for all $r \in E$ and for Lebesgue almost all $x \in \mathbb{R}^n$ we have

$$f(x) = \int_{S(x,r)} f\,d\sigma, \tag{1}$$

where $S(x,r)$ is the sphere with center x and radius r in $\mathbb{R}^n$ equipped with the standard normalized measure $d\sigma$. For what E does this imply that

$$\Delta f = \sum_{m=1}^{n} \frac{\partial^2 f}{\partial x_m^2} = 0$$

in the sense of distributions? The well-known Delsarte two-circle theorem asserts that $\Delta f = 0$ if E consists of two numbers r_1 and r_2 such that r_1/r_2 is not a ratio of roots of the entire function

$$\eta(z) = 1 - 2^{n/2-1}\Gamma\Big(\frac{n}{2}\Big)\frac{J_{n/2-1}(z)}{z^{n/2-1}}$$

([1]; see also [2]). Simple examples (e.g., see [2]) show that the cited condition on r_1/r_2 is necessary.

The Delsarte theorem was further developed and improved in numerous papers (see [2–6] and the survey [7], which contains an extensive bibliography). The "local" version of this theorem, in which a function f satisfying condition (1) is given in the ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $R > r_1 + r_2$, is of particular interest ([5]; see also [7]). Note that an analog of the Delsarte theorem holds for a function $f \in C^\infty(B_R)$ satisfying condition (1) even if $R = r_1 + r_2$ (see [5]). In the general situation, so far it was not known whether f is harmonic for $R \leq r_1 + r_2$.

The case $R \leq r_1 + r_2$ is completely studied in the present paper. Let us state the definitive version of the two-circle theorem.

Theorem 1. *Let $A = \{\alpha/\beta : \eta(\alpha) = \eta(\beta) = 0\}$, $E = \{r_1, r_2\}$, and $R > \max(r_1, r_2)$. Suppose that $f \in L_{\text{loc}}(B_R)$ satisfies Eq. (1) for all $r \in E$ and for almost all $x \in B_{R-r}$. Then the following conditions hold:*

1. *If $r_1 + r_2 < R$ and $r_1/r_2 \notin A$, then $\Delta f = 0$.*
2. *If $r_1 + r_2 = R$, $r_1/r_2 \notin A$, and $f \in C^\infty(B_R)$, then $\Delta f = 0$.*
3. *If $r_1 + r_2 = R$, then for each integer $s \geq 0$ there exists a nonharmonic function $f \in C^s(B_R)$ that satisfies the assumptions of the theorem.*
4. *If $r_1 + r_2 > R$, then there exists a nonharmonic function $f \in C^\infty(B_R)$ that satisfies the assumptions of the theorem.*
5. *If $r_1/r_2 \in A$, then there exists a nonharmonic real-analytic function f that is defined on the entire space $\mathbb{R}^n$ and satisfies condition (1) for all $r \in E$ and all $x \in \mathbb{R}^n$.*

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As was already mentioned, assertions 1, 2, and 5 are known (see [2, 5, 7]). Assertions 3 and 4 are new; they are proved in §5.

§2. Main notation

Let $S = \{x \in \mathbb{R}^n : |x| = 1\}$, and let (ρ, σ) be the polar coordinates on $\mathbb{R}^n$ (for each $x \in \mathbb{R}^n$ we have $\rho = |x|$, and if $x \neq 0$, then $\sigma = x/\rho \in S$). As usual, $\mathrm{SO}(n)$ is the rotation group of the space $\mathbb{R}^n$ equipped with the normalized Haar measure dg. The quasiregular representation $T(g)$ (for each $f \in L^2(S)$ we set $(T(g)f)(\sigma) = f(g^{-1}\sigma)$, where $\sigma \in S$ and $g \in \mathrm{SO}(n)$) is known to be the direct sum of pairwise nonequivalent irreducible unitary representations $T^k(g)$ acting on the spaces $\mathcal{H}_k$ of kth-order homogeneous harmonic polynomials [8, p. 426 of the Russian translation]). Let $\{Y_l^{(k)}(\sigma)\}$, $1 \leq l \leq a_k$, be an orthonormal basis in the space $\mathcal{H}_k$ regarded as a subspace in $L^2(S)$, and let $\{t_{lp}^k(g)\}$, $1 \leq l, p \leq a_k$, be the matrix of the representation $T^k(g)$; thus,

$$Y_l^{(k)}(g^{-1}\sigma) = \sum_{p=1}^{a_k} t_{lp}^k(g) Y_p^{(k)}(\sigma).$$

To each function $f \in L_{\mathrm{loc}}(B_R)$ there corresponds a Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{kl}(\rho) Y_l^{(k)}(\sigma),$$

where

$$f_{kl}(\rho) = \int_S f(\rho\sigma) \overline{Y_l^{(k)}(\sigma)}\, d\sigma. \tag{2}$$

Note the identity

$$f_{kl}(\rho) Y_p^{(k)}(\sigma) = a_k \int_{\mathrm{SO}(n)} f(g^{-1}x) \overline{t_{lp}^k(g)}\, dg \tag{3}$$

(for the proof, see [6]). In the following, we use the standard symbols J_λ and N_λ for the Bessel and the Neumann functions of index λ, respectively. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $[x] = x_1$.

Let $r > 0$ be fixed. For $R > r$, by $H_r(B_R)$ we denote the set of functions $f \in L_{\mathrm{loc}}(B_R)$ that satisfy (1) for almost all $x \in B_{R-r}$. For any nonnegative integer m we set

$$H_r^m(B_R) = H_r(B_R) \cap C^m(B_R).$$

§3. Properties of the roots of $\eta(z)$

We need some results [6] concerning the roots of $\eta(z)$. It follows from the general statements established in the theory of entire functions that η has infinitely many roots. Furthermore, all these roots except for $z = 0$ are simple, whereas the root $z = 0$ is of multiplicity 2. Moreover, η does not have real or pure imaginary roots other than $z = 0$ [6].

Let $\Lambda = \{\nu_1, \nu_2, \ldots\}$ be the sequence of all roots of η in the half-plane $\operatorname{Re} z > 0$ arranged in ascending order of their absolute values (for roots with equal absolute values the numbering is chosen arbitrarily).

Lemma 1. a) *For any $\varepsilon > 0$ one has $\sum_{q=1}^{\infty} |\nu_q|^{-1-\varepsilon} < \infty$.*

b) *Let $\rho > 0$ and $m \geq 0$. Then*

$$\rho^{n/2-1} N_{n/2-1}(\nu_q \rho) = O\big(|\nu_q|^{(n-1)\rho/2}\big), \qquad J_m(\nu_q\rho) = O\big(|\nu_q|^{(n+1)\rho/2}\big),$$

where the constants in the O-estimates are independent of q and ρ.

c) *As $q \to \infty$, one has*

$$|J_{n/2}(\nu_q)| = \frac{|\nu_q/2|^{n/2-1}}{\pi\Gamma(n/2)} + O\big(|\nu_q|^{n/2-2}\big).$$

d) *As $q \to \infty$, one has*

$$\sqrt{\pi}|\nu_q|^{(n-1)/2} = e^{|\operatorname{Im}\nu_q|} 2^{(n-3)/2} \Gamma\left(\frac{n}{2}\right) + O\big(|\nu_q|^{(n-3)/2}\big).$$

e) *Let $\alpha > 0$ and $\alpha \neq 1$. Then $|\eta(\alpha\nu_q)| > |\nu_q|^{1-n/2}$ for all sufficiently large q.*

Proof. Assertions a)–d) are proved in [6]. Assertion e) follows from d) and from the asymptotic expansions of the Bessel functions as $z \to \infty$ (e.g., see [9, p. 175 of the Russian translation]). □

Next, let

$$u_q(t) = \frac{J_{n/2-1}(\nu_q t)}{(\nu_q t)^{n/2-1}}, \qquad v_q(t) = (\nu_q t)^{n/2}\big(J_{n/2-1}(\nu_q t)N_{n/2-1}(\nu_q) - J_{n/2-1}(\nu_q)N_{n/2-1}(\nu_q t)\big).$$

Furthermore, let $v_0(t) = t - t^n$. Then [6]

$$\int_0^1 u_q(t)v_m(t)\,dt = \begin{cases} 0 & \text{if } q \neq m, \\ \frac{1}{\pi}J_{n/2}(\nu_q) & \text{if } q = m. \end{cases} \tag{4}$$

§4. Examples of functions of class $H_r(B_R)$

The classical mean value theorem for the Helmholtz equation (e.g., see [10, p. 289 of the Russian translation]) asserts that a necessary and sufficient condition for a function $f \in C(\mathbb{R}^n)$ to satisfy the equation $\Delta f + \lambda^2 f = 0$ is that

$$\int_{S(x,r)} f\,d\sigma = f(x)\big(1 - \eta(\lambda r)\big)$$

for all $x \in \mathbb{R}^n$ and $r > 0$. In particular, this equation holds if $f(x)$ is equal to

$$\psi_\lambda(x) = J_{n/2+k-1}(\lambda\rho)(\lambda\rho)^{1-n/2}Y_l^{(k)}(\sigma).$$

Thus, if $\eta(\lambda r) = 0$, then the functions $\sin(\lambda[x])$, $\cos(\lambda[x])$, and ψ_λ belong to $H_r(\mathbb{R}^n)$. Moreover, if $\varphi \in L(\mathbb{R}^n)$ is a compactly supported function that depends only on ρ, then the above equations imply the following expression for the convolution of φ and ψ_λ:

$$(\varphi * \psi_\lambda)(x) = (2\pi)^{n/2}\psi_\lambda(x)\widehat{\varphi}(\lambda), \tag{5}$$

where $\widehat{\varphi}(\lambda) = \int_0^\infty \varphi(\rho)\rho^{n/2}\lambda^{(2-n)/2}J_{n/2-1}(\lambda\rho)\,d\rho$ is the Fourier transform of φ (see [11, p. 176 of the Russian translation]).

Lemma 2. *For each $l \in \mathbb{N}$ there exists a nonconstant even function $h \in C^l(-R, R)$ with the following properties*:

1) *h is a polynomial on $[-r, r]$;*
2) *$h([x]) \in H_r^l(B_R)$.*

Proof. Let $m, q \in \mathbb{N}$, $p(t) = t^{2m}(1-t^2)^m$, and $\mu = \sqrt{\pi}2^{n/2-2} \times \Gamma((n-1)/2)$. Since ν_q are simple roots of η, it follows that $J_{n/2}(\nu_q) \neq 0$. Set

$$c_q = \frac{\pi}{\mu J_{n/2}(\nu_q)}\int_0^1 p(t)v_q(t)\,dt. \tag{6}$$

From the Bessel differential equation we have

$$v_q(t) = \left(\frac{n+1}{t^2}v_q(t)\frac{n-1}{t}v_q'(t) - v_q''(t)\right)\nu_q^{-2}.$$

Let us use this identity to integrate (6) by parts $m-1$ times. Then from the estimates in Lemma 1 and from the equalities $p^{(s)}(0) = p^{(s)}(1) = 0$, $0 \le s \le m-1$, we obtain

$$c_q = O\big(|\nu_q|^{n+2-2m}\big) \quad \text{as } q \to \infty. \tag{7}$$

Let $l \in \mathbb{N}$ and $m > (n+1)R/2 + l + n + 2$. We set

$$g(t) = \sum_{q=1}^{\infty} \cos(\nu_q t) c_q.$$

Then it follows from Lemma 1 that $g([x]) \in H_1^l(B_R)$. From the Poisson formula [11, p. 174 of the Russian translation], we obtain

$$\int_0^1 (1-u^2)^{(n-1)/2} g(tu)\, du \sum_{q=1}^{\infty} \mu c_q u_q(t). \tag{8}$$

We evaluate the c_q in (8) by using (4) and compare the result with (6); then it follows from the completeness of the system $\{v_q\}_{q=0}^{\infty}$ [6] that

$$\int_0^1 (1-u^2)^{(n-1)/2} g(tu)\, du = p(t) + c,$$

where c is a complex constant. We solve this integral equation for g [12, p. 126 of the Russian translation] and find that g is a polynomial of nonzero degree on $[-1, 1]$. Then the function $h = g(t/r)$ satisfies the desired conditions. □

Lemma 3. *Suppose that $k \in \mathbb{N}$, $\delta > 1$, and $\{c_q\}$ is a sequence of complex numbers such that*

$$\sum_{q=1}^{\infty} |c_q \nu_q|^{(n+1)\delta/2} < \infty.$$

If the function

$$w_k(t) = \sum_{q=1}^{\infty} c_q J_{n/2+k-1}(\nu_q t) \tag{9}$$

vanishes identically on $[0, \delta]$, then $c_q = 0$ for all q.

Proof. It follows from the condition of Lemma 3 and from statement b) of Lemma 1 that the series (9) uniformly converges on $[0, \delta]$, so that the function w_k is well defined. Let $0 < \varepsilon < \delta - 1$, and let φ be a radial function of class $C^\infty(\mathbb{R}^n)$ with support in B_ε. Since φ is smooth, it follows from the estimates proved in Lemma 1 that $|\widehat{\varphi}(\nu_q)|$ (see (5)) decreases as $q \to \infty$ more rapidly than any power of $|\nu_q|$. Set $f(x) = \rho^{1-n/2} w_k(\rho) Y_l^{(k)}(\sigma)$. Since the series (9) is uniformly convergent and Eq. (5) is valid, we have

$$(f * \varphi)(x) = \sum_{q=1}^{\infty} (2\pi)^{n/2} c_q \widehat{\varphi}(\nu_q) \psi_{\nu_q}(x).$$

It follows from the condition of the lemma that $(f * \varphi)(x) = 0$ for $x \in B_1$. Then

$$\sum_{q=1}^{\infty} c_q \widehat{\varphi}(\nu_q) J_{n/2+k-1}(\nu_q \rho) = 0, \qquad 0 \le \rho \le 1. \tag{10}$$

By applying the differentiation operator $(d/\rho\, d\rho)^k \rho^{n/2+k-1}$, which reduces the index of the Bessel function (e.g., see [9, p. 24]), to (10), we obtain

$$\sum_{q=1}^{\infty} c_q \widehat{\varphi}(\nu_q) \nu_q^{k+n/2} u_q(\rho) = 0, \qquad 0 \le \rho \le 1.$$

In conjunction with (4), this implies that $c_q \widehat{\varphi}(\nu_q) = 0$ for all q. Since φ is arbitrary, we obtain the desired statement. □

§5. Proof of Theorem 1

First, note that the first assertion of the theorem follows from the second assertion by the standard smoothing method (e.g., see [13, p. 409 of the Russian translation]). The proof of these assertions can be found in [5]. A different proof of assertion 2 can be obtained from the description [6] of the space of solutions to (1) for fixed r. Next, if $r_1/r_2 \in A$, then $r_1/r_2 = \alpha/\beta$ for some $\alpha, \beta \in \Lambda$. Set $\lambda = \alpha/r_1 = \beta/r_2$; then it follows from the results of §4 that the function $\sin(\lambda[x])$ satisfies the requirements in assertions 3–5 of Theorem 1. Therefore, in the following we assume that $r_1/r_2 \notin A$.

Proof of assertion 3. Let $r_1 + r_2 = R$. We set

$$\alpha_q \int_{S(0,r_2)} \cos\left(\frac{\nu_q[x]}{r_1}\right) d\sigma - 1 = -\eta\left(\frac{\nu_q r_2}{r_1}\right). \tag{11}$$

It follows from assertion e) in Lemma 1 and from (11) that $1/\alpha_q = O(|\nu_q|^{n/2-1})$ as $q \to \infty$. Let $s \in \mathbb{N}$ and $m > (n+1)R + s + 4n$. In the proof of Lemma 2 we constructed a nonzero function

$$h([x]) = \sum_{q=1}^{\infty} c_q \cos\left(\frac{\nu_q[x]}{r_1}\right), \qquad c_q = O(|\nu_q|^{n+2-2m}),$$

which coincides with a polynomial in B_{r_1}. Let

$$F([x]) = \sum_{q=1}^{\infty} \frac{c_q}{\alpha_q} \cos\left(\frac{\nu_q[x]}{r_1}\right). \tag{12}$$

Suppose that $g \in \mathrm{SO}(n)$. Then, by (12), (11), and the definition of h, we have

$$h([g^{-1}x]) = \int_{S(g^{-1}x,r_2)} F\, d\sigma.$$

Let us multiply this equation by $\overline{t_{lp}^{(k)}(g)}$ and integrate over $\mathrm{SO}(n)$; then we obtain

$$\int_{\mathrm{SO}(n)} h([g^{-1}x])\overline{t_{lp}^{(k)}(g)}\, dg = \int_{S(x,r_2)} G_k\, d\sigma, \tag{13}$$

where

$$G_k(x) = \int_{\mathrm{SO}(n)} F([g^{-1}x])\overline{t_{lp}^{(k)}(g)}\, dg.$$

Furthermore, it follows from (12) and from the behavior of c_q and α_q as $q \to \infty$ that $F([x]) \in H^s_{r_1}(B_R)$. Then from the definition of G_k and (1) we obtain $G_k \in H^s_{r_1}(B_R)$. Let k be greater than the order of the polynomial h in B_{r_1}. For these k, the support of the function on the left-hand side in (13) does not intersect B_{r_1} (this follows from Eqs. (2) and (3) for h and from the fact that harmonics of different orders are orthogonal on S [11, p. 161 of the Russian translation]). Since $r_1 + r_2 = R$, it follows from (13) that $G_k \in H^s_{r_2}(B_R)$. Suppose that $\Delta G_k = 0$. Then from the definition of G_k, (12), and (2), (3) we have

$$0 = \sum_{q=1}^{\infty} \frac{c_q}{\alpha_q} \nu_q^2 \int_{\mathrm{SO}(n)} \overline{t_{lp}^{(k)}(g)} \cos\left(\frac{\nu_q[g^{-1}x]}{r_1}\right) dg = \sum_{q=1}^{\infty} \frac{c_q \nu_q^2}{\alpha_q a_k} \int_S \cos\left(\frac{\nu_q[\rho\tau]}{r_1}\right) \overline{Y_l^{(k)}(\tau)}\, d\tau\, Y_p^{(k)}(\sigma)$$

for all $x = \rho\sigma \in B_R$. For odd k it follows that [12, p. 40 of the Russian translation])

$$\sum_{q=1}^{\infty} \frac{c_q \nu_q^2}{\alpha_q} J_{n/2+k-1}\left(\frac{\nu_q \rho}{r_1}\right) Y_p^{(k)}(\mathrm{e}) = 0$$

for $0 \le \rho < R$, where $\mathrm{e} = (1, 0, \ldots, 0) \in S$. If $Y_p^{(k)}(\mathrm{e}) \ne 0$, then, by Lemma 3, $c_q = 0$ for all q, which contradicts the definition of h. Thus, for all sufficiently large even k the function G_k satisfies the requirements of assertion 3 for any l, $1 \le l \le a_k$, and for at least one p (for which $Y_p^{(k)}(\mathrm{e}) \ne 0$). □

Proof of assertion 4. Let $r_1 + r_2 = r > R$. By assertion 3, there exists a nonharmonic function $g \in C^2(B_r)$ that satisfies the conditions of Theorem 1 in B_r. Let $\varphi \in C^\infty(\mathbb{R}^n)$ and $\varphi = 0$ outside B_{r-R}. Then the function $f = g * \varphi \in C^\infty(B_R)$ satisfies the conditions of Theorem 1. It remains to note that, for appropriate φ, the function f is not harmonic, since $\Delta f = (\Delta g) * \varphi$ and $\Delta g \neq 0$. □

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