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Unified Description of the Early Universe

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A homogeneous and isotropic cosmological model of the universe is studied in which the parameter gamma of the "gamma-law" equation of state $p = (\gamma - 1)\rho$ varies continuously with cosmological time. A unified description of the early evolution of the universe is given in which an inflationary period is followed by a radiation dominated phase. Comparing the dynamics of the model with those models endowed with matter creation or vacuum energy decay, allows us to establish the time dependence of the creation rate and of the cosmological constant in terms of the varying γ index.

1. INTRODUCTION

In cosmology, the evolution of the universe is described by Einstein's equations together with an equation of state for the matter content. In the standard model (Weinberg, 1972), the history of the universe begins with a radiation era and then evolves to the present matter-dominated era. These two phases can be described by an equation of state relating the pressure p and the density ρ for an ideal gas known as the "gamma-law" equation of state $p = (\gamma - 1)\rho$. In the radiation phase we have $\gamma = 4/3$ ($p = \rho/3$), whereas for the pressureless, matter-dominated phase $\gamma = 1$.

In order to overcome some of the difficulties met by the standard model, such as the horizon problem and the flatness problem, an inflationary phase was proposed by Guth (1981). This would happen prior to the radiation-dominated era. This period can be described by the same equation of state provided the parameter $\gamma < 2/3$ in the case of a power-law inflation and, in particular, $\gamma = 0$ for an exponential inflation ($p = -\rho$).

Usually, the field equations are solved and analyzed separately for the different epochs, although some authors have given unified solutions. For instance, Israelit and Rosen (1989, 1993) use a different equation of state in

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which the pressure varies continuously from $-\rho$ to its value during the radiation era ($p = \rho/3$).

In this paper we consider a model to study the evolution of the universe as it goes from an inflationary phase to a radiation-dominated era. In this model, we keep the gamma-law form of the equation of state, but let the parameter γ vary continuously as the universe expands. Our approach is similar in some aspects to that of Israelit and Rosen, although it has distinct features. We also show that models of the universe endowed with decaying vacuum energy density (varying cosmological constant) or matter creation have dynamical behavior similar to the one studied here. Since there is no complete theory that establishes uniquely the time dependence of the cosmological constant or the matter creation rate, we find appropriate to determine them in terms of the time-dependent γ-index proposed here. As a consequence, a new class of decaying vacuum model arises in which the cosmological constant has the form $\Lambda \propto R^a H^3$.

In Section 2 we present the basic model, while in Section 3 we give the solutions to the field equation. In Section 4 we study briefly the dynamical behavior of models with varying cosmological constant and matter creation, and by comparing them with our model we calculate both the creation rate and the time dependence of the cosmological constant. The main conclusions are given in Section 5.

2. THE MODEL

Let us consider the homogeneous and isotropic Friedmann–Robertson–Walker (FRW) line element

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin\theta^2 \cdot d\phi^2)\right] \tag{1}$$

where $k = 0, \pm 1$ is the curvature parameter and R the scale factor (hereafter we take $c = 1$). If we take that the universe is filled with a perfect fluid whose energy density is $\rho(t)$ and the pressure is $p(t)$, the time and space components of Einstein's field equations $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ give, respectively,

$$3\ddot{R} = -4\pi G(\rho + 3p)R \tag{2}$$

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi G(\rho - p)R^2 \tag{3}$$

where a dot denotes time derivative. Eliminating $\ddot{R}$ from (2) and (3), we get

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{4}$$

Equations (4) and (2) can be rewritten in terms of the Hubble parameter

$$H = \frac{\dot{R}}{R}$$

to give, respectively,

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{5}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \tag{6}$$

We shall suppose that the pressure p and the density ρ are related through the "gamma-law" equation of state

$$p = (\gamma - 1)\rho \tag{7}$$

Substituting this into (6) and using (5), we finally obtain

$$\dot{H} + \frac{3}{2}\gamma H^2 + \left(\frac{3}{2}\gamma - 1\right)\frac{k}{R^2} = 0 \tag{8}$$

Our aim is to study how the adiabatic parameter should vary so that in the course of its evolution the universe goes through a transition from an inflationary to a radiation-dominated phase. The function $\gamma(R)$ thus must be such that when the scale factor is less than a certain reference value R_*, we have the inflationary phase ($\gamma < 2/3$). As the scale factor increases, γ also increases to reach the value 4/3 for $R >> R_*$, when we have the radiation phase. Among the many possible functional forms, we choose one which, despite being quite simple, will give both an exponential and power-law inflation. These conditions are satisfied by a one-parameter function

$$\gamma(R) = \frac{4}{3}\frac{A(R/R_*)^2 + (a/2)(R/R_*)^a}{A(R/R_*)^2 + (R/R_*)^a} \tag{9}$$

where A is a constant and a is a free parameter. The above expression is an increasing function of R. In the limit $R \to 0$ we have

$$\gamma(R) = \frac{2a}{3}$$

so that 1 is the required maximum value of a for an inflation epoch to exist ($\gamma < 2/3$). As we shall see below, the parameter a is related to the power of the cosmic time t during the inflationary era, when we have $R \propto t^{1/a}$, and for a approaching zero we have an exponential inflation ($\gamma = 0$ at $R = 0$). Therefore, a must lie in the interval

$$0 \leq a < 1$$

To solve equation (8) we rewrite it in the form

$$HH' + \frac{3}{2}\gamma\frac{H^2}{R} + \left(\frac{3}{2}\gamma - 1\right)\frac{k}{R^3} = 0 \qquad (10)$$

where $H' = dH/dR$.

We shall investigate models in which the curvature of the universe is zero, that is, $k = 0$. Equation (10) becomes

$$H' = -\frac{3}{2}\gamma\frac{H}{R} \qquad (11)$$

and its integral is

$$H = B\exp\left(-\frac{3}{2}\int\frac{\gamma(R)\,dR}{R}\right) \qquad (12)$$

3. SOLUTION OF THE FIELD EQUATIONS

Substituting (9) into (12) and integrating, we have that the Hubble parameter is given by

$$H = \frac{B}{A(R/R_*)^2 + (R/R_*)^a} \qquad (13)$$

where B is a constant. Since for $R = R_*$ we have $H = H_*$, a relation between A and B can be written in the form

$$B = (1 + A)H_* \qquad (14)$$

Using (13), we can integrate $H = \dot{R}/R$ to obtain an expression for t in terms of the scale factor R, which for $a \neq 0$ is given by

$$Bt = \frac{1}{a}\left(\frac{R}{R_*}\right)^a + \frac{1}{2}A\left(\frac{R}{R_*}\right)^2 \qquad (15)$$

(The case $a = 0$ will be examined later.) For $R << R_*$, the first term in the right-hand side dominates and one has a phase of power-law inflation $R \propto t^{1/a}$. On the other hand, for $R >> R_*$ we have

$$R = \left(\frac{2B}{A}t\right)^{1/2} R_*$$

If we compare this with the solution of a pure radiation phase, that is,

$$R = (2H_o t)^{1/2} R_o \qquad (16)$$

where H_o and R_o are the present values of H and R respectively, we obtain

$$B = AH_o\left(\frac{R_o}{R_*}\right)^2$$

The above expression together with (14) gives

$$A = \left[\left(\frac{R_o}{R_*}\right)^2 \frac{H_o}{H_*} - 1\right]^{-1} \tag{17}$$

and

$$B = \left[1 - \left(\frac{R_*}{R_o}\right)^2 \frac{H_*}{H_o}\right]^{-1} H_* \tag{18}$$

We now study the solution in the limit $a \to 0$. In this case the integral (12) becomes

$$H = \frac{H_I}{A(R/R_*)^2 + 1} \tag{19}$$

where H_I is the initial value of H at $R = 0$. Integrating $H = \dot{R}/R$, we obtain

$$H_I t = \ln\left(\frac{R}{R_*}\right) + \frac{1}{2} A\left(\frac{R}{R_*}\right)^2 \tag{20}$$

Again, the radiation phase is described by the solution in the limit $R >> R_*$, that is,

$$R = \left(\frac{2H_I}{A} t\right)^{1/2} R_*,$$

and comparing this with (16), we get

$$A = \left(\frac{R_*}{R_o}\right)^2 \frac{H_I}{H_o} \tag{21}$$

Once again, in the limit of very small R ($R << R_*$), the logarithm term dominates and one has an exponential inflation phase with $R = R_* \exp(H_I t)$.

For $a > 0$ the model is singular with the density varying for $R << R_*$ as

$$\rho = \frac{3B^2}{8\pi G}\left(\frac{R}{R_*}\right)^{-a}$$

with B given by (18). The deceleration parameter varies from $q_I = a - 1$ at $R = 0$ to 1 for $R >> R_*$.

For $a = 0$ the universe is infinitely old, since $R \to 0$ for $t = -\infty$. However, there is no physical singularity, since the density assumes a finite value as $R \to 0$. It can be inferred from equations (5) and (19) that for $R = 0$

$$\rho_I = \frac{3H_I^2}{8\pi G}$$

The deceleration parameter has the form

$$q = \frac{A(R/R_*)^2 - 1}{A(R/R_*)^2 + 1}$$

and varies from $q = -1$ at $R = 0$ to $q = +1$ for $R >> R_*$, as expected.

4. MATTER CREATION AND DECAYING VACUUM ENERGY

Cosmological models with matter creation (Prigogine *et al.*, 1989; Lima *et al.*, 1995) and decaying vacuum energy density (varying cosmological constant) (Lima and Trodden, 1995) have a dynamical behavior similar to the models with time-dependent γ index studied here. In the absence of a model that gives some clue regarding the time dependence of the cosmological constant Λ, various work so far has adopted a phenomenological approach (Chen and Wu, 1990; Carvalho *et al.*, 1992; Lima and Trodden, 1995). The same is true for the matter creation rate, which is an unknown quantity in most models.

On the other hand, in both cases, by introducing an effective index γ, which, as in the present case, turns out to be time dependent, we find that the Einstein field equations assume the general FRW-type form as in equations (5) and (8). It seems therefore opportune to compare these two approaches with the present models with the purpose of determining both the creation rate and time dependence of the cosmological constant by requiring that the models have the same dynamical behavior.

4.1. The Matter Creation Rate

Following Lima *et al.* (1995), we introduce the matter creation rate ψ so that instead of equation (10) we get

$$HH' + \frac{3}{2}\left(1 - \frac{\psi}{3nH}\right)\gamma_o \frac{H^2}{R} + \left[\frac{3}{2}\left(1 - \frac{\psi}{3nH}\right)\gamma_o - 1\right]\frac{k}{R^3} = 0 \quad (22)$$

where γ_o is the constant asymptotic limit of the adiabatic index, which in our case is the limiting value of γ for $R >> R_*$ during the radiation era, that is, 4/3, and n is the particle number density related to the energy density through (Lima *et al.*, 1995)

$$n = n_*\left(\frac{\rho}{\rho_*}\right)^{1/\gamma_o}$$

If we compare equations (10) and (22), we conclude that

$$\gamma = \left(1 - \frac{\psi}{3nH}\right)\gamma_o$$

from which we get

$$\psi = 3\left(1 - \frac{\gamma}{\gamma_o}\right)nH \tag{23}$$

By combining (9) and (13) we obtain

$$\psi = 3\left(1 - \frac{a}{2}\right)\frac{n}{B}\left(\frac{R}{R_*}\right)^a H^2$$

Note that (23) has the limit $\psi = 0$ as γ approaches γ_o for $R >> R_*$. Thus, during the radiation era the creation of matter is negligible. The maximum rate occurs at $R = 0$, when we have $\gamma(0) = 2a/3$, and therefore

$$\psi_I = 3nH\left(1 - \frac{a}{2}\right)$$

This shows how the value of a affects ψ at the beginning of the inflationary period, the maximum value being $3nH$ for exponential inflation ($a = 0$) and the minimum value $3nH/2$ for $a = 1$.

4.2. Time-Dependent Cosmological Constant

We now study the time dependence of the cosmological constant. The field equations (5) and (8) with a Λ term are, respectively,

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda \tag{24}$$

and

$$\dot{H} + \frac{3}{2}\gamma_o H^2 + \left(\frac{3}{2}\gamma_o - 1\right)\frac{k}{R^2} - \frac{\gamma_o}{3}\Lambda = 0 \tag{25}$$

In order to compare these equations with (5) and (8), we first rewrite Λ in a convenient way, namely

$$\Lambda = 8\pi G\rho\lambda \tag{26}$$

Equation (24) becomes

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho(1 + \lambda) \tag{27}$$

Substituting Λ given by (26) into (25) and using (27) to eliminate ρ, we finally obtain

$$\dot{H} + \frac{3}{2}\frac{\gamma_o}{1 + \lambda}H^2 + \left(\frac{3}{2}\frac{\gamma_o}{1 + \lambda} - 1\right)\frac{k}{R^2} = 0 \tag{28}$$

Comparing the above equation with (8), we conclude that

$$\gamma = \frac{\gamma_o}{1 + \lambda} \tag{29}$$

Combining this expression with (26) and (27) gives then

$$\Lambda = 3\left(H^2 + \frac{k}{R^2}\right)\left(1 - \frac{\gamma}{\gamma_o}\right) \tag{30}$$

In case of the flat model discussed in Section 3, for $a = 0$, equations (9) and (19) give

$$1 - \frac{\gamma}{\gamma_o} = \frac{H}{H_I}$$

and (30) reduces to

$$\Lambda = 3\frac{H^3}{H_I} \tag{31}$$

We note that this is a particular case, $\beta = 0$, of the law used by Lima and Maia (1994).

For $a > 0$ we can combine (9) and (13) to get

$$1 - \frac{\gamma}{\gamma_o} = \left(1 - \frac{a}{2}\right)\left(\frac{R}{R_*}\right)^a \frac{H}{B}$$

Now, substituting this into (30) provides a general expression for Λ in the case of a flat universe, namely

$$\Lambda = 3\left(1 - \frac{a}{2}\right)\left(\frac{R}{R_*}\right)^a \frac{H^3}{B}$$

This represents an entirely new class of model for the decaying vacuum since the above form of Λ, to our knowledge, has not been examined before. Previous work has used either a power of R [R^{-2} as in Chen and Wu (1990)] or a power of H [H^n as in Lima and Maia (1994)] or a combination of both [$\beta H^2 + \alpha R^{-n}$ as in Carvalho *et al.* (1992) and Lima and Maia (1993)].

5. CONCLUSION

We have investigated homogeneous and isotropic cosmological models with zero curvature in which the parameter γ of the equation of state varies continuously as a function of the scale factor. In this way it is possible to have a unified description of the early evolution of the universe. A class of models is obtained according to the value of the parameter a in equation (9). For $a = 0$ the parameter γ varies from 0 for $R = 0$ to $+4/3$ when $R >> R_*$. The universe is infinitely old and for $R < R_*$ we have an exponential inflation phase. For $R > R_*$ it enters a radiation phase. There is no real singularity, the density being always finite. For a in the range $0 < a < 1$, γ slowly increases from $(2/3)a$ for $R = 0$ to 4/3, again when $R >> R_*$. The first period of evolution can be described as a power-law inflation with the scale factor varying according $R \propto T^{1/a}$. This is then followed by a radiation era when $R >> R_*$.

We have also pointed out the similarity between the dynamical behavior of this model and models that incorporate either matter creation or varying cosmological constant. Rewriting the field equations of the latter in terms of an effective, time-dependent adiabatic index, we were able to calculate both the matter creation rate and the cosmological constant as a function of the scale factor.

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