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FULL-INFORMATION ITEM BI-FACTOR ANALYSIS

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A plausible s-factor solution for many types of psychological and educational tests is one that exhibits a general factor and $s - 1$ group or method related factors. The bi-factor solution results from the constraint that each item has a nonzero loading on the primary dimension and at most one of the $s - 1$ group factors. This paper derives a bi-factor item-response model for binary response data. In marginal maximum likelihood estimation of item parameters, the bi-factor restriction leads to a major simplification of likelihood equations and (a) permits analysis of models with large numbers of group factors; (b) permits conditional dependence within identified subsets of items; and (c) provides more parsimonious factor solutions than an unrestricted full-information item factor analysis in some cases.

Key words: bi-factor model, marginal maximum likelihood, EM algorithm, item analysis, dichotomous factor analysis.

Introduction

Consider a set of n test items for which an s-factor solution exists with one general factor and $s - 1$ group or method related factors. The bi-factor solution constrains each item j to have a nonzero loading α_{j1} on the primary dimension and a second loading (α_{jk}, $k = 2, \ldots, s$) on not more than one of the $s - 1$ group factors. For four items, the bi-factor pattern matrix might be

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & 0 & \alpha_{33} \\ \alpha_{41} & 0 & \alpha_{43} \end{bmatrix}.$$

This structure, which Holzinger and Swineford (1937) termed the "bi-factor" solution, also appears in the inter-battery factor analysis of Tucker (1958) and is one confirmatory factor analysis model considered by Jöreskog (1969). In these applications, the model is restricted to test scores assumed continuously distributed. But it is easy to conceive of situations where the bi-factor pattern might also arise at the item level (Muthén, 1989). It is plausible for paragraph comprehension tests, for example, where the primary dimension describes the targeted process skill and additional factors describe content area knowledge within paragraphs. In this context, items would be conditionally independent between paragraphs, but conditionally dependent within paragraphs. Tests consisting of items tapping different content areas also are suitable for this type of analysis.

The purpose of this paper is to derive an item-response model for binary response data exhibiting the bi-factor structure and to develop a practical parameter estimation

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method. As we show, the bi-factor restriction leads to a major simplification of likelihood equations that (1) permits analysis of models with large numbers of group factors, (2) permits conditional dependence among identified subsets of items, and (3) provides more parsimonious factor solutions than an unrestricted full-information item factor analysis in some cases (e.g., Bock & Aitkin, 1981). In the following sections, the marginal likelihood and its first derivatives will be derived so that an EM solution to item bi-factor analysis may be obtained.

Likelihood Evaluation

To begin, consider Thurstone's (1947) multiple factor model for item j, ($j = 1, \ldots, n$)

$$y_j = \alpha_{j1}\theta_1 + \alpha_{j2}\theta_2 + \cdots + \alpha_{js}\theta_s + \varepsilon_j. \tag{1}$$

The latent variable y_j is assumed to be distributed $N(0, 1)$ and the underlying abilities ($\boldsymbol{\theta}$) as $N(\mathbf{0}, \mathbf{I})$. These assumptions imply uncorrelated underlying abilities and residuals distributed as

$$\varepsilon_j \sim N\left(0,\ 1 - \sum_{k=1}^{s} \alpha_{jk}^2\right).$$

Subject i is assumed to score $x_{ij} = 1$ on item j if

$$y_{ij} = \sum_{k=1}^{s} \alpha_{jk}\theta_{ik} + \varepsilon_{ij}$$

exceeds the threshold γ_j; otherwise, $x_{ij} = 0$. The bi-factor restriction requires that only one of the $k = 2, \ldots, s$ values of α_{jk} be nonzero in addition to α_{j1}. As will be shown, this restriction remarkably simplifies the numerical integration that is required in the marginal maximum likelihood solution of the unrestricted multiple item-factor model.

Returning to the unrestricted case, the probability of a correct response for subject i to item j, conditional on abilities $\boldsymbol{\theta}$ is,

$$P(x_{ij} = 1|\boldsymbol{\theta}_i) = \frac{1}{(2\pi)^{1/2}} \int_{-z_j(\boldsymbol{\theta}_i)}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt = \Phi_j(\boldsymbol{\theta}_i), \tag{2}$$

where

$$-z_j(\boldsymbol{\theta}_i) = \frac{\gamma_j - \sum_{k=1}^{s} \alpha_{jk}\theta_{ki}}{\sigma_j}.$$

Using the multidimensional extension of the conditional independence assumption (i.e., responses are independent conditional on all $\boldsymbol{\theta}$), the probability of subject i responding in pattern $\mathbf{x}_i = [x_{i1}, x_{i2}, \ldots, x_{in}]$ conditional on abilities $\boldsymbol{\theta}$ is

$$P(\mathbf{x} = \mathbf{x}_i|\boldsymbol{\theta}) = \prod_{j=1}^{n} [\Phi_j(\boldsymbol{\theta})]^{x_{ij}}[1 - \Phi_j(\boldsymbol{\theta})]^{1 - x_{ij}} = L_i(\boldsymbol{\theta}). \tag{3}$$

For a random subject sampled from a population with continuous ability distributions $g(\boldsymbol{\theta})$, the unconditional probability of response pattern $\mathbf{x}_i$ is

$$P(\mathbf{x}=\mathbf{x}_i)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}L_i(\boldsymbol{\theta})_{g(\theta_1)}(\theta_1)\cdots g(\theta_s)\,d\theta_1\cdots d\theta_s, \tag{4}$$

given the conditional independence assumption and taking $g(\boldsymbol{\theta})$ to be multivariate normal $N(\mathbf{0}, \mathbf{I})$. We can employ an s-dimensional Gauss-Hermite quadrature to approximate (4) as

$$P(\mathbf{x}=\mathbf{x}_i)=\sum_{qs}^{Q}\cdots\sum_{q2}^{Q}\sum_{q1}^{Q}P(\mathbf{x}=\mathbf{x}_i|X_{q1}, X_{q2},\cdots X_{qs})A(X_{q1})A(X_{q2})\cdots A(X_{qs}). \tag{5}$$

The multidimensional conditional independence assumption permits evaluation of the s integrals in (4) by a q^s point quadrature for the unrestricted multiple factor model. Of course, as s gets larger than 4 or 5, this computation becomes intractable for even small numbers of quadrature points per dimension. This is not the case, however, for the bi-factor model. The unconditional probability in (4) is obtained by evaluating the probability of dimensions $2, \ldots, s$, and integrating with respect to the distribution of θ_1. The bi-factor restriction reduces the s-dimensional integral in (4) to a two-dimensional integral, one for θ_1 and one for $\theta_2, \ldots, \theta_s$. The reduction formula is due to Stuart (1958), who showed that if n variables follow a standardized multivariate normal distribution where the correlation $\rho_{j'j} = \Sigma_{k=1}^{s}\alpha_{j'k}\alpha_{jk}$ and α_{jk} is nonzero for only one k, then the probability that respective variables are simultaneously less than γ_j is given by

$$P=\prod_{k=1}^{s}\int_{-\infty}^{\infty}\left\{\prod_{j=1}^{n}\left[\Phi\left(\frac{\gamma_j-\alpha_{jk}y}{(1-\alpha_{jk}^2)^{1/2}}\right)\right]^{c_{jk}}\right\}g(y)\,dy, \tag{6}$$

where $c_{jk} = 1$ denotes a nonzero loading of item j on dimension k ($k = 1, \ldots, s$), and $c_{jk} = 0$ otherwise. Note that for item j, $c_{jk} = 1$ for only one k.

Equation (6) follows from the fact that if each variate is related to a single dimension only, then the s dimensions are unconditionally independent, and the joint probability is the product of s unidimensional probabilities. In this context, the result applies only to the $s - 1$ "nuisance" dimensions (i.e., $k = 2, \ldots, s$). If a primary dimension exists, it will not be unconditionally independent of the other $s - 1$ dimensions, since each item now loads on each of two dimensions. Therefore, this probability requires a two-dimensional generalization of Stuart's (1958) original result for computation.

To derive the two-dimensional result, we note that probability of the primary dimension can be obtained using the unidimensional form of (4), which is originally due to Dunnett and Sobel (1955), and adapted to Item Response Theory (IRT) by Bock and Lieberman (1970). In this context, the unidimensional equation is

$$P=\int_{-\infty}^{\infty}\left[\prod_{j=1}^{n}\Phi\left(\frac{\gamma_j-\alpha_{j1}\theta_1}{(1-\alpha_{j1}^2)^{1/2}}\right)\right]g(\theta_1)\,d\theta_1, \tag{7}$$

which is valid as long as $\rho_{j'j} = \alpha_{j'}\alpha_j$, as is assumed here for the primary dimension. Integration of (6) over the probability distribution function in (7), leads to the desired unconditional probability,

$$P = \int_{-\infty}^{\infty} \left\{ \prod_{k=2}^{s} \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} \left(\Phi \left[\frac{\gamma_j - \alpha_{j1}\theta_1 - \alpha_{jk}\theta_k}{(1 - \alpha_{j1}^2 - \alpha_{jk}^2)^{1/2}} \right] \right)^{c_{jk}} \right] g(\theta_k)\, d\theta_k \right\} g(\theta_1)\, d\theta_1. \tag{8}$$

Here, $c_{jk} = 1$ for only one k ($k = 2, \ldots, s$) due to the bi-factor structure. Equation (8) can be approximated to any practical degree of accuracy using two-dimensional Gauss-Hermite quadrature (Stroud & Sechrest, 1966). As in the unrestricted multiple factor model, an important consequence of (8) is the implication that primary and secondary factors are distributed independently in the examinee population. In many instances, this assumption should prove to be reasonable.

For example, if $y_j = \Sigma_{k=1}^{s} \alpha_{jk}\theta_k + \varepsilon_j$ and the above distribution assumptions apply, the unconditional probability of observing score pattern $\mathbf{x} = \mathbf{x}_l$ is

$$P_l = \int_{-\infty}^{\infty} \left\{ \prod_{k=2}^{s} \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} ([\Phi_{jk}(\theta_1, \theta_k)]^{x_{lj}} [1 - \Phi_{jk}(\theta_1, \theta_k)]^{1 - x_{lj}})^{c_{jk}} \right] g(\theta_k)\, d\theta_k \right\} g(\theta_1)\, d\theta_1, \tag{9}$$

which can be approximated by

$$\hat{P}_l \cong \sum_{q1}^{Q} \left\{ \prod_{k=2}^{s} \sum_{q_k}^{Q} \left[\prod_{j=1}^{n} ([\Phi_{jk}(X_{q1}, X_{qk})]^{x_{lj}} [1 - \Phi_{jk}(X_{q1}, X_{qk})]^{1 - x_{lj}})^{c_{jk}} \right] A(X_{qk}) \right\} A(X_{q1}), \tag{10}$$

where

$$\Phi_{jk}(\theta_1, \theta_k) \cong \Phi_{jk}(X_{q1}, X_{qk}) = \Phi\left(-\frac{\gamma_j - \alpha_{j1}X_{q1} - \alpha_{jk}X_{qk}}{(1 - \alpha_{j1}^2 - \alpha_{jk}^2)^{1/2}} \right),$$

and $\mathbf{X}_q$ and $A(\mathbf{X}_q)$ are the nodes and corresponding weights of a Gauss-Hermite quadrature.

Marginal Maximum Likelihood Estimation

The parameters of the item bi-factor analysis model can be estimated by the marginal maximum likelihood method using a variation of the EM algorithm described by Bock and Aitkin (1981). The parameters of this model include n "thresholds" (γ_j), n primary factor loadings (α_{j1}) and a total of n factor loadings on the $k = 2, \ldots, s$ additional dimensions (i.e., α_{jk} for $k > 1$, where $\Sigma_{k=2}^{s}\Sigma_{j=1}^{n} c_{jk} = n$). The likelihood equations are derived as follows.

Denoting the k-th subset of the components of $\boldsymbol{\theta}$ as $\boldsymbol{\theta}_k^* = [{}^{\theta_1}_{\theta_k}]$, let

$$P_l = P(\mathbf{x} = \mathbf{x}_l) = \int_{\theta_1} \left\{ \prod_{k=2}^{s} \int_{\theta_k} \left[\prod_{j=1}^{n} ([\Phi_{jk}(\boldsymbol{\theta}_k^*)]^{x_{lj}} \right. \right.$$

$$\left. \left. [1 - \Phi_{jk}(\boldsymbol{\theta}_k^*)]^{1-x_{lj}})^{c_{jk}} \right] g(\theta_k)\, d\theta_k \right\} g(\theta_1)\, d\theta_1$$

$$= \int_{\theta_1} \left\{ \prod_{k=2}^{s} \int_{\theta_k} L_{lk}(\boldsymbol{\theta}_k^*) g(\theta_k)\, d\theta_k \right\} g(\theta_1)\, d\theta_1, \quad (11)$$

where

$$L_{lk}(\boldsymbol{\theta}_k^*) = \prod_{j=1}^{n} ([\Phi_{jk}(\boldsymbol{\theta}_k^*)]^{x_{lj}} [1 - \Phi_{jk}(\boldsymbol{\theta}_k^*)]^{1-x_{lj}})^{c_{jk}}.$$

Then the log likelihood is

$$\log L = \sum_{l=1}^{S} r_l \log P_l, \quad (12)$$

where S denotes number of unique response patterns. The derivative of the log marginal likelihood to a general item parameter ν_j follows. Let

$$E_{lk}(\theta_1) = \frac{\prod_{h=2}^{s} \int_{\theta_h} L_{lh}(\boldsymbol{\theta}_h^*) g(\theta_h)\, d\theta_h}{\int_{\theta_k} L_{lk}(\boldsymbol{\theta}_k^*) g(\theta_k)\, d\theta_k}, \quad (13)$$

then

$$\frac{\partial \log L}{\partial \nu_j} = \sum_{l}^{S} \frac{r_l}{P_l} \left(\frac{\partial P_l}{\partial \nu_j} \right) = \sum_{l=1}^{S} \frac{r_l}{P_l} \int_{\theta_1} \sum_{k=2}^{s} c_{jk} E_{lk}(\theta_1)$$

$$\times \left\{ \int_{\theta_k} \left(\frac{x_{lj} - \Phi_{jk}(\boldsymbol{\theta}_k^*)}{\Phi_{jk}(\boldsymbol{\theta}_k^*)[1 - \Phi_{jk}(\boldsymbol{\theta}_k^*)]} \right) L_{lk}(\boldsymbol{\theta}_k^*) \frac{\partial \Phi_{jk}(\boldsymbol{\theta}_k^*)}{\partial \nu_j} g(\theta_k)\, d\theta_k \right\} g(\theta_1)\, d\theta_1. \quad (14)$$

Following Bock and Aitkin (1981), marginal likelihood equations can be solved, using the EM algorithm of Dempster, Laird, and Rubin (1977), by replacing the integrals with Gauss-Hermite quadratures and rearranging terms into the two-dimensional form:

$$\frac{\partial \log L}{\partial \nu_j} \cong \sum_{q1}^{Q} \sum_{k=2}^{s} c_{jk} \sum_{qk}^{Q} \frac{\bar{r}_{jk}(\mathbf{X}) - \bar{N}_k(\mathbf{X})\Phi_{jk}(\mathbf{X})}{\Phi_{jk}(\mathbf{X})[1 - \Phi_{jk}(\mathbf{X})]} \left(\frac{\partial \Phi_{jk}(\mathbf{X})}{\partial \nu_j} \right) A(X_{qk}) A(X_{q1}), \quad (15)$$

where

$$\bar{r}_{jk}(\mathbf{X}) = \sum_{l=1}^{s} r_l x_{lj}[E_{lk}(X_{q1})] \frac{L_{lk}(X_{q1}, X_{qk})}{P_l}, \tag{16}$$

$$\bar{N}_k(\mathbf{X}) = \sum_{l=1}^{s} r_l[E_{lk}(X_{q1})] \frac{L_{lk}(X_{q1}, X_{qk})}{P_l}, \tag{17}$$

and $\mathbf{X} = [{}^{X_{q1}}_{X_{qk}}]$. These equations are similar to those in the unrestricted case, except that in the bi-factor case, the conditional probability of response pattern x_{lk} (i.e., responses to items $j = 1, \ldots, n_k$ in subsection k for response pattern l) is weighted by the factor, $E_{lk}(X_{q_1})$. Furthermore, since each item appears in one subsection only (k), $\bar{r}$ and $\bar{N}$ now vary with k, in contrast to the unrestricted case. $\bar{N}_k$ denotes the effective sample size for subset k at quadrature point (X_{q_1}, X_{q_k}) and $\bar{r}_{jk}$ the corresponding expected number of positive responses. When weighted by $A(\mathbf{X})$ and summed over quadrature nodes for each subsection, $\bar{N}_k$ yields total number of respondents, whereas corresponding weighting and summation for $\bar{r}_{jk}$ yields total number of respondents answering item j correctly.

From provisional parameter values, each E-step yields $\bar{r}_{jk}$ and $\bar{N}_k$, (expectations of complete data statistics computed conditionally on incomplete data, see Bock, Gibbons, & Muraki, 1988). The subsequent M-step solves (15) using conventional maximum likelihood multiple probit analysis, substituting provisional expectations of $\bar{r}_{jk}$ and $\bar{N}_k$ (see Bock & Jones, 1968).

Illustration

To illustrate application of the bi-factor IRT model, we have evaluated 20 items selected from an ACT natural science test for a random sample of 1000 examinees (we are indebted to Terry Ackerman and Mark Reckase for these data). This test involves a series of questions regarding each of four paragraphs. For this illustration, we selected the first five items from each of four paragraphs. Table 1 displays the unrestricted Promax-rotated 4-factor solution, which adequately fits these data (improvement in fit of a four-factor model over a three-factor model was $\chi^2_{17} = 31.59$, $p < .02$; improvement in fit of five factors over four factors was not significant ($\chi^2_{16} = 18.44$, $p < .30$)). In this size sample with 20 items, most patterns were realized only once, and expected frequencies are near zero. In such cases, the usual chi-square approximation for the distribution of the multinomial goodness of fit statistic is inaccurate (e.g., Table 4). Haberman (1977) has shown, however, that the difference in fit statistics for alternative models is distributed in large samples as chi-square, with degrees of freedom equal to the difference in numbers of degrees of freedom, even when the frequency table is sparse. These difference statistics will be used in the comparison of alternate models.

Inspection of Table 1 reveals that each factor is dominated by items from a particular paragraph. In contrast, the estimated factor loadings for the bi-factor model (see Table 2) with $s = 5$ (i.e., one primary dimension and four paragraph-specific dimensions) revealed a strong general ability dimension, as well as appreciable within paragraph associations. Fit of the restricted model was not significantly different from the fit of either the four-factor ($\chi^2_{45} = 23.83$, $p < .99$) or the five-factor ($\chi^2_{60} = 43.22$, $p < .95$) unrestricted models. Inspection of loadings within each paragraph reveals that intra-paragraph item associations are quite variable.

The numerical precision of the bi-factor solution represents a major computational improvement over the unrestricted solution. Given the bi-factor solution requires only approximation of a two-dimensional integral, it was possible to use 100 quadrature points (i.e., ten in each dimension) instead of the 243 quadrature points used in the

TABLE 1

Full-Information Item Factor Analysis - Unrestricted Promax Solution
ACT Natural Science Test - 20 items and 1000 subjects

Item	γ_j	α_{j1}	α_{j2}	α_{j3}	α_{j4}
1	-.215	**.401**	-.005	-.036	.216
2	-.385	**.185**	-.019	-.007	.105
3	-.356	**.667**	-.070	-.081	-.081
4	-.098	**.619**	.013	.044	-.022
5	-.029	**.562**	-.092	-.059	.119
6	-.582	.129	.068	**.256**	.030
7	-.585	.184	-.211	**.419**	.102
8	-.137	-.037	-.061	**.025**	.172
9	-.246	.238	.063	**.362**	-.284
10	-.089	-.224	.128	**.620**	.060
11	-.049	.182	.135	-.034	**.311**
12	-.407	-.024	-.065	.124	**.320**
13	-.265	.247	.082	.020	**.173**
14	-.051	.137	.005	.007	**.585**
15	.040	.224	.129	-.045	**.295**
16	.345	.153	**.289**	-.122	-.109
17	.167	-.007	**.682**	.089	-.044
18	.172	-.096	**.520**	-.024	.120
19	.543	.008	**.500**	.067	.091
20	.672	-.073	**-.010**	.004	.163

unrestricted five factor solution (i.e., three in each dimension). Five factors probably represents the highest dimensional solution that is computational tractable at this time. Parameters of the unrestricted models were estimated using the TESTFACT program (Wilson, Wood, & Gibbons, 1984). This limitation of the unrestricted multiple factor model suggests that to some extent, the previously described comparison of chi-square statistics is questionable since it represents differences in model parameterization and accuracy in numerical approximation. Although Bock and Aitkin (1981) have shown the latter effect is reasonably small, it illustrates an additional benefit of the bi-factor restriction, where sufficiently large numbers of quadrature points per dimension are always available, and accuracy of numerical integration is never at issue.

A Simple Structure Model

Consider an orthogonal simple structure factor model in which each item loads on only one of s dimensions. This satisfies a complete simple structure model as defined by

TABLE 2

Full-Information Item Bi-Factor Analysis
ACT Natural Science Test - 20 items and 1000 subjects

Item	γ_j	α_{j1}	α_{j2}	α_{j3}	α_{j4}	α_{j5}
1	-.230	.524	.129			
2	-.392	.232	.115			
3	-.370	.411	.427			
4	-.118	.548	.278			
5	-.046	.489	.338			
6	-.593	.311		.277		
7	-.600	.376		.314		
8	-.138	.087		-.019		
9	-.259	.207		.390		
10	-.103	.226		.476		
11	-.062	.484			.141	
12	-.413	.261			.135	
13	-.277	.423			.199	
14	-.066	.573			.187	
15	.025	.492			.260	
16	.340	.112				.261
17	.150	.306				.662
18	.160	.240				.571
19	.528	.340				.493
20	.671	.061				.031

Thurstone (1947), which could be evaluated using methods for confirmatory factor analysis for measurement data (Jöreskog, 1969). This is a simplification of the bi-factor model in which there is no primary dimension. In this case, the unconditional probability in (10) is reduced to the unidimensional form,

$$P_l \cong \prod_{k=1}^{s} \left\{ \sum_{qk}^{Q} \left[\prod_{j=1}^{n} ([\Phi_{jk}(X_{qk})]^{x_{lj}}[1 - \Phi_{jk}(X_{qk})]^{1-x_{lj}})^{c_{jk}} \right] A(X_{qk}) \right\}, \tag{18}$$

where

$$\Phi_{jk}(X_{qk}) = \Phi\left(\frac{\gamma_j - \alpha_{jk}X_{qk}}{(1 - \alpha_{jk}^2)^{1/2}}\right);$$

that is, (10) reduces to the product of the s independent unidimensional probabilities. The likelihood equations in (15) can then be approximated by

$$\frac{\partial \log L}{\partial \nu_j} \cong \sum_{k=1}^{s} c_{jk} \sum_{qk}^{Q} \frac{\bar{r}_{jk}(X_{qk}) - \bar{N}_k(X_{qk})\Phi_{jk}(X_{qk})}{\Phi_{jk}(X_{qk})[1 - \Phi_{jk}(X_{qk})]} \left(\frac{\partial \Phi_{jk}(X_{qk})}{\partial \nu_j}\right) A(X_{qk}), \quad (19)$$

where

$$\bar{r}_{jk}(X_{qk}) = \sum_{l=1}^{S} \frac{r_l x_{lj} L_{lk}(X_{qk})}{e_{lk}}, \quad (20)$$

and

$$\bar{N}_k(X_{qk}) = \sum_{l=1}^{S} \frac{r_l L_{lk}(X_{qk})}{e_{lk}}. \quad (21)$$

In this case, e_{lk} represents the constant

$$e_{lk} = \sum_{qk}^{Q} L_{lk}(X_{qk}) A(X_{qk}),$$

and

$$P_l = \prod_{k=1}^{s} e_{lk};$$

$\bar{r}_{jk}$ and $\bar{N}_k$ now contain information from the specific subset of items (k) only, for which item j is a member. This is due to independence between subsets resulting from the simple structure.

Application of the simple structure model to the ACT natural science test example yields item-parameters displayed in Table 3. Inspection of parameter estimates in Table 3 reveals that removal of the primary factor increases the magnitude of loadings on the individual paragraph dimensions. For model fit, the bi-factor model ($\chi^2_{20} = 336$, $p < .0001$) and the unrestricted four-factor model ($\chi^2_{65} = 361$, $p < .0001$) provide significant improvements in fit over the simple structure model, indicating the test in fact measures a primary ability dimension and not four independent knowledge realms.

An Example from Psychiatric Research

Psychometric properties of psychiatric symptom rating scales have not been rigorously studied (see Gibbons, Clark, & Cavenaugh, 1985, for a review). Psychiatric symptom rating scales are inherently multidimensional. For example, the Hamilton Depression Rating Scale (HDRS) is a widely-used instrument for assessment of depression severity, and contains symptom-items ranging from mild psychological impairment (e.g., depressed mood) to severe psychopathology (e.g., suicidal ideation and/or attempts). The HDRS includes several dimensions of depressive illness, including sleep disorder, vegetative signs such as motor retardation, and anxiety. Full-information item factor analysis of the 17-item version of the HDRS obtained from a sample of 351 drug-free patients with major depression (see Table 4) revealed a five-factor structure.

TABLE 3

Full-Information Simple Structure Item Factor Analysis
ACT Natural Science Test - 20 items and 1000 subjects

Item	γ_j	α_{j1}	α_{j2}	α_{j3}	α_{j4}
1	-.224	.482			
2	-.391	.251			
3	-.368	.571			
4	-.111	.612			
5	-.040	.585			
6	-.592		.408		
7	-.597		.467		
8	-.138		.032		
9	-.258		.429		
10	-.102		.509		
11	-.056			.489	
12	-.412			.297	
13	-.273			.449	
14	-.058			.591	
15	.031			.566	
16	.341				.282
17	.157				.732
18	.163				.616
19	.534				.597
20	.671				.057

(We are grateful to David Kupfer of Western Psychiatric Institute at the University of Pittsburgh for providing us with these data). In addition to a primary depressive dimension, there appear to be dimensions of sleep disorder, loss of insight, motoric retardation, and appetite disturbance. Table 5 presents results of a full-information item bi-factor analysis of these data using one primary dimension and 4 subdimensions. Bi-factor loadings were virtually identical to dominant loadings for the unrestricted five-factor solution. This is in part expected, since items comprising the four subdimensions were selected on the basis of dominant loadings of Factors 2 through 5 of the unrestricted five-factor solution. The difference in fit between the bi-factor and unrestricted five-factor models was significant ($\chi^2_{41} = 111$, $p < .0001$), which suggests some symptom-items load on more than one subdimension. Practically speaking, interpretation based on either model is highly similar, and the bi-factor solution provides

TABLE 4

Five-Factor Model
Hamilton Depression Rating Scale Data ($N = 351$)

Symptom	Threshold	Factor					
		1	2	3	4	5	6
Depressed Mood	-1.8	**.5**	.0	.2	**.6**	.2	
Guilt	0.0	**.5**	.1	.1	.2	.2	
Suicidal	0.4	**.5**	.1	.0	.2	.1	
Initial Insomnia	-0.5	.1	**.2**	-.1	-.1	**-.3**	
Middle Insomnia	-0.4	.0	**.6**	.0	-.1	-.1	
Late Insomnia	-0.3	.0	**.8**	.2	.0	**-.3**	
Problem Working	-1.0	**.5**	.2	.2	**.5**	-.1	
Motor Retardation	0.4	.1	-.1	-.2	**.7**	.0	
Agitation	0.0	**.5**	-.1	.0	.1	-.2	
Psychic Anxiety	-0.8	**.7**	-.1	.0	.0	-.2	
Somatic Anxiety	0.0	**.7**	**-.3**	.0	-.1	.0	
Appetite	-0.3	.0	-.1	.1	.1	**.8**	
Anergia	-1.4	.1	.2	.2	**.7**	-.1	
Loss of Sex Interest	-0.5	**.4**	.1	**.3**	**.3**	.1	
Hypochondriasis	0.3	.1	**-.2**	-.1	.0	.0	
Loss of Insight	1.0	.0	-.1	**-.8**	-.1	.0	
Weight Loss	0.5	.1	-.1	**.5**	.0	**.5**	
χ^2		2616	2518	2466	2431	2401	2383
df		316	300	285	271	258	246
Change in χ^2			98	53	34	30	18
df			16	15	14	13	12
probability			.000	.000	.002	.004	.113

enormous computational savings relative to the unrestricted solution (i.e., 2-dimensional versus s-dimensional quadrature approximation).

The simple structure model was also fit to these data to determine if a primary depressive dimension exists (see Table 6). Dominant items on each dimension from the unrestricted five-factor solution were used to define five independent dimensions of the simple structure model. The bi-factor model provided a significant improvement in fit

TABLE 5

Bi-Factor Model (Five-Dimensional)
Hamilton Depression Rating Scale Data ($N = 351$)

Symptom	Threshold	Factor				
		1	2	3	4	5
Depressed Mood	-1.8	.7			.5	
Guilt	0.0	.6				-.2
Suicidal	0.4	.5			.0	
Initial Insomnia	-0.5	.0	.3			
Middle Insomnia	-0.5	-.1	.8			
Late Insomnia	-0.4	.1	.6			
Problem Working	-1.0	.6			.5	
Motor Retardation	0.4	.2			.5	
Agitation	0.0	.5				.1
Psychic Anxiety	-0.8	.6				.1
Somatic Anxiety	0.0	.5	.3			
Appetite	-0.3	.1				.7
Anergia	-1.4	.2			.7	
Loss of Sex Interest	-0.5	.5		.3		
Hypochondriasis	0.3	.0	.2			
Loss of Insight	1.0	-.2		-.7		
Weight Loss	0.5	.2				.6

Note: $\chi^2_{299} = 2512$

over the simple structure model ($\chi^2_{17} = 75$, $p < .0001$), suggesting that a primary depressive dimension is required to fully describe these data.

Analysis results suggest that depression (measured by the HDRS) is a multidimensional disorder consisting of a primary depressive severity dimension and four subdimensions reflecting sleep disturbance, loss of insight, motoric retardation, and eating disorder. The unrestricted five-factor model provided a significant fit improvement over the bi-factor model suggesting the four subdimensions are not independent, as the bi-factor model assumes.

Discussion

This bi-factor model provides a natural alternative to the traditional conditionally-independent unidimensional IRT model. When conditional dependence is suspected, as in the case of paragraph comprehension tests or tests in which two or more methods of

TABLE 6

Simple Sructure Model
Hamilton Depression Rating Scale Data ($N = 351$)

Symptom	Threshold	Factor 1	2	3	4	5
Depressed Mood	-1.8				.9	
Guilt	0.0	.5				
Suicidal	0.4	.5				
Initial Insomnia	-0.5		.3			
Middle Insomnia	-0.5		.8			
Late Insomnia	-0.4		.6			
Problem Working	-1.0				.8	
Motor Retardation	0.4				.5	
Agitation	0.0	.5				
Psychic Anxiety	-0.8	.6				
Somatic Anxiety	0.0	.6				
Appetite	-0.3					.7
Anergia	-1.4				.7	
Loss of Sex Interest	-0.5	.4				
Hypochondriasis	0.3		.2			
Loss of Insight	1.0			.5		
Weight Loss	0.5					.7

Note: $\chi^2_{316} = 2587$

item presentation are involved, the item bi-factor solution provides an excellent alternative. An attractive by-product of this model is that it requires only the evaluation of a two-dimensional integral, regardless of the number of subtests, paragraphs, or content areas. Of course, subsections (e.g., paragraphs) must be known in advance.

In certain situations (for example, psychiatric measurement) existence of a primary dimension (e.g., depression), is itself at question. In this case, comparison of the bi-factor and simple factor solutions can help answer whether depression is a unitary disorder or a series of qualitatively distinct abnormalities; a question that has long plagued psychiatric researchers. Using analysis of the HDRS as an illustration, it suggests depression consists of a common primary dimension and a series of correlated subdimensions. It should be clinically interesting that the dimension characterized by sleep disturbance (i.e., insomnia items), seems little related to anything else. In the bi-factor solution, these items have nearly zero loadings on the primary factor. Further research should focus on the relationship of insomnia items to the depressive factors.

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