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General function spaces. III
(Spaces $B^{g(x)}_{p,q}$ and $F^{g(x)}_{p,q}$, $1<p<\infty$: basic properties)

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Contents

1. Introduction

2. Definitions, preliminaries
   2.1. Classes M and $\overset{+}{M}$
   2.2. Spaces Z and Z'
   2.3. B-spaces, F-spaces, and H-spaces
   2.4. Multipliers in L_p

3. Fundamental properties
   3.1. Independence, completeness
   3.2. Density
   3.3. Comparison
   3.4. Translation invariance

4. Interpolation
   4.1. General interpolation formulas
   4.2. Concrete interpolation formulas

5. Spaces $H^{g(x)}_p$ and $\overset{+}{H}{}^{g(x)}_p$
   5.1. Representation
   5.2. Comparison theorems for H-spaces. Examples

6. Inclusions
   6.1. Inclusion theorems
   6.2. Comparison theorems for B-spaces

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1. Introduction

It is a known fact that decomposition methods are very useful in spaces of functions and distributions defined in R_n. The methods can be described as follows. If f is a given function or distribution in R_n, then one asks for the approximation of f by entire analytic functions of exponential type (rate of convergence, etc.). By the Paley—Wiener—Schwartz theorem this problem can be reformulated. One asks for the approximation of the Fourier transform Ff of f by distributions with compact support. This is essentially the same as the problem of decomposition of Ff by distributions with compact support: Ff is represented as an infinite sum of distributions of this type. The supports of these distributions are determined by corresponding decompositions of R_n in subsets. Systematic treatments of (isotropic and anisotropic) Sobolev—Lebesgue (=Bessel-potential)—Besov (=Lipschitz)-spaces on the basis of such decomposition methods may be found in [10] and [15, Chapter 2]. But these methods work also in other spaces [8, 14, 15] (the spaces $\Lambda^s_{p,q}$ and $F^s_{p,q}$). For all these considerations multiplier theorems of Michlin—Hörmander—Marcinkiewicz type for L_p-spaces are very important. This has the consequence that the above mentioned decompositions of R_n must be related to decompositions of R_n appearing in the corresponding multiplier theorems (dyadic decompositions). From this point of view one has at least two decompositions of R_n which seem to be of peculiar interest:

(i) R_n is decomposed in differences of parallelepipeds centered at the origin (this is essentially the method in the above cited papers and books and gives isotropic and anisotropic spaces of the above type. A short description is also given at the beginning of Subsection 6.2).

(ii) R_n is decomposed in parallelepipeds of type $\{x: 2^{k_j} < \varepsilon_j x_j \leqq 2^{k_j+1}; \varepsilon_j = \pm 1\}$, (this includes isotropic and anisotropic spaces, spaces with dominating mixed derivatives, and more general spaces related to spaces considered in [5], and [17, Appendix]).

In both cases one has again two possibilities: for instance, in case (ii) one can consider the above parallelepipeds, where $k_j = 0, \pm 1, \pm 2, \ldots$ (homogeneous spaces); or one can consider only the parallelepipeds where $k_j = 0, 1, 2, \ldots$ (obviously, here one must modify the decomposition near the axes) (non-homogeneous spaces). Although cases (i) and (ii) (and their homogeneous and non-homogeneous subcases) are mutually independent, we shall be concerned here only with (homogeneous and non-homogeneous) decompositions of type (ii). Of peculiar interest seems to be the homogeneous case. It includes spaces where the norms are given by $\|D^\alpha f\|_{L_p}$ or $\|F^{-1}|x|^\beta Ff\|_{L_p}$, where β is real. The complex interpolation $[\cdot,\cdot]_{1/2}$ of the spaces characterized by $\partial^2 f/\partial x_1^2$ and $\partial^2 f/\partial x_2^2$, respectively, gives the space characterized by $\partial^2 f/\partial x_1 \partial x_2$, see (56). In order to consider these spaces we must generalize the notion of distributions.

This paper is the third part of a series started with [16, Part I]. In the first part motivations are given for decomposition methods in the framework of spectral theory. The second part deals with basic spaces from which more general spaces considered here are built up: L_p-spaces, where the supports of the corresponding Fourier transforms are contained in a given compact subset of R_n. The third part presented here is self-contained. Here are proved the basic properties for spaces $B^{g(x)}_{p,q}$, $\overset{+}{B}{}^{g(x)}_{p,q}$, $F^{g(x)}_{p,q}$, and $\overset{+}{F}{}^{g(x)}_{p,q}$: density of smooth functions, completeness, equivalence of norms, inclusion properties, comparisons between the different types of spaces, interpolation. The fourth part of this series will be a direct continuation of this paper. For the spaces treated here there will be considered special properties: representation theorems, duality, multipliers, embeddings, traces. In the fifth part (again essentially self-contained) there will be considered the spaces $B^{g(x)}_{p,q}$ and $F^{g(x)}_{p,q}$ for $0<p\leqq 1$ and $p=\infty$. Although some results proved here remain true, the situation will change completely: the methods will be closely related to the methods developed by Fefferman and Stein [13] in connection with the real variable approach to Hardy spaces.

Insignificant positive constants will be denoted by $c, c', c_1, \ldots$ which does not indicate that these numbers are equal in different occurences.

2. Definitions, preliminaries

2.1. *Classes M and $\overset{+}{M}$*

In the sequel R_n is the n-dimensional real Euclidean space; its general point is denoted by $x=(x_1, \ldots, x_n)$. Let $\varepsilon=(\varepsilon_1, \ldots, \varepsilon_n)$ where ε_j is either 1 or -1. There are 2^n vectors of this type. The set of all these vectors is denoted by E. Let N_n be the set of all vectors $k=(k_1, \ldots, k_n)$, where the k_j's are integers. N_n^* is the subset of N_n where $k_j=0, 1, 2, \ldots$ for $j=1, \ldots, n$. Then

$$Q_{k,\varepsilon}=\{x: 2^{k_j}<\varepsilon_j x_j<2^{k_j+1};\quad j=1, \ldots, n\} \tag{1}$$

gives a decomposition of $\overset{+}{R}_n$:

$$\left\{x: \prod_{j=1}^{n} x_j \neq 0\right\}=\overset{+}{R}_n=\bigcup_{k\in N_n, \varepsilon\in E} \bar{Q}_{k,\varepsilon}. \tag{2}$$

Let

$$P_{k,\varepsilon}=\{x:\ \eta_{0,k_j}2^{k_j}<\varepsilon_j x_j<2^{k_j+1};\quad j=1, \ldots, n\}, \tag{3}$$

where $k \in N_n^*$. Here $\eta_{0,k_j}=0$ for $k_j=0$ and $\eta_{0,k_j}=1$ for $k_j=1, 2, \ldots$. We have

(4) $$R_n = \bigcup_{k \in N_n^*, \varepsilon \in E} \bar{P}_{k,\varepsilon}.$$

The decomposition (2) leads to homogeneous spaces (the fact that the hyper-planes $\{x\colon x_j=0\}$, j is fixed, do not appear in the decomposition (2) will be very important), the decomposition (4) leads to non-homogeneous spaces.

Definition 2.1/1. $\overset{+}{M}$ is the set of all positive functions $g(x)$ defined in $\overset{+}{R}_n$ and n times continuously differentiable in $\overset{+}{R}_n$ for which there exists a positive number c such that for all multi-indices $\gamma=(\gamma_1, \ldots, \gamma_n)$ where $\gamma_j = 0$ or 1, and for all $x \in R_n$

(5) $$\left| \prod_{j=1}^{n} x_j^{\gamma_j} D^\gamma g(x) \right| \leqq c g(x)$$

holds.

Remark 2.1/1. A simple application of the mean value theorem to $\log g(x)$ shows that there exists a positive number c such that for all parallelepipeds $Q_{k,\varepsilon}$

(6) $$\max_{x \in \bar{Q}_{k,\varepsilon}} g(x) \leqq c \min_{x \in \bar{Q}_{k,\varepsilon}} g(x), \quad k \in N_n, \quad \varepsilon \in E.$$

Definition 2.1/2. M is the set of all positive functions $g(x)$ defined in R_n and n times continuously differentiable in R_n for which there exists a positive number c such that for all multi-indices $\gamma=(\gamma_1, \ldots, \gamma_n)$ where $\gamma_j = 0$ or 1, and for all $x \in R_n$

(7) $$\left| \prod_{j=1}^{n} (1+x_j^2)^{\gamma_j/2} D^\gamma g(x) \right| \leqq c g(x).$$

Remark 2.1/2. Similarly to Remark 2.1/1 one obtains that there exists a positive number c such that for all parallelepipeds $P_{k,\varepsilon}$

(8) $$\max_{x \in \bar{P}_{k,\varepsilon}} g(x) \leqq c \min_{x \in \bar{P}_{k,\varepsilon}} g(x), \quad k \in N_n^*, \quad \varepsilon \in E.$$

Remark 2.1/3. The classes M and $\overset{+}{M}$ are related to the differentiability properties of the spaces under consideration. For instance, in the "classical" counterpart the smoothness index s in the Besov spaces $B_{p,q}^s$ would correspond to $g(x)= =(1+|x|^2)^{s/2}$.

Lemma 2.1/1. (i) *Let* $g(x) \in \overset{+}{M}$, $g_1(x) \in \overset{+}{M}$, $g_2(x) \in \overset{+}{M}$; $\lambda > 0$ *and* $\varkappa$ *real numbers. Then*

$$\lambda g(x) \in \overset{+}{M}, \quad g^\varkappa(x) \in \overset{+}{M}, \quad g_1(x)+g_2(x) \in \overset{+}{M}, \quad g_1(x) g_2(x) \in \overset{+}{M}, \quad \frac{g_1(x)}{g_2(x)} \in \overset{+}{M}.$$

(ii) *If* $g(x) \in \overset{+}{M}$, *then there exist three positive numbers* c, s_1, *and* s_2 *such that*

$$g(x) \leqq c(1+|x|^2)^{s_1} \left| \prod_{j=1}^{n} x_j \right|^{-s_2} \quad \textit{for} \quad x \in \overset{+}{R}_n. \tag{9}$$

Lemma 2.1/2. (i) *The assertion of Lemma* 2.1/1 (i) *remains true if one replaces* $\overset{+}{M}$ *by* M.

(ii) *If* $g(x) \in M$, *then there exist two positive numbers c and s such that*

$$g(x) \leqq c(1+|x|^2)^s \quad \textit{for} \quad x \in R_n. \tag{10}$$

Proof. Part (i) of either lemma can be proved by direct computation. Let us prove (9). Let us set $k^0=(0, 0, ..., 0)$ and $k^1=(1, 0, ..., 0)$. If c has the same meaning as in (6), then it follows that

$$\max_{x \in \bar{Q}_{k^1,\varepsilon}} g(x) \leqq c \min_{x \in \bar{Q}_{k^1,\varepsilon}} g(x) \leqq c \max_{x \in \bar{Q}_{k^0,\varepsilon}} g(x).$$

Here $\varepsilon \in E$ is fixed. If $|k| = \sum_{j=1}^{n} |k_j|$, then it follows by induction that

$$\max_{x \in \bar{Q}_{k,\varepsilon}} g(x) \leqq c^{|k|} \max_{x \in \bar{Q}_{k^0,\varepsilon}} g(x).$$

But for $x \in \bar{Q}_{k,\varepsilon}$

$$c^{|k|} = 2^{c'|k|} \leqq c'' \prod_{j=1}^{n} \left(|x_j| + \frac{1}{|x_j|}\right)^{c'}$$

holds. This proves (9). (10) can be proved in the same way.

Examples. With the aid of the above lemmas one can construct immediately a large variety of examples of functions belonging to M and $\overset{+}{M}$. Since $|x_j|$ belongs to $\overset{+}{M}$, so do also $|x_j|^{\varkappa_j}$ ($\varkappa_j$ real) and every rational function of $|x_j|^{\varkappa_j}$ with positive coefficients. Furthermore, $\log(2+|x_j|) \in \overset{+}{M}$, ... Of peculiar interest are the following functions belonging to $\overset{+}{M}$:

$|x|^s$, s real (isotropic spaces),

$\sum_{j=1}^{n} |x_j|^{s_j}$, s_j real (anisotropic spaces),

$\prod_{j=1}^{n} |x_j|^{s_j}$, s_j real (spaces with dominating mixed derivatives),

$|x|^s |\log(2+|x|)|^t$, s and t real,

where the related spaces seem to be of interest in the theory of pseudo-differential operators. If one replaces in the above functions $|x_j|$ by $(1+x_j^2)^{1/2}$ and $|x|$ by $(1+|x|^2)^{1/2}$, one obtains functions belonging to M.

2.2. *Spaces Z and Z'*

D is the space of all complex-valued infinitely differentiable functions, defined in R_n, with compact support. S is the well-known Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions defined in R_n. Both spaces will be considered as locally convex spaces equipped in the usual way with a topology. D' and S' are the dual spaces (distributions and tempered distributions, respectively). What is needed here is a generalization of D'. Let $\overset{+}{D}$ be the subset of D consisting of all functions of D with compact support in $\overset{+}{R}_n$. Again, $\overset{+}{D}$ will be considered as a locally convex space equipped in the usual way with a topology (see, for instance, [9, p. 76]). The dual space is denoted by $\overset{+}{D}'$ (distributions over $\overset{+}{R}_n$). Let F be the Fourier transform, defined on S by

$$(Ff)(x) = (2\pi)^{-n/2} \int_{R_n} e^{ix\xi} f(\xi)\, d\xi, \quad x\xi = \sum_{j=1}^{n} x_j \xi_j.$$

The inverse Fourier transform F^{-1} is given by a corresponding formula, where i is replaced by $-i$. As is known, F is a one-to-one map from S onto S. The image of $\overset{+}{D}$ by such a map is denoted by Z. Hence $Z = F\overset{+}{D}$. By the Paley—Wiener theorem Z consists of entire analytic functions of exponential type. The topology of Z is to be taken over from $\overset{+}{D}$ by this mapping. Let Z' be the corresponding dual space. If $f \in Z'$, then the Fourier transform of f, denoted by Ff, is defined as an element of $\overset{+}{D}'$ by

$$(Ff)(\varphi) = f(F\varphi) \quad \text{for all} \quad \varphi \in \overset{+}{D}.$$

As usually, the Fourier transform in S' is defined in the same way if $\overset{+}{D}$ is replaced by S. We have

$$FZ' = \overset{+}{D}', \quad FS' = S'. \tag{11}$$

(The inverse image under the Fourier transform of a given distribution $g \in \overset{+}{D}'$ is $f \in Z'$ defined by $f(\varphi) = g(F^{-1}\varphi)$). For the later considerations it will be useful to note that we also have

$$Z = F^{-1}\overset{+}{D} \quad \text{and} \quad F^{-1}Z' = \overset{+}{D}',$$

where $F^{-1}f$ ($f \in Z'$) is defined in an obvious way. All the dual spaces, considered here, are equipped with the strong topology.

2.3. B-spaces, F-spaces, and H-spaces

First we need an extension of the definition of the parallelepipeds $Q_{k,\varepsilon}$ and $P_{k,\varepsilon}$ given in Subsection 2.1. Assume that ε has the same meaning as explained there. Let L be a natural number. Then

(12) $$Q^{(L)}_{k,\varepsilon} = \{x\colon\ 2^{k_j-L} < \varepsilon_j x_j < 2^{k_j+L+1};\quad j = 1, \dots, n\},\quad k \in N_n,\quad \varepsilon \in E,$$

and

(13) $$P^{(L)}_{k,\varepsilon} = \{x\colon\ \eta'_{0,k_j} 2^{k_j-L} < \varepsilon_j x_j < 2^{k_j+L+1};\quad j = 1, \dots, n\},\quad k \in N^*_n,\quad \varepsilon \in E.$$

Here $\eta'_{0,k_j}=1$ for $k_j=1, 2, \dots,$ and $\eta'_{0,0}=-1$. There exists a number N (depending on L) such that each point $x \in \overset{+}{R}_n$, respectively $x \in R_n$, belongs to at most N parallelepipeds $\bar{Q}^{(L)}_{k,\varepsilon}$, respectively $\bar{P}^{(L)}_{k,\varepsilon}$.

Definition 2.3/1. If $L=1, 2, \dots,$ then $\overset{+}{\Phi}_L$ is the set of all systems $\{\varphi_{k,\varepsilon}\}_{k\in N_n, \varepsilon\in E}$ of functions with the following properties:

(14) (i) $$\varphi_{k,\varepsilon}(x) \in \overset{+}{D},\quad \operatorname{supp} \varphi_{k,\varepsilon} \subset Q^{(L)}_{k,\varepsilon};$$

(ii) for each multi-index $\gamma=(\gamma_1, \dots, \gamma_n)$ there exists a positive number c_γ such that for all $k \in N_n$, all $\varepsilon \in E$, and all $x \in R_n$,

(15) $$|(D^\gamma \varphi_{k,\varepsilon})(x)| \leqq C_\gamma 2^{-\gamma_1 k_1 - \dots - \gamma_n k_n};$$

(iii) there exist two positive numbers c and C such that for all $x \in \overset{+}{R}_n$

(16) $$c \leqq \sum_{k\in N_n, \varepsilon\in E} \varphi_{k,\varepsilon}(x) \leqq C.$$

Let us set

$$\overset{+}{\Phi} = \bigcup_{L=1}^{\infty} \overset{+}{\Phi}_L.$$

Definition 2.3/2. If $L=1, 2, \dots,$ then Φ_L is the set of all systems $\{\varphi_{k,\varepsilon}\}_{k\in N^*_n, \varepsilon\in E}$ of functions with the following properties:

(i) $$\varphi_{k,\varepsilon}(x) \in D,\quad \operatorname{supp} \varphi_{k,\varepsilon} \subset P^{(L)}_{k,\varepsilon};$$

(ii) for each multi-index $\gamma=(\gamma_1, \dots, \gamma_n)$ there exists a positive number c_γ such that for all $k \in N^*_n$, all $\varepsilon \in E$, and all $x \in R_n$ (15) holds;

(iii) there exist two positive numbers c and C such that for all $x \in R_n$ (16) holds.

Let us set

$$\Phi = \bigcup_{L=1}^{\infty} \Phi_L.$$

Remark 2.3/1. This is the counterpart to corresponding systems of functions used in [14] and [15, 2.3.1] in connection with the Besov spaces $B^s_{p,q}$ and the spaces

$F^s_{p,q}$. (There is an irrelevant difference in comparison with the systems in [14, 15]: there we used the Fourier transform of the functions in the systems instead of the functions themselves). The assumptions for the functions $\varphi_{k,\varepsilon}$ may be weakened: for instance, it will be sufficient if (15) holds for multi-indices $\gamma=(\gamma_1, \dots, \gamma_n)$ where all γ_j are either 0 or 1.

It is not difficult to see that Φ_L and $\overset{+}{\Phi}_L$ are non-empty. Let $\varphi \in D$ be such that

$$\varphi(x) = 1 \quad \text{for} \quad |x_j| < 1/2 \quad (j = 1, \dots, n), \quad \varphi \geqq 0,$$

and $\varphi(x)=0$ outside a small neighbourhood of $\{y:|y_j|\leqq 1/2\}$. Let $x^{k,\varepsilon}$ be the centre of $Q_{k,\varepsilon}$. Then the system of functions $\{\varphi_{k,\varepsilon}\}_{k\in N_n, \varepsilon\in E}$ defined by

$$\varphi_{k,\varepsilon}(x) = \varphi\big(2^{-k_1}(x_1 - x_1^{k,\varepsilon}), \dots, 2^{-k_n}(x_n - x_n^{k,\varepsilon})\big)$$

belongs to $\overset{+}{\Phi}_L$. After a small modification one obtains in the same way systems belonging to Φ_L. Furthermore, $\{\psi_{k,\varepsilon}\}_{k\in N_n, \varepsilon\in E} \in \overset{+}{\Phi}_L$, where

$$\psi_{k,\varepsilon}(x) = \varphi_{k,\varepsilon}(x)\Big[\sum_{l\in N_n, \lambda\in E} \varphi_{l,\lambda}(x)\Big]^{-1}, \quad x \in \overset{+}{R}_n.$$

This system has the advantage that (16) can be replaced by

$$\sum_{k\in N_n, \varepsilon\in E} \psi_{k,\varepsilon}(x) = 1 \quad \text{for} \quad x \in \overset{+}{R}_n. \tag{17}$$

Corresponding assertions hold for Φ_L. (17) is a *partition of unity.*

Now we can define the B-spaces, F-spaces, and H-spaces considered here. If $1<p<\infty$, then L_p has the usual meaning: it is the Banach space of all complex-valued Lebesgue-measurable functions in R_n such that

$$\|f\|_{L_p} = \Big(\int_{R_n} |f(x)|^p dx\Big)^{1/p} < \infty.$$

Definition 2.3/3. *(Homogeneous spaces.)* Let $g(x)\in \overset{+}{M}$ and $1<p<\infty$. Let $x^{k,\varepsilon}$ be the centre of $Q_{k,\varepsilon}$. Let $\{\varphi_{k,\varepsilon}\}_{k\in N_n, \varepsilon\in E} \in \overset{+}{\Phi}$.

(i) If $1\leqq q\leqq\infty$, then

$$\overset{+}{B}{}^{g(x)}_{p,q} = \{f: f\in Z', \quad \|f\|_{\overset{+}{B}{}^{g(x)}_{p,q}} = \Big(\sum_{k\in N_n, \varepsilon\in E} \|g(x^{k,\varepsilon}) F^{-1}\varphi_{k,\varepsilon} Ff\|^q_{L_p}\Big)^{1/q} < \infty\}. \tag{18}$$

(If $q=\infty$, then the l_q-norm in (18) must be replaced by the l_∞- norm).

(ii) If $1<q<\infty$, then

$$\overset{+}{F}{}^{g(x)}_{p,q} = \{f: f\in Z', \quad \|f\|_{\overset{+}{F}{}^{g(x)}_{p,q}} = \Big\|\Big(\sum_{k\in N_n, \varepsilon\in E} |g(x^{k,\varepsilon})(F^{-1}\varphi_{k,\varepsilon} Ff)(x)|^q\Big)^{1/q}\Big\|_{L_p} < \infty\}. \tag{19}$$

(iii) $\overset{+}{H}{}^{g(x)}_p = \overset{+}{F}{}^{g(x)}_{p,2}$.

Definition 2.3/4. *(Non-homogeneous spaces.)* Let $g(x)\in M$ and $1<p<\infty$. Let $x^{k,\varepsilon}$ be the centre of $P_{k,\varepsilon}$. Let $\{\varphi_{k,\varepsilon}\}_{k\in N_n^*,\varepsilon\in E}\in \Phi$.

(i) If $1\leqq q\leqq\infty$, then

$$(20)\qquad B^{g(x)}_{p,q}=\{f: f\in S',\quad \|f\|_{B^{g(x)}_{p,q}}=\Big(\sum_{k\in N_n^*,\varepsilon\in E}\|g(x^{k,\varepsilon})F^{-1}\varphi_{k,\varepsilon}Ff\|^q_{L_p}\Big)^{1/q}<\infty\}.$$

(If $q=\infty$, then the l_q-norm in (20) must be replaced by the l_∞-norm).

(ii) If $1<q<\infty$, then

$$(21)\qquad F^{g(x)}_{p,q}=\{f: f\in S',\quad \|f\|_{F^{g(x)}_{p,q}}\Big\|\Big(\sum_{k\in N_n^*,\varepsilon\in E}|g(x^{k,\varepsilon})(F^{-1}\varphi_{k,\varepsilon}Ff)(x)|^q\Big)^{1/q}\Big\|_{L_p}<\infty\}.$$

(iii) $H^{g(x)}_p=F^{g(x)}_{p,2}$.

Remark 2.3/2. The norms (18)—(21) obviously depend on the choice of the system $\{\varphi_{k,\varepsilon}\}\in\overset{+}{\Phi}$, respectively $\{\varphi_{k,\varepsilon}\}\in\Phi$. But we did not stress this on the left-hand side of (18)—(21): it will be shown later on that all the norms (18) (respectively (19), (20), or (21)) for different choices of systems belonging to $\overset{+}{\Phi}$ are mutually equivalent and so, from our point of view here, are not essentially different.

Remark 2.3/3. For $f\in Z'$ the expression $F^{-1}\varphi_{k,\varepsilon}Ff$ is meaningful. By (11) we have $Ff\in\overset{+}{D}'$. Hence, by an appropriate interpretation, $\varphi_{k,\varepsilon}Ff\in S'$ and so $F^{-1}\varphi_{k,\varepsilon}Ff\in S'$. Furthermore, by the Paley—Wiener—Schwartz theorem, all the distributions $F^{-1}\varphi_{k,\varepsilon}Ff$ are entire analytic functions of exponential type.

2.4. Multipliers in L_p

For the convenience of the reader we formulate here two multiplier theorems in L_p which will be useful in later considerations.

Theorem 2.4/1. *Let $1<p<\infty$ and $1<q<\infty$. Let μ_j be complex measures in R_n $(j=1,2,\ldots)$ with bounded variations:*

$$(22)\qquad \operatorname{Var}\mu_j=|\mu_j|(R_n)\leqq B,$$

where B is independent of j. Let

$$(23)\qquad m_j(x)=\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_n}d\mu_j=\mu_j(\{y:\ -\infty<y_l<x_l;\ \ l=1,\ldots,n\}).$$

Then there exists a positive number c depending only on p, q, and n such that for all systems $\{f_j\}_{j=1}^{\infty}, f_j \in S$,

(24) $f_j(x) \equiv 0$ *with the exception of a finite number of these functions,*

$$\left\| \left(\sum_{j=1}^{\infty} |(F^{-1} m_j Ff_j)(x)|^q \right)^{1/q} \right\|_{L_p} \leqq cB \left\| \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{1/q} \right\|_{L_p} \tag{25}$$

holds.

Remark 2.4/1. The case $n=1$ is due to SCHWARTZ [11, Lemma 11] (see also [1, 11.11, Lemma 24]). The general case, inclusively the above formulation, is due to LIZORKIN [6, p. 241]. (25) is meaningful: $Ff_j \in S$, $m_j Ff \in S'$, and hence $F^{-1} m_j Ff_j \in S'$. By (25) $F^{-1} m_j Ff_j$ belongs to L_p considered as a subspace of S'. This inequality can be extended by completion arguments to systems of functions $\{f_j\}_{j=1}^{\infty}$ for which the right-hand side of (25) is finite.

Examples. 1. Let $\{\varphi_{k,\varepsilon}\}_{k \in N_n, \varepsilon \in E} \in \overset{+}{\Phi}$. Let

$$d\mu_{k,\varepsilon} = \frac{\partial^n \varphi_{k,\varepsilon}}{\partial x_1 \dots \partial x_n} dx.$$

Then

$$\varphi_{k,\varepsilon}(x) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} d\mu_{k,\varepsilon}$$

and

$$|\mu_{k,\varepsilon}|(R_n) \leqq c \int_{R_n} \left| \frac{\partial^n \varphi_{k,\varepsilon}}{\partial x_1 \dots \partial x_n} \right| dx \leqq c' \int_{Q_{k,\varepsilon}^{(L)}} 2^{-k_1 - \dots - k_n} dx \leqq c'',$$

c'' is independent of $k \in N_n$ and $\varepsilon \in E$. Hence, after introducing an appropriate new enumeration, the functions m_j in (25) can be identified with $\varphi_{k,\varepsilon}$. In the same way it follows that the system $\{\varphi_{k,\varepsilon}\}_{k \in N_n^*, \varepsilon \in E} \in \Phi$ satisfies also the hypotheses of the theorem.

2. Let $Q = \{x: 0 < x_j \leqq 1;\ j = 1, \dots, n\}$ be the standard cube. Let $\delta_{(\lambda_1, \dots, \lambda_n)}$ be the Dirac-measure with respect to the point $(\lambda_1, \dots, \lambda_n)$ where λ_j is either 0 or 1. If

$$\mu = \sum_{(\lambda_1, \dots, \lambda_n)} (-1)^{\lambda_1 + \dots + \lambda_n} \delta_{(\lambda_1, \dots, \lambda_n)},$$

then $|\mu|(R_n) = 2^n$ and

$$m(x) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} d\mu = \chi(x),$$

the characteristic function of Q. The arguments hold true for an arbitrary parallelepiped in R_n with edges parallel to the axes. Hence Theorem 2.4/1 holds true if

$m_j(x)=\chi_j(x)$ is an arbitrary set of characteristic functions of parallelepipeds with edges parallel to the axes.

Theorem 2.4/2. *Let* $m_{j,l}(x)$ $(j, l=1, 2, \ldots)$ *be n times continuously differentiable functions defined in* $\overset{+}{R}_n$. *Let*

$$\left|\prod_{r=1}^{n} x_r^{\gamma_r}\right| \left(\sum_{j,l=1}^{\infty} |D^\gamma m_{j,l}(x)|^2\right)^{1/2} \leqq B \quad \text{for} \quad x \in \overset{+}{R}_n \tag{26}$$

and for $\gamma=(\gamma_1, \ldots, \gamma_n)$, γ_r *is either* 0 *or* 1. *If* $1<p<\infty$, *then there exists a positive number c, depending only on p and n, such that for all systems* $\{f_j\}_{j=1}^{\infty}$, $f_j \in S$, *satisfying* (24),

$$\left\|\left(\sum_{j=1}^{\infty} |(F^{-1} \sum_{l=1}^{\infty} m_{j,l} Ff_l)(x)|^2\right)^{1/2}\right\|_{L_p} \leqq cB \left\|\left(\sum_{j=1}^{\infty} |f_j(x)|^2\right)^{1/2}\right\|_{L_p} \tag{27}$$

holds.

Remark 2.4/2. This is essentially the Hilbert space version of Marcinkiewicz's multiplier theorem. We refer to [13, IV. 6] and [7].

3. Fundamental properties

3.1. Independence, completeness

The aim of this subsection is to show that the spaces defined in 2.3 are independent of the choice of the systems belonging to $\overset{+}{\Phi}$, respectively Φ, and that all these spaces are Banach spaces.

Theorem 3.1/1. *The spaces* $\overset{+}{B}{}^{g(x)}_{p,q}$ *(where* $g \in \overset{+}{M}$, $1<p<\infty$, *and* $1 \leqq q \leqq \infty$*) and the spaces* $\overset{+}{F}{}^{g(x)}_{p,q}$ *(where* $g \in \overset{+}{M}$, $1<p<\infty$, *and* $1<q<\infty$*) from Definition* 2.3/3 *are Banach spaces. They are independent of the choice of* $\{\varphi_{k,\varepsilon}\}_{k \in N_n, \varepsilon \in E} \in \overset{+}{\Phi}$ *(equivalent norms).*

Theorem 3.1/2. *The spaces* $B^{g(x)}_{p,q}$ *(where* $g \in M$, $1<p<\infty$, *and* $1 \leqq q \leqq \infty$*), and the spaces* $F^{g(x)}_{p,q}$ *(where* $g \in M$, $1<p<\infty$ *and* $1<q<\infty$*) from Definition* 3.3/4 *are Banach spaces. They are independent of the choice of* $\{\varphi_{k,\varepsilon}\}_{k \in N_n^*, \varepsilon \in E} \in \Phi$ *(equivalent norms).*

Proof. We shall prove the first theorem, the second one may be proved in the same way.

Step 1. Let $\{\varphi_{k,\varepsilon}\}\in\overset{+}{\Phi}$ and $\{\psi_{k,\varepsilon}\}\in\overset{+}{\Phi}$. Assume that the spaces $\overset{+}{B}{}^{g(x)}_{p,q}$ and $\overset{+}{F}{}^{g(x)}_{p,q}$ are defined by the first system. Let $f\in\overset{+}{F}{}^{g(x)}_{p,q}$. We have

$$(\psi_{k,\varepsilon}Ff)(x)=\frac{\psi_{k,\varepsilon}(x)}{\sum\limits_{l\in N_n}\varphi_{l,\varepsilon}(x)}F\Big[\sum_{r\in N_n}F^{-1}\varphi_{r,\varepsilon}\cdot Ff\Big](x).$$

Here $\varepsilon\in E$ is fixed, the summation $\sum\limits_{r\in N_n}$ can be restricted to a finite number of terms, $|r-k|\leqq A$, independent of k and ε. Example 1 to Theorem 2.4/1 shows that the system $\{\psi_{k,\varepsilon}(\sum\limits_{l\in N_n}\varphi_{l,\varepsilon})^{-1}\}_{k\in N_n,\varepsilon\in E}$ satisfies the hypotheses of Theorem 2.4/1. Furthermore, we shall use the fact that $g(x^{k,\varepsilon})/g(x^{r,\varepsilon})$, where $|r-k|\leqq A$, can be estimated from below and from above by positive constants, independent of k, r, and ε. Now it follows from Theorem 2.4/1 that

$$\Big\|\Big(\sum_{k\in N_n,\varepsilon\in E}|(g(x^{k,\varepsilon})(F^{-1}\psi_{k,\varepsilon}Ff)(x)|^q\Big)^{1/q}\Big\|_{L_p}\leqq$$
$$\leqq c\,\Big\|\Big(\sum_{k\in N_n,\varepsilon\in E}|g(x^{k,\varepsilon})(F^{-1}\varphi_{k,\varepsilon}Ff)(x)|^q\Big)^{1/q}\Big\|_{L_p}.$$

This proves the independence of $\overset{+}{F}{}^{g(x)}_{p,q}$ from the choice of $\{\varphi_{k,\varepsilon}\}\in\overset{+}{\Phi}$. For the proof of the corresponding assertion for the spaces $\overset{+}{B}{}^{g(x)}_{p,q}$ one needs only the scalar case of Theorem 2.4/1.

Step 2. It is not hard to see that $\|f\|_{\overset{+}{B}{}^{g(x)}_{p,q}}$ and $\|f\|_{\overset{+}{F}{}^{g(x)}_{p,q}}$ are norms. ($\|f\|=0$ if and only if $f=0$, is also a consequence of the considerations below). Let us prove the completeness. Let $\{\varphi_{k,\varepsilon}\}\in\overset{+}{\Phi}$ be such that

(28) $$\sum_{k\in N_n,\varepsilon\in E}\varphi_{k,\varepsilon}(x)=1\quad\text{for}\quad x\in\overset{+}{R}_n$$

(Remark 2.3/1 ensures the existence of such a system), and let $\{\psi_{k,\varepsilon}\}\in\overset{+}{\Phi}$ be such that

$$\psi_{k,\varepsilon}(x)=1\quad\text{for}\quad x\in\operatorname{supp}\varphi_{k,\varepsilon}.$$

Further, $L_p(l_q)$ denotes the vector-valued L_p-space consisting of all functions $v(x)$, defined in R_n, with values in l_q such that $\|v(x)\|_{l_p}\in L_p$. The operator $\mathcal{S}$, defined by

(29) $$\mathcal{S}f=\{g(x^{k,\varepsilon})F^{-1}\varphi_{k,\varepsilon}Ff\}_{k\in N_n,\varepsilon\in E},$$

gives a linear and bounded map from $\overset{+}{F}{}^{g(x)}_{p,q}$ into $L_p(l_q)$. The operator $\mathcal{R}$, defined by

(30) $$\mathcal{R}\{h_{k,\varepsilon}\}=\sum_{k\in N_n,\varepsilon\in E}g^{-1}(x^{k,\varepsilon})F^{-1}\psi_{k,\varepsilon}Fh_{k,\varepsilon},$$

gives a linear and bounded map from $L_p(l_p)$ into $\overset{+}{F}{}^{g(x)}_{p,q}$:

(i) The convergence in (30) must be understood in the sense of Z';

(ii) One must again use Theorem 2.4/1 in the same manner as above.

If $f \in \overset{+}{F}{}^{g(x)}_{p,q}$, then it follows

$$\mathscr{R}\mathscr{S}f = \sum_{k \in N_n, \varepsilon \in E} F^{-1}\varphi_{k,\varepsilon}(x) Ff = f$$

(convergence and equality in Z'). Furthermore,

$$(\mathscr{S}\mathscr{R})^2 = \mathscr{S}\mathscr{R}\mathscr{S}\mathscr{R} = \mathscr{S}\mathscr{R}$$

is a projection in $L_p(l_q)$. In particular, the image of $\mathscr{S}\mathscr{R}$, denoted by $I(\mathscr{S}\mathscr{R})$, is a complemented subspace of $L_p(l_q)$. It follows that $\mathscr{S}$ is a one-to-one map from $\overset{+}{F}{}^{g(x)}_{p,q}$ onto $I(\mathscr{S}\mathscr{R})$. Since $I(\mathscr{S}\mathscr{R})$ is a Banach space, so is $\overset{+}{F}{}^{g(x)}_{p\,q}$. In the same way one proves that $\mathscr{S}$ is an isomorphic map from $\overset{+}{B}{}^{g(x)}_{p,q}$ onto a complemented subspace of $l_q(L_p)$.

3.2. Density

Theorem 3.2/1. *Let $g \in \overset{+}{M}$ and $1 < p < \infty$.*

(i) *If $1 \leqq q < \infty$, then Z is dense in $\overset{+}{B}{}^{g(x)}_{p,q}$.*

(ii) *If $1 < q < \infty$, then Z is dense in $\overset{+}{F}{}^{g(x)}_{p,q}$.*

Theorem 3.2/2. *Let $g \in M$, $1 < p < \infty$, and $FD = \{f : Ff \in D\}$.*

(i) *If $1 \leqq q < \infty$, then FD is dense in $B^{g(x)}_{p,q}$.*

(ii) *If $1 < q < \infty$, then FD is dense in $F^{g(x)}_{p,q}$.*

Proof. Prove the first theorem (the proof of the second one is essentially the same). Let $f \in \overset{+}{F}{}^{g(x)}_{p,q}$. Let $\{\varphi_{k,\varepsilon}\} \in \overset{+}{\Phi}$. Assume that (28) is statisfied. Let

$$\chi_l(x) = \sum_{|k| \leqq l, \varepsilon \in E} \varphi_{k,\varepsilon}(x) \quad (l = 1, 2, \ldots). \tag{31}$$

Let $\{\psi_{k,\varepsilon}\} \in \overset{+}{\Phi}$ be an arbitrary system. Then it follows from Theorem 2.4/1 (and Example 1 to this theorem) that

$$\|f - F^{-1}\chi_l Ff\|_{\overset{+}{F}{}^{g(x)}_{p,q}} = \Big\|\Big(\sum_{k \in N_n, \varepsilon \in E} |g(x^{k,\varepsilon})(F^{-1}(1-\chi_l)\psi_{k,\varepsilon}Ff)(x)|^q\Big)^{1/q}\Big\|_{L_p} \leqq$$

$$\leqq c\,\Big\|\Big(\sum_{|k| \geqq l', \varepsilon \in E} |g(x^{k,\varepsilon})(F^{-1}\psi_{k,\varepsilon}Ff)(x)|^q\Big)^{1/q}\Big\|_{L_p}.$$

Here $l' \to \infty$ for $l \to \infty$. But by Lebesgue's convergence theorem the right-hand side of the last inequality tends to zero for $l' \to \infty$. This proves that $F^{-1}\chi_l Ff$ approximates f. The same arguments may be applied to $\overset{+}{B}{}^{g(x)}_{p,q}$ provided that $q < \infty$. Hence, in both cases it will be sufficient to approximate a function $f \in L_p$ where $\operatorname{supp} Ff$ is a compact subset of $\overset{+}{R}_n$ in L_p. Let $\eta \in S$ be such that

$$\eta(0) = 1, \quad \operatorname{supp} F\eta \subset \{y : |y| \leqq 1\}.$$

Let $\eta_h(x) = \eta(hx)$, $h > 0$. Then $f_h(x) = \eta(hx) f(x)$ has the desired properties:
(i) $f_h(x) \to f(x)$ in L_p for $h \downarrow 0$;
(ii) $Ff_h = Ff * h^{-n}(F\eta)(x/h)$, and hence

$$\operatorname{supp} Ff_h \subset \operatorname{supp} Ff + \{y : |y| \leqq h\} \subset \overset{+}{R}_n,$$

provided h is small enough;
(iii) $Ff_h \in \overset{+}{D}$, provided h is small enough.

Remark 3.2/1. The last part of the proof coincides essentially with the statement in [16, II, Theorem 4.2]. But to be self-contained, we included the above proof here.

Remark 3.2/2. It is possible to show that not only FD, but also S is contained in all spaces $B^{g(x)}_{p,q}$, where $1 \leqq q \leqq \infty$, and in all spaces $F^{g(x)}_{p,q}$, where $1 < q < \infty$. (Obviously, $FD \subset S$). This will be an easy consequence of the inclusion properties proved later (see Remark 6.1/1).

3.3. Comparison

Here are considered two types of spaces: homogeneous spaces (Definition 2.3/3) and non-homogeneous spaces (Definition 2.3/4). The question arises whether these two types of spaces can be compared. Furthermore, what are the relations between the "classical" spaces H^s_p (Lebesgue spaces = Bessel potential spaces = Liouville spaces), $B^s_{p,q}$ (Besov spaces = Lipschitz spaces), and the spaces $F^s_{p,q}$ introduced in [14] and the spaces considered here. From the view-point of Section 5 and Subsection 6.2 it will be clear that of peculiar interest is a comparison between H^s_p, $H^{g(x)}_p$, and $\overset{+}{H}{}^{g(x)}_p$. Recall the definition of H^s_p: If $-\infty < s < \infty$ and $1 < p < \infty$, then

$$H^s_p = \{f : f \in S', \quad \|f\|_{H^s_p} = \|F^{-1}(1+|x|^2)^{s/2} Ff\|_{L_p} < \infty\}. \tag{32}$$

If $m = s = 1, 2, \ldots$, then $H^s_p = W^m_p$ are the usual Sobolev spaces, $H^0_p = L_p$ (see [15, 2.3.3]).

But a serious problem arises. The continuous embedding $Z \subset S$ holds if both spaces are equipped with their natural topology, but Z is not dense in S (this is an easy consequence of the fact that $\overset{+}{D}$ is not dense in S). So it is not possible to compare the dual spaces S' and Z' on the basis of the usual procedure (that is, if $A \subset B$

is a densely continuous embedding, then $f \in B'$ can be interpreted, by restriction to A, as an element of A'. This is a one-to-one correspondence. Hence $B' \subset A'$). We describe here a possibility how to overcome this difficulty and give examples. A discussion of the questions described above will be given later in Subsections 5.2 and 6.2.

Let A be a locally convex space (for our purpose here it would be sufficient to assume that A is a Banach space). Let

$$S \subset A, \quad Z \subset A \tag{33}$$

be dense and continuous embeddings. Then it is meaningful to interpret the dual space of A either as a subspace of S', denoted by A'_S, or as a subspace of Z', denoted by A'_Z. Because Z is dense in S in the topology of A, there is a natural one-to-one correspondence between the elements of A'_S and the elements of A'_Z. In this case spaces A'_S and A'_Z can be identified: $A'_S = A'_Z = A'$. In other words: there is no essential difference, if A' is considered as a subspace of S' or as a subspace of Z'. If this situation happens then we write: $A' \in \Pi$. The notation $B \in \Pi$ means that there is a space A with the above properties such that $A' = B$.

Theorem 3.3. *If $-\infty < s < \infty$ and $1 < p < \infty$, then $H^s_p \in \Pi$.*

Proof. Let $A = L_q$ where $1 < q < \infty$, or $A = C$, the space of all complex-valued continuous functions, defined in R_n, vanishing at infinity. Then (33) is satisfied and S is dense in A. We must show that Z is also dense in A. Let $\chi_l(x)$ be the function defined in (31). If $f \in S$ then $F^{-1}\chi_l Ff \in Z$. We have

$$\|f - F^{-1}\chi_l Ff\|_C \leqq \int_{R_n} |1-\chi_l(x)|\,|(Ff)(x)|\,dx \to 0 \tag{34}$$

for $l \to \infty$. Hence $A = C$ satisfies the above hypotheses. Let $1 < r < q < \infty$. Then

$$\|f - F^{-1}\chi_l Ff\|_{L_q} \leqq \|f - F^{-1}\chi_l Ff\|_C^{1-r/q}\, \|f - F^{-1}\chi_l Ff\|_{L_r}^{r/q}. \tag{35}$$

By Marcinkiewicz's multiplier theorem (the scalar case of Theorem 2.4/1 and the first example to this theorem)

$$\|F^{-1}\chi_l Ff\|_{L_r} \leqq c\|f\|_{L_r}$$

holds, where c is independent of l. Using this fact, it follows from (34) and (35) that the left-hand side of (35) tends to zero for $l \to \infty$. This proves that also $A = L_q$, where $1 < q < \infty$, satisfies the above hypotheses. Hence $A' \in \Pi$. By the above interpretation $A' = (L_q)' = L_p$, provided that $1 < p < \infty$ and $1/p + 1/q = 1$. If $A = H_q^{-s}$, where $1 < q < \infty$, then again S is dense in H_q^{-s}. If $f \in S$, then

$$\|f - F^{-1}\chi_l Ff\|_{H_q^{-s}} = \|g - F^{-1}\chi_l Fg\|_{L_q}, \quad g = F^{-1}(1+|x|^2)^{-s/2} Ff \in S.$$

Hence, by the above arguments, Z is dense in H_q^{-s}. This shows that $(H_q^{-s})'\in\Pi$. By the above interpretation $(H_q^{-s})'=H_p^s$, provided $1/p+1/q=1$ (see [15, 2.6.1]).

Remark 3.3/1. The proof shows that $C'\in\Pi$ holds. However, the dual space to C, by the above interpretation, coincides with the space of all complex Radon measures with finite variation (Riesz's representation theorem).

Remark 3.3/2. The theorem can be generalized in several directions. Let $B_{p,q}^s$ be the usual isotropic Besov spaces, where $-\infty<s<\infty$, $1<p<\infty$, $1\leqq q\leqq\infty$, which can be defined by

$$B_{p,q}^s=(H_p^{s_0},H_p^{s_1})_{\theta,q},\quad s_0<s<s_1,\quad s=(1-\theta)s_0+\theta s_1.$$

(Here $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation method, see [15].) We have the dense and continuous embedding

$$S\subset H_p^{s+\varepsilon}\subset B_{p,q}^s,\quad q<\infty.$$

Since Z is dense in $H_p^{s+\varepsilon}$, Z is also dense in $B_{p,q}^s$. Hence, the above hypotheses for $A=B_{p,q}^s$ are satisfied. We have $A'=(B_{p,q}^s)'=B_{p',q'}^{-s}$ [15, 2.6.1]. Consequently,

$$B_{p,q}^s\in\Pi \text{ for } -\infty<s<\infty,\ 1<p<\infty,\quad\text{and}\quad 1<q\leqq\infty.$$

This assertion holds true also for $q=1$ (here one must use $A=\mathring{B}_{p,\infty}^s$ [15, 2.6.1]) and for the spaces $F_{p,q}^s$ ([14] or [15, 2.6.2]).

Remark 3.3/3. It is not very hard to see that the theorem cannot be extended to L_∞. If L_∞ is interpreted as the dual space of L_1, then $L_\infty\notin\Pi$. Obviously, (33) holds true for L_1. So we must show that Z is not dense in L_1. Let $\varphi\in S$ be such that $(F\varphi)(0)=1$ and let $\psi\in Z$. In particular $(F\psi)(0)=0$. Hence

$$1=|(F\varphi)(0)-(F\psi)(0)|\leqq\int_{R_n}|\varphi(x)-\psi(x)|\,dx=\|\varphi-\psi\|_{L_1}.$$

Hence φ cannot be approximated in L_1 by functions belonging to Z.

3.4. Translation invariance

Let $h\in R_n$. If $f\in Z'$ (or $f\in S'$), then T_hf is defined by

$$(T_hf)(\varphi(x))=f(\varphi(x-h))$$

(translation operator). Here $\varphi\in Z$ (or $\varphi\in S$). It is not hard to see that $\varphi(x)\to\varphi(x-h)$ is an isomorphic map from S onto S and from Z onto Z. Hence T_h is an isomorphic map from S' onto S' and from Z' onto Z'.

Theorem 3.4/1. *Let* $g(x)\in \overset{+}{M}$ *and* $1<p<\infty$.

(i) *If* $1\leqq q\leqq\infty$, *then* T_h *is an isomorphic map from* $\overset{+}{B}{}^{g(x)}_{p,q}$ *onto* $\overset{+}{B}{}^{g(x)}_{p,q}$.

(ii) *If* $1<q<\infty$, *then* T_h *is an isomorphic map from* $\overset{+}{F}{}^{g(x)}_{p,q}$ *onto* $\overset{+}{F}{}^{g(x)}_{p,q}$.

Theorem 3.4/2. *Theorem* 3.4/1 *remains valid if one replaces* $\overset{+}{M}$ *by* M, $\overset{+}{B}$ *by* B, *and* $\overset{+}{F}$ *by* F.

Proof. If $f\in Z'$ (or $f\in S'$), then

$$FT_h f = e^{ixh} Ff,$$

where xh is the scalar product in R_n. If f belongs to one of the above spaces, then both theorems follow from the equalities

$$(F^{-1}\varphi_{k,\varepsilon}FT_h f)(x) = \big(F^{-1}\varphi_{k,\varepsilon}(y)e^{iyh}Ff\big)(x) = (F^{-1}\varphi_{k,\varepsilon}Ff)(x+h)$$

and from the definition of these spaces.

4. Interpolation

4.1. General interpolation formulas

If $1<p<\infty$ and $1\leqq q\leqq\infty$, then the spaces $L_p(l_q^g)$ and $l_q^g(L_p)$ are defined below either by (i) or by (ii), depending as we are concerned with homogeneous spaces or with non-homogeneous spaces.

(i) Let $g(x)\in\overset{+}{M}$. Then $L_p(l_q^g)$ is the space of all sequences $\{a_{k,\varepsilon}(x)\}_{k\in N_n,\varepsilon\in E}$ such that

$$\|a_{k,\varepsilon}\|_{L_p(l_q^g)} = \Big\|\Big(\sum_{k\in N_n,\varepsilon\in E}|g(x^{k,\varepsilon})a_{k,\varepsilon}(x)|^q\Big)^{1/q}\Big\|_{L_p}<\infty.$$

Here $x^{k,\varepsilon}$ has the same meaning as in Definition 2.3/3. $l_q^g(L_p)$ is the space of all sequences $\{a_{k,\varepsilon}(x)\}_{k\in N_n,\varepsilon\in E}$ such that

$$\|a_{k,\varepsilon}\|_{l_q^g(L_p)} = \Big(\sum_{k\in N_n,\varepsilon\in E}\|g(x^{k,\varepsilon})a_{k,\varepsilon}(x)\|^q_{L_p}\Big)^{1/q}<\infty.$$

For $q=\infty$ one must modify in the usual way.

(ii) Let $g(x)\in M$. Then $L_p(l_q^g)$ and $l_q^g(L_p)$ are defined for sequences $\{a_{k,\varepsilon}(x)\}_{k\in N_n^*,\varepsilon\in E}$ in the same way: N_n is replaced by N_n^*, and $x^{k,\varepsilon}$ has the meaning of Definition 2.3/4.

Furthermore, we shall use the notation of interpolation theory [15, Chapter 1]. Although the theorem below is formulated for an arbitrary interpolation functor Ψ, one may assume that Ψ is either the complex interpolation functor $[\cdot,\cdot]_\theta$ or the real interpolation functor $(\cdot,\cdot)_{\theta,r}$, where $0<\theta<1$ and $1\leqq r\leqq\infty$.

Theorem 4.1/1. *Let* $g_0(x)\in \overset{+}{M}$, $g_1(x)\in \overset{+}{M}$, $1<p_0<\infty$, *and* $1<p_1<\infty$. *Let* Ψ *be an arbitrary interpolation functor.*

(i) *If* $1\leqq q_0\leqq\infty$ *and* $1\leqq q_1\leqq\infty$, *then*

$$\Psi(\overset{+}{B}{}^{g_0(x)}_{p_0,q_0}, \overset{+}{B}{}^{g_1(x)}_{p_1,q_1}) = \{f\colon f\in Z',\quad \|F^{-1}\varphi_{k,\varepsilon}Ff\|_{\Psi(l^{g_0}_{q_0}(L_{p_0}),\, l^{g_1}_{q_1}(L_{p_1}))}<\infty\}. \tag{36}$$

(ii) *If* $1<q_0<\infty$ *and* $1<q_1<\infty$, *then*

$$\Psi(\overset{+}{F}{}^{g_0(x)}_{p_0,q_0}, \overset{+}{F}{}^{g_1(x)}_{p_1,q_1}) = \{f\colon f\in Z',\quad \|F^{-1}\varphi_{k,\varepsilon}Ff\|_{\Psi(L_{p_0}(l^{g_0}_{q_0}),\, L_{p_1}(l^{g_1}_{q_1}))}<\infty\}. \tag{37}$$

Theorem 4.1/2. *Let* $g_0(x)\in M$, $g_1(x)\in M$, $1<p_0<\infty$, and $1<p_1<\infty$. *Let* Ψ *be an arbitrary interpolation functor.*

(i) *If* $1\leqq q_0\leqq\infty$ *and* $1\leqq q_1\leqq\infty$, *then*

$$\Psi(B^{g_0(x)}_{p_0,q_0}, B^{g_1(x)}_{p_1,q_1}) = \{f\colon f\in S',\quad \|F^{-1}\varphi_{k,\varepsilon}Ff\|_{\Psi(l^{g_0}_{q_0}(L_{p_0}),\, l^{g_1}_{q_1}(L_{p_1}))}<\infty\}. \tag{38}$$

(ii) *If* $1<q_0<\infty$ *and* $1<q_1<\infty$, *then*

$$\Psi(F^{g_0(x)}_{p_0,q_0}, F^{g_1(x)}_{p_1,q_1}) = \{f\colon f\in S',\quad \|F^{-1}\varphi_{k,\varepsilon}Ff\|_{\Psi(L_{p_0}(l^{g_0}_{q_0}),\, L_{p_1}(l^{g_1}_{q_1}))}<\infty\}. \tag{39}$$

Proof. Let us prove the first theorem. (The proof of the second one is the same.) Modify the operators $\mathscr{S}$ and $\mathscr{R}$ from (29) and (30):

$$\mathscr{S}f = \{F^{-1}\varphi_{k,\varepsilon}Ff\}_{k\in N_n,\, \varepsilon\in E},\quad \mathscr{R}\{h_{k,\varepsilon}\} = \sum_{k\in N_n,\, \varepsilon\in E} F^{-1}\psi_{k,\varepsilon}Fh_{k,\varepsilon}. \tag{40}$$

By the same arguments as in the second step to the proof of Theorem 3.1/1 it follows that $\mathscr{R}\mathscr{S}f=f$. Furthermore, $\mathscr{S}$ is an isomorphic map from $\overset{+}{F}{}^{g_0(x)}_{p_0,q_0}$ onto a complemented subspace of $L_{p_0}(l^{g_0}_{q_0})$. Corresponding assertions hold true for $\overset{+}{F}{}^{g_1(x)}_{p_1,q_1}$, $\overset{+}{B}{}^{g_0(x)}_{p_0,q_0}$, and $\overset{+}{B}{}^{g_1(x)}_{p_1,q_1}$. By the interpolation property, this remains valid for all interpolation spaces obtained from these spaces. (36) and (37) are examples of this general statement.

4.2. *Concrete interpolation formulas*

The last two theorems show that each concrete interpolation theorem for the spaces $L_p(l^g_q)$ and $l^g_q(L_p)$ gives a corresponding interpolation theorem for the B-spaces and F-spaces.

Theorem 4.2/1. *Let* $g_0(x)\in \overset{+}{M}$, $g_1(x)\in \overset{+}{M}$, $1<p_0<\infty$, *and* $1<p_1<\infty$. *Let*

$$g(x) = g_0^{1-\theta}(x)\, g_1^{\theta}(x),\quad \frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \tag{41}$$

where $0<\theta<1$.

(i) *If* $1 \leqq q_0 < \infty$, $1 \leqq q_1 < \infty$, *and*

(42) $$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then

(43) $$[\overset{+}{B}{}^{g_0(x)}_{p_0,q_0}, \overset{+}{B}{}^{g_1(x)}_{p_1,q_1}]_\theta = \overset{+}{B}{}^{g(x)}_{p,q}.$$

(ii) *If* $1 < q_0 < \infty$, $1 < q_1 < \infty$, *and if q is determined by* (42), *then*

(44) $$[\overset{+}{F}{}^{g_0(x)}_{p_0,q_0}, \overset{+}{F}{}^{g_1(x)}_{p_1,q_1}]_\theta = \overset{+}{F}{}^{g(x)}_{p,q}.$$

(iii) *If* $1 \leqq q_0 < \infty$, $1 \leqq q_1 < \infty$, *if q is determined by* (42), *and if* $p = q$, *then*

(45) $$(\overset{+}{B}{}^{g_0(x)}_{p_0,q_0}, \overset{+}{B}{}^{g_1(x)}_{p_1,q_1})_{\theta,p} = \overset{+}{B}{}^{g(x)}_{p,p}.$$

Theorem 4.2/2. *Theorem* 4.2/1 *remains valid if one replaces* $\overset{+}{M}$ *by* M, $\overset{+}{B}$ *by* B, *and* $\overset{+}{F}$ *by* F.

Proof. Prove the first theorem. (36) and (37) show that the following formulas are needed:

(i) for the proof of (43)

(46) $$[l^{g_0}_{q_0}(L_{p_0}), l^{g_1}_{q_1}(L_{p_1})]_\theta = l^g_q(L_p),$$

(ii) for the proof of (44)

(47) $$[L_{p_0}(l^{g_0}_{q_0}), L_{p_1}(l^{g_1}_{q_1})]_\theta = L_p(l^g_q),$$

(iii) and for the proof of (45)

(48) $$(l^{g_0}_{q_0}(L_{p_0}), l^{g_1}_{q_1}(L_{p_1}))_{\theta,p} = l^q_p(L_p).$$

(46) and (48) are consequences of Theorem 1.18.1 in [15] (interpolation theorem for vector-valued l_q-spaces). (47) follows from Theorem 1.18.3 in [15] and

$$[l^{g_0}_{q_0}, l^{g_1}_{q_1}]_\theta = l^g_q.$$

The last formula can be also obtained from Theorem 1.18.1 in [15].

Remark 4.2/1. The most interesting formula seems to be the special case of (44), where $q_0 = q_1 = q = 2$:

(49) $$[\overset{+}{H}{}^{g_0(x)}_{p_0}, \overset{+}{H}{}^{g_1(x)}_{p_1}]_\theta = \overset{+}{H}{}^{g(x)}_{p}.$$

As will be seen later, this formula has some curious consequences mentioned in the introduction, see 5.2.

Remark 4.2/2. One can compare the above formulas with corresponding interpolation formulas for the isotropic Besov spaces $B^s_{p,q}$, Lebesgue spaces H^s_p, and the $F^s_{p,q}$-spaces, see [14] or [15, 2.4]. One of the most interesting formulas for these spaces is

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} = B^s_{p,q},$$

where $s_0 \neq s_1$, $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$, $1\leqq q_0, q_1, q \leqq \infty$. Here $B^{s_0}_{p,q_0}$ can be replaced by $H^{s_0}_p$ and/or $B^{s_1}_{p,q_1}$ by $H^{s_1}_p$. But as will be seen later (Remark 6.2/2) in general there is no counterpart of this formula for the spaces considered here.

Remark 4.2/3. The above theorems contain a large variety of special cases. For the homogeneous spaces:

(i) $g_0(x)=|x|^{s_0}$, $g_1(x)=|x|^{s_1}$, s_0 and s_1 real (isotropic homogeneous spaces),

(ii) $$g_0(x) = \prod_{j=1}^{n} |x_j|^{s_{0,j}}, \quad g_1(x) = \prod_{j=1}^{n} |x_j|^{s_{1,j}},$$

$s_{0,j}$ and $s_{1,j}$ real (homogeneous spaces with dominating mixed derivatives).

For the second case one may find corresponding interpolation formulas in GRISVARD [4, p. 180] and SPARR [12, pp. 302—306]. On the basis of the papers by SPARR [12] and FREITAG [2, 3] it seems to be possible to prove further interpolation theorems (and also non-interpolation theorems).

5. Spaces $H^{g(x)}_p$ and $\overset{+}{H}{}^{g(x)}_p$

5.1. Representations

By Definitions 2.3/3 and 2.3/4 the spaces $\overset{+}{H}{}^{g(x)}_p$ and $H^{g(x)}_p$ are special cases of the spaces $\overset{+}{F}{}^{g(x)}_{p,q}$ and $F^{g(x)}_{p,q}$, respectively. We give here a representation formula for these spaces in the sense of the Paley-Littlewood theorems.

Theorem 5.1/1. *Let $g(x)\in \overset{+}{M}$ and $1<p<\infty$. Then there exist two positive numbers c_1 and c_2 such that for all $f\in Z$*

(50) $$c_1\|f\|_{\overset{+}{F}{}^{g(x)}_{p,2}} \leqq \|F^{-1}g(x)Ff\|_{L_p} \leqq c_2\|f\|_{\overset{+}{F}{}^{g(x)}_{p,2}}.$$

Theorem 5.1/2. *Let $g(x)\in M$ and $1<p<\infty$. Then there exist two positive numbers c_1 and c_2 such that for all $f\in FD$* (defined in Theorem 3.2/2)

(51) $$c_1\|f\|_{F^{g(x)}_{p,2}} \leqq \|F^{-1}g(x)Ff\|_{L_p} \leqq c_2\|f\|_{F^{g(x)}_{p,2}}.$$

Proof. We shall prove the first theorem. The proof of the second one is the same.

Step 1. Let $\{\varphi_{k,\varepsilon}\}\in \overset{+}{\Phi}$ and $\{\psi_{k,\varepsilon}\}\in \overset{+}{\Phi}$ be the two systems described at the beginning of the second step in the proof of Theorem 3.1/1. Let $f\in Z$. Then it follows that

$$F^{-1}g(x)\,Ff = \sum_{k\in N_n,\,\varepsilon\in E} F^{-1}g(x)\,\varphi_{k,\varepsilon}(x)\,Ff =$$

$$= \sum_{k\in N_n,\,\varepsilon\in E} F^{-1}\left[\frac{g(x)}{g(x^{k,\varepsilon})}\,\psi_{k,\varepsilon}F\big(F^{-1}g(x^{k,\varepsilon})\,\varphi_{k,\varepsilon}Ff\big)\right].$$

Now one can apply Theorem 2.4/2. The functions $m_{1,l}(x)$ appearing there are identified with $\psi_{k,\varepsilon}(x)g(x)/g(x^{k,\varepsilon})$ (after an appropriate new enumeration). This is the first row of the matrix. The other rows are zeros. (27) yields

$$\|F^{-1}g(x)\,Ff\|_{L_p} \leqq c\,\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p(l_2^{\varepsilon})}$$

(in the notation of 4.1). This proves the right-hand side of (50).

Step 2. Suppose again that $f\in Z$. Let $\{m_{j,l}(x)\}$ be the matrix of Theorem 2.4/2, where the first column $m_{j,1}(x)$ is given by $\varphi_{k,\varepsilon}(x)g(x^{k,\varepsilon})/g(x)$ (after an appropriate new enumeration). The other columns are zeros. The vector $\{f_j\}$ appearing in Theorem 2.4/2 is identified here with the vector $(F^{-1}g(x)\,Ff, 0, 0, \ldots)$. Then Theorem 2.4/2 yields

$$\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p(l_2^{\varepsilon})} \leqq c\,\|F^{-1}g(x)\,Ff\|_{L_p}.$$

This proves the left-hand side of (50).

Remark 5.1. By Theorem 3.2/1, Z is dense in $\overset{+}{H}{}_p^{g(x)}$. Hence Theorem 5.1/1 can be extended by continuity to $\overset{+}{H}{}_p^{g(x)}$. Using Theorem 3.2/2 it follows that (51) can be extended by continuity to $H_p^{g(x)}$.

5.2. *Comparison theorems for H-spaces. Examples*

In Subsection 3.3 we described a method to identify some subspaces of S' with corresponding subspaces of Z'. The theorem below must be understood in this sense. We recall the definition of the classical Lebesgue spaces H_p^s, see (32).

Theorem 5.2/1. *If* $-\infty<s<\infty$ *and* $1<p<\infty$, *then*

$$H_p^s = H_p^{(1+|x|^2)^{s/2}} = \overset{+}{H}{}_p^{(1+|x|^2)^{s/2}}. \tag{52}$$

Proof. *Step* 1. It is well-known that S is dense in H_p^s ([15, Theorems 2.3.2(b) and 2.3.3(a)]). Here is a short direct proof. If $f\in H_p^s$, then $F^{-1}(1+|x|^2)^{s/2}Ff\in L_p$ can be approximated in L_p by $g\in S$. Then $F^{-1}(1+|x|^2)^{-s/2}Fg\in S$ yields the desired approximation in H_p^s. But FD is dense in S and consequently also in H_p^s. Hence by (32), Theorem 5.1/2, and Theorem 3.2/2 it follows that H_p^s and $H_p^{(1+|x|^2)^{s/2}}$ coincide on the dense subset FD. Completion yields the first equality in (52).

Step 2. By Theorem 3.3, H_p^s belongs to the class Π. Since S is dense in H_p^s, Z is also dense in H_p^s. Then the second equality of (52) is a consequence of (32) and Theorem 5.1/1 (and Remark 5.1).

Remark 5.2/1. In particular, it follows that the usual spaces L_p, $1<p<\infty$, are special cases of the spaces considered here. The same holds true for the Sobolev spaces $W_p^m = H_p^m$, $1<p<\infty$, $m=1,2,\ldots$. But for the Besov spaces $B_{p,q}^s$ there arise some difficulties, see Subsection 6.2.

If $f\in Z'$ and α is an arbitrary multi-index, then $D^\alpha f\in Z'$ is defined in the usual way

$$(D^\alpha f)(\varphi) = (-1)^{|\alpha|} f(D^\alpha \varphi) \quad \text{for all} \quad \varphi\in Z.$$

Theorem 5.2/2. *Let* $1<p<\infty$. *If* $\alpha=(\alpha_1,\ldots,\alpha_n)$ *is a multi-index, then it follows that*

$$\overset{+}{H}{}_p^{\Pi|x_j|^{\alpha_j}} = \{f\colon f\in Z', D^\alpha f\in L_p\}, \tag{53}$$

and $\|D^\alpha f\|_{L_p}$ *is an equivalent norm in* $\overset{+}{H}{}_p^{\Pi|x_j|^{\alpha_j}}$.

Proof. *Step* 1. If $f\in Z$, then it follows that

$$\|D^\alpha f\|_{L_p} = \|F^{-1}\prod_{j=1}^n x_j^{\alpha_j} Ff\|_{L_p}. \tag{54}$$

By Marcinkiewicz's multiplier theorem it follows that $\Pi x_j^{\alpha_j}/\Pi|x_j|^{\alpha_j}$ and its inverse are multipliers in L_p. Hence, by (54),

$$\|D^\alpha f\|_{L_p} \quad \text{and} \quad \|F^{-1}\prod_{j=1}^n |x_j|^{\alpha_j} Ff\|_{L_p} \tag{55}$$

are equivalent norms on Z (this follows also from the simpler fact that the characteristic functions of $\{x\colon 0<\varepsilon_j x_j<\infty;\ j=1,\ldots,n\}$, $\varepsilon_j=\pm 1$, are multipliers in L_p). By Theorem 5.1/1 it follows that both the spaces in (53) coincide on Z. Since Z is dense in $\overset{+}{H}{}_p^{\Pi|x_j|^{\alpha_j}}$ one must prove that Z is also dense in the space on the right-hand side of (53).

Step 2. Prove the last assertion. We have in Z' that

$$D^\alpha f = F^{-1}FD^\alpha f = F^{-1}\Pi x_j^{\alpha_j} Ff.$$

If $\chi_l(x)$ has the same meaning as in (31) and if $D^\alpha f\in L_p$, then

$$D^\alpha(F^{-1}\chi_l Ff) = F^{-1}\chi_l F(F^{-1}\Pi x_j^{\alpha_j} Ff)\in L_p.$$

Repeating the arguments in the proof to Theorem 3.3 it follows that

$$\|D^\alpha f - D^\alpha(F^{-1}\chi_l Ff)\|_{L_p} \to 0 \quad \text{for} \quad l \to \infty.$$

On the other hand, $b = F^{-1}\chi_l Ff \in S'$, and $\operatorname{supp} Fb$ is compact in $\overset{+}{R}_n$. A small modification of the end of the proof to Theorem 3.2/2 shows that $D^\alpha b$ can be approximated in L_p by $D^\alpha u$, $u \in Z$. (If one uses Nikol'skiĭ's inequality $\|D^\alpha b\|_{L_p} \leqq c\|b\|_{L_p}$, then one can apply the arguments at the end of the proof to Theorem 3.2/2 without changes, provided that the supports of the Fourier transforms of all the approximating functions are in a small neighbourhood of $\operatorname{supp} Fb$. But this is ensured by the above procedure.) This proves the theorem.

Remark 5.2/2. Let us denote the space in (53) by $\overset{+}{W}{}_p^{D^\alpha}$ (this resembles the usual notation for the Sobolev spaces). The last theorem and (49) have some remarkable consequences. For instance

(56) $$[\overset{+}{W}{}_p^{\partial^2/\partial x_1^2}, \overset{+}{W}{}_p^{\partial^2/\partial x_2^2}]_{1/2} = \overset{+}{W}{}_p^{\partial^2/\partial x_1 \partial x_2}.$$

Obviously, the counterpart to (53) can be proved also for other special polynomials $P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ with constant coefficients. But (56) is only an example of a large variety of similar formulas. There are also other special cases which seem to be of interest, for instance the spaces $\overset{+}{H}{}_p^{|x|^\sigma}$, where σ is an arbitrary real number. If $-n<\sigma<0$, then (by an appropriate interpretation)

(57) $$\overset{+}{H}{}_p^{|x|^\sigma} = \{f: f \in Z', \quad \left\| \int_{R_n} \frac{f(y)}{|x-y|^{n+\sigma}}\, dy \right\|_{L_p} < \infty\}.$$

(Here we use $F|x|^\sigma = c\,|x|^{-\sigma-n}$ for $-n<\sigma<0$.)

6. Inclusions

6.1. Inclusion theorems

Theorem 6.1/1. *Let* $g(x) \in \overset{+}{M}$ *and* $1<p<\infty$.

(i) *If* $1<q<\infty$, *then*

(58) $$\overset{+}{B}{}^{g(x)}_{p,\min(p,q)} \subset \overset{+}{F}{}^{g(x)}_{p,q} \subset \overset{+}{B}{}^{g(x)}_{p,\max(p,q)} \quad (\textit{continuous embedding}).$$

(ii) *If* $1 \leqq q_1 < q_2 \leqq \infty$, *then*

(59) $$Z \subset \overset{+}{B}{}^{g(x)}_{p,q_1} \subset \overset{+}{B}{}^{g(x)}_{p,q_2} \subset Z' \quad (\textit{continuous embedding}).$$

A corresponding relation holds for the $\overset{+}{F}$*-spaces, where the values* $q_1=1$ *and* $q_2=\infty$ *are excluded.*

Theorem 6.1/2. *Let* $g(x)\in M$ *and* $1<p<\infty$.

(i) *If* $1<q<\infty$, *then*

(60) $$B^{g(x)}_{p,\min(p,q)} \subset F^{g(x)}_{p,q} \subset B^{g(x)}_{p,\max(p,q)} \quad \textit{(continuous embedding).}$$

(ii) *If* $1\leqq q_1<q_2\leqq\infty$, *then*

(61) $$S \subset B^{g(x)}_{p,q_1} \subset B^{g(x)}_{p,q_2} \subset S' \quad \textit{(continuous embedding).}$$

A corresponding relation holds for the F-spaces, where the values $q_1=1$ *and* $q_2=\infty$ *are excluded.*

Proof. *Step* 1. We shall prove (58) (the proof of (60) is the same). Let $a_{k,\varepsilon}=g(x^{k,\varepsilon})F^{-1}\varphi_{k,\varepsilon}Ff$. First assume that $1<q\leqq p<\infty$. Then (58) follows from

$$\|a_{k,\varepsilon}\|_{l_p(L_p)} \leqq \|a_{k,\varepsilon}\|_{L_p(l_q)} = \Big\|\sum_{k\in N_n,\varepsilon\in E}|a_{k,\varepsilon}|^q\Big\|^{1/q}_{L_{p/q}} \leqq$$

$$\leqq \Big(\sum_{k\in N_n,\varepsilon\in E}\||a_{k,\varepsilon}|^q\|_{L_{p/q}}\Big)^{1/q} = \|a_{k,\varepsilon}\|_{l_q(L_p)}.$$

If $1<p<q<\infty$, then (58) follows from

$$\|a_{k,\varepsilon}\|_{l_q(L_p)} = \Big\|\int_{R_n}|a_{k,\varepsilon}|^p\,dx\Big\|^{1/p}_{l_{q/p}} \leqq$$

$$\leqq \Big(\int_{R_n}\||a_{k,\varepsilon}|^p\|_{l_{q/p}}\,dx\Big)^{1/p} = \|a_{k,\varepsilon}\|_{L_p(l_q)} \leqq \|a_{k,\varepsilon}\|_{l_p(L_p)}.$$

Step 2. The middle parts of (59) and (61) follow from the monotony of the l_q-spaces.

Step 3. Prove the first inclusion in (59) (by appropriate changes a similar proof yields the first inclusion in (61)). By Step 2 we may assume that $q_1=1$. (By (58) this includes also a corresponding assertion for the $\overset{+}{F}$-spaces.) Let $f\in Z$. Using (9) it follows that

(62) $$\|f\|_{\overset{+}{B}{}^{g(x)}_{p,1}} \leqq c\sum_{k\in N_n,\varepsilon\in E}2^{s|k|}\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p},$$

where s is an appropriate positive number. We have

$$\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p} \leqq c\|(1+|x|^{2m})F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_\infty} = c\|F^{-1}(1+(-\Delta)^n)[\varphi_{k,\varepsilon}Ff]\|_{L_\infty} \leqq$$

(63) $$\leqq c'\|(1+(-\Delta)^n)[\varphi_{k,\varepsilon}Ff]\|_{L_1} \leqq c''2^{t|k|}\sup_{x\in Q^{(L)}_{k,\varepsilon}}\sum_{|\alpha|\leqq 2n}|D^\alpha(Ff)(x)|,$$

where again t is an appropriate positive number. ($Q^{(L)}_{k,\varepsilon}$ was defined in (12).) Putting (63) in (62) it follows

$$(64)\qquad \|f\|_{\overset{+}{B}{}^{g(x)}_{p,1}} \leqq c \sum_{k\in N_n,\varepsilon\in E} 2^{(s+t)|k|} \sup_{x\in Q^{(L)}_{k,\varepsilon}} \sum_{|\alpha|\leqq 2n} |D^\alpha(Ff)(x)|.$$

The norm on the right-hand side determines a neighbourhood in Z (more precisely, the set of all $f\in Z$ such that this norm is less than a given positive ε is a neighbourhood for the zero in Z).

Step 4. Prove the last inclusion in (59) (a similar argument yields the last inclusion in (61)). One may assume $q_2=\infty$. Use the two systems $\{\varphi_{k,\varepsilon}\}\in\overset{+}{\Phi}$ and $\{\psi_{k,\varepsilon}\}\in\overset{+}{\Phi}$ described in Step 2 of the proof to Theorem 3.1/1. If $f\in\overset{+}{B}{}^{g(x)}_{p,\infty}$ and $\varphi\in Z$, then it follows that

$$|f(\varphi)| = |\sum_{k\in N_n,\varepsilon\in E} F^{-1}\psi_{k,\varepsilon}Ff(F\varphi_{k,\varepsilon}F^{-1}\varphi)| \leqq$$

$$\leqq \sup_{k,\varepsilon}\|g(x^{k,\varepsilon})F^{-1}\psi_{k,\varepsilon}Ff\|_{L_p} \sum_{k,\varepsilon}\|g^{-1}(x^{k,\varepsilon})F\varphi_{k,\varepsilon}F^{-1}\varphi\|_{L_{p'}},$$

where $1/p+1/p'=1$. Hence

$$(65)\qquad |f(\varphi)| \leqq \|f\|_{\overset{+}{B}{}^{g(x)}_{p,\infty}} \|\varphi\|_{\overset{+}{B}{}^{g^{-1}(x)}_{p',1}}.$$

Now it follows from Step 3 that the embedding $\overset{+}{B}{}^{g(x)}_{p,\infty}\subset Z'$ is continuous (Z' is equipped with the strong topology).

Remark 6.1/1. By (60) and (61), S is contained in $B^{g(x)}_{p,q}$ and $F^{g(x)}_{p,q}$. This question was left open (see Remark 3.2/2).

Remark 6.1/2. Of peculiar interest is the embedding

$$(66)\qquad \overset{+}{B}{}^{g(x)}_{p,\min(2,p)} \subset \overset{+}{H}{}^{g(x)}_{p} \subset \overset{+}{B}{}^{g(x)}_{p,\max(2,p)}$$

and its counterpart for $H^{g(x)}_p$.

6.2. *Comparison theorems for B-spaces*

The question arises whether (52) can be carried over to the B-spaces. The answer is in the negative. The method developed below works for the non-homogeneous spaces and for the homogeneous spaces. However, we restrict ourselves here mainly to the non-homogeneous spaces.

First we must recall the following characterization of $B^s_{p,q}$, [14] or [15, 2.3]. Let

$$q_l = \{x\colon |x_j| < 2^l;\ j=1,\ldots,n\}\quad (l=0,1,2,\ldots).$$

Let $\{\eta_l\}_{l=0}^{\infty}$ be a system of functions belonging to D with the following properties:

(i) $\operatorname{supp}\eta_l \subset q_{l+2}-q_{l-1}$ for $l=1, 2, \ldots$, $\operatorname{supp}\eta_0 \subset q_1$;

(ii) for each multi-index γ there exists a positive number c_γ such that for all $l=0, 1, 2, \ldots$ and $x\in R_n$

$$|D^\gamma \eta_l(x)| \leqq c_\gamma 2^{-l|\gamma|}$$

holds;

(iii) there exist two positive numbers c and C such that for all $x\in R_n$

$$c \leqq \sum_{l=0}^{\infty} \eta_l(x) \leqq C.$$

Obviously, this is the counterpart to the system $\{\varphi_{k,\varepsilon}\}$ in Subsection 2.3. If $-\infty<s<\infty$, $1<p<\infty$, and $1\leqq q\leqq\infty$, then

$$B^s_{p,q}=\{f\colon f\in S', \|2^{ls}F^{-1}\eta_l Ff\|_{l_q(L_p)}<\infty\}.$$

For the proof of the fact that $B^s_{p,q}$ and $B^{(1+|x|^2)^{s/2}}_{p,q}$ are not equal we consider only the case $p=2$.

Theorem 6.2/1. *If* $-\infty<s<\infty$ *and* $q\neq 2$, *then*

$$B^s_{2,q}\neq B^{(1+|x|^2)^{s/2}}_{2,q}.$$

Proof. Let $L(l)$ be the number of parallelepipeds $P_{k,\varepsilon}$ (formula (3)) which are contained in $q_{l+1}-q_l$. We have $L(l)\to\infty$ as $l\to\infty$. Let $f\in S'$ be given by

$$(Ff)(x)=\begin{cases} a_{k,\varepsilon}(=\text{const}) & \text{for } x\in P_{k,\varepsilon}\subset q_{l+1}-q_l,\\ 0 & \text{otherwise.}\end{cases}$$

Then it follows (by an appropriate choice of η_l) that

$$\|f\|_{B^s_{2,q}}=2^{ls}\Big(\sum_{P_{k,\varepsilon}\subset q_{l+1}-q_l}|a_{k,\varepsilon}|^2|P_{k,\varepsilon}|\Big)^{1/2}. \tag{67}$$

On the other hand,

$$\|f\|_{B^{(1+|x|^2)^{s/2}}_{2,q}}\sim 2^{ls}\Big(\sum_{P_{k,\varepsilon}\subset q_{l+1}-q_l}|a_{k,\varepsilon}|^q|P_{k,\varepsilon}|^{q/2}\Big)^{1/2}. \tag{68}$$

(Here "$\sim$" means equivalence, and the corresponding constants are independent of l.) Since $a_{k,\varepsilon}$ can be chosen arbitrarily, it follows that the right-hand sides of (67) and (68) cannot be equivalent to each other $(q\neq 2,\ L(l)\to\infty$ as $l\to\infty)$.

Remark 6.2/1. The same argument yields

$$B^s_{2,q}\neq \overset{+}{B}{}^{(1+|x|^2)^{s/2}}_{2,q},\quad B^{(1+|x|^2)^{s/2}}_{2,q}\neq \overset{+}{B}{}^{(1+|x|^2)^{s/2}}_{2,q},$$

provided that $q\neq 2$.

Theorem 6.2/2. *Let* $-\infty<s<\infty$, $\delta>0$, $1<p<\infty$, *and* $1\leqq q\leqq\infty$. *Then*

(69) $$B^{s+\delta}_{p,q}\subset B^{(1+|x|^2)^{s/2}}_{p,1}\subset B^s_{p,1}\subset B^s_{p,\infty}\subset B^{(1+|x|^2)^{s/2}}_{p,\infty}\subset B^{s-\delta}_{p,q}$$

and

(70) $$B^{(1+|x|^2)^{s/2}}_{p,\min(2,p)}\subset B^s_{p,p}\subset B^{(1+|x|^2)^{s/2}}_{p,\max(2,p)}.$$

Proof. *Step* 1. We prove the first inclusion in (69). One obtains as an easy consequence of the definition of the B-spaces that

$$\|f\|_{B^{(1+|x|^2)^{s/2}}_{p,1}}\leqq c\|f\|_{B^{(1+|x|^2)^{(s+\delta')/2}}_{p,\infty}},\quad \delta'>0.$$

Hence

(71) $$B^{(1+|x|^2)^{(s+\delta')/2}}_{p,\infty}\subset B^{(1+|x|^2)^{s/2}}_{p,1}\qquad(\text{similarly, } B^{s+\delta'}_{p,\infty}\subset B^s_{p,1}).$$

Consequently, by (60) and (61) (the corresponding formulas for $B^s_{p,q}$ and H^s_q), and (52) it follows that

$$B^{s+\delta}_{p,q}\subset H^{s+\delta/2}_p=H^{(1+|x|^2)^{(2s+\delta)/4}}_p\subset B^{(1+|x|^2)^{(2s+\delta)/4}}_{p,\infty}\subset B^{(1+|x|^2)^{s/2}}_{p,1}.$$

In the same way one may prove the last inclusion in (69).

Step 2. The second inclusion follows from

$$\|F^{-1}\eta_l Ff\|_{L_p}\leqq c\sum_{P_{k,\varepsilon}\subset q_{l+2}-q_{l-1}}\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p}.$$

(Here we used again a multiplier theorem.) Similarly, it follows the last but one inclusion from

$$\sup_{P_{k,\varepsilon}\subset q_{l+2}-q_{l-1}}\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p}\leqq c\|F^{-1}\eta_l Ff\|_{L_p}.$$

Hence (69) is proved.

Step 3. Let us prove (70). Let $1<p\leqq 2$. Using a Paley—Littlewood theorem it follows for fixed l ($l=0,1,2,\ldots$) that

$$\|F^{-1}\eta_l Ff\|_{L_p}\leqq c\|F^{-1}\eta_l\varphi_{k,\varepsilon}Ff\|_{L_p(l_2)}\leqq c'\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p(l_2)}\leqq c''\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p(l_p)},$$

where $L_p(l_2)$ and $L_p(l_p)$ indicate summation with respect to k and ε over values such that $P_{k,\varepsilon}\subset q_{l+2}-q_{l-1}$. This proves the left-hand side of (70). The right-hand side follows from

(72) $$B^s_{p,p}\subset H^s_p=H^{(1+|x|^2)^{s/2}}_p\subset B^{(1+|x|^2)^{s/2}}_{p,2}.$$

(Here we used (52) and (66).) Let $2<p<\infty$. Then the left-hand side can be obtained similarly to (72). The right-hand side is a consequence of

$$\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p(l_p)}\leqq\|F^{-1}\varphi_{k,\varepsilon}Ff\|_{L_p(l_2)}\leqq c\|F^{-1}\eta_l Ff\|_{L_p},$$

where again $L_p(l_p)$ and $L_p(l_2)$ must be understood in the above way, for fixed l. This proves (70).

Remark 6.2/2. (69) has the following consequence: *In general, the two spaces*

$$(73)\qquad \left(B_{p,q_0}^{(1+|x|^2)^{s_0/2}},\ B_{p,q_1}^{(1+|x|^2)^{s_1/2}}\right)_{\theta,q} \quad and \quad B_{p,q}^{(1+|x|^2)^{s/2}},$$

where $s_0<s<s_1$, $s=(1-\theta)s_0+\theta s_1$, $1\leqq q_0, q_1, q\leqq\infty$, *do not coincide* (as one would expect in comparison with the spaces $B_{p,q}^s$, see Remark 4.2/1). Assume that the two spaces in (73) coincide for all values of the parameters. Then it follows from (69) and the reiteration theorem of interpolation theory that

$$B_{p,q}^s = (B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} = B_{p,q}^{(1+|x|^2)^{s/2}}$$

(with the same values of the parameters as above). By Theorem 6.2/1 this is impossible.

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Общие функциональные пространства. III
(Пространства $B^{g(x)}_{p,q}$ и $F^{g(x)}_{p,q}$, $1<p<\infty$: основные свойства)

Х. ТРИБЕЛЬ

Настоящая работа (а также ее продолжение — работа «Общие функциональные пространства, IV») посвящена исследованию банаховых пространств $B^{g(x)}_{p,q}$ и $F^{g(x)}_{p,q}$ распределений (обобщенных) в R_n. В специальных случаях эти пространства родственны известным классам Соболева—Лебега—Бесова, изотропным и анизотропным. Здесь рассматриваются следующие свойства: плотность гладких функций, эквивалентные нормы, интерполяция, включения и сравнения.

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