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Quantum Extensions of Semigroups Generated by Bessel Processes

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ABSTRACT. We construct a quantum extension of the Markov semigroup of the classical Bessel process of order $\nu \geq 1$ to the noncommutative von Neumann algebra $\mathcal{B}(L^2(0,+\infty))$ of bounded operators on $L^2(0,+\infty)$.

KEY WORDS: quantum stochastic process, unitary evolution, noncommutative Markov process, Bessel process, quantum dynamical semigroup.

§1. Introduction

The notion of a *quantum stochastic process* on a $*$-algebra $\mathcal{A}$ ranging in another $*$-algebra $\mathcal{B}$ defined as a family $(j_t)_{t\geq 0}$ of $*$-homomorphisms $j_t : \mathcal{B} \to \mathcal{A}$, was introduced by Accardi, Frigerio and Lewis [1]. It generalizes the concept of classical stochastic process $(x_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ ranging in a measurable space $(E, \mathcal{E})$. Indeed, such a process x_t can be defined via the family of homomorphisms

$$j_t \colon L^\infty(E,\mathcal{E}) \to L^\infty(\Omega,\mathcal{F},\mathbb{P}), \qquad j_t(f) = f(x_t).$$

Just as in classical probability theory, in quantum probability theory one can describe Markov processes via their semigroups, which are ultraweakly continuous completely positive semigroups on (noncommutative) von Neumann algebras [2]. The quantum Markov process thus obtained is called a *quantum extension* of a classical Markov process if the restriction of the completely positive semigroup to an Abelian subalgebra (an algebra of functions) coincides with the classical Markov semigroup.

Quantum extensions of several classical Markov processes were described in [3–6] in terms of the quantum stochastic calculus. In this case, the quantum stochastic process satisfies a quantum stochastic differential equation given in explicit form. However, a number of analytical difficulties are encountered in dealing with the simplest classical processes whose domain has a boundary [7] because it is not clear how the classical boundary conditions affect the domain of the infinitesimal generator of the completely positive semigroup.

In this paper, following the general scheme developed in [3, 7] (see also [4]), we construct a quantum extension of semigroups corresponding to the classical Bessel processes of order $\nu \geq 1$ to the von Neumann algebra $\mathcal{B}(L^2(0,+\infty))$. This scheme can be applied to a wide class of classical Markov processes, since it does not depend on special properties of a specific process. However, the scheme requires a deep analysis of operators related to the infinitesimal generator. In §4 we perform this analysis and obtain a quantum dynamical semigroup which is likely to be the desired extension for all real values $\nu > 0$ of the dimension parameters. Using the sufficient conservativity condition [8] for quantum dynamical semigroups, we show that this is indeed the case for $\nu \geq 1$. In the last section, we describe the relationship of the quantum dynamical semigroup that extends the classical semigroup of the Bessel process of the integer dimension n with the quantum dynamical semigroup extending the semigroup of the classical n-dimensional Brownian motion. This result is a generalization to operator algebras of a well-known classical formula.

A noncommutative analog of the Bessel semigroup for integer dimensions was constructed in [9] on the C^*-algebra of the Heisenberg group on the basis of some special properties of quantum Brownian motion. Our approach is based on a general construction of quantum extensions; however, it also applies to a wider class of classical processes and allows one to explicitly construct quantum flows that extend classical Bessel processes.

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§2. A quantum extension of the transition semigroup of a classical Markov process

Let E be a closed subspace of the Euclidean space $\mathbb{R}^n$, and let $L^\infty(E)$ be the Banach space of bounded complex-valued Borel functions on E endowed with the norm $\|f\|_\infty = \sup_{x\in E}|f(x)|$.

Definition 2.1. A semigroup T_t on $L^\infty(E)$ is called a *transition semigroup* if for every $t > 0$ the operator T_t is a positive identity-preserving contraction.

It is well known that every classical Markov process with state space E canonically defines a transition semigroup via its family of transition probabilities.

Let T_t be a transition semigroup on $L^\infty(E)$, and let $\mathcal{C}$ be a closed subspace of continuous functions in $L^\infty(E)$.

Definition 2.2. We say that T_t has the *Feller property* with respect to the subspace $\mathcal{C} \subseteq L^\infty(E)$ if

1) the subspace $\mathcal{C}$ is T_t-invariant for every $t \geq 0$,
2) the restriction of T_t to $\mathcal{C}$ is a strongly continuous contraction semigroup.

If the transition semigroup T_t has the Feller property with respect to a suitable subspace $\mathcal{C}$ of continuous functions in $L^\infty(E)$, one usually considers the restriction of T_t to $\mathcal{C}$ and defines the infinitesimal generator of T_t as the operator $A : D(A) \to \mathcal{C}$ given by

$$D(A) = \{f \in \mathcal{C} \mid \lim_{t\to 0} t^{-1}(T_t - I)f \in \mathcal{C}\}, \qquad Af = \lim_{t\to 0} t^{-1}(T_t - I)f.$$

Some authors (e.g., see [10]) define the Feller property using a specific subspace of $L^\infty(E)$, namely the subspace $C_0(E)$ of all continuous functions vanishing at infinity. For our purposes, it is necessary that the subspace $\mathcal{C}$ contains the characteristic functions of Borel subsets E. The proof of the following result can be easily adapted from [10, Chapter III, (2.4), p. 84] with slight modifications.

Theorem 2.1. *The transition semigroup T_t has the Feller property with respect to a closed subspace $\mathcal{C} \subset L^\infty(E)$ if and only if*

1) *$\mathcal{C}$ is T_t-invariant for every $t \geq 0$;*
2) *for any $f \in \mathcal{C}$ and $x \in X$ one has $\lim_{t\to 0} T_t f(x) = f(x)$.*

In the sequel, the semigroup T_t is assumed to satisfy the Feller property with respect to a closed subspace $\mathcal{C} \subset L^\infty(E)$ that is a unital (i.e., containing the identity operator) algebra of continuous functions.

Let h be the Hilbert space $L^2(E)$ of all square integrable complex-valued Borel functions on E endowed with the inner product $\langle f, g\rangle = \int_E \bar{f}(\mathrm{x})g(\mathrm{x})\,d\mathrm{x}$, where $d\mathrm{x}$ is the Lebesgue measure. The Banach space $L^\infty(E)$ is isomorphic to a maximal Abelian unital von Neumann algebra in $\mathcal{B}(h)$; the isomorphism is given by the identification of each element of $L^\infty(E)$ with the corresponding multiplication operator acting on h. Therefore, it seems natural to extend classical Markov semigroups to semigroups acting on $\mathcal{B}(h)$.

Definition 2.3. A *quantum dynamical semigroup* (q.d.s.) is an ultraweakly continuous completely positive semigroup $\mathcal{T}_t$ on $\mathcal{B}(h)$. A q.d.s. $\mathcal{T}_t$ is said to be *conservative* if $\mathcal{T}_t(I) = I$ for every $t \geq 0$, where I is the identity operator on h.

Definition 2.4. A q.d.s. $\mathcal{T}_t$ on $\mathcal{B}(h)$ is said to be a *quantum extension* of the transition semigroup T_t if the restriction of $\mathcal{T}_t$ to the unital $*$-subalgebra $\mathcal{C} \subset \mathcal{B}(h)$ coincides with T_t.

We recall some results from [8] in a form suitable for our applications.

Theorem 2.2. *Let G and $(L_l)_{l=1}^\infty$ be operators on h satisfying the following assumptions:*

1) *G is the infinitesimal generator of a strongly continuous contraction semigroup P_t on h;*
2) *the common domain of the operators L_l contains the domain $D(G)$ of G, and for all $u, v \in D(G)$ we have*

$$\langle v, Gu\rangle + \langle Gv, u\rangle + \sum_{l=1}^{\infty}\langle L_l u, L_l v\rangle = 0,$$

where the series is absolutely convergent.

Then there exists a q.d.s. $\mathcal{T}_t$ satisfying the equation

$$\langle v, \mathcal{T}_t(X)u\rangle = \langle v, Xu\rangle + \int_0^t \langle v, \mathcal{L}(\mathcal{T}_s(X))u\rangle \, ds \tag{1}$$

for all $v, u \in D(G)$ and $X \in \mathcal{B}(h)$, where

$$\langle v, \mathcal{L}(X)u\rangle = \langle v, XGu\rangle + \langle Gv, Xu\rangle + \sum_{l=1}^{\infty}\langle L_l v, XL_l u\rangle. \tag{2}$$

Note that the form $\mathcal{L}(X)$ coincides with the operator

$$w^*\text{-}\lim_{t\to 0} t^{-1}\big(\mathcal{T}_t(X) - X\big)$$

whenever X belongs to the domain of the infinitesimal generator of $\mathcal{T}_t$.

The following result provides us with a powerful condition, which we shall apply to prove uniqueness and conservativity. We refer the reader to [8, Theorem 4.2] for the proof.

Theorem 2.3. *Under the assumptions of Theorem 2.2, suppose also that:*

1) *there exists a core D of G and a positive self-adjoint operator C such that $D \subseteq D(C^{1/2})$ and for all $v, u \in D$ we have*
$$\langle C^{1/2}v, C^{1/2}u\rangle = -\langle v, Gu\rangle - \langle Gv, u\rangle;$$

2) *there exists a positive constant b such that any $u \in D$ belongs to $D(C^{-1})$ and satisfies*
$$-2\langle Gu, C^{-1}u\rangle \le b\|u\|^2,$$
$$\sum_{l=1}^{\infty}\int_0^{+\infty} \exp(-\lambda t)\langle L_l P_t u, C_\varepsilon L_l P_t u\rangle \, dt \le \|C^{1/2}u\|^2 + b\|u\|^2, \varepsilon > 0,$$

where $C_\varepsilon = C\,(I + \varepsilon C)^{-1}$ is the bounded regularization of C for $\varepsilon > 0$.

Then the q.d.s. *satisfying* (1) *is unique and conservative.*

As shown in [7, Theorem 2.4], the representation of the infinitesimal generator of a transition semigroup in Lindblad form is the first step in constructing quantum extensions.

Definition 2.5. We say that the infinitesimal generator $A : D(A) \to C$ of the transition semigroup T_t ***can be represented in Lindblad form*** if there exist operators G and L_l satisfying assumptions 1 and 2 in Theorem 2.2 and the identity

$$\langle v, Afu\rangle = \langle v, fGu\rangle + \langle Gv, fu\rangle + \sum_{l=1}^{\infty}\langle L_l v, fL_l u\rangle$$

for all $u, v \in D(G)$ and every $f \in D(A)$. In this case, for each $X \in \mathcal{B}(h)$, we by $\mathcal{L}_A(X)$ denote the form on h defined by (2) with domain $D \times D$.

We recall a fundamental result on quantum extensions (see [7, Theorem 2.4]).

Theorem 2.4. *Assume that the infinitesimal generator A of the transition semigroup T_t can be represented in Lindblad form by means of the operators G and L_l. If there exists a (unique) conservative q.d.s. $\mathcal{T}_t$ whose infinitesimal generator $\mathcal{L}_A$ defines the form $\mathcal{L}_A(X)$ for every $X \in \mathcal{B}(h)$, then $\mathcal{T}_t$ is the quantum extension of the transition semigroup T_t.*

§3. The classical Bessel processes BES^{ν}

The classical Bessel processes BES^{ν} of "dimension" $\nu > 0$ are among the most interesting Markov processes with the state space $[0, +\infty[$.

For integer dimensions ($\nu = n$), the Bessel process can be constructed as the Euclidean norm of the corresponding n-dimensional Brownian motion. If $0 < \nu < 2$, then the point 0 is *attained a.s.* and *instantaneous reflection* occurs, whereas for $\nu \geq 2$ the point 0 is *polar*. For dimensions $0 < \nu \leq 2$, the processes BES^{ν} are *recurrent* and for $\nu > 2$ they are *transient* (cf. [10, Chap. XI]).

The transition semigroup $T_t^{(\nu)}$ of BES^{ν} acts on $L^{\infty}(0, +\infty)$ as follows:

$$T_t^{(\nu)} f(x) = \int_0^{+\infty} f(y) p_t^{(\nu)}(x\,;\,y)\, dy,$$

where $\left(p_t^{(\nu)}\right)_{t>0}$ is the family of densities defined by

$$p_t^{(\nu)}(x\,;\,y) = \begin{cases} 2^{1-\nu/2}\Gamma\left(\dfrac{\nu}{2}\right)^{-1} y^{\nu-1} t^{-\nu/2} \exp\left(-\dfrac{y^2}{2t}\right) & \text{for } x = 0, \\ x^{1-\nu/2} y^{\nu/2} t^{-1} \exp\left(-\dfrac{x^2+y^2}{2t}\right) I_{\nu/2-1}\left(\dfrac{xy}{t}\right) & \text{for } x > 0, \end{cases}$$

Γ is the gamma-function, and $I_{\nu/2-1}(z)$ is the modified Bessel function of index $\nu/2 - 1$.

Recall that the function $I_{\nu}(z)$ and the Macdonald function $K_{\nu}(z)$ form a pair of fundamental solutions of the modified Bessel equation

$$z^2 u'' + z u' - (\nu^2 + z^2) u = 0. \tag{3}$$

As we shall see, the main properties of these functions (see [11] for exhaustive information) play a crucial role in the proof of our results. For instance, in view of Theorem 2.1, a straightforward computation using the properties of the function $I_{\nu}(z)$ (see [1]2 for details), gives the following theorem.

Theorem 3.1. *For each $\nu > 0$, the transition semigroup $T_t^{(\nu)}$ has the Feller property with respect to the closed subspace $C[0, +\infty] \subset L^{\infty}(0, +\infty)$ of all continuous functions converging to finite limits as $x \to 0$ and as $x \to +\infty$.*

Moreover, following the scheme suggested in [10, Chap. VII, (1.10), p. 264] for Brownian motion and using the properties of the functions $I_{\nu}(z)$ and $K_{\nu}(z)$ [12], we can prove the following theorem.

Theorem 3.2. *For each $\nu > 0$ the infinitesimal generator $A_{\nu} : D(A_{\nu}) \to C[0, +\infty]$ of $T_t^{(\nu)}$ is given by*

$$D(A_{\nu}) = \{f \in C^2[0, +\infty[\,|\, f'(0) = 0\}, \qquad (A_{\nu} f)(x) = \frac{1}{2} f''(x) + \frac{\nu - 1}{2x} f'(x),$$

where $C^2[0, +\infty] \subset C^0[0, +\infty]$ is the subspace of all twice differentiable functions with the first and the second derivatives in $C^0[0, +\infty]$.

§4. A quantum extension of BES^{ν}

4.1. The Lindblad form of the infinitesimal generator of BES^{ν}. Our first step in constructing a quantum extension of BES^{ν} is the representation theorem for the infinitesimal generator A_{ν} of the transition semigroup $T_t^{(\nu)}$ in Lindblad form. To this end, following the discussion in [3, Sec. 5], we consider the following differential operators in the Hilbert space $h = L^2(0, +\infty)$:

$$D(L_{\nu}) = \left\{u \in h \,\middle|\, u' - \frac{(\nu - 1)u}{2x} \in h\right\}, \qquad L_{\nu} u = u' - \frac{\nu - 1}{2x} u, \tag{4}$$

$$D(G_{\nu}) = \{u \in h \mid u \in D(L_{\nu}), L_{\nu} u \in D(L_{\nu}^*)\}, \qquad G_{\nu} = -\frac{1}{2} L_{\nu}^* L_{\nu}. \tag{5}$$

The operator L_ν is densely defined and closed [13, Corollary of Theorem 2, p. 196]. It can be easily seen that the adjoint is the minimal differential operator

$$v \to -v' - \frac{\nu - 1}{2x} v,$$

whose domain $C_c^\infty(0, +\infty) \subset h$ is the dense linear submanifold of all infinitely differentiable functions with compact support. The operator G_ν is negative and self-adjoint by the von Neumann theorem [13, Theorem 2, p. 200]. Hence, it is the infinitesimal generator of a strongly continuous contraction semigroup $P_t^{(\nu)}$ on h [13, Corollary of the Phillips–Lumer Theorem, p. 251].

We need some properties of the operators L_ν and G_ν for various values of the parameter ν. Let $AC_{\text{loc}}(0, +\infty) \subset h$ be the linear submanifold of locally absolutely continuous functions.

Proposition 4.1. *Let* $u \in D(L_\nu)$. *Then*

1) *for all* $\nu > 0$ *we have* $\lim_{x \to +\infty} u(x) = 0$;
2) *if* $0 < \nu < 2$, *then there exists a limit* $\lim_{x \to 0} u(x)/x^{(\nu-1)/2}$;
3) *if* $\nu > 1$, *then* $\lim_{x \to 0} u(x) = 0$;
4) *if* $\nu > 1$, *then the function* $x \to u(x)/\sqrt{x}$ *belongs to* h.

Proof. Obviously, u' belongs to $L^2(1, +\infty)$, since $x^{-1}u$ belongs to $L^2(1, +\infty)$ and $u \in D(L_\nu)$. Thus, the function $\bar{u}u'$ is integrable on $(1, +\infty)$ and

$$|u(x)|^2 - |u(y)|^2 = 2 \operatorname{Re} \int_y^x \bar{u}(s) u'(s)\, ds$$

for all $x, y \in (1, +\infty)$. This implies the first statement.

Since $u \in AC_{\text{loc}}(0, +\infty)$, by setting $\widetilde{u} = u' - u/(2x)$ and by integrating the differential equation

$$u' - \frac{\nu - 1}{2x} u = \widetilde{u},$$

we find that

$$\frac{u(x)}{x^{(\nu-1)/2}} - \frac{u(y)}{y^{(\nu-1)/2}} = \int_y^x \frac{\widetilde{u}(t)}{t^{(\nu-1)/2}}\, dt \tag{6}$$

for all $x, y \in]0, +\infty[$. The function $\widetilde{u}(t)/t^{(\nu-1)/2}$ is integrable on $(0, 1)$, since both $t^{(1-\nu)/2}$ and $\widetilde{u}$ are square integrable on $(0, 1)$ for $0 < \nu < 2$. Therefore, the second statement follows as x, y tend to 0.

Finally, under our assumptions we have

$$\lim_{x \to 0} \int_x^{+\infty} \operatorname{Re}\big[\bar{u}(t)(L_\nu u)(t)\big]\, dt = \operatorname{Re}\langle u, L_\nu u \rangle.$$

On the other hand, for each $x > 0$ we can write

$$\begin{aligned}
\int_x^{+\infty} \operatorname{Re}\big(\bar{u}(t)(L_\nu u)(t)\big)\, dt &= \int_x^{+\infty} \operatorname{Re}\big(\bar{u}(t) u'(t)\big)\, dt - \frac{\nu - 1}{2} \int_x^{+\infty} \frac{|u(t)|^2}{t}\, dt \\
&= \frac{1}{2} \int_x^{+\infty} \frac{d}{dt} |u(t)|^2\, dt - \frac{\nu - 1}{2} \int_x^{+\infty} \frac{|u(t)|^2}{t}\, dt \\
&= -\left(\frac{1}{2} |u(x)|^2 + \frac{\nu - 1}{2} \int_x^{+\infty} \frac{|u(t)|^2}{t}\, dt \right);
\end{aligned}$$

therefore, the limit

$$\lim_{x \to 0} \left(\frac{1}{2} |u(x)|^2 + \frac{\nu - 1}{2} \int_x^{+\infty} \frac{|u(t)|^2}{t}\, dt \right)$$

exists and is finite. Since for $\nu > 1$ the second term in the sum is positive nondecreasing and the first term is positive, it follows from the existence of the limit of the sum as $x \to 0$ that the limits of both terms exist. Specifically, the limits

$$\lim_{x\to 0} |u(x)|, \qquad \lim_{x\to 0} \int_x^{+\infty} \frac{|u(t)|^2}{t}\, dt$$

necessarily exist. Thus, $x \to u/\sqrt{x}$ is square integrable and $\lim_{x\to 0} |u(x)| = 0$. □

Note that if $\nu < 1$, then $\lim_{x\to 0} u(x)$ need not exist for a function $u(x)$ satisfying the assumption of Proposition 4.1. Indeed, the function $u(x) = x^{(\nu-1)/2}e^{-x}$ satisfies the assumption, but $\lim_{x\to 0} u(x) = +\infty$.

Proposition 4.2. *The adjoint operator L_ν^* is given by*

$$D(L_\nu^*) = \left\{ v \in h \;\middle|\; v' + \frac{\nu-1}{2x}v \in h,\ \lim_{x\to 0+} v(x)u(x) = 0\ \forall u \in D(L_\nu) \right\},$$
$$(L_\nu^* v)(x) = -v'(x) - \frac{\nu-1}{2x}v(x).$$

Proof. Let $v \in h$ be such that $v' + (\nu-1)v/(2x)$ is square integrable. We can show just as in Proposition 1.4 that v tends to zero at infinity. Clearly, v belongs to $AC_{\mathrm{loc}}(0, +\infty)$ and, for every $u \in D(L_\nu)$, we have

$$\begin{aligned}\langle v, L_\nu u\rangle &= \int_0^{+\infty} \bar{v}(t)\Big(u'(t) - \frac{\nu-1}{2t}u(t)\Big)\, dt = \lim_{x\to 0, y\to+\infty}\Big(\int_x^y \bar{v}(t)u'(t)dt - \int_x^y \frac{\nu-1}{2t}\bar{v}(t)u(t)dt\Big) \\ &= \lim_{x\to 0, y\to+\infty}\Big([\bar{v}(t)u(t)]_x^y - \int_x^y \big(\bar{v}'(t) + \frac{\nu-1}{2t}\bar{v}(t)\big)u(t)\, dt\Big). \qquad (7)\end{aligned}$$

Now, if $\lim_{x\to 0} v(x)u(x) = 0$ for $u \in D(L_\nu)$, then we obtain the formula

$$\langle v, L_\nu u\rangle = -\int_0^{+\infty} \Big(\bar{v}'(t) + \frac{\nu-1}{2t}\bar{v}(t)\Big)u(t)\, dt.$$

This shows that v belongs to the domain of L_ν^* and that $L_\nu^* v$ has the desired form. Conversely, if v belongs to the domain of L_ν^*, then there exists a $w \in h$ such that for every $u \in D(L_\nu)$ we have

$$\langle v, L_\nu u\rangle = \langle w, u\rangle. \qquad (8)$$

Integrating by parts, we find that v, treated as a distribution, must satisfy the differential equation

$$v' + \frac{\nu-1}{2x}v = -w.$$

Thus, $v \in AC_{\mathrm{loc}}(0, +\infty)$ and this, together with (7), implies that

$$\langle v, L_\nu u\rangle = \langle w, u\rangle - \lim_{x\to 0+} \bar{v}(x)u(x).$$

The limit is zero for all $u \in D(L_\nu)$ by virtue of (8). □

To represent the operator A_ν in Lindblad form, let us prove the following auxiliary Lemma.

Lemma 4.1. *For all* $u, v \in D(G_\nu)$ *and every* $f \in C^2[0, +\infty]$ *such that* $f'(0) = 0$, *we have*

$$\lim_{x\to 0} \bar{v}(x)f'(x)u(x) = 0 \tag{9}$$

and

$$\lim_{x\to 0}\left[\frac{\nu-1}{2x}\bar{v}(x)f(x)u(x) - \frac{1}{2}\bar{v}'(x)f(x)u(x) - \frac{1}{2}\bar{v}(x)f(x)u'(x)\right] = 0. \tag{10}$$

Proof. When $0 < \nu < 1$, by applying de l'Hôpital's rule, we obtain

$$\lim_{x\to 0} x^{\nu-1}f'(x) = \lim_{x\to 0} x^\nu f''(x) = 0.$$

This, together with Proposition 4.1, 2), yields

$$\lim_{x\to 0}\bar{v}(x)f'(x)u(x) = \lim_{x\to 0} x^{\nu-1}f'(x)\cdot\frac{\bar{v}(x)}{x^{(\nu-1)/2}}\cdot\frac{u(x)}{x^{(\nu-1)/2}} = 0$$

for every ν with $0 < \nu \le 1$. When $\nu > 1$, the limit (9) readily follows from Proposition 4.1, 3).
This proves the first assertion. To prove the second, it suffices to rewrite (10) as

$$\frac{1}{2}f(x)\left[\left(\frac{\nu-1}{2x}u(x) - u'(x)\right)\bar{v}(x) + \left(\frac{\nu-1}{2x}\bar{v}(x) - \bar{v}'(x)\right)u(x)\right]$$

and apply Proposition 4.2 since both

$$\frac{\nu-1}{2x}u(x) - u'(x) \qquad \text{and} \qquad \frac{\nu-1}{2x}\bar{v}(x) - \bar{v}'(x)$$

belong to the domain of L_ν^*. □

Theorem 4.1. *The infinitesimal generator* A_ν *of* BES^ν *can be represented in Lindblad form*

$$\langle v, (A_\nu f)u\rangle = \langle v, fG_\nu u\rangle + \langle G_\nu v, fu\rangle + \langle L_\nu v, fL_\nu u\rangle \tag{11}$$

for $u, v \in D(G_\nu)$ *and* $f \in D(A_\nu)$.

Proof. A straightforward calculation shows that (11) holds if and only if

$$\lim_{x\to 0, y\to+\infty}\left[\frac{1}{2}\bar{v}(t)f(t)u'(t)\Big|_x^y + \frac{1}{2}\bar{v}'(t)f(t)u(t)\Big|_x^y - \frac{\nu-1}{2t}\bar{v}(t)f(t)u(t)\Big|_x^y - \frac{1}{2}\bar{v}(t)f'(t)u(t)\Big|_x^y\right] = 0$$

for $u, v \in D(G_\nu)$ and $f \in D(A_\nu)$.

By applying the same argument as in Proposition 4.1 to the function u', we can show that for every $u \in D(G_\nu)$ the derivative u' vanishes at infinity. Thus each of the terms tends to 0 as $y \to +\infty$.

The sum vanishes also as x tends to 0. Indeed, the sum of the first three terms vanishes because of (10), and the last term vanishes because of (9). □

We refer the reader to [12] for a detailed study of the operators L_ν, L_ν^* and G_ν.

4.2. Conservativity of the q.d.s. We use Theorem 2.3 to prove the conservativity of the q.d.s. $\mathcal{T}_t^{(\nu)}$ constructed for the operators G_ν and L_ν introduced in the previous subsection. As the first step, we find a core of the operator G_ν.

Proposition 4.3. *The linear manifold* $C_c^\infty(0, +\infty)$ *is a core of* G_ν *if and only if* $\nu \ge 4$. *If* $0 < \nu < 4$, *then the linear span of this manifold and the Macdonald function* $x^{3/2}K_{2-\nu/2}(x)$ *is a core of* G_ν.

Proof. Since G_ν is a negative self-adjoint operator, it follows that the point $1/2$ is in the resolvent set of G_ν and the operator $\left(\frac{1}{2}I - G_\nu\right)^{-1}$ is everywhere defined and bounded on h. Therefore, a dense linear submanifold $D_\nu \subset h$ contained in $D(G_\nu)$ is a core of G_ν if and only if the linear manifold $(I/2 - G_\nu)(D_\nu)$ is dense in h [14, III, 3, Proposition 5.19, p. 166].

Let $u \in h$ be a vector orthogonal to $(I/2 - G_\nu)(C_c^\infty(0, +\infty))$. An easy computation shows that u, treated as a distribution, must be a solution in h of the differential equation

$$u'' - \left(1 + \frac{(\nu-1)(\nu-3)}{4x^2}\right)u = 0. \tag{12}$$

On the other hand, it is well known that such a solution can be represented, neglecting a set of measure zero, by a function belonging to $C^\infty(0, +\infty)$. Therefore we can seek classical solutions of Eq. (12) belonging to $D(G_\nu)$. The change of variables $u \to x^{1/2}u$, $\nu \to (1 - \nu/2)$ reduces Eq. (12) to the homogeneous modified Bessel equation (3). The fundamental solution of this equation is

$$c_1 x^{1/2} I_{1-\nu/2}(x) + c_2 x^{1/2} K_{1-\nu/2}(x)$$

for arbitrary constants c_1 and c_2. If $\nu \geq 4$, then these functions are not square integrable unless $c_1 = c_2 = 0$. It follows that the manifold $C_c^\infty(0, +\infty)$ is a core of G_v for $\nu \geq 4$.

If $0 < \nu < 4$, then the only solutions in h of Eq. (12) is the function

$$u(x) = cx^{1/2}K_{1-\nu/2}(x)$$

with an arbitrary constant c. Thus, the core of G_v is given by the linear span of $C_c^\infty(0, +\infty)$ and possibly of solutions in $D(G_\nu)$ of the differential equation

$$u'' - \left(1 + \frac{(\nu-1)(\nu-3)}{4x^2}\right)u = x^{1/2}K_{1-\nu/2}(x). \tag{13}$$

The only solution of this differential equation is the function

$$u(x) = x^{1/2}\left(cK_{1-\nu/2}(x) - \frac{1}{4}x\big(K_{2-\nu/2}(x) + K_{\nu/2}(x)\big)\right),$$

where c is an arbitrary constant. Now the definition of $D(G_\nu)$ forces us to choose $c = (\nu/2 - 1)/2$, thus obtaining

$$u(x) = -\frac{1}{2}x^{3/2}K_{2-\nu/2}(x)$$

(see [12] for details). This completes the proof. □

In the sequel we need the following property of the core D_ν.

Proposition 4.4. *Let $u(x) = x^{3/2}K_{2-\nu/2}(x)$ and $v = L_\nu u$. For $0 < \nu < 4$, the functions $x^{-1}v$ and v' belong to h, and*

$$L_\nu(D_\nu) \subseteq D(L_\nu) \cap D(L_\nu^*).$$

Proof. Indeed, $v(x) = -x^{3/2}K_{1-\nu/2}(x)$, whence we have the following asymptotic behavior as $x \to 0$.

$$x^{-1}v \sim \begin{cases} -2^{\nu/2-2}\Gamma\left(\frac{\nu}{2} - 1\right)x^{(3-\nu)/2} & \text{for } \nu > 2, \\ x^{1/2}\ln x & \text{for } \nu = 2, \\ -2^{-\nu/2}\Gamma\left(1 - \frac{\nu}{2}\right)x^{(\nu-1)/2} & \text{for } \nu < 2. \end{cases}$$

We see that $x^{-1}v$ is in h, since the Macdonald functions exponentially decay at infinity. Moreover, we have $v' + (\nu-1)v/(2x) \in h$ since $u \in D(G_\nu)$, and so v' is in h. □

Now we establish now three technical lemmas.

Lemma 4.2. *For every* $u \in D_\nu$ *we have*

$$\langle L_\nu L_\nu u, L_\nu L_\nu u\rangle = 4\langle G_\nu u, G_\nu u\rangle - (\nu - 1)\left\langle \frac{1}{x}L_\nu u, \frac{1}{x}L_\nu u\right\rangle.$$

Proof. For $v = L_\nu u$, Proposition 4.4 implies $L_\nu v = L_\nu^* v + 2v'$; therefore,

$$\begin{aligned}\langle L_\nu^2 u, L_\nu^2 u\rangle &= \left\langle \left(L_\nu^* + 2\frac{d}{dx}\right)L_\nu u, \left(L_\nu^* + 2\frac{d}{dx}\right)L_\nu u\right\rangle \\ &= \langle L_\nu^* L_\nu u, L_\nu^* L_\nu u\rangle + 4\,\mathrm{Re}\left\langle \frac{d}{dx}L_\nu u, L_\nu^* L_\nu u + \frac{d}{dx}L_\nu u\right\rangle \\ &= 4\langle G_\nu u, G_\nu u\rangle - 2(\nu-1)\,\mathrm{Re}\left\langle \frac{d}{dx}L_\nu u, \frac{1}{x}L_\nu u\right\rangle.\end{aligned}$$

Since

$$\lim_{x\to 0,\, y\to+\infty} t^{-1}|v(t)|^2\Big|_x^y = 0$$

for $v \in C_c^\infty(0,+\infty)$ or $v(x) = L_\nu u = -x^{3/2}K_{1-\nu/2}$ if $0 < \nu < 4$, we have

$$\begin{aligned}2\,\mathrm{Re}\left\langle v', \frac{1}{x}v\right\rangle &= \int_0^{+\infty} \frac{1}{x}2\,\mathrm{Re}\big(\bar{v}(x)v'(x)\big)\,dx = \int_0^{+\infty}\frac{1}{x}\frac{d}{dx}|v(x)|^2\,dx \\ &= \int_0^{+\infty}\frac{1}{x^2}|v(x)|^2\,dx = \left\langle \frac{1}{x}v, \frac{1}{x}v\right\rangle,\end{aligned}$$

and the desired assertion follows. □

Let us introduce the positive self-adjoint operator $C_\nu = I - 2G_\nu$ with the domain $D(C_\nu) = D(G_\nu)$, and let $C_\varepsilon^{(\nu)} = C_\nu(I + \varepsilon C_\nu)^{-1}$.

Lemma 4.3. *For every* $u \in D_\nu$ *we have*

$$\big\langle L_\nu u, C_\varepsilon^{(\nu)} L_\nu u\big\rangle \le -2\langle u, G_\nu u\rangle + \langle L_\nu L_\nu u, L_\nu L_\nu u\rangle.$$

Proof. First, note that the operator $(I + \varepsilon C_\nu)^{-1}$ commutes with the positive operator $L_\nu^* L_\nu$ and with $(\lambda I - L_\nu^* L_\nu)^{-1}$ if $\lambda = -(1+\varepsilon)/\varepsilon$ [14, Chap.III, Theorem 6.5, p. 173]. Consequently, $(I+\varepsilon C_\nu)^{-1}$ also commutes with the self-adjoint positive operator $|L_\nu| = (L_\nu^* L_\nu)^{1/2}$ that has the domain $D(|L_\nu|) = D(L_\nu)$ [14, Theorem 3.35, p. 281]. Consider the polar decomposition $L_\nu = U_\nu |L_\nu|$ of L_ν where U_ν is a partial isometry [14, Chap.VI, p. 334]. Taking into account Proposition 4.4, for every $u \in D_\nu$ we have

$$\begin{aligned}\langle L_\nu u, C_\varepsilon^{(\nu)} L_\nu u\rangle &= \big\langle L_\nu u, C_\nu(I+\varepsilon C_\nu)^{-1}L_\nu u\big\rangle = \big\langle L_\nu u, (I + L_\nu^* L_\nu)(I+\varepsilon C_\nu)^{-1}L_\nu u\big\rangle \\ &\le \langle L_\nu u, L_\nu u\rangle + \big\langle |L_\nu|L_\nu u, (I+\varepsilon C_\nu)^{-1}|L_\nu|L_\nu u\big\rangle \le \langle L_\nu u, L_\nu u\rangle + \big\langle |L_\nu|L_\nu u, |L_\nu|L_\nu u\big\rangle \\ &= -2\langle u, G_\nu u\rangle + \big\langle U_\nu|L_\nu|L_\nu u, U_\nu|L_\nu|L_\nu u\big\rangle = -2\langle u, G_\nu u\rangle + \langle L_\nu L_\nu u, L_\nu L_\nu u\rangle.\end{aligned}$$

This proves the Lemma. □

Lemma 4.4. *Suppose* $\nu \ge 1$. *Then for* $\varepsilon > 0$ *and* $u \in D(G_\nu)$ *we have*

$$\big\langle L_\nu u, C_\varepsilon^{(\nu)} L_\nu u\big\rangle \le -2\langle u, G_\nu u\rangle + 4\langle G_\nu u, G_\nu u\rangle. \tag{14}$$

Proof. Inequality (14) holds for $u \in D_\nu$ by Lemma 4.2 and Lemma 4.3. Since D_ν is a core of G_ν, it follows that this inequality holds for every $u \in D(G_\nu)$. □

Theorem 4.2. *Suppose $\nu \geq 1$. Then the q.d.s. $\mathcal{T}_t^{(\nu)}$ is conservative.*

Proof. Let us introduce the operators

$$\widetilde{G}_\nu = -\frac{1}{2}I + G_\nu, \qquad \widetilde{L}_{\nu,1} = I, \qquad \widetilde{L}_{\nu,2} = L_\nu,$$

which, together with $C_\nu = -2\widetilde{G}_\nu$, satisfy assumptions 1) and 2) of Theorem 2.2. The theorem will be proved once we show that these operators also satisfy the assumptions of Theorem 2.3 (see [8, Sect. 3]). Clearly, assumption 1) is satisfied.

By the definition of $C_\varepsilon^{(\nu)}$, for $u \in D(G_\nu)$ we have

$$\langle \widetilde{P}_t^{(\nu)}u, C_\varepsilon^{(\nu)}\widetilde{P}_t^{(\nu)}u\rangle \leq -2\langle \widetilde{P}_t^{(\nu)}u, \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u\rangle.$$

Furthermore, it follows from Lemma 4.4 that

$$\begin{aligned}\langle L_\nu\widetilde{P}_t^{(\nu)}u, C_\varepsilon^{(\nu)}L_\nu\widetilde{P}_t^{(\nu)}u\rangle &\leq -2\langle \widetilde{P}_t^{(\nu)}u, G_\nu\widetilde{P}_t^{(\nu)}u\rangle + 4\langle G_\nu\widetilde{P}_t^{(\nu)}u, G_\nu\widetilde{P}_t^{(\nu)}u\rangle \\ &= 2\langle \widetilde{P}_t^{(\nu)}u, \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u\rangle + 4\langle \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u, \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u\rangle;\end{aligned}$$

hence

$$\begin{aligned}\sum_{n=1}^{2}\int_0^{+\infty}\exp(-t)\langle \widetilde{L}_{\nu,n}\widetilde{P}_t^{(\nu)}u, C_\varepsilon^{(\nu)}\widetilde{L}_{\nu,n}\widetilde{P}_t^{(\nu)}u\rangle\,dt &\leq 4\int_0^{+\infty}\exp(-t)\langle \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u, \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u\rangle\,dt \\ &= 2\int_0^{+\infty}\exp(-t)\frac{d}{dt}\langle \widetilde{P}_t^{(\nu)}u, \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u\rangle\,dt \\ &= -2\langle u, \widetilde{G}_\nu u\rangle + 2\int_0^{+\infty}\exp(-t)\langle \widetilde{P}_t^{(\nu)}u, \widetilde{G}_\nu\widetilde{P}_t^{(\nu)}u\rangle\,dt \\ &\leq \langle u, C_\nu u\rangle.\end{aligned}$$

This proves that assumption 2) is also satisfied. □

§5. The relationship with the quantum extension of classical n-dimensional Brownian motion

In this section we show that for a positive integer n the quantum extension of the semigroup BES^n can be obtained from the quantum extension of the semigroup of n-dimensional Brownian motion [3]. Let BM^n (resp., BES^n) denote all objects related to classical n-dimensional Brownian motion (resp., to the classical n-dimensional Bessel process).

The transition semigroup $T_t^{\mathrm{BM}^n}$ of classical n-dimensional Brownian motion acts on $L^\infty(\mathbb{R}^n)$ as follows:

$$T_t^{\mathrm{BM}^n}f(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y})p_t^{\mathrm{BM}^n}(\mathbf{x};\mathbf{y})\,d\mathbf{y},$$

where $(p_t^{\mathrm{BM}^n})$ is the family of densities defined by

$$p_t^{\mathrm{BM}^n}(\mathbf{x};\mathbf{y}) = (2\pi t)^{-n/2}\exp\left(-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{2t}\right)$$

for $t > 0$, and the infinitesimal generator of $T_t^{\mathrm{BM}^n}$ is a linear operator $A_{\mathrm{BM}^n} : D(A_{\mathrm{BM}^n}) \to C_b^0(\mathbb{R}^n)$ such that

$$C_b^2(\mathbb{R}^n) \subseteq D(A_{\mathrm{BM}^n}), \qquad A_{\mathrm{BM}^n}f = \frac{1}{2}\sum_{k=1}^{n}\partial_k^2 f,$$

where ∂_k is the usual partial derivative operator (see [10, Chap. VII]).

For each $f \in L^\infty(\mathbb{R}^n)$, the function $\tilde{f}(x) = f(\|x\|) \in L^\infty(0, +\infty)$ is essentially bounded and it is well known [10, Prop. 3.1, p. 232] that $T_t^{\mathrm{BES}^n} \tilde{f}(x) = T_t^{\mathrm{BM}^n} f(\|x\|)$.

Consider the Laplace operator $\Delta : D(\Delta) \to L^2(\mathbb{R}^n, d\mathbf{x})$, where

$$D(\Delta) = \{u \in L^2(\mathbb{R}^n, d\mathbf{x}) : \Delta u \in L^2(\mathbb{R}^n, d\mathbf{x})\} \qquad \text{and} \qquad \Delta u = \sum_{k=1}^n \partial_k^2 u.$$

The paper [3] shows how to construct BM^n as a commutative quantum flow in the boson Fock space over $L^2(\mathbb{R}^n, dx)$ with the associated q.d.s. $\mathcal{T}_t^{\mathrm{BM}^n}$, that extends the corresponding classical semigroup. The semigroup $\mathcal{T}_t^{\mathrm{BM}^n}$ satisfies the Eq. (1) with

$$\langle \mathbf{v}, \mathcal{L}_{\mathrm{BM}^n}(\mathbf{X})\mathbf{u}\rangle = \frac{1}{2}\langle \mathbf{v}, \mathbf{X}\Delta\mathbf{u}\rangle + \frac{1}{2}\langle \Delta\mathbf{v}, \mathbf{X}\mathbf{u}\rangle + \sum_{k=1}^n \langle \partial x_k u, \mathbf{X}\partial x_k u\rangle$$

for all $\mathbf{u}, v \in D(\Delta)$ and every $\mathbf{X} \in \mathcal{B}(L^2(\mathbb{R}^n, d\mathbf{x}))$.

Let us introduce spherical coordinates and write

$$\begin{aligned} x_1 &= \rho \sin\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{n-2} \sin\vartheta_{n-1}, \\ x_2 &= \rho \sin\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{n-2} \cos\vartheta_{n-1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1} &= \rho \sin\vartheta_1 \cos\vartheta_2, \\ x_n &= \rho\cos\vartheta_1, \end{aligned}$$

and $S^{n-1} = [0, \pi] \times [0, 2\pi[\times \cdots \times [0, 2\pi[$. Consider the Hilbert spaces $L^2(\mathbb{R}_+, dx)$ and $L^2(S^{n-1}, \omega)$, where μ and ω are the Borel measures defined by

$$\mu(B) = \int_B \rho^{n-1} d\rho \qquad \omega(B) = \int_B \sin^{n-2}\vartheta_1 \sin^{n-3}\vartheta_2 \cdots \sin\vartheta_{n-2} d\vartheta_1 d\vartheta_2 \cdots d\vartheta_{n-1}.$$

The Hilbert space $h = L^2(\mathbb{R}_+, dx)$ is unitarily isomorphic to $L^2(\mathbb{R}_+, \mu)$,

$$Uu(x) = \rho^{-(n-1)/2} u(\rho), \quad UU^* = U^*U = I,$$

and the tensor product $L^2(\mathbb{R}_+, \mu) \otimes L^2(S^{n-1}, \omega)$ of Hilbert spaces is isomorphic to the Hilbert space $L^2(\mathbb{R}^n, d\mathbf{x})$ by means of the unique unitary extension of the operator $V(f \otimes g) = f \diamond g$, where

$$f \diamond g(\mathbf{x}) = f(\rho) g(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-2}, \vartheta_{n-1})$$

for every $f \in L^2(\mathbb{R}_+, \mu)$ and every $g \in L^2(S^{n-1}, \omega)$.

Let φ be the pure state on $\mathcal{B}(L^2(S^{n-1}, \sigma))$ corresponding to the density operator $|\varphi\rangle\langle\varphi|$ associated with the normalized constant function of the sphere S^{n-1}, such that for $B \in \mathcal{B}(L^2(S^{n-1}, \sigma))$ one has

$$\varphi(B) = \mathrm{Tr}(B|\varphi\rangle\langle\varphi|) = \langle \varphi, B\varphi\rangle.$$

Let

$$E_\varphi : \mathcal{B}(L^2(\mathbb{R}_+, \mu) \otimes L^2(S^{n-1}, \omega)) \to \mathcal{B}(L^2(\mathbb{R}_+, \mu))$$

be the conditional expectation with respect to the state φ, and let $I_{L^2(S^{n-1}, \omega)}$ be the identity operator on $L^2(S^{n-1}, \omega)$.

A natural candidate for the q.d.s. $\mathcal{T}_t^{\mathrm{BES}^n}(X)$ is the q.d.s. $\mathcal{T}_t^{(n)}(X)$ defined by

$$\mathcal{T}_t^{(n)}(X) \to U^* E_\varphi\big[V^* \mathcal{T}_t^{\mathrm{BM}^n}\big(V(UXU^* \otimes I_{L^2(S^{n-1},\omega)})V^*\big)V\big]U$$

for $t \geq 0$ and $X \in B\big(L^2(\mathbb{R}_+, dx)\big)$. Note that both semigroups are conservative. Hence, to show that $\mathcal{T}_t^{(n)}(X)$ coincides with $\mathcal{T}_t^{\mathrm{BES}^n}(X)$, it suffices to verify that both semigroups satisfy the Eq. (1) with the same form (2)of $\mathcal{L}(X)$.

Indeed, we intend to show that the q.d.s. $\mathcal{T}_t^{(n)}(X)$ satisfies Eq. (1) with the infinitesimal generator $\mathcal{L}_{\mathrm{BES}^n}$ of $\big(\mathcal{T}_t^{\mathrm{BES}^n}(X)\big)$. Namely, we must show that

$$\begin{aligned}\big\langle v, \mathcal{T}_t^{(n)}(X)u\big\rangle &= \langle v, Xu\rangle + \int_0^t \big\langle v, \mathcal{L}_{\mathrm{BES}^n}\big(\mathcal{T}_t^{(n)}(X)\big)u\big\rangle\, ds \\ &= \langle v, Xu\rangle + \int_0^t \big\langle v, \mathcal{T}_t^{(n)}(X)G_n u\big\rangle\, ds \\ &\quad + \int_0^t \big\langle G_n v, \mathcal{T}_t^{(n)}(X)u\big\rangle\, ds + \int_0^t \big\langle L_n v, \mathcal{T}_t^{(n)}(X)L_n u\big\rangle\, ds \end{aligned} \tag{15}$$

for $u, v \in L^2(\mathbb{R}_+, dx)$ and $X \in \mathcal{B}\big(L^2(\mathbb{R}_+, dx)\big)$. To this end, first we note that for every $u \in L^2(\mathbb{R}_+, dx)$ one has

$$\Delta V(Uu \otimes \varphi) = V(V^*\Delta V)(Uu \otimes \varphi),$$

where $V^*\Delta V$ is the Laplace operator in the spherical coordinates with the radial component

$$(V^*\Delta V)_r = \rho^{-(n-1)}\frac{\partial}{\partial\rho}\left(\rho^{(n-1)}\frac{\partial}{\partial\rho}\right)$$

and the spherical component $(V^*\Delta V)_s$, so that

$$\begin{aligned}(V^*\Delta V)(Uu \otimes \varphi) &= (V^*\Delta V)_r(Uu) \otimes \varphi + \rho^{-(n-2)}Uu \otimes (V^*\Delta V)_s\varphi \\ &= \rho^{-(n-1)}\frac{\partial}{\partial\rho}\left(\rho^{(n-1)}\frac{\partial}{\partial\rho}Uu\right) = 2UG_n u.\end{aligned}$$

Note that we also have

$$\partial_k V(Uu \otimes \varphi) = V(V^*\partial_k V)(Uu \otimes \varphi),$$

where $(V^*\partial_k V)$ is the kth partial derivative operator in the spherical coordinates,

$$\begin{aligned}(V^*\partial_k V)(Uu \otimes \varphi) &= \left(\frac{\rho}{\partial_k}\frac{\partial}{\partial\rho} + \sum_{j=1}^{n-1}\frac{\partial\vartheta_j}{\partial x_k}\frac{\partial}{\partial\vartheta_j}\right)(Uu \otimes \varphi) \\ &= \left(\frac{\partial\rho}{\partial x_k}\frac{\partial}{\partial\rho}Uu\right) \otimes \varphi + Uu \otimes \sum_{j=1}^{n-1}\frac{\partial\vartheta_j}{\partial x_k}\frac{\partial}{\partial\vartheta_j}\varphi = \frac{\partial\rho}{\partial x_k}(UL_n u) \otimes \varphi.\end{aligned}$$

Now let

$$\mathbf{X} = V\big(UXU^* \otimes I_{L^2(S^{n-1},\omega)}\big)V^*, \qquad \mathbf{u} = V(Uu \otimes \varphi), \qquad \mathbf{v} = V(Uv \otimes \varphi).$$

Straightforward computations (see [12] for details) show that

$$\begin{aligned}\big\langle v, \mathcal{T}_t^{(n)}(X)x\big\rangle &= \big\langle v, U^*E_\varphi\big[V^*\mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})V\big]Uu\big\rangle = \mathrm{Tr}\Big(V^*\mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})V\big(|Uu\rangle\langle Uv| \otimes |\varphi\rangle\langle\varphi|\big)\Big) \\ &= \big\langle Uv \otimes \varphi, V^*\mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})V(Uu \otimes \varphi)\big\rangle = \big\langle \mathbf{u}, \mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})\mathbf{v}\big\rangle.\end{aligned}$$

Similarly, we can prove that $\langle v, Xu\rangle = \langle \mathbf{v}, \mathbf{X}\mathbf{u}\rangle$ and

$$\begin{aligned} \langle v, \mathcal{T}_t^{(n)}(X)G_n u\rangle &= \frac{1}{2}\langle \mathbf{v}, \mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})\Delta\mathbf{u}\rangle, \\ \langle G_n v, \mathcal{T}_t^{(n)}(X)u\rangle &= \frac{1}{2}\langle \Delta\mathbf{v}, \mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})\mathbf{u}\rangle, \\ \langle L_n v, \mathcal{T}_t^{(n)}(X)L_n u\rangle &= \sum_{k=1}^{n}\langle \partial_k\mathbf{v}, \mathcal{T}_t^{\mathrm{BM}^n}(\mathbf{X})\partial_k\mathbf{u}\rangle \end{aligned}$$

for $\sum_{k=1}^{n}\left(\frac{\partial\rho}{\partial x_k}\right)^2 = 1$. It follows that both $\mathcal{T}_t^{(n)}(X)$ and $\mathcal{T}_t^{\mathrm{BES}^n}(X)$ satisfy (15). Thus, by the uniqueness argument, they coincide and $\sum_{k=1}^{n}(\partial\rho/\partial x_k)^2 = 1$.

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