PSYCHOMETRIKA—VOL. 66, NO. 1, 99–107
MARCH 2001

ROTATION IN THE DYNAMIC FACTOR MODELING OF MULTIVARIATE STATIONARY TIME SERIES

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A special rotation procedure is proposed for the exploratory dynamic factor model for stationary multivariate time series. The rotation procedure applies separately to each univariate component series of a q-variate latent factor series and transforms such a component, initially represented as white noise, into a univariate moving-average. This is accomplished by minimizing a so-called state-space criterion that penalizes deviations of the rotated solution from a generalized state-space model with only instantaneous factor loadings. Alternative criteria are discussed in the closing section. The results of an empirical application are presented in some detail.

Key words: dynamic factor model, identifiability, polynomial division, moving-average, rotation criteria.

Application of the common factor model to multivariate time series obtained from a single case (subject, system) has been established in the psychological literature for over half a century (e.g., P-technique factor analysis, Cattell, Cattell, & Rhymer, 1947). One also finds related applications in the social science literature (Engle & Watson, 1981; Geweke & Singleton, 1981). It has also long been known that adaptations of the traditional common factor model are required in order to exploit the riches of time series data, for example, to explain the lagged covariance structure of manifest variables (Anderson, 1963; Cattell, 1963; Holtzman, 1963). Molenaar (1985) introduced a dynamic factor model that can handle stationary time series and can be fitted by means of standard structural equation modeling (SEM) software (see also Nesselroade & Molenaar, 1999; Wood & Brown, 1994). Although these more advanced modeling techniques have been in existence now for well over a decade, their ability to accommodate some of the more subtle features of factor analysis, especially exploratory factor analysis, has been greatly limited. In this article we will address one of the classical exploratory factor analysis problems rotation by presenting a special form of rotation for exploratory dynamic factor analysis.

On the one hand, the rotation method we present is special in that it applies separately to each of the q univariate latent factor series in a dynamic q-factor model. In particular, the method applies to a dynamic 1-factor model and hence involves a kind of rotation for which there is no analog in traditional factor analysis. On the other hand, our rotation method accomplishes two results that are in keeping with those of rotation in traditional exploratory factor analysis. First, the rotation "simplifies" the factor loading pattern. Second, the accompanying implied transformation of the factor scores induces properties into the latter that render them compatible with the new loading pattern. To denote the fact that the transformation we develop and apply is, strictly speaking, not a rotation of axes in the usual factor analytic sense, we will consistently refer to it as a special rotation. A merit of our proposal is that the properties induced into the

This research was supported by the Institute for Developmental and Health Research Methodology, University of Virginia.

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0033-3123/2001-1/1998-0627-A $00.75/0

implicit factor scores obviates some criticisms of the way they are specified in Molenaar's (1985) dynamic factor model.

1. Statement of the Problem

Let $\mathbf{y}(t)$ be a p-variate stationary time series with mean function $\mathbf{E}[\mathbf{y}(t)] = \mathbf{c}_y$ and lagged covariance function $\text{cov}[\mathbf{y}(t), \mathbf{y}(t+u)] = \mathbf{C}_y(u)$, $u = 0, \pm 1, \ldots$ (bold lowercase letters denote column vectors and bold uppercase letters denote matrices and operators). Consider the following dynamic q-factor model for $\mathbf{y}(t)$, following the notation in (Molenaar, 1985):

$$\mathbf{y}(t) = \sum_{u=0}^{s} \mathbf{\Lambda}(u)\boldsymbol{\eta}(t-u) + \boldsymbol{\epsilon}(t), \tag{1}$$

where $\sum_{u=0}^{s}$ denotes summation over the index $u = 0, 1, \ldots, s$, $\mathbf{\Lambda}(u)$ is a sequence of $s+1$ lagged (p, q)-dimensional matrices of factor loadings, $\boldsymbol{\eta}(t)$ is a q-variate, zero-mean latent factor series with covariance function $\text{cov}[\boldsymbol{\eta}(t), \boldsymbol{\eta}(t+u)] = \mathbf{\Psi}(u)$, $u = 0, \pm 1, \ldots,$ and $\boldsymbol{\epsilon}(t)$ is a p-variate, zero-mean measurement error series with covariance function $\text{cov}[\boldsymbol{\epsilon}(t), \boldsymbol{\epsilon}(t+u)] = \text{diag}\,\mathbf{\Theta}(u)$, $u = 0, \pm 1, \ldots.$ That $\mathbf{\Theta}(u)$ is defined as a diagonal matrix implies that the p univariate component series in $\boldsymbol{\epsilon}(t)$ have arbitrary lagged autocovariances, but lack any cross-covariances.

Molenaar (1985) proved that the latent factor covariance function $\mathbf{\Psi}(u)$ is not identified if the sequence of lagged factor loadings $\mathbf{\Lambda}(u)$ is unconstrained. In contrast, it was also proved that if s, the maximum lag in (1), is 0, then (1) reduces to a generalized state-space model in which $\mathbf{\Phi}(u)$ is identifiable (see also Molenaar, de Gooijer, & Schmitz, 1992). Given these conclusions, in order to arrive at an identifiable dynamic factor model with $s > 0$ and unconstrained $\mathbf{\Lambda}(u)$, $u = 0, 1, \ldots, s$ (denoted as an exploratory model), the lagged covariances $\mathbf{\Psi}(u)$ have to be fixed. One convenient way to fix $\mathbf{\Psi}(u)$ is to conceive of $\boldsymbol{\eta}(t)$ as a q-variate, zero-mean white noise series with covariance function $\mathbf{\Psi}(u) = \delta(u)\mathbf{I}_q$, where $\mathbf{I}_q$ denotes the (q, q)-dimensional unity matrix and where Kronecker's delta, $\delta(u)$, is equal to 1 if $u = 0$ and $\delta(u) = 0$ otherwise (see, e.g., Molenaar, 1994b, for alternative ways to arrive at an identifiable dynamic factor model).

Although the restriction $\mathbf{\Psi}(u) = \delta(u)\mathbf{I}_q$ should be regarded only as a minimal identifiability constraint (see Molenaar, 1985), similar in nature to constraints that fix the scales of common latent factors in standard factor analysis (see Jöreskog & Sörbom, 1993), it has given rise to questions about the plausibility of specifications that latent factor series underlying psychological time series constitute white noise. We therefore introduce a special rotation procedure for the exploratory model (1) that yields an equivalent solution (in the goodness-of-fit sense) in which the latent factor covariance function no longer is associated with a white noise series, but with a q-variate moving-average model (to be defined below). The substantive implications of this alternative are important for several psychological and behavioral concepts, including the representation of processes and other kinds of changes.

2. A Special Rotation Procedure

We now introduce a rotation procedure with the unusual feature that it applies separately to each of the q univariate latent factor series in (1). Hence even if $q = 1$, this rotation procedure applies. Consequently, to simplify the presentation, we can without loss of generality restrict attention to the case in which the dimension of the latent factor series in (1) is $q = 1$.

Let $\mathbf{B}$ denote the backward shift operator defined by $\mathbf{B}\boldsymbol{\eta}(t) = \boldsymbol{\eta}(t-1)$. Then (1) can be rewritten as:

$$\mathbf{y}(t) = \sum_{u=0}^{s} [\mathbf{\Lambda}(u)\mathbf{B}^{u}]\boldsymbol{\eta}(t) + \boldsymbol{\epsilon}(t) = \mathbf{\Lambda}(\mathbf{B})\boldsymbol{\eta}(t) + \boldsymbol{\epsilon}(t), \tag{2}$$

where $\mathbf{\Lambda}(\mathbf{B}) = \sum_{u=0}^{s}[\mathbf{\Lambda}(u)\mathbf{B}^u]$ is a p-variate, vector-valued polynomial of finite order s in $\mathbf{B}$.[1] Restricting attention to the relevant part of (2), we seek a rotated solution $\Lambda^r(\mathbf{B})$ that is implicitly defined as:

$$\mathbf{\Lambda}(\mathbf{B}) = \mathbf{\Lambda}(\mathbf{B})\phi^{-1}(\mathbf{B})\phi(\mathbf{B}) = \mathbf{\Lambda}^r(\mathbf{B})\phi(\mathbf{B}) \tag{3}$$

where $\mathbf{\Lambda}^r(\mathbf{B}) = \mathbf{\Lambda}(\mathbf{B})\phi^{-1}(\mathbf{B})$ and $\phi(\mathbf{B})$ is some scalar polynomial of finite order v in $\mathbf{B}$.

Under the identifiability constraint $\psi(u) = \delta(u)$, the communal part of the covariance function associated with the dynamic factor model (1) equals $\Lambda(\mathbf{B})\Lambda(\mathbf{B})'$, where $'$ denotes transposition. The transformation defined by (3) is a special rotation in that it leaves this communal part invariant: $\Lambda(\mathbf{B})\Lambda(\mathbf{B})' = \Lambda^r(\mathbf{B})\phi(\mathbf{B})\phi(\mathbf{B})'\Lambda^r(\mathbf{B})'$. This point will be further elaborated below.

As in all factor rotation procedures, a suitable criterion has to be defined. In the present context a natural criterion is that the rotated loadings in $\mathbf{\Lambda}^r(\mathbf{B})$ at lags $u > 0$ are vanishing. Heuristically speaking, this criterion implies that the rotated dynamic factor model resembles as closely as possible (in some metric) a generalized state-space model in which $s = 0$ in (1). Hence we will refer to this criterion as a state-space criterion.

More specifically, if $s = 0$ in the dynamic 1-factor model (2), implying that there are no lagged factor loadings, then the autocovariance function $\psi(u)$, $u = 0, \ldots,$ of the univariate latent factor series $\eta(t)$ is identifiable (Molenaar, 1985). It then follows from the Wald decomposition theorem (Hannan, 1970, p. 137) that $\eta(t)$ can be represented as an identifiable moving average model: $\eta(t) = \phi(\mathbf{B})\zeta(t)$, where $\phi(\mathbf{B})$ is defined as in (3) and $\text{cov}[\zeta(t), \zeta(t+u)] = \delta(u)\sigma^2$. This implies that for $s = 0$, (2) can be rewritten as a linear state-space model: $y(t) = \Lambda\mathbf{x}(t) + \epsilon(t)$; $\mathbf{x}(t+1) = \mathbf{H}\mathbf{x}(t) + \mathbf{w}(t)$ where $\mathbf{x}(t)' = [\eta(t), \zeta(t-1), \ldots]$, $\mathbf{w}(t)' = [\zeta(t), 0, \ldots]$, and Λ and $\mathbf{H}$ are matrices of appropriate dimensions with entries depending upon, respectively, $\Lambda(0)$ and $\phi(\mathbf{B})$ (see appendix in Molenaar, 1985, for a complete exposition of rewriting dynamic factor models as state-space models).

The implications of a state-space rotation criterion can be understood in the following way. Let $\lambda_i(\mathbf{B})$ denote the scalar polynomial of order s associated with the i-th manifest univariate series in $\mathbf{y}(t)$. Then each $\lambda_i(\mathbf{B})$ can be factored into the form:

$$\lambda_i(\mathbf{B}) = (\nu_{i1} + \beta_{i1}\mathbf{B})(\nu_{i2} + \beta_{i2}\mathbf{B})\ldots(\nu_{is} + \beta_{is}\mathbf{B}), i = 1, \ldots, p. \tag{4}$$

Suppose that the p forms in (4) share a common factor, for example, $(\nu_{ij} + \beta_{ij}\mathbf{B})$ for some fixed j. Then taking $\phi(\mathbf{B}) = \nu_{ij} + \beta_{ij}\mathbf{B}$ in (3) yields a rotated solution $\mathbf{\Lambda}^r(\mathbf{B})$ of maximum lag $s - 1$ (the maximum lag of $\mathbf{\Lambda}(\mathbf{B})$ is s).

In general, a state-space rotation procedure for (3) can be implemented by division of $\mathbf{\Lambda}(\mathbf{B})$ by a scalar polynomial $\phi(\mathbf{B})$ of order v, $\phi(\mathbf{B}) = 1 + \phi_1\mathbf{B} + \cdots + \phi_v\mathbf{B}^v$, in which the coefficients ϕ_k, $k = 1, \ldots, v$ are chosen so as to minimize (see next section for details of the minimization procedure)

$$\sum_{i=1}^{p}\sum_{u=1}^{w}\lambda_i^r(u)^2, \tag{5}$$

where w is the order of $\mathbf{\Lambda}^r(\mathbf{B})$ associated with the number of lags that in practical applications still contribute nonnegligibly.

The rotated latent factor series $\boldsymbol{\eta}^r(t)$ is no longer white noise, but a moving-average of order v (MA[v]):

$$\boldsymbol{\eta}^r(t) = \phi(\mathbf{B})\boldsymbol{\eta}(t) = \boldsymbol{\eta}(t) + \phi_1\boldsymbol{\eta}(t-1) + \cdots + \phi_v\boldsymbol{\eta}(t-v), \tag{6}$$

where $\text{cov}[\boldsymbol{\eta}(t), \eta(t+u)] = \delta(u)$. For instance, if $v = 1$ then the covariance function of the rotated latent factor series is: $\psi^r(0) = 1 + \phi_1^2$, $\psi^r(1) = \psi^r(-1) = \phi_1$, and $\psi^r(u) = 0$ if

[1] In what follows, an expression such as $a(\mathbf{B})$ always refers to the polynomial in the backward shift operator $\mathbf{B}$ which is associated with the sequence $a(u)$, indexed by lag $u = 0, 1, \ldots$.

$|u| > 1$. It follows from (6) that positive-definiteness inheres in the covariance function of the rotated latent factor series from the covariance function of the unrotated latent factor series. Consequently, the block-Toeplitz matrix associated with a rotated q-variate latent factor series obtained from (1) will be block-diagonal and has no eigenvalues ≤ 0.

In conclusion of this section, it is noted that the special rotation (3) is defined at the level of $(p, 1)$-dimensional vector-valued polynomials $\Lambda(\mathbf{B})$ representing the lagged factor loadings in the dynamic 1-factor model (2). That is, the $(p, 1)$-dimensional polynomial $\Lambda(\mathbf{B})$ is rotated by a (1,1)-dimensional scalar polynomial $\phi(\mathbf{B})$. In the general dynamic q-factor model (1) this special rotation is applied to each of the q columns of the (p, q)-dimensional matrix-valued polynomial $\Lambda(\mathbf{B})$. Hence, special rotation in the dynamic q-factor model involves rotation by a diagonal (q, q)-dimensional matrix-valued polynomial $\Phi(\mathbf{B})$.

3. Implementation and Validation of the Rotation Algorithm

First, a straightfoward implementation of the rotation procedure (3) is outlined and accompanied by a discussion of some details related to applying the procedure. Then the results of an application to simulated data are presented in order to show the performance of the algorithm. The Fortran source code of our implementation can be obtained via ftp.[2]

The basic ingredients to implement the rotation procedure (3) are an algorithm to carry out polynomial division and a general optimization algorithm. It is convenient to use an optimization algorithm that uses numerical derivatives such as that employed for IMSL subroutine ZXMIN (*IMSL Library Reference Manual*, 1980). A suitable polynomial division algorithm is given by Robinson (1967, p. 31) with which each row $\lambda_i(B) = \lambda_i(0) + \lambda_i(1)B + \cdots + \lambda_i(s)B^s$, $i = 1, 2, \ldots, p$, of $\mathbf{\Lambda}(\mathrm{B})$ in (3) can be divided by the same polynomial $\phi(\mathbf{B}) = 1+\phi_1\mathbf{B}+\cdots+\phi_\upsilon\mathbf{B}^\upsilon$. The coefficients of $\phi(\mathbf{B})$ then are determined by means of the quasi-Newton method in such a way that the criterion (5) is minimized. Notice that the zero-th order coefficient in $\phi(\mathbf{B})$ is fixed at 1.0. This implies that the zero-th order coefficient $\lambda_i(0)$ in each $\lambda_i(B)$, $i = 1, 2, \ldots, p$, remains invariant after division by $\phi(\mathbf{B})$. If the zero-th order coefficient in $\phi(\mathbf{B})$ would be defined as an additional free parameter ϕ_0, that is, $\phi(\mathbf{B}) = \phi_0 + \phi_1\mathbf{B} + \cdots + \phi_\upsilon\mathbf{B}^\upsilon$, then minimization of (5) would lead to the nonsensical result that the value of ϕ_0 becomes arbitrarily large while each rotated $\lambda_i^r(B)$ becomes arbitrarily small.

Two of the details associated with the implementation and application of the given rotation algorithm require special attention. The first one concerns the determination of the optimal order υ of the polynomial $\phi(\mathbf{B})$. In principle, the minimum value of the criterion (5) is a nonincreasing function of this order, yet taking υ too large can create numerical problems (division by zero) due to finite precision computation. Hence in our implementation the order υ is increased from $\upsilon = 2$ by steps of 1 until the decrease in the minimum value of (5) becomes too small (due to a vanishing coefficient of maximum order υ) to warrant further increases in υ. Second, the maximum order w in criterion (5) should be taken as large as possible without creating numerical problems. Again, a stepwise procedure is followed in our implementation in which the value of w is increased until the rotated loadings at maximum lag w become vanishingly small.

To illustrate the rotation procedure, let i denote the imaginary unit, $i = \sqrt{-1}$ and consider the following instance of (4) for $p = 4$:

$$\begin{aligned}
\lambda_1(\mathrm{B}) &= (1 + [.8 - .7i]\mathrm{B})(1 + [.8 + .7i]\mathrm{B})(1 + .6\mathrm{B}) = 1 + 2.2\mathrm{B} + 2.09\mathrm{B}^2 + .678\mathrm{B}^3 \\
\lambda_2(\mathrm{B}) &= (1 + [.8 - .7i]\mathrm{B})(1 + [.8 + .7i]\mathrm{B})(1 + .5\mathrm{B}) = 1 + 2.1\mathrm{B} + 1.93\mathrm{B}^2 + .565\mathrm{B}^3 \\
\lambda_3(\mathrm{B}) &= (1 + [.8 - .7i]\mathrm{B})(1 + [.8 + .7i]\mathrm{B})(1 + .4\mathrm{B}) = 1 + 1.2\mathrm{B} + .49\mathrm{B}^2 - .452\mathrm{B}^3 \\
\lambda_4(\mathrm{B}) &= (1 + [.8 - .7i]\mathrm{B})(1 + [.8 + .7i]\mathrm{B})(1 + .3\mathrm{B}) = 1 + 1.3\mathrm{B} + .65\mathrm{B}^2 - .339\mathrm{B}^3. \qquad (7)
\end{aligned}$$

[2]The implementation can be found at ftp.kiptron.psyc.virginia.edu

In (7) the coefficients of the $\lambda_i(\mathrm{B})$, $i = 1, 2, 3, 4$, are the lagged factor loadings associated with the i-th manifest series in $\mathbf{y}(t)$ which share a pair of complex conjugated factors $(1 + [.8 - .7i]\mathrm{B})(1 + [.8 + .7i]\mathrm{B}) = 1 + 1.6\mathrm{B} + 1.13\mathrm{B}^2$. Hence we expect that the rotation procedure will yield $\phi(\mathbf{B}) = 1 + 1.6\mathrm{B} + 1.13\mathrm{B}^2$ in combination with the following rotated loadings:

$$\begin{aligned}
\lambda_1^r(\mathrm{B}) &= 1 + .6\mathrm{B} \\
\lambda_2^r(\mathrm{B}) &= 1 + .5\mathrm{B} \\
\lambda_3^r(\mathrm{B}) &= 1 + .4\mathrm{B} \\
\lambda_4^r(\mathrm{B}) &= 1 + .3\mathrm{B}.
\end{aligned} \tag{8}$$

This result is indeed obtained.

To show the performance of the rotation algorithm with estimated lagged factor loadings, a small-scale simulation study was carried out. Using the lagged factor loadings as specified by (7), 10 independent realizations of a 4-variate time series of length $T = 200$ were generated according to (2). In each realization $\boldsymbol{\eta}(t)$ was a univariate Gaussian white noise series, while the component series in $\boldsymbol{\varepsilon}(t)$ were autocorrelated Gaussian series. Each realization was subjected to an exploratory dynamic factor analysis as specified in (Molenaar, 1985). The (24,24)-dimensional input block-Toeplitz matrices consisted of blocks of lagged (4,4)-dimensional covariance matrices up to lag 5. The mean estimated lagged factor loadings and the associated standard deviations (in parentheses) across 10 realizations are:

$$\lambda_1^r(\mathrm{B}) = \begin{matrix} 1.163 & +2.290\mathrm{B} & +1.953\mathrm{B}^2 & +.625\mathrm{B}^3 \\ (.125) & (.112) & (.254) & (.177) \end{matrix}$$

$$\lambda_2^r(\mathrm{B}) = \begin{matrix} 1.141 & +2.176\mathrm{B} & +1.792\mathrm{B}^2 & +.511\mathrm{B}^3 \\ (.149) & (.103) & (.228) & (.146) \end{matrix}$$

$$\lambda_3^r(\mathrm{B}) = \begin{matrix} 1.132 & +1.148\mathrm{B} & - + .345\mathrm{B}^2 & -.376\mathrm{B}^3 \\ (.120) & (.097) & (.191) & (.091) \end{matrix}$$

$$\lambda_4^r(\mathrm{B}) = \begin{matrix} 1.144 & +1.270\mathrm{B} & +.518\mathrm{B}^2 & -.291\mathrm{B}^3 \\ (.125) & (.076) & (.206) & (.077) \end{matrix}$$

The mean rotated factor loadings (up to lag 3) and the associated standard deviations (in parentheses) across the 10 realizations are:

$$\lambda_1^r(\mathrm{B}) = \begin{matrix} 1.163 & +.638\mathrm{B} & +.003\mathrm{B}^2 & -.011\mathrm{B}^3 \\ (.125) & (.093) & (.131) & (.075) \end{matrix}$$

$$\lambda_2^r(\mathrm{B}) = \begin{matrix} 1.141 & +.558\mathrm{B} & -.026\mathrm{B}^2 & -.010\mathrm{B}^3 \\ (.149) & (.097) & (.094) & (.090) \end{matrix}$$

$$\lambda_3^r(\mathrm{B}) = \begin{matrix} 1.132 & -.460\mathrm{B} & -.015\mathrm{B}^2 & -.005\mathrm{B}^3 \\ (.120) & (.134) & (.076) & (.033) \end{matrix}$$

$$\lambda_4^r(\mathrm{B}) = \begin{matrix} 1.144 & -.355\mathrm{B} & -.002\mathrm{B}^2 & -.034\mathrm{B}^3 \\ (.125) & (.107) & (.057) & (.091) \end{matrix}$$

It appears that the rotation procedure yields satisfactory results, both with ideal loading patterns and with lagged loadings estimated from realizations of finite length. In an application of the rotation procedure, the dimension p of the manifest series is of course given, as is the maximum s in (1) of the estimated lagged loadings obtained in the exploratory dynamic factor analysis of this manifest series. The optimal order v of $\phi(\mathbf{B})$ in (3) and (6) as well as the maximum order w in criterion (5) can be determined according to the stepwise procedures outlined above.

4. Alternative Implementation

The implicit definition (3) of special rotation in the dynamic 1-factor model can be replaced by the following equivalent definition:

$$\mathbf{\Lambda}(\mathbf{B}) = \mathbf{\Lambda}(\mathbf{B})\gamma(\mathbf{B})\gamma^{-1}(\mathbf{B}) = \mathbf{\Lambda}^r(\mathbf{B})\gamma^{-1}(\mathbf{B}). \tag{9}$$

It is seen that (3) and (9) are related by

$$\phi^{-1}(\mathbf{B}) = \gamma(\mathbf{B}) \tag{10}$$

State-space rotation according to (9) implies that the coefficients of the scalar polynomial $\gamma(\mathbf{B})$ of order v are chosen in such a way that the product of $\Lambda(\mathbf{B})$ and $\gamma(\mathbf{B})$ minimizes (5). In addition, the rotated latent factor series $\eta^r(t)$ is an autoregression of order v:

$$\gamma(\mathbf{B})\eta^r(t) = \eta(t) \tag{11}$$

where $\text{cov}[\eta(t), \eta(t+u)] = \delta(u)$.

Whereas the special rotation according to (3) is obtained by division of $\Lambda(\mathbf{B})$ by $\phi(\mathbf{B})$ of order v, yielding a moving-average of order v for the rotated latent factor series, the same special rotation defined by (9) is obtained by multiplication of $\Lambda(\mathbf{B})$ by $\gamma(\mathbf{B})$ of order v, yielding an autoregression of order v for the rotated latent factor series. In view of (10) these solutions are equivalent in that they yield the same value for (5) and the same values for the rotated loadings. The source code of this alternative implementation also can be obtained via ftp.

5. Empirical Example

The use of dynamic factor rotation in empirical research will be illustrated with a dynamic factor analysis of multilead electroencephalographic (EEG) registrations obtained with a single subject. The EEG registrations were obtained in a large-scale quantitative genetical study of MZ and DZ twins (Beijsterveldt, Molenaar, de Geus, & Boomsma, 1996). For a randomly selected subject we analyze the EEG (eyes closed) during 6 seconds, sampling rate 50 Hz (Herz or cycles per second), at four locations on the head. The four locations (F3, C3, P3, O1 in the 10–20 system; (see Regan, 1989, p. 12) are linearly arranged at equal distances on a small circle of the left hemisphere running from the front (F3) to the back (O1) of the head parallel to the midline. This yields a 4-variate manifest series of length $T = 300$: $y_1(t)$ is EEG at the frontal pole F3, $y_2(t)$ is EEG at the central pole C3, $y_3(t)$ is EEG at the parietal pole P3 and $y_4(t)$ is EEG at the occipital pole O1.

This 4-variate EEG series was standardized and subjected to a dynamic factor analysis as described in (Molenaar, 1985). The (32, 32)-dimensional input block-Toeplitz matrix consists of blocks of lagged (4, 4)-dimensional correlation matrices up to lag 7. The selected model is an exploratory dynamic 1-factor model in which the univariate latent factor series $\eta(t)$ is a white noise series with loadings on $\mathbf{y}(t)$ up to lag 6. The lagged loadings are presented in Table 1.

State-space rotation of this solution yields the pattern of lagged loadings shown in Table 2.

A comparison of Tables 1 and 2 shows that the special state-space rotation yields a substantial reduction of loadings at nonzero lags. The value of criterion (5) is 2.13 for the initial solution presented in Table 1, whereas the value of (5) for the rotated solution in Table 2 is .28. In contrast to the initial solution, the largest rotated loadings occur at lag zero and quickly decay to small absolute values at increasing lags. This general pattern of rotated loadings has one notable exception, however. At lags 5 and 6 the loadings on F3 (−.20 at lag 5; .22 at lag 6), C3 (−.15

TABLE 1.
Lagged factor loadings of 4-variate EEG series

	Lagged Loading						
Time Series	λ(0)	λ(1)	λ(2)	λ(3)	λ(4)	λ(5)	λ(6)
F3	.20	.39	.50	.37	.12	−.22	−.06
C3	.35	.55	.58	.45	.22	−.08	−.05
P3	.27	.35	.27	.15	.08	.00	.03
O1	.46	.46	.26	.13	.14	.26	.12

TABLE 2.
State-space rotated factor loadings of 4-variate EEG series

Time	Lagged Loading										
Series	λ(0)	λ(1)	λ(2)	λ(3)	λ(4)	λ(5)	λ(6)	λ(7)	λ(8)	λ(9)	λ(10)
F3	.20	.15	.14	−.02	−.07	−.20	.22	−.07	−.03	.03	.00
C3	.35	.12	.11	.05	−.04	−.15	.11	.01	−.04	.02	.00
P3	.27	.02	.00	.01	.02	−.04	.04	−.02	.00	.00	.00
O1	.46	−.10	−.04	.06	.07	.12	−.15	.04	.02	−02	.00

at lag 5; .11 at lag 6) and O1 (.13 at lag 5; −.15 at lag 6) show a substantial transient increase in absolute value. This transient increase does not occur for the loadings on P3 at lag 5 (−.04) and lag 6 (.04). Because the sampling rate of the EEG series is 50 Hz, loadings at lags 5 and 6 correspond to 10 and 12 Hz in real time. The dominant frequency of oscillation of EEG in an eyes closed condition also is 10–12 Hz (so-called alpha band; (see Regan, 1989). Hence the state-space rotated loadings show a transient increase at the lag corresponding to the dominant alpha band of frequencies of the EEG. No such transient pattern is visible in the initial solution.

The transient increase in absolute value of the specially rotated loadings at lags 5 and 6 on F3, C3, and O1, on the one hand, and the lack of such a transient increase of the rotated loading on P3 at lags 5 and 6, on the other hand, induces a lead-lag pattern in the cross-covariances between EEG at P3 and at the remaining locations. That is, this transient pattern of lagged loadings will give rise to covariance between EEG activity at P3 (lead) and EEG activity at the remaining locations 0.08–0.10 second later (lag), but not in reverse. Such a directional lead-lag relationship is suggestive of the presence of an EEG source or generator in the neighborhood of P3. In a previous analysis of multilead EEG obtained with a single subject, using dynamic factor analysis in the frequency domain (i.e., after discrete Fourier transformation of the data), similar evidence was found for the presence of an EEG source located at about the same brain region (see Molenaar, 1994b), for an extensive description of the method and results).

A possible further interpretation of the remarkable pattern of state-space rotated loadings in Table 2 can be given in terms of Nunez's (1981, 1995) influential EEG wave model for the spatiotemporal structure of electrocortical potential fields. Under simplifying assumptions (e.g., a spherical homogeneous head model; see Nunez, 1995, for detailed discussion) an empirical application of the EEG wave model boils down to a dynamic factor analysis in the frequency domain (i.e., after discrete Fourier transformation of the data) of multilead EEG registrations (see Molenaar, 1987; Molenaar, 1994b, for derivations and methodological improvements). In this dynamic factor model in the frequency domain the loadings on leads at various locations on the head represent a spatial filter associated with the normal modes (spherical harmonics) of potential fields. Because the dynamic factor model in the frequency domain constitutes a one-to-one transformation of the dynamic factor model in the time domain, the lagged loadings in the latter model also can be interpreted in terms of a spatial filter. Accordingly, the transient increase in absolute value of loadings at the lag corresponding to the dominant alpha band of EEG could be interpreted as a resonant frequency of the neocortex conceived of as a spherical biophysical surface. In fact, (Nunez, 1981) has put forward theoretical biophysical arguments implying a similar interpretation of the spatiotemporal coherence of EEG registrations.

Whereas the latent factor series $\eta(t)$ in the initial solution is defined as white noise lacking any sequential covariance, the specially rotated latent factor series associated with the solution presented in Table 2 is a fourth-order moving average MA[4]: $\eta^r(t) = \eta(t) + 1.22\eta(t-1) + .92\eta(t-2) + .45\eta(t-3) + .14\eta(t-4)$ where $\eta(t)$ is white noise with mean zero and variance equal to 1.

6. Discussion and Conclusion

The option of rotating initial solutions to mathematically equivalent solutions that can be more meaningfully characterized from a substantive point of view has long been considered a desirable feature of exploratory factor analysis (Cattell, 1952). Although the same holds true for exploratory *dynamic* factor analysis, until now rotation has not been a viable option for investigators using the method. The proposed special rotation procedure for the exploratory dynamic factor model introduced in (Molenaar, 1985) yields equivalent solutions in which the latent factor series is no longer a white noise series, but a moving-average. The rotation is defined for each univariate component of a q-variate latent factor series and hence also applies if $q = 1$.

Defining a mathematical criterion to maximize or minimize is perhaps the most critical step of specifying a rotation procedure. Traditional exploratory factor analysis has not lacked either orthogonal or oblique criteria. For dynamic exploratory factor analysis, the state-space criterion presented above would seem to be a natural criterion in order to obtain rotated solutions which, according to the measure defined in (5), resemble as closely as possible a generalized state-space model which only has nonzero factor loadings at lag zero. The analogy of this criterion to a traditional conception of "simple structure" is in the sense that the descriptions of the manifest time series in terms of the latent time series are made as simple as possible in terms of lagged relationships. Yet, the criterion in (5) can be modified in arbitrary ways. For instance, an alternative criterion could be defined in which deviations of lagged factor loadings from zero at lags $u > 0$ are weighted by the lag u. This would induce larger penalties for deviations from zero of factor loadings occurring at larger lags. Hence rotated solutions thus obtained would have their factor loadings concentrated at small lags. This alternative criterion could be called a *maximum cumulative energy* criterion, that is, the cumulative sum of squared factor loadings as function of lag u is maximum among all possible equivalent solutions. The maximum cumulative energy criterion is well-known in engineering applications (see Robinson & De Silva, 1978), for further discussion of this criterion). An analogous criterion is used by (Molenaar, 1987) for rotation in the dynamic factor model in the frequency domain, that is, the Fourier transform of (1). Another possibly suitable modification of (5) would be to penalize deviations from zero of factor loadings at a lag larger than, say, $u = 2$. More generally, one could define an arbitrary fixed pattern matrix for the rotated lagged loadings, thus creating a Procrustes version of the special rotation procedure.

Further possible extensions at a different level can be made by referring to our earlier definition of the special rotation procedure for dynamic q-factor models in terms of transformation by $\mathbf{\Phi}(\mathbf{B})$, that is, a diagonal (q, q)-dimensional matrix-valued polynomial in the lag operator $\mathbf{B}$. If it is allowed that $\mathbf{\Phi}(\mathbf{B})$ also has off-diagonal elements for $q > 1$ then the special rotation procedure can be combined with ordinary (e.g., varimax) rotation for the dynamic q-factor model. This extension involves the product of special rotation by means of a diagonal (q, q)-dimensional matrix-valued polynomial and ordinary rotation by means of a full (q, q)-dimensional matrix-valued polynomial. This possibility remains to be elaborated, in particular with respect to the conditions which should be met by the full (q, q)-dimensional matrix-valued polynomial defining ordinary rotation.

Extensions such as outlined above as well as other possible rotational criteria for the special rotation procedure will no doubt appeal to readers interested in modeling process and change via time series data. The method we have presented lays the ground work for specifying, implementing, and evaluating a number of different approaches to dealing with this difficult but critical aspect of exploratory dynamic factor analysis. We strongly encourage investigators who seek more powerful ways to model process to develop and test alternative specifications that can be evaluated against empirical data by means of the general modeling approach presented here.

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Manuscript received 27 JAN 1998
Final version received 25 JAN 2000