PSYCHOMETRIKA—VOL. 55, NO. 4, 707–720
DECEMBER 1990

SOCIAL NETWORKS WITH TWO SETS OF ACTORS

DAWN IACOBUCCI

DEPARTMENT OF MARKETING
KELLOGG GRADUATE SCHOOL OF MANAGEMENT
NORTHWESTERN UNIVERSITY

STANLEY WASSERMAN

DEPARTMENTS OF PSYCHOLOGY AND STATISTICS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Traditional network research analyzes relational ties within a single group of actors; the models presented in this paper involve relational ties that exist between two distinct sets of actors. Statistical models for traditional networks in which relations are measured within a group simplify when modeling unidirectional relations measured between groups. The traditional paradigm results in a one-mode sociomatrix; the network paradigm considered in this paper results in a two-mode sociomatrix. A statistical model is presented, illustrated on a sample data set, and compared to its traditional counterpart. Extensions are discussed, including those that model multivariate relations simultaneously, and those that allow for the inclusion of attributes of the individuals in the group.

Key words: relational data, bipartite graphs, sociomatrix, categorical data analysis.

Introduction

Most standard statistical techniques are designed for subjects-by-variables data sets, where the focus of the analysis is to model the associations among the variables; for example, analysis of variance (Scheffé, 1959), regression (Draper & Smith, 1981), multidimensional scaling (Kruskal & Wish, 1978; Torgerson, 1958), and so on. These analyses generally aggregate over subjects who are assumed to provide independent observations, although exceptions exist. For example, the primary focus in factor analysis is generally the modeling of the covariance structure among the measures, but subjects may also be modeled, as in Q-factor analysis (Gorsuch, 1983), three-mode factor analysis (Tucker 1963, 1966, 1972), or in estimating subject-factor scores (Harman, 1976; McDonald, 1985). Another exception would be those methods developed for social networks. Briefly, a network is defined to be a group of actors, usually persons, who may interact in some way (e.g., social friendship ties, communication e-mail links, etc.). Social network models (described in Hubert & Baker, 1978; Kenny & LaVoie, 1984; Knoke & Kuklinski, 1982; Wasserman, 1987; Wasserman & Faust, 1990; Wasserman & Iacobucci, 1988) attempt to examine the structure of the interrelationships that exist among the actors, including the incorporation of information on properties of the actors to help understand the structural relations present (e.g., Fienberg & Wasserman, 1981; Wasserman & Iacobucci, 1986).

Usually, social network models and applications are considered for relations mea-

We thank the Editor and two anonymous reviewers for their helpful comments. We are also grateful to George Barnett for allowing us to analyze his data.

Requests for reprints should be sent to Dawn Iacobucci, Department of Marketing, Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL, 60208.

0033-3123/90/1200-89134$00.75/0

sured within a single group of actors. For example, classic studies in sociometry measured the friendship ties among all the children in a classroom. Thus, the network consists of a single closed set of actors and the friendship links among them. In this paper, we consider the case where relational ties exist between distinct groups of actors. For example, we might be interested in modeling relations that originate with adults and are directed toward children (e.g., scolds, helps, etc.) or from buyers to sellers (e.g., makes payment, requests information, etc.); the network relations flow from one set of actors (e.g., adults, buyers) to another (children, sellers). We discuss the two-group case in detail, but briefly describe the extension to three or more groups. In the first section of this paper, we distinguish the new two-group situation from the standard one-group case. A statistical model for the two-group situation is presented in the second section, and demonstrated on an example data set in the third. The fourth section discusses extensions of the model.

Terminology and Notation

To differentiate the current discussion from previous research, we refer to the modeling of unidirectional relations observed in a two-mode network, or on a bipartite directed graph. Each term will now be defined.

Standard social network research (and even early sociometry) tabulates relational ties between actors in a sociomatrix $\mathbf{X}$ with entries:

$$\begin{aligned} X_{ij} &= 1 \quad \text{if actor } i \text{ relates to actor } j, \text{ for } i, j = 1, 2, \ldots, g; \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{1}$$

A sociomatrix has a row and column for each of g actors in the set of all actors G so that $\mathbf{X}$ is $g \times g$. The diagonal is usually taken to be zero; relations such as friendship or communication ties are not considered to be reflexive. Equation (1) defines the simplest sociomatrices, those representing binary relations—the presence or absence of a link between two actors in the network. Alternatively, the strength of that relation may be measured, and the sociomatrix could be defined more generally as:

$$\begin{aligned} X_{ij} &= k \quad \text{if actor } i \text{ relates to actor } j \text{ at strength } k; \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{2}$$

These standard sociomatrices may be referred to as one-mode matrices, drawing from Tucker's factor analysis terminology (Tucker 1966, 1972). A two-dimensional matrix may be one-mode or two-mode. A one-mode matrix contains the same entities as indices of both the rows and columns. A two-mode matrix has two different sets of entities as indices. For example, a subjects-by-variables data matrix is two-mode, since the rows delineate subjects and the columns, variables. A correlation matrix among variables is one-mode, since both the rows and columns represent a single set of entities—the variables.

Similarly, in the social network context, the terms one- and two-mode may be used to describe different network paradigms. Standard sociomatrices, defined by (1) or (2), are one-mode matrices since the single set of actors in G form both the rows ($i = 1, 2, \ldots, g$) and columns ($j = 1, 2, \ldots, g$). In contrast, we will focus not on the relations sent within a single set of actors, but on those relations that are sent between two sets of actors. The first set of actors is denoted by G ($i = 1, 2, \ldots, g$) and the second by H ($j = 1, 2, \ldots, h$), where g is generally not equal to h. The entries in a sociomatrix that tabulates the relations sent from actors in G to their partners in H are defined as follows:

$$X_{ij} = k \quad \text{if actor } i \text{ in set } G \text{ relates to actor } j \text{ in set } H \text{ with strength } k;$$
$$= 0 \quad \text{otherwise.} \tag{3}$$

The matrix defined in (3) is $g \times h$ and is two-mode, because the actors in G form the rows of $\mathbf{X}$ and the partners in H form the columns. Note that now there is no main diagonal and reflexive relations may not even be considered. (Some researchers might also call the matrix in (3) rectangular, but this term is not truly descriptive, since g may equal h, even though the sets of actors may still be distinct.)

Social network researchers who are not familiar with the mode terminology may instead use related graph theory concepts, since many graph theoretic concepts form the foundation of much of social network methodology (see Hage & Harary, 1983; Wong, 1989). In the graph theoretical perspective, actors are represented as points and the relational ties between any pair of actors is represented by a link or arc between the points (e.g., Bondy & Murty, 1976). A bipartite graph is a special type of graph in which the actors may be partitioned into two groups in such a way that all links are between-group links, and no within-group links exist (see Wilson, 1982, and Fararo and Doreian, 1984, for related issues on bipartite and tripartite graphs). These between-group links are those tabulated in two-mode matrices. For example, in the network where scolding is measured, the relation would flow from the group of adults to the group of children. We would expect to see few scoldings among the adults or scoldings among the children, and therefore would probably not even code for such interactions.

In this same context as above, few if any scoldings would originate with the group of children directed toward the group of adults, reflecting the property of unidirectionality. In two-mode networks, the relation flows in one direction—from the row actor ($i \in G$) to the column actor ($j \in H$), and not the reverse. For example, an adult might scold a child but a child would not scold an adult. Or, a corporation might make donations to a nonprofit organization, but the reverse would not be true. In more standard one-mode networks, the relation may flow in either direction—the actor i may relate to the actor j, and the actor j may reciprocate and relate to actor i. For example, actor i may claim j as a friend, and j might also claim actor i as a friend. Note that if a different relation is measured on the sets of actors, the relation might be unidirectional but flowing in the opposite direction (e.g., a child asks for help of an adult, or a nonprofit organization solicits support from an organization). Note also that the definition of a unidirectional relation tabulated in two-mode sociomatrices as defined in (3) would be represented by a *directed* bipartite graph, where the arcs originate in set G and flow to set H.

The Model

Modeling a single unidirectional relation in a two-mode network follows analogously to modeling the standard one-mode case, and in fact, is parametrically simpler. The model introduced by Holland and Leinhardt (1981) for the standard one-mode network measured on a single binary relation, is referred to as p_1. The model presented here, which might be labeled p_2, is a close relative of p_1, and is applicable to a network with two sets of actors. (The subscripts on p_1 and p_2 may serve as a mnemonic for one-mode and two-mode network models.)

To present the model, the likelihood function, and the method of parameter estimation and testing, a contingency table is defined, which is termed a **Y**-array and used in many different types of applications (beginning with Fienberg & Wasserman, 1981):

$$Y_{ijk} = 1 \quad \text{if actor } i \in G \text{ relates to actor } j \in H \text{ at strength } k;$$
$$= 0 \quad \text{otherwise.} \tag{4}$$

For binary data, k equals 0 or 1, and the **Y**-array is of size $g \times h \times 2$. For discrete or ordinal data in general, $k = 1, 2, \ldots, C$, and the **Y**-array is a $g \times h \times C$ contingency table. For each dyad, there will be a single "1" in one of the $k = 1, 2, \ldots, C$ cells, and the remainder will be zero.

These data are modeled using two primary sets of effects—one for the actors in G and one for the actors in H:

$$\ln P\{Y_{ijk} = 1\} = \lambda_{ij} + \theta_k + \alpha_{ik} + \beta_{jk}, \tag{5}$$

with the following constraints: $\theta_k = 0$ for $k = 0$, $\alpha_{ik} = 0$ for $k = 0$, $\Sigma_i\ \alpha_{ik} = 0$ for all k, $\beta_{jk} = 0$ for $k = 0$, $\Sigma_j\ \beta_{jk} = 0$ for all k. The $\{\lambda_{ij}\}$ ensure the probabilities sum to one over k for each dyad. The $\{\theta_k\}$ reflect the volume of the relations, like the density of the choices over the entire network, and will vary depending on whether the relations from G to H tend to be at low or high strengths (k). The two sets of effects of primary interest in typical network applications will be the $\{\alpha_{ik}\}$ and $\{\beta_{jk}\}$; the former represent the expansiveness of the actors in G—the tendency for each actor i to make relational ties at the various strengths k; the latter represent the attractiveness of the actors in H—tendencies for each actor j to receive relational ties at strength k. (Note that in contrast to the standard model p_1, there are no α's for actors in H nor β's for actors in G since the relation in the current application cannot originate in H and extend back to G. Furthermore, given the unidirectionality of the flow of the relation, there is no opportunity for reciprocal behavior. Thus, the reciprocity parameter of p_1 is irrelevant in (5).)

To make (5) mirror p_1 more closely, consider a binary relation ($k = 1, 2$; $C = 2$) and constrain $\{\alpha_{ik}\}$ and $\{\beta_{jk}\}$ so that the first columns in these $g \times 2$ and $h \times 2$ tables of parameters are zero. That is, propensities to *not* send a relation or receive a relation ($k = 1$) are forced to be zero, while propensities to send or receive relations ($k = 2$) are the nonzero expansiveness or attractiveness parameters.The model in (5) simplifies to:

$$\ln P\{Y_{ij1} = 1\} = \lambda_{ij};$$
$$\ln P\{Y_{ij2} = 1\} = \lambda_{ij} + \theta + \alpha_i + \beta_j. \tag{6}$$

(Note that the dependence on k is dropped, which no longer appears as a subscript when $C = 2$, because of the constraints that $\theta_k = 0$ for $k = 0$, $\alpha_i = 0$ for $k = 0$, $\beta_j = 0$ for $k = 0$. Model (5) is the more general statement for $C \geq 2$.)

The likelihood function that will enable parameter estimation and the fitting of models to data, is derived by assuming dyads to be independent. The assumption of independence is made to simplify the mathematics, and we consider its plausibility in the last section of this article. The log of the product of these dyadic probabilities defines the log likelihood function for the entire network:

$$\ln L(\{\lambda_{ij}\}, \{\theta_k\}, \{\alpha_{ik}\}, \{\beta_{jk}\} | \{y_{ijk}\})$$
$$= \sum_{i,j} \lambda_{ij} y_{ij+} + \sum_k \theta_k y_{++k} + \sum_i \sum_k \alpha_{ik} y_{i+k} + \sum_j \sum_k \beta_{jk} y_{+jk}. \tag{7}$$

TABLE 1

Models Fit to the 3-d Y-array on the Corporate-Promotions Data

Model	Margins	ΔG^2	Δdf	Tests
$\ell n\ P(Y_{ijk}=1) =$				
1) $\lambda_{ij} + \theta_k + \alpha_{ik} + \beta_{jk}$	[12][13][23]			
2) $\lambda_{ij} + \theta_k + \beta_{jk}$	[12][23]	$G^2(2)-G^2(1)=321.01^*$	164	H_0: all $\{\alpha_{ik}\}=0$
3) $\lambda_{ij} + \theta_k + \alpha_{ik}$	[12][13]	$G^2(3)-G^2(1)=141.56^*$	54	H_0: all $\{\beta_{jk}\}=0$

$^*p<.01$

Maximum likelihood estimates for log linear models are obtained by setting the sufficient statistics (the margins $\{y_{ij+}\}$, $\{y_{i+k}\}$, and $\{y_{+jk}\}$) equal to their expected values under (5). The log linear model for y can be denoted by [12] [13] [23], which is Fienberg's (1980) notation for the interactions between Variables 1 and 2, 1 and 3, and 2 and 3, as well as all lower-order terms, since these models are hierarchical. Statistical tests for the significance of the contributions of the $\{\alpha_{ik}\}$'s to (5) are conducted by comparing the fit of (5) (i.e., [12] [13] [23]) with the special case of (5) where all $\{\alpha_{ik}\} = 0$ (i.e., [12] [23]). Alternatively, to test H_0: all $\{\beta_{jk}\}$'s $= 0$, the fit of (5) is compared to the fit of the model [12] [13]. Parameter estimation and testing, and the likelihood ratio test statistic ($G^2 = \Sigma_{ijk} y_{ijk} \ln (y_{ijk}/\hat{y}_{ijk})$, where $\hat{y}$ is the corresponding fitted cell value), is described in more detail in Wasserman and Weaver (1985).

An Example

As an example of model fitting, parameter estimation and testing, and interpretation, we use a data set collected on a two-mode network where the set of actors in G are corporations and the set of actors in H are advertising agencies and public relations firms. The relational tie measured was a simple binary indicator of whether corporation i employed agency j. There were 825 corporations selected to represent service (e.g., banking, transportation) and industrial (e.g., pharmaceuticals, metal manufacturing) companies (Friedland, Barnett, & Danowski, 1988), and 125 promotional organizations selected from 1985 directories for advertising agencies and public relations firms. Because the y-array for these data would be $825 \times 125 \times 2$, and computationally unmanageable, a pseudo-random sample of 165 was selected of the 825 corporations ($g = 165$), and 25 advertising agencies and 30 public relations firms ($h = 55$). The y-array is now $165 \times 55 \times 2$, but still large enough that we have not provided a table containing this array. Briefly, for each of the 165×55 dyads (corporation i and promotional agency j), there are two cells, the first of which contains a 1 if actor i did not hire actor j, the second of which contains a 1 if actor i did in fact hire actor j. Note that a promotional agency j (those actors in H) would not hire a corporation i (those actors in G) so the relation of "hiring" is indeed unidirectional, and the network is two-mode.

The models fit to this 3-dimensional contingency table are listed in Table 1, along

TABLE 2

Frequency Distribution of Parameter Estimates

$\hat{\alpha}_{ik}$	frequency	$\hat{\beta}_{jk}$	frequency
5.1 - 5.5	3		
4.6 - 5.0	5		
4.1 - 4.5	16		
3.6 - 4.0	11		
3.1 - 3.5	35		
2.6 - 3.0	0		
2.1 - 2.5	0		
1.6 - 2.0	0		
1.1 - 1.5	0	1.1 - 1.5	5
.6 - 1.0	0	.6 - 1.0	14
.1 - .5	0	.1 - .5	27
-.4 - .0	0	-.4 - .0	0
-.9 - -.5	0	-.9 - -.5	0
-1.4 - -1.0	0	-1.4 - -1.0	0
-1.9 - -1.5	0	-1.9 - -1.5	0
-2.4 - -2.0	0	-2.4 - -2.0	0
-2.9 - -2.5	95	-2.9 - -2.5	9

with the test statistics. The significance tests of the $\{\alpha_{ik}\}$ and the $\{\beta_{jk}\}$ are also given, and show that both sets of parameters contribute significantly to the fit of the model. This result is true even given the size and sparseness of the corporate relations' array, and suggests heterogeneity of corporate expansiveness and promotion attractiveness. The estimates are given in Table 2.

The majority of the expansiveness parameters (α) are negative, reflecting the fact that many of the corporations have no relations to any of the promotional agencies. (We are treating these absences of links as sampling rather than structural zeros, since at least in theory, any corporation may hire any promotional firm.) Among those corporations with links to promotional agencies, the histogram indicates a greater variety of expansive tendencies. The beta estimates present a similar picture, with some promotional agencies not being used, and others used to different extents. In practice, we might have additional information (e.g., corporate industry, promotional agency size, etc.) that may be correlated with these estimates to further our interpretation.

Model Extensions

In this final section, we generalize (5) to apply to more network research settings. The first two extensions are the two-mode counterparts to models already presented for the more standard one-mode case—the inclusion of attribute variables (e.g., Fienberg & Wasserman, 1981) and multiple relations (e.g., Fienberg, Meyer, & Wasserman, 1985; Iacobucci, 1989; Iacobucci & Wasserman, 1988). The remaining two extensions are solutions to problems that exist only for the two-mode case.

Attribute Variables

Although the primary focus of network research is on the structural ties between actors, researchers may also find it useful to use a more traditional statistical focus, and study attributes of the actors themselves. With these two sources of information, patterns of relational ties may be understood as a function of the different characteristics of the actors. For example, in studying friendship choices in young children, gender should be a critical attribute. Attribute information has been treated in various ways, but the most common assumption is that actors who share similar properties (e.g., education level of adults, size of corporation, sex of children, etc.) will behave similarly. Specifically, the assumption is that the pattern of relations is expected to show structural equivalence (see Lorrain & White, 1971; White, Boorman, & Breiger, 1976).

Briefly, two actors are structurally equivalent if they relate to all other actors in the network in the same way, and in turn are related to by all other actors in the same way. For example, two structurally equivalent boys in a network of friendship ties would claim the same other children as friends, and those children who chose one of the boys would also choose the other boy. More recently, researchers have relaxed this assumption to allow for probabilistic tendencies toward equivalence (Wasserman & Anderson, 1987), rather than the more demanding and exact structural equivalence. In our previous analysis of standard one-mode networks, individuals with the same characteristics were aggregated and the resulting subgroups were modeled rather than the individual actors. A similar strategy is taken here.

Before presenting the models that use this information, equivalence must be redefined for a two-mode network. For one-mode networks, in which bidirectional relations are measured, equivalence considers both the sending and receiving perspectives. That is, for actors i and j to be equivalent, all choices from i must resemble the choices from j, and all choices directed to i must resemble the choices to j. In two-mode networks, the definition of equivalence can be simplified, since the relations are inherently unidirectional.

Equivalence is defined separately for the actors in G and for those in H. Two actors in G, i and i', are equivalent if the choices they make to actors in H are identical. Two actors in H, j and j', are equivalent if the choices they receive from actors in G are identical. Thus, one evaluates (or assumes) the equivalence of actors in G separately from the consideration of equivalence of actors in H. Seeking equivalence in the standard one-mode case requires the simultaneous analysis of the rows and columns of the sociomatrix (i.e., the columns of the sociomatrix and its transpose as in Arabie, Boorman, & Levitt, 1978). Analyzing a two-mode network for equivalence of actors in G would require focusing on the rows of the sociomatrix; analyzing the network for equivalence of actors in H would require focusing on the columns of the sociomatrix.

The assumption of stochastic equivalence is imposed when actors who share similar characteristics are aggregated. The number of subgroups formed by the actors in G is denoted by S. For example, if the sex of actors in G is noted, and we aggregated over boys and girls, $S = 2$. The number of subgroups formed for the actors in h is denoted by T. The actors in G ($i = 1, 2, \ldots, g$) and H ($j = 1, 2, \ldots, h$) are mapped into one of the mutually exclusive and exhaustive categories using mapping functions $s(\cdot)$ and $t(\cdot)$ ($s(i) = 1, 2, \ldots, S$ for actors in G and $t(j) = 1, 2, \ldots, T$ for actors in H). This mapping results in aggregating over the $g \times h \times C$ contingency table defined in (4) to form a $S \times T \times C$ **W**-array defined below:

$$W_{s(i)t(j)k} = \sum_{i \in s(i)}^{S} \sum_{j \in t(j)}^{T} y_{ijk}. \tag{8}$$

Note that special cases of **W** are defined when there is attribute information on one set of actors but not on the other. We can either define a one-to-one mapping or (8) can be simplified in the following manner:

$$\text{For attributes in } G \text{ but not } H: \quad W_{s(i)jk} = \sum_{i \in s(i)}^{S} y_{ijk}; \tag{9}$$

$$\text{for attributes in } H \text{ but not } G: \quad W_{it(j)k} = \sum_{j \in t(j)}^{T} y_{ijk}. \tag{10}$$

Model (5) can be rewritten as it would apply to the **W**-array; its appearance is similar, but the parameterization is simpler. Stochastic equivalence is assumed so that actors in G who are equivalent share an α, and actors in H who are equivalent share a β:

$$\ln P\{Y_{ijk} = 1\} = \lambda^{s(i)t(j)} + \theta_k + \alpha_k^{s(i)} + \beta_k^{t(j)}, \tag{11}$$

with the following constraints: $\theta_k = 0$ for $k = 0$, $\alpha_k^{s(i)} = 0$ for $k = 0$,

$$\sum_{s(i)} \alpha_k^{s(i)} = 0 \text{ for all } k, \quad \beta_k^{t(j)} = 0 \text{ for } k = 0, \quad \sum_{t(j)} \beta_k^{t(j)} = 0 \text{ for all } k.$$

The model is still written at the level of the dyad—the individual actors i and j. However, the model has fewer parameters, since they are shared by all actors i and i' in $s(i)$, and j and j' in $t(j)$. In (5), there are $(g - 1)(C - 1)$ and $(h - 1)(C - 1)$ degrees of freedom for the α and β parameters, since a set of $(C - 1)$ parameters is estimated for each individual actor i and j. In (11), there are fewer parameters, given the simplifying assumption of stochastic equivalence. The degrees of freedom for the α and β of (11) are $(S - 1)(C - 1)$ and $(T - 1)(C - 1)$, respectively. S and T can be far fewer than g and h; for example when sex is an attribute variable ($T = 2$) for a group of 100 children ($h = 100$).

To demonstrate the use of attributes in the modeling, we return to the corporate-promotion data. There are many attributes to choose from, such as the size of the organization (using, say pre-tax annual income—Galaskiewicz, 1985), its industry, characteristics of its parent corporation, and so on. We have chosen to continue to model the actors in G (the corporations) as individual actors, so $S = g = 165$. The actors in H come in one of two varieties: 25 advertising agencies and 30 public relations firms. Thus, $T = 2$, which is much less than $h = 55$. Since we are not aggregating over corporations, the assumption of stochastic equivalence among them is not needed in their choices of promotional agencies hired. However, forming two subgroups of the promotional agencies requires that different ad agencies are chosen with equal tendencies, and similarly that all the public relations firms are chosen with equal tendencies. This view may be too simplistic, and additional attributes of the ad agencies and public relations firms may be required for a better fitting model and a more complete under-

TABLE 3

Models Fit to the 3-d W-array

Model	Margins	ΔG^2	Δdf	Tests
$\ell n\, P\{Y_{ijk}=1\} =$				
$\lambda^{it(j)} + \theta_k +$				
1) $+ \alpha_{ik} + \beta_k^{t(j)}$	[12][13][23]			
2) $+ \beta_k^{t(j)}$	[12][23]	$G^2(2)-G^2(1)=314.50^*$	164	H_0: all $\{\alpha_{ik}\}=0$
3) $+ \alpha_{ik}$	[12][13]	$G^2(3)-G^2(1)=$ 1.47	1	H_0: all $\{\beta_k^{t(j)}\}=0$

		$\beta_k^{t(j)}$ estimate	
		k=0	k=1
ADVERTISING	t(j)=1	0	.093
PUBLIC RELATIONS	t(j)=2	0	-.093

$^*p<.01$

standing of the data. But we continue with the two subgroups on H to demonstrate fitting model (11) to the **W**-array defined in (10).

Table 3 contains the models fit to the **W**-array formed from these data. The fit statistics are also listed and indicate that the expansiveness parameters are significant. (These estimates are highly correlated ($r = .99$) with the alpha estimates resulting from fitting (5) to the **Y**-array in the previous modeling.) The β estimates are not significant suggesting that ad agencies are no more or less attractive than public relations firms when a corporation makes the decision to support promotional campaigns. The β parameters require only $(T - 1)(C - 1) = (2 - 1)(2 - 1) = 1$ degree of freedom; the 2×2 table of the parameter estimate is also given in Table 3. (There is no correlation between the two sets of beta estimates from the two different models, due to the presence of little variation in the beta estimates in the second modeling (i.e., 1 df). When each ad agency is represented by the $\hat{\beta}$ they share because of the assumption of equivalence (.093 for $t(j) = 1$), and each public relation firm is represented by the $\hat{\beta}$ they share ($-.093$ for $t(j) = 2$), the correlation coefficient is .039.)

Multivariate Models

In this section, (5) and (11) for the **Y**- and **W**-arrays are extended to allow for two or more (in general, R) relations to be measured on the same groups of actors. For example, we might observe the following ties between the corporations and the promotional agencies: corporation i in G hires agency j in H, corporation i pays agency j, corporation i sends referrals to agency j, and so on. The first relation between actors i and j is denoted by the subscript k_1, where $k_1 = 1, 2, \ldots, C_1$. The remaining relations may be denoted by $k_2 = 1, 2, \ldots, C_2$, $k_3 = 1, 2, \ldots, C_3$, and so on. The Y- or W-arrays would be $(2 + R)$-dimensional; for $R = 3$, these are sizes $g \times h \times C_1 \times C_2 \times C_3$, and $S \times T \times C_1 \times C_2 \times C_3$, respectively. The **Y**-array for three relations has elements:

$$Y_{ijk_1k_2k_3} = 1 \quad \text{if actor } i \text{ in } G \text{ relates to actor } j \text{ in } H \text{ at strength } k_1 \text{ on the first relation, } k_2 \text{ on the second, and } k_3 \text{ on the third;}$$

$$= 0 \quad \text{otherwise.} \tag{12}$$

Model (5) can be extended to include parameters that reflect the associations among these different relational ties. One example is:

$$\ln P\{Y_{ijk_1k_2k_3} = 1\} = \lambda_{ij} + \theta_{(1)k_1} + \theta_{(2)k_2} + \theta_{(3)k_3} + \alpha_{(1)ik_1} + \alpha_{(2)ik_2} + \alpha_{(3)ik_3}$$
$$+ \beta_{(1)jk_1} + \beta_{(2)jk_2} + \beta_{(3)jk_3} + \theta_{(12)k_1k_2} + \theta_{(13)k_1k_3} + \theta_{(23)k_2k_3}, \tag{13}$$

with constraints similar to those for (5). Model (13) contains an expansiveness parameter (α) for each of the three relations, since the tendencies for actor i to hire, pay, or send referrals to j may differ. The three β's represent the attractiveness parameters of the three different relational structures. The remaining three θ's are known as multiplexity parameters (e.g., Fienberg et al. 1985), and are pairwise associations between the relations.

The log linear model for (13) is the model that contains all two-way interactions: [12] [13] [14] [15] [23] [24] [25] [34] [35] [45]. Higher-order associations might also be fit. For example, a parameter such as $\theta_{(123)k_1k_2k_3}$ (with its sufficient statistic [345]) would suggest a three-way interaction among the three relational variables. A parameter such as $\theta_{ik_1k_2}$ would suggest that the association among the first two relations (k_1 and k_2) varies depending on the actors i in G.

We have been careful to choose three relations that all flow from actors in G to actors in H. If there were unidirectional relations that flowed in the opposite direction, such as promotional agency j in H sends a bill to corporation i in G, one of two strategies may be followed. The first and simplest is to reword the relation with its passive form so that all relations flow in the same direction (e.g., corporation i in G receives a bill from agency j in H). Then, associative parameters between other relations and the billing relation would simply be translated in the interpretation. For example, assuming payment and billing are highly associated, the parameter could be interpreted as a multiplex relation: corporation i receives a bill and pays the bill to agency j, or as an exchange parameter with the two original directions of the relational flows: agency j billed corporation i and in exchange, corporation i paid agency j. Note that the meaning of the α and β parameters would reverse, so that the α's would represent the differential tendencies for actors in G in attracting bills, and the β's would represent the differential tendencies for actors in H to send bills.

The alternative to simply rewording the relations would be to create a one-mode network by concatenating the sets of actors in G and H. However, the size of the

Y-array will become even larger. A two-mode network for a single binary relation on these data would be of size $165 \times 55 \times 2$. The comparable one-mode Y-array would be of size $220 \times 220 \times 2 \times 2$. The researcher should simply reword the relation or use the methods in Wasserman and Iacobucci (1989), to be discussed shortly.

Multivariate models can be easily postulated and fit for as many unidirectional (in either direction) relations that are measured and whose simultaneous analysis is desired. The subgroupings of actors in G and/or H becomes even more useful in multivariate models to keep the size of the contingency tables relatively small. The relations need not be substantively different relations, but alternatively, may be the same relation measured at different points in time, or some combination (e.g., Allison & Liker, 1982; Iacobucci & Wasserman, 1988). Thus, one may ask correlational questions, such as: are the ties on the first relation associated with the ties on the second relation?; or predictive questions such as: is the network structure on the relation at time 1 useful in predicting the network structure at time 2? (Wasserman, 1987; Wasserman & Iacobucci, 1988).

One-Mode and Two-Mode Networks Modeled Simultaneously

In Wasserman and Iacobucci (in press), models are described for the case where different relations are measured within G and within H, from G to H, and from H to G. For example, the two sets of within-group relations would form one-mode networks if studied in isolation (e.g., corporation i in G shares a board member with corporation i' in G, and agency j in H is represented at the same trade shows as agency j' in H). The between-group relations would form two-mode networks in isolation, and either may be modeled using the methods described here. In Wasserman and Iacobucci (1989), methods are presented for the simultaneous modeling of these four mini-networks or some subset, since relations in one of the networks may indeed affect the practices in relations of another sort.

More Than Two Groups of Actors

A researcher might wish to extend the models in this paper to the case of three or more groups of actors, and the models desired would be appropriate for a k-partite directed graph, where the actors may be partitioned into k groups, $G_1, G_2, \ldots, G_k$, and relational ties exist only between groups. When three or more groups are involved, the simplification from a one-mode, sparse sociomatrix to a two-mode sociomatrix is no longer possible. Relations from say G_1 to G_2 could be modeled separately from relations from G_2 to G_3 using methods described here, but to simultaneously model those sets of between-group relations requires building a matrix where the sets of actors are concatenated, so that both the rows and columns of the matrix represent all the actors ($i, j = 1, 2, \ldots, g_1, g_1 + 1, g_1 + 2, \ldots, g_1 + g_2$, etc.).

Final Considerations

We conclude with the discussion of the important assumption of dyadic independence. The assumption is made for mathematical convenience: it allows us to write the likelihood as a simple product of the probabilities of all dyads (or, the log likelihood as a simple additive function), greatly facilitating the estimation procedure. Specifically, the assumption requires that the probabilities of the social relations between any two actors i and j are independent of the probabilities of the social relations between any other two actors, q and z. When these actors are all distinct, the assumption of statistical independence is quite reasonable. For an example in a one-mode network, a clinical psychologist studying patterns of dyadic interactions between husbands and wives (e.g., Gottman, 1979) would have a social network in which i = husband 1, j = wife 1, q = husband 2, z = wife 2, and so on, where no husband or wife interacts with any other membes in this sparse network. For an example in a two-mode network, if corporations i and i' in G were competitors, they would hire distinct advertising agencies j and j' in H to prevent conflicts of interest, and it would be reasonable that these dyads were assumed to be statistically independent.

For other applications (i.e., when the actors are not distinct), the assumption of dyadic independence is more questionable. For a one-mode network example, children i and j liking each other might affect the probability that children i and q like each other. The plausibility of the assumption might even vary with networks. For example, we might assume that adults, unlike children, are capable of choosing their friends without considering the balance of relations with other friends. For a two-mode network example, a corporation i in G with a finite budget for advertising, might hire agency j in H, which would likely affect how many other agencies in H that corporation i might hire.

We also note that as g increases in a one-mode network (or g and/or h in a two-mode network), the assumption of independent dyads becomes more reasonable, for a fixed density (proportion of existant ties in the network). Conversely, for a fixed number of actors, the assumption of independent dyads becomes more reasonable as the network becomes less dense, such as in the case of the marital interactions described above.

Recently, researchers have begun investigating the properties of models such as the ones discussed here. In particular, Strauss and Ikeda (1988) investigate a pseudo-likelihood estimation procedure—a generalization of maximum likelihood using an approximate likelihood function that does not assume dyadic independence—for the standard, one-mode network case. The theoretical foundations of their work is found in Frank and Strauss (1986). The pseudo-likelihood is written as a function of each data point x_{ij}, conditional on the remainder of the data, unlike the methods using the true likelihood (as in this paper) in which the modeling unit is the dyad $D_{ij} = (x_{ij}, x_{ji})$, and not the components of the dyad. This conditioning enables the pseudolikelihood estimation procedure to model any interdependencies, and the assumption of independence is not needed. Their methods should also be applicable to this newer, two-mode network case. This application should be more straightforward because actors in H do not choose actors in G, so that dyads have fewer associations than in one-mode networks.

Strauss and Ikeda (1988) compared the performance of the standard maximum likelihood estimates (MLEs) to their maximum pseudolikelihood estimates (MPEs) in a simulation study, and in the analysis of the "like" relation measured on Sampson's monastery (Sampson, 1968). Under all conditions for which both MLEs and MPEs could be estimated, the two performed similarly. This comparison addresses the issue of how well the maximum likelihood estimation of model parameters performs even under conditions where the assumption of dyadic independence is known to be violated. Thus, this research suggests the assumption of dyadic independence might not be so restrictive—the simpler MLE methods can be used with less concern that violations of the assumption will greatly affect the nature of the results. Furthermore, the MLE and MPE parameter estimates were highly correlated in the reported analysis of the Sampson data. The main advantage in the use of MPEs is that there are conditions under which one can estimate the MPEs and not the MLEs. Thus, the MPE approach expands the applicability of the p_1 and its relatives, such as p_2.

References

Allison, P. D., & Liker, J. K. (1982). Analyzing sequential categorical data on dyadic interactions: A comment on Gottman. *Psychological Bulletin, 91*, 393–403.

Arabie, P. A., Boorman, S. A., & Levitt, P. R. (1978). Constructing blockmodels: How and why. *Journal of Mathematical Psychology, 17*, 21–63.

Bondy, J. A., & Murty, U. S. R. (1976). *Graph theory with applications*. New York: North-Holland.

Draper, N., & Smith, H. (1981). *Applied regression analysis* (2nd ed.). New York: Wiley.

Fararo, T. J., & Doreian, P. (1984). Tripartite structural analysis: Generalizing the Breiger-Wilson formalism. *Social Networks, 6*, 141–175.

Fienberg, S. E. (1980). *The analysis of cross-classified categorical data* (2nd ed.). Cambridge, MA: The MIT Press.

Fienberg, S. E., Meyer, M. M., & Wasserman, S. (1985). Statistical analysis of multiple sociometric relations. *Journal of the American Statistical Association, 80*, 51–67.

Fienberg, S. E., & Wasserman, S. (1981). Categorical data analysis of single sociometric relations. In S. Lienhardt (Ed.), *Sociological methodology* (pp. 156–192). San Francisco: Jossey-Bass.

Frank, O., & Strauss, D. (1986). Markov graphs. *Journal of the American Statistical Association, 81*, 832–842.

Friedland, M. H., Barnett, G. A., & Danowski, J. A. (1988, February). *The network structure of American industrial and service organizations based upon shared public relations and advertising firms*. Paper presented to the Sunbelt Social Networks Conference, San Diego, CA.

Galaskiewicz, J. (1985). *Social organizations of an urban grant economy: A study of business philanthropy and nonprofit organizations*. Orlando, FL: Academic Press.

Gorsuch, R. L. (1983). *Factor analysis* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum.

Gottman, J. M. (1979). Detecting cyclicity in social interaction. *Psychological Bulletin, 86*, 338–348.

Hage, P., & Harary, F. (1983). *Structural models in anthropology*. Cambridge: Cambridge University Press.

Harman, H. H. (1976). *Modern factor analysis* (3rd ed.). Chicago: The University of Chicago Press.

Holland, P. W., & Leinhardt, S. (1981). An exponential family of probability distributions for directed graphs. *Journal of the American Statistical Association, 76*, 33–50.

Hubert, L. J., & Baker, F. B. (1978). Evaluating the conformity of sociometric measurements. *Psychometrika, 43*, 31–41.

Iacobucci, D. (1989). Modeling multivariate sequential dyadic inteactions. *Social Networks, 11*, 315–362.

Iacobucci, D., & Wasserman, S. (1988). A general framework for the statistical analysis of sequential dyadic interaction data. *Psychological Bulletin, 103*, 379–390.

Kenny, D. A., & LaVoie, L. (1984). The social relations model. In L. Berkowitz (Ed.), *Advances in experimental social psychology, 18* (pp. 141–182). New York: Academic Press.

Knoke, D., & Kuklinski, J. H. (1982). *Network analysis*. Beverly Hills: Sage.

Kruskal, J. B., & Wish, M. (1978). *Multidimensional scaling*. Beverly Hills: Sage.

Lorrain, F., & White, H. C. (1971). The structural equivalence of individuals in social networks. *Journal of Mathematical Sociology, 1*, 49–80.

McDonald, R. P. (1985). *Factor analysis and related methods*. Hillsdale, NJ: Lawrence Erlbaum.

Sampson, S. F. (1968). *A novitiate in a period of change: An experimental and case study of relationships*. Unpublished doctoral dissertation, Cornell University, Department of Sociology.

Scheffé, H. (1959). *The analysis of variance*. New York: Wiley.

Strauss, D., & Ikeda, M. (1990). Pseudolikelihood estimation for social networks. *Journal of the American Statistical Association, 85*, 204–212.

Torgerson, W. S. (1958). *Theory and methods of scaling*. New York: Wiley.

Tucker, L. R. (1963). Implications of factor analysis of three-way matrices for measurement of change. In C. W. Harris (Ed.), *Problems in measuring change* (pp. 122–137). Madison: University of Wisconsin Press.

Tucker, L. R. (1966). Some mathematical notes on three-mode factor analysis. *Psychometrika, 31*, 279–311.

Tucker, L. R. (1972). Relations between multidimensional scaling and three-mode factor analysis. *Psychometrika, 37*, 3–27.

Wasserman, S. (1987). Conformity of two sociometric relations. *Psychometrika, 52*, 3–18.

Wasserman, S., & Anderson, C. (1987). Stochastic a posteriori blockmodels: Construction and assessment. *Social Networks, 9*, 1–36.

Wasserman, S., & Faust, K. (1990). *Social network analysis: Methods and applications*. New York: Cambridge University Press.

Wasserman, S., & Iacobucci, D. (1986). Statistical analysis of discrete relational data. *British Journal of Mathematical and Statistical Psychology, 39*, 41–64.

Wasserman, S., & Iacobucci, D. (1988). Sequential social network data. *Psychometrika, 53*, 261–282.

Wasserman, S., & Iacobucci, D. (in press). Statistical modeling of one-mode and two-mode networks: Simultaneous analysis of graphs and bipartite graphs. *British Journal of Mathematical and Statistical Psychology.*

Wasserman, S., & Weaver, S. O. (1985). Statistical analysis of binary relational data: Parameter estimation. *Journal of Mathematical Psychology*, *29*, 406–427.

White, H. C., Boorman, S. A., & Breiger, R. L. (1976). Social structure from multiple networks I: Blockmodels of roles and positions. *American Journal of Sociology*, *81*, 730–780.

Wilson, T. P. (1982). Relational networks: An extension of sociometric concepts. *Social Networks*, *4*, 105–116.

Wong, G. (1989, April). *Maximum likelihood estimation of the p_1 model.* Paper presented at the Stockholm Conference on Random Graphs and Applications. Stockholm, Sweden.

Manuscript received 6/21/89
Final version received 12/11/89