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EQUIVALENT MODELS IN COVARIANCE STRUCTURE ANALYSIS

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Defining equivalent models as those that reproduce the same set of covariance matrices, necessary and sufficient conditions are stated for the local equivalence of two expanded identified models M_1 and M_2 when fitting the more restricted model M_0. Assuming several regularity conditions, the rank deficiency of the Jacobian matrix, composed of derivatives of the covariance elements with respect to the union of the free parameters of M_1 and M_2 (which characterizes model M_{12}), is a necessary and sufficient condition for the local equivalence of M_1 and M_2. This condition is satisfied, in practice, when the analysis dealing with the fitting of M_0, predicts that the decreases in the chi-square goodness-of-fit statistic for the fitting of M_1 or M_2, or M_{12} are all equal for any set of sample data, except on differences due to rounding errors.

Key words: equivalent models, covariance structure analysis, LISREL, EQS, model modification, modification index, Lagrange multiplier statistic, identification, regular point.

Introduction

Covariance structure analysis can be used to analyze models for linear relationships. At present, there are several programs available, such as LISREL 7 (Jöreskog & Sörbom, 1988), EQS (Bentler, 1989), and COSAN (McDonald, 1978), that can be used to estimate the strength of interrelationships between both observed and latent variables in a model, and to test whether a model fits the sample data.

Frequently, a researcher has to decide between several competing models. In this paper we will concentrate on a special class of competing models: the equivalent models. Jöreskog and Sörbom (1988, p. 224) define models to be equivalent when they represent different (conceptual) parameterizations of the same covariance matrix. Stelzl (1986) developed rules on how to generate equivalent models from the one under investigation. In this paper the phenomenon of equivalent models is approached in the light of a model modification process. Often an investigator starts with some baseline model (Luijben, Boomsma, & Molenaar, 1988), and if it does not fit the data well, tries to modify it. The discussion here considers models that form a nested sequence. Therefore, each model modification considered to improve the fit of a model implies a parameter relaxation. In general, both theoretical and statistical considerations are involved in the decision of which parameter has to be relaxed. It may happen, however, that statistical considerations are of no use in this choice, as in the case when two different parameter relaxations result in two equivalent models.

Sörbom (1989) developed the modification index (MI), which is an estimate of the decrease in the chi-square goodness-of-fit statistic when a single constraint is relaxed (both the Lagrange multiplier statistic, Bentler, 1986; Lee & Bentler, 1980; Silvey, 1959, and Score statistic, Satorra, 1989, p. 133, are equivalent). Assume that the MI's

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for two parameters θ_i and θ_j are equal. Such an equality suggests that relaxation of either θ_i or θ_j results in the same improvement in fit. A researcher fitting the baseline model would like to know whether the equality of the MI's is a necessary and/or sufficient condition for the equivalence of the expanded models. MacCallum (1986) implicitly suggested this by stating: "When MI's for all solutions were examined, an interesting, but not unusual phenomenon occurred: MI's for two different structural parameters ... were almost identical in each sample. In fact freeing either of these parameters would have the same effect on the fit of the model, and any observed difference in their MI's would be due only to rounding error" (p. 114). It turns out that equality of the MI's is a necessary condition for the equivalence of the expanded models but not a sufficient condition. In contrast, necessary and sufficient conditions for model equivalence are developed below with particular attention to the practical possibilities of checking these conditions. The discussion is limited to the case of one parameter added at a time to a model, because this case is the simplest and clearest; moreover, a researcher frequently wants to stay as close as possible to the original model. Extensions to applications where more parameters are added at a time can be derived fairly simply after this case has been worked out.

The outline of the paper is as follows. The next section defines formally the concept of model equivalence and displays an example. Section 3 gives a new theorem that provides a necessary and sufficient condition for the local equivalence of two models when a more constrained model is fitted. Section 4 gives practical guidelines on how to conclude that the conditions of the theorem are satisfied. The paper closes with a discussion.

An Example and Definitions

An example is given below that illustrates several aspects of equivalence. Although very simple, it has the advantage that the equivalence of the different models can be checked easily, and can serve an illustrative value throughout the whole paper. We do note that in this set of equivalent models, an unrealistic aspect is sometimes encountered by fixing the variance of a measurement error to zero, but this is done to keep this example as simple as possible. Throughout this paper the LISREL terminology and notation (Jöreskog & Sörbom, 1988) will be followed.

Example

Consider the factor analysis model with one common latent factor ξ_1 for two observed variables, x_1 and x_2:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_{11} \\ \lambda_{21} \end{pmatrix} \xi_1 + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix},$$

where the parameters λ_{11} and λ_{21} denote the factor loadings, δ_1 and δ_2 the error terms with variances $\theta_{\delta_{11}}$ and $\theta_{\delta_{22}}$ and covariance $\theta_{\delta_{21}}$, and ϕ_{11} the variance of ξ_1. Assume that in Model A_1, the parameters λ_{11}, λ_{21}, and $\theta_{\delta_{11}}$ are free, $\theta_{\delta_{22}}$ and $\theta_{\delta_{21}}$ are fixed at zero, and ϕ_{11} at one. The covariance matrix of this model is given in Table 1. Note that besides this factor-analytic description, A_1 can also be regarded as a regression model. Of course, the assumption that $\theta_{\delta_{22}}$ is zero is somewhat unrealistic implying that variable ξ_1 is measured without error by x_2. The scale of ξ_1 in A_1 is directly determined by fixing the variance ϕ_{11} of ξ_1 at one. A different way of scaling is when one of the λ_{ij} is fixed at a certain value, and ϕ_{11} is a free parameter. Such a model is A_2, where $\lambda_{11} = 1$ and ϕ_{11} is free. Model A_2 is also displayed in Table 1.

TABLE 1

The Four Models A_1 to A_4

Model	λ_{11}	λ_{21}	$\theta_{\delta_{11}}$	$\theta_{\delta_{22}}$	$\theta_{\delta_{21}}$	ϕ_{11}	σ_a^2	σ_{ab}	σ_b^2
A_1	λ_{11}	λ_{21}	$\theta_{\delta_{11}}$	0	0	1	$\lambda_{11}^2+\theta_{\delta_{11}}$	$\lambda_{11}\lambda_{21}$	λ_{21}^2
A_2	1	λ_{21}	$\theta_{\delta_{11}}$	0	0	ϕ_{11}	$\phi_{11}+\theta_{\delta_{11}}$	$\lambda_{21}\phi_{11}$	$\lambda_{21}^2\phi_{11}$
A_3	λ_{11}	1	$\theta_{\delta_{11}}$	0	0	ϕ_{11}	$\lambda_{11}^2\phi_{11}+\theta_{\delta_{11}}$	$\lambda_{11}\phi_{11}$	ϕ_{11}
A_4	λ	λ	$\theta_{\delta_{11}}$	$\theta_{\delta_{22}}$	0	1	$\lambda^2+\theta_{\delta_{11}}$	λ^2	$\lambda^2+\theta_{\delta_{22}}$

Note: The left part of the table gives the six possible parameters and those that are free and fixed for each model. The free parameters are given by the corresponding Greek letter. The right part gives the nonduplicated covariance elements for each model.

Two additional Models A_3 and A_4 will also be considered. In A_3, it is assumed that the variance of ξ_1 is fixed at the variance of x_2. This is done by setting λ_{21} to one and $\theta_{\delta_{22}}$ to zero, and corresponds, in our view, to equalizing the unit of measurement of ξ_1 and x_2. Further, $\theta_{\delta_{21}} = 0$. In A_4, λ_{11} and λ_{21} are constrained to be equal: $\lambda_{11} = \lambda_{21} = \lambda$; λ, $\theta_{\delta_{11}}$, and $\theta_{\delta_{22}}$ are free parameters; $\theta_{\delta_{21}} = 0$ and $\phi_{11} = 1$. All four models are given in Table 1.

A formal definition of the equivalence of M_1 and M_2 could be: For each parameter vector of M_1, there is one parameter vector of M_2, both producing the same covariance matrix. Note that both A_1 and A_3 can produce any positive-definite covariance matrix, which would mean that A_1 and A_3 are equivalent (saturated) models. However, for any parameter vector of A_3, there are two parameter vectors, $(\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}})$ and $(-\lambda_{11}, -\lambda_{21}, \theta_{\delta_{11}})$, of A_1 that produce the same covariance matrix. This is a consequence of the fact that A_1 is globally nonidentified in the sense of Definition 1 given below.

Definitions

Define $\sigma(\theta) = \text{vecs}\,(\Sigma(\theta))$, the vector of $m = 2^{-1}p(p+1)$ non-duplicated covariance elements of Σ. Moreover, let $\sigma_i(\theta) = \text{vecs}\,(\Sigma_i(\theta))$, where $\Sigma_i(\theta)$ the covariance matrix of model M_i, and denote the parameter space by Ω_i.

Definition 1. A model M_i is *globally identified* when for θ^a, $\theta^b \in \Omega_i$, $\sigma_i(\theta^a) = \sigma_i(\theta^b)$ implies $\theta^a = \theta^b$.

Although Model A_1 is globally nonidentified, it is not overparameterized (which means that at least one parameter is redundant), and in this sense one often considers this model still as identified. Wald (1950, p. 239) states that a model is identified when only a finite number of parameter vectors give the same (covariance) matrix. Hurwicz (1950, p. 248) allows even a denumerably infinite number of vectors (the "model" $\mu = \sin(\theta)$ would be identified by his definition). In general, one states that a model is identified when there is a subset $\tilde{\Omega}_i$ of Ω_i that has a positive Lebesgue measure (Apostol, 1957, p. 228) in Ω_i, implying that $\tilde{\Omega}_i$ has the same dimension as Ω_i, on which the model is uniquely identified. For example, in A_1,

$$\Omega_{A_1} = \{(\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}}, \theta_{\delta_{22}}, \theta_{\delta_{21}}, \phi_{11}) | \lambda_{21} \neq 0, \theta_{\delta_{11}} > 0, \theta_{\delta_{22}} = 0, \theta_{\delta_{21}} = 0, \phi_{11} = 1\}$$

is a hypersurface of dimension 3 in $\mathbb{R}^6$, and

$$\tilde{\Omega}_{A_1} = \{(\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}}, \theta_{\delta_{22}}, \theta_{\delta_{21}}, \phi_{11}) | \lambda_{21} > 0, \theta_{\delta_{11}} > 0, \theta_{\delta_{22}} = 0, \theta_{\delta_{21}} = 0, \phi_{11} = 1\}$$

is the set that has a positive Lebesgue measure in Ω_{A_1} and on which A_1 is uniquely defined. Consequently,

Definition 2. Let $IM(\sigma_i)$ be the image of σ_i, consisting of all positive-definite symmetric matrices (written as a vector) that can be produced by σ_i: $IM(\sigma_i) = \{x \in \mathbb{R}^m | x = \sigma_i(\theta) \text{ for some } \theta \in \Omega_i\}$.

Definition 3. The model M_i is said to be *identified* when a subset $\tilde{\Omega}_i$ of Ω_i exists that has a positive Lebesgue measure in Ω_i such that for each $x \in IM(\sigma_i)$, there is one and only one $\theta \in \tilde{\Omega}_i$ with $\sigma_i(\theta) = x$ (see, e.g., Sen, 1979, p. 1021, Assumption A1).

Note that for an overparameterized model, a subset $\tilde{\Omega}_i$ of Ω_i exists that has zero Lebesgue measure in Ω_i, such that for each $x \in IM(\sigma_i)$, there is one and only one $\theta \in \tilde{\Omega}_i$ with $\sigma_i(\theta) = x$

Now Model A_1 is identified following Definition 3. The equivalence of A_1 and A_3 can thus be established by the fact that there is a one-one correspondence between each element of $\tilde{\Omega}_{A_1}$ and

$$\tilde{\Omega}_{A_3} = \{(\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}}, \theta_{\delta_{22}}, \theta_{\delta_{21}}, \phi_{11}) | \lambda_{21} = 1, \theta_{\delta_{11}} > 0, \theta_{\delta_{22}} = 0, \theta_{\delta_{21}} = 0, \phi_{11} > 0\},$$

producing the same covariance matrices.

Assume that M_i^* has one more free parameter compared with M_i even though M_i^* reproduces the same set of covariance matrices as M_i. Consequently, M_i^* is a non-identified (overparameterized) model that is equivalent with M_i. We will not consider this kind of trivial equivalence, and more generally, will restrict equivalence of models to identified models that are, moreover, the ones usually examined in practice. Now a formal definition of equivalence is given.

Definition 4. Two models M_i and M_j are called *equivalent* (notation $M_i \sim M_j$) when:

i. M_i and M_j are both identified;
ii. there is a 1:1 function $g_{ji}: \tilde{\Omega}_i \rightarrow \tilde{\Omega}_j$ with $\sigma_i(\theta) = \sigma_j(g_{ji}(\theta))$ for all $\theta \in \tilde{\Omega}_i$.

A common name for the 1:1 function g_{ji} is a reparameterization (g_{ji} reparameterizes the parameters of M_i into the parameters of M_j). It can be found by $g_{ji}(\theta) = \sigma_j^{-1} \circ \sigma_i(\theta)$ (the symbol $\circ$ denotes the composition of two functions).

In the Example above, Models A_1 and A_3 are equivalent because both identified models can reproduce any 2×2 positive-definite covariance matrix. Luijben (1989, sec. 6.2.4) gives an example of equivalent models that are not saturated. To generate the reparameterization, the parameters of this factor-analytics model are denoted by $\theta_A = (\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}}, \theta_{\delta_{22}}, \theta_{\delta_{21}}, \phi_{11})$; take $\bar{\Omega}_{A_1}$ and $\bar{\Omega}_{A_3}$ as defined above, and derive g_{13} by calculating $\sigma_{A_1}^{-1} \circ \sigma_{A_3}$. Let

$$\sigma_{A_1} = (\lambda_{11}^2 + \theta_{\delta_{11}}, \lambda_{11}\lambda_{21}, \lambda_{21}^2) = (\sigma_a^2, \sigma_{ab}, \sigma_b^2). \tag{1}$$

Then,

$$\sigma_{A_1}(\lambda_{11} = \sigma_{ab}(\sigma_b^2)^{-1/2}, \lambda_{21} = (\sigma_b^2)^{-1/2}, \theta_{\delta_{11}} = \sigma_a^2 - \sigma_{ab}^2(\sigma_b^2)^{-1}, \theta_{\delta_{22}} = 0,$$
$$\theta_{\delta_{21}} = 0, \phi_{11} = 1) = (\sigma_a^2, \sigma_{ab}, \sigma_b^2). \tag{2}$$

Therefore, define

$$\sigma_{A_1}^{-1}(\sigma_a^2, \sigma_{ab}, \sigma_c) = (\sigma_{ab}(\sigma_b^2)^{-1/2}, (\sigma_b^2)^{-1/2}, \sigma_a^2 - \sigma_{ab}^2(\sigma_b^2)^{-1}, 0, 0, 1), \tag{3}$$

and insert $\sigma_{A_3} = (\lambda_{11}^2\phi_{11} + \theta_{\delta_{11}}, \lambda_{11}\phi_{11}, \phi_{11})$ into (3), giving

$$g_{13}(\lambda_{11}, \lambda_{21}, = 1, \theta_{\delta_{11}}, \theta_{\delta_{22}} = 0, \theta_{\delta_{21}} = 0, \phi_{11})$$
$$= (\lambda_{11}(\phi_{11})^{1/2}, (\phi_{11})^{-1/2}, \theta_{\delta_{11}}, 0, 0, 1). \tag{4}$$

Analogously,

$$g_{31}(\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}}, \theta_{\delta_{22}} = 0, \theta_{\delta_{21}} = 0, \phi_{11} = 1) = (\lambda_{11}\lambda_{21}, 1, \theta_{\delta_{11}}, 0, 0, \lambda_{21}^{-2}). \tag{5}$$

This example shows directly that the estimates for the parameters that are present in both equivalent models can differ considerably. For example, the value of λ_{11} is the same in A_1 and A_3 only when $\lambda_{21}^{-2} = 1$ in A_1, and thus, $\phi_{11} = 1$ in A_3.

In Example A, Models A_1 and A_2 are not equivalent because A_2 cannot produce positive-definite covariance matrices with a zero off-diagonal element. This set of matrices, however, has Lebesgue measure zero in the set of 2×2 covariance matrices, and the set $\bar{\Omega}_{A_1}^c = \{(\lambda_{11}, \lambda_{21}, \theta_{\delta_{11}}, \theta_{\delta_{22}}, \theta_{\delta_{21}}, \phi_{11}) | \theta_{\delta_{22}} = 0, \theta_{\delta_{21}} = 0, \phi_{11} = 1, \lambda_{21} > 0, \theta_{\delta_{11}} > 0$ and $\lambda_{11} = 0\}$ that produces the positive-definite covariance matrices with a zero off-diagonal element in A_1, has Lebesgue measure zero in $\bar{\Omega}_{A_1}$. This results in the following definition.

Definition 5. Two models M_i and M_j are called *almost equivalent* (notation $M_i \bar{a} M_j$) when:

i. M_i and M_j are both identified;

ii. there exists subsets $\bar{\Omega}_i^c$ and $\bar{\Omega}_j^c$ of $\bar{\Omega}_i$ and $\bar{\Omega}_j$, respectively, with Lebesgue measure zero (in $\bar{\Omega}_i$ and $\bar{\Omega}_j$, respectively), and a 1:1 function $g_{ji}: \bar{\Omega}_i \backslash \bar{\Omega}_i^c \rightarrow \bar{\Omega}_j \backslash \bar{\Omega}_j^c$ with $\sigma_i(\theta) = \sigma_j(g_{ji}(\theta))$ for all $\theta \in \bar{\Omega}_i \backslash \bar{\Omega}_i^c$. (The expression $A \backslash B$, where A and B are sets, denotes the exclusion of set B from set A.)

One can see immediately that A_4 cannot produce covariance matrices with a negative off-diagonal element, or those with a positive off-diagonal element in which one of the diagonal elements is smaller than the off-diagonal one, because this would imply a negative variance of the corresponding measurement error (i.e., a Heywood case). Model A_4 shows that equivalence of models can depend on the values that are admissible, and if negative variances are admissible, A_4 can produce all covariance matrices with a nonnegative off-diagonal element. However, one could then suggest that the imaginary numbers are also part of the admissible values. To avoid these cases, the admissible values considered in this paper are the proper ones in the sense that they are meaningfully interpretable. This implies that A_4 is not equivalent with A_1, A_2, and A_3 since the matrices that cannot be reproduced by A_4 have a positive Lebesgue measure. One could state, however, that A_4 and A_1 are locally equivalent implying there are open sets U_4 and U_1 of $\bar{\Omega}_4$ and $\bar{\Omega}_1$, respectively, on which the models are equivalent, and this will be worked out in detail in the next section.

Local Equivalence

Consider the baseline Model A_0 where λ_{11} and λ_{21} are constrained to be equal to λ, $\theta_{\delta_{11}}$ is a free parameter, ϕ_{11} fixed at one, and $\theta_{\delta_{22}}$ and $\theta_{\delta_{21}}$ are fixed at zero. For any given sample covariance matrix, the LISREL-output gives $\text{MI}(\lambda_{11}) = \text{MI}(\lambda_{21}) = \text{MI}(\theta_{\delta_{22}})$, but the two expanded Models A_1 and A_4 are not equivalent because A_4 could only perfectly fit a sample covariance matrix with a positive off-diagonal element while A_1 can fit any covariance matrix perfectly. This example shows that the equality of the MI does not guarantee the equivalence of the expanded models. Intuitively, this is understandable considering that at a certain point of the parameter space (e.g., the maximum likelihood estimate), only local information is available in the form of the first and second-order derivatives. From calculus, it is known that such local information is often sufficient to prove theorems that hold in a neighborhood of the point from which the information is obtained. It will be investigated in this section which information is available at a certain point, and whether this information is sufficient to obtain the local equivalence of models. A comparison is made with the related topic of conditions of local identification.

A Theorem for the Local Equivalence of Models

Let $\theta = (\theta_1, \ldots, \theta_{q+2}) \in \bar{\Omega}_{12} \subseteq \mathbb{R}^{q+2}$ be the $q + 2$ free parameters of a model M_{12}. Assume that M_0, M_1 and M_2 are nested in M_{12} with q, $q + 1$ and $q + 1$ free parameters, respectively, with M_0 is nested both within M_1 and M_2. Assume that M_1 and M_2 are defined by fixing θ_{q+2} and θ_{q+1}, respectively, at zero; M_0 is then defined by fixing both parameters θ_{q+1} and θ_{q+2} at zero (the value zero is completely arbitrary and could be replaced by any other value, but in practice, fixing a parameter at zero is very common as it denotes a superfluous parameter). Nesting in a different way by constraining parameters to be equal will not be discussed but can be reduced to this situation by a transformation. The nesting of these three models in M_{12} will be described by the following functions:

$$n_0 : \bar{\Omega}_0 \subseteq \mathbb{R}^q \to \mathbb{R}^{q+2} \text{ with } n_0(\theta_1, \ldots, \theta_q) = (\theta_1, \ldots, \theta_q, 0, 0); \tag{6}$$

$$n_1 : \bar{\Omega}_1 \subseteq \mathbb{R}^{q+1} \to \mathbb{R}^{q+2} \text{ with } n_1(\theta_1, \ldots, \theta_{q+1}) = (\theta_1, \ldots, \theta_q, \theta_{q+1}, 0); \tag{7}$$

$n_2:\tilde{\Omega}_2 \subseteq \mathbb{R}^{q+1} \rightarrow \mathbb{R}^{q+2}$ with $n_2(\theta_1, \ldots, \theta_q, \theta_{q+2})$

$$= (\theta_1, \ldots, \theta_q, 0, \theta_{q+2}). \quad (8)$$

The notation of the function: $\sigma_{12}:\mathbb{R}^{q+2} \rightarrow \mathbb{R}^m$ is

$$\sigma_{12}(\theta_1, \ldots, \theta_{q+2}) = (\sigma_{12}^1(\theta_1, \ldots, \theta_{q+2}), \ldots, \sigma_{12}^m(\theta_1, \ldots, \theta_{q+2})). \quad (9)$$

The σ functions for the models M_0, M_1, and M_2 are related to σ_{12} by the expression

$$\sigma_i:\mathbb{R}^{q+\min(i,1)} \rightarrow \mathbb{R}^m, \text{ with } \sigma_i = \sigma_{12} \circ n_i (i = 0, 1, 2). \quad (10)$$

Denote θ^i as an arbitrary parameter point of M_i *(i = 0, 1, 2, 12)*.

Definition 6. M_0 is *locally identified at* θ^0 when there is a set U_0, open in $\tilde{\Omega}_0 \subseteq \mathbb{R}^q$ with $\theta^0 \in U_0$, such that $\theta^* \in U_0$, $\sigma_0(\theta^*) = \sigma_0(\theta^0)$ implies $\theta^* = \theta^0$.

Definition 7. M_1 and M_2 are *locally equivalent at* (θ^1, θ^2) (denoted as $M_1(\theta^1) \sim_l M_2(\theta^2)$) if M_1 and M_2 are locally identified at θ^1 and θ^2, respectively, and there exist sets U_1 and U_2, both open in $\tilde{\Omega}_1 \subseteq \mathbb{R}^{q+1}$ and $\tilde{\Omega}_2 \subseteq \mathbb{R}^{q+1}$, respectively, with $\theta^1 \in U_1$ and $\theta^2 \in U_2$, and a 1:1 function $g_{21}:U_1 \rightarrow U_2$ with $\sigma_1(\theta) = \sigma_2(g_{21}(\theta))$ for all $\theta \in U_1$.

There is a natural correspondence between the condition under which models are locally identified and locally equivalent. This follows from the next definitions and conditions.

Define $\Delta_{12} = \partial\sigma_{12}/\partial\theta$ the $m \times (q + 2)$ Jacobian matrix, and $\Delta_0 = \partial\sigma_0/\partial\theta$ the $m \times q$ Jacobian matrix, and analogously, Δ_1 and Δ_2. Further, $\Delta_i(\theta^i)$ is the evaluation of Δ_i at θ^i $(i = 0, 1, 2, 12)$.

Definition 8. The point θ^0 is called a *regular point* of M_0 if there is a set U_0, open in $\tilde{\Omega}_0 \subseteq \mathbb{R}^q$ with $\theta^0 \in U_0$, where for each $\theta^* \in U_0$, the rank of $\Delta_0(\theta^*)$ equals the rank of $\Delta_0(\theta^0)$.

The following theorem gives necessary and sufficient conditions for the local identification of models when the following assumption is satisfied (see Shapiro & Browne, 1983, for a short discussion).

Assumption 1: σ_{12} is a continuously differentiable function.

Theorem 1. M_0 is locally identified at θ^0, and θ^0 is a regular point, if and only if the rank of $\Delta_0(\theta^0)$ equals q, the number of free parameters.

Proof. See Fisher (1966, p. 163); Dijkhuizen (1978, p. 30), Bekker and Pollock (1986, p. 105).

Let $\bar{\theta}^{12} = (\bar{\theta}_1, \ldots, \bar{\theta}_q, 0, 0) = (\bar{\theta}^0, 0, 0)$ be any point of M_{12} that satisfies the conditions of M_0. Then $\bar{\theta}^1 = \bar{\theta}^2 = (\bar{\theta}^0, 0)$, $n_1(\bar{\theta}^1) = n_2(\bar{\theta}^2)$, and thus, $\sigma_1(\bar{\theta}^1) = \sigma_2(\bar{\theta}^2)$.

The following assumptions (and Assumption 1) are made and assumed throughout the remainder of the paper.

Assumption 2. The points $\bar{\theta}^{12}$, $\bar{\theta}^1$ and $\bar{\theta}^2$ are regular points of M_{12}, M_1 and M_2, respectively.

Assumption 3. M_1 and M_2 are locally identified at $\bar{\theta}^1$ and $\bar{\theta}^2$, respectively.

Note 1. Assumptions 1 and 2, and Theorem 1 imply that the number of non-duplicated covariance elements m is larger than or equal to the number of free parameters $q + 1$ of M_1 and M_2.

Note 2. If $\Delta_{12}(\bar{\theta}^{12})$ is of deficient rank, the previous three assumptions and Theorem 1 imply that its rank is $q + 1$.

The following (new) theorem gives necessary and sufficient conditions for the local equivalence of models. Note that in correspondence with the conditions for local identification, the Jacobian matrix Δ is of crucial importance.

Theorem 2. $\Delta_{12}(\bar{\theta}^{12})$ is of deficient rank if and only if the models M_1 and M_2 are locally equivalent at $(\bar{\theta}^1, \bar{\theta}^2)$.

Proof: see the Appendix.

Note 3. A direct consequence of Theorem 2 is that if M_1 and M_2 are locally equivalent at $(\bar{\theta}^1, \bar{\theta}^2)$, then M_{12} is locally nonidentified at $\bar{\theta}^{12}$ because of the rank deficiency of $\Delta_{12}(\bar{\theta}^{12})$ and Theorem 1.

Note 4. From the proof of Theorem 2, it follows that the reparameterization g and its inverse are both locally differentiable functions when the stated assumptions and conditions of Theorem 2 are fulfilled.

Note 5. One might think that two locally identified models M_1 and M_2 with different number of free parameters can reproduce the same set of covariance matrices as well. This would mean that the function g is a bijection between two sets of different dimensions, which is possible. But a consequence of Theorem 2 (and thus of the assumptions) is that the reparameterization g and its inverse are differentiable, and in particular, g is a local homeomorphism (a function is a homeomorphism when the function and its inverse are one-to-one, open, and continuous—Lipschutz, 1965, p. 100; the subsets $U_1^* \subset \bar{\Omega}_1$ and $U_2^* \subset \bar{\Omega}_2$ are then called homeomorphic. An important theorem from topology can now be applied, stating that U_1^* and U_2^* can only be homeomorphic when the dimensions of both subsets are equal (Brouwer, 1911). This implies that whenever locally identified models M_1 and M_2 reproduce the same set of covariance matrices, both have the same number of free parameters. Consequently, Definition 7 is not too restrictive when assuming that U_1 and U_2 are both open in $\mathbb{R}^{q+1}$, implying that U_1 and U_2 have equal dimensions.

Note 6. The point $\bar{\theta}^{14} = (\lambda_{11} = a, \lambda_{21} = a, \theta_{\delta_{11}} = b, \theta_{\delta_{21}} = 0, \theta_{\delta_{22}} = 0, \phi_{11} = 1)$ with $a, b > 0$, is a regular point in Model A_{14}, where the Models A_0, A_1, and A_4 are identified and the rank of the gradient $\Delta_{14}(\bar{\theta}^{14})$ equals 3 (A_0 is nested in both A_1 and A_4, consequently, $\lambda_{11} = \lambda_{21} = \lambda$, $\theta_{\delta_{11}}$ is free, and the other parameters are fixed; A_{14} is the union of A_1 and A_4). Thus, following Theorem 2, A_1 and A_4 are locally equivalent at $(\bar{\theta}^1, \bar{\theta}^4)$. This implies that the points in the neighborhood U_1 (which is open in $\bar{\Omega}_1$) of $\bar{\theta}^1$ correspond to points in the neighborhood U_4 (which is open in $\bar{\Omega}_4$)

of $\bar{\theta}^4$. But U_4 contains points that have a negative $\theta_{\delta_{22}}$, consequently, some proper values of A_1 correspond with improper values of A_4 although the same covariance matrix is reproduced. This is a direct result of the use of the implicit function theorem in the proof of Theorem 2. This theorem deals with open subsets of $\mathbb{R}^{q+1}$ and may cause interpretation problems when the real line (as a subset of $\mathbb{R}^{q+1}$) corresponds to values of a variance or a correlation.

Note 7. One could ask whether all these aspects of equivalent modeling also occur when the model is simplified. In particular, one may think that omitting two different parameters from a locally identified model M results in two restricted models that are locally equivalent. This cannot be the case because the union of those two models is M, which was locally identified, and consequently, the rank of its gradient matrix is (locally) full. Thus, two models that are both simplified models of an identified one can never be equivalent.

How to Test Local Equivalence in Practice

Theorem 2 shows that if there is a dependence between the columns of Δ_{12} that is structural—which means that the rank of Δ_{12} is deficient for all parameter values in an open $U_{12} \subset \bar{\Omega}_{12}$ of M_{12}, except for a set of irregular points that has Lebesgue measure zero in $\bar{\Omega}_{12}$—then the identified models M_1 and M_2 are locally equivalent. Hence, a researcher wants to know whether Δ_{12} is not of full rank when expansion of the fitted model M_0 is considered. Seidel and Eicheler (1990) developed the LISRAN program that determines the rank of any Jacobian matrix, for example Δ_{12}, by searching for dependencies between the columns of Δ_{12} only using parameter equations—so without any data. This is a straightforward procedure to determine the rank of Δ_{12} without problems of irregular points. There is no such option in LISREL and EQS, and even the Jacobian matrices $\Delta_1(\bar{\theta}^1)$, $\Delta_2(\bar{\theta}^2)$, and $\Delta_{12}(\bar{\theta}^{12})$ cannot be obtained, though this could easily be established. There is an alternative approach (only in EQS), however, to determine the rank deficiency of $\Delta_{12}(\bar{\theta}^{12})$. The multivariate modification index (or equivalently, the multivariate Lagrange multiplier statistic) is an estimate of the decrease in the chi-square goodness-of-fit statistic when several parameters are relaxed simultaneously. The Jacobian matrix $\Delta_{12}(\bar{\theta}^{12})$ is involved in the formula of the multivariate modification index $MI_{12}(\bar{\theta}^{12})$ when M_0 is fitted and θ_{q+1} and θ_{q+2} are the ones to relax. The rank deficiency of $\Delta_{12}(\bar{\theta}^{12})$ results in a singular information matrix $I_{12}(\bar{\theta}^{12})$ and a nonexistent $MI_{12}(\bar{\theta}^{12})$, because the inverse of $I_{12}(\bar{\theta}^{12})$ is involved in $MI_{12}(\bar{\theta}^{12})$ (see, e.g., Luijben, 1989, p. 163). EQS, however, uses a generalized inverse of a singular $I_{12}(\bar{\theta}^{12})$, to calculate a "generalized" modification index $MI^*_{12}(\bar{\theta}^{12})$. Denote $MI_1(\bar{\theta}^{12})$ and $MI_2(\bar{\theta}^{12})$ as the MI for θ_{q+1} and θ_{q+2}, respectively. It turns out that $\Delta_{12}(\bar{\theta}^{12})$ is of deficient rank if and only if $MI^*_{12}(\bar{\theta}^{12}) = MI_1(\bar{\theta}^{12}) = MI_2(\bar{\theta}^{12})$, except for a set of data matrices S that have Lebesgue measure zero in the Wishart distribution of S, given Σ (the population covariance matrix). Thus, we can conclude—with probability one—that the rank of $\Delta_{12}(\bar{\theta}^{12})$ is deficient when the EQS-output denotes that the estimated decreases in chi-square are identical when either θ_{q+1} or θ_{q+2}, or both θ_{q+1} and θ_{q+2} are added as free parameters to model M_0.

Can we now safely conclude that the identified models M_1 and M_2 are locally equivalent when the rank of $\Delta_{12}(\bar{\theta}^{12})$ is deficient? We recognize one problem. The parameter point $\bar{\theta}^{12} = (\bar{\theta}^0, 0, 0)$ can be an irregular point such that the rank of Δ_{12} is only deficient in $\bar{\theta}^{12}$ but full in any point of an open neighborhood U_{12} of $\bar{\theta}^{12}$. What is the probability that a θ^0 is obtained—when M_0 is fitted—such that $(\theta^0, 0, 0)$ is irregular in M_{12}? Can we conclude, in general, that this probability equals zero because the set of parameter estimates θ^0, for which $(\theta^0, 0, 0)$ is irregular in M_{12}, has Lebesgue measure zero in M_0? The answer is no, as pointed out by an anonymous reviewer. The reason is that the two parameters θ_{q+1} and θ_{q+2} are a priori fixed at zero and only θ_1 up to θ_q are estimated freely, which can cause a disappearance of several elements from Δ_{12} that are nonzero in general. Luijben (1989, p. 160–161) gives an example where $(\theta^0, 0, 0)$ is irregular in M_{12} for even all parameter estimates θ^0. The main conclusion is that there are models for which a positive probability exists that a θ^0 is obtained, when M_0 is fitted, such that $(\theta^0, 0, 0)$ is irregular in M_{12}. Hence, there is a positive probability, for such models, that the deficiency of the rank of $\Delta_{12}(\bar{\theta}^{12})$ is not sufficient to conclude that M_1 and M_2 are locally equivalent. Note, that this situation is different from the situation that the identification of M_{12} is determined by examining the rank of Δ_{12} at, for example, the maximum likelihood estimate $\hat{\theta}^{12}$ obtained by free estimation of θ_1 up to θ_{q+2}. The probability that $\hat{\theta}^{12}$ is irregular in M_{12} when M_{12} is fitted, is equal to zero when σ_{12} is an analytical function (Fisher, 1966, p. 167), which is the case in covariance structure analysis (see below the subsection). Therefore, if $\Delta(\hat{\theta}^{12})$ is not of full rank $q + 2$, then M_{12} is not locally identified with probability one (see Theorem 1).

In our view, it is difficult to tell in general for which models the set with elements θ^0 (and analogously, θ^1 and θ^2), such that $(\theta^0, 0, 0)$ is irregular in M_{12}, has a nonzero Lebesgue measure in M_0. However, no problems of irregular points occur for models where the (co-)variances of the measurement errors (elements of θ_δ and θ_ε) are the ones to relax, because these parameters vanish always in all columns of δ_{12}. A suggestion—given in the discussion below—could be the basis of a general procedure.

Regarding Assumption 1

The assumption that σ_{12} is a continuous differentiable function is fulfilled in covariance structure analysis (see Shapiro, 1986, p. 145). This can directly be observed from the general form of the covariance matrix given in Jöreskog and Sörbom (1988, p. 5). The covariance elements are usually simple polynomials except for the elements in which the matrix

$$(\mathrm{I} - B)^{-1} \tag{11}$$

is involved. Matrix (11), however, equals

$$\mathrm{I} + B + B^2 + B^3 + B^4 + \ldots, \tag{12}$$

when all eigenvalues of B are within the unit circle (Jöreskog & Sörbom, 1988, p. 35), and a sufficient condition is that the largest eigenvalue of BB' is less than one. This holds in LISREL 7 when the "stability index" is smaller than one. This means that all covariance elements are analytical functions (can be developed into power series) when the parameter space is restricted such that (11) is equal to (12). This restriction implies that only stable (or nonexplosive) models are considered (see Hayduk, 1987, p. 258). Note that an analytical function is differentiable an infinite number of times, so Assumption 1 is satisfied when stable models are considered.

Note 8. If M_1 and M_2 are locally equivalent, then the MI for θ_{q+1} is equal to the MI for θ_{q+2}. Thus, heuristically, the estimated decrease in the chi-square goodness-of-fit statistic is identical when either θ_{q+1} or θ_{q+2} is added as a free parameter to M_0, or θ_{q+1} or θ_{q+2} simultaneously, if M_1 and M_2 are locally equivalent. This result is the well-known fact that the Lagrange multiplier statistic is invariant under reparameterization of a model (see Cox & Hinkley, 1974, p. 339).

Note 9. If the MI for θ_{q+1} is unequal to the MI for θ_{q+2}, then M_1 and M_2 are not (locally) equivalent.

Discussion

This paper presents a necessary and sufficient condition for two expanded models M_1 and M_2 to be locally equivalent under the somewhat strong condition that $(\overline{\theta}^0, 0, 0)$ is a regular point in M_{12}. We give one suggestion on how the regularity problem could be approached in future research. Fit M_{12}, and assume that $\Delta(\hat{\theta}^{12})$ is of deficient rank at the maximum-likelihood-estimate $\hat{\theta}_{12} = (\hat{\theta}_1, \ldots, \hat{\theta}_q, \hat{\theta}_{q+1}, \hat{\theta}_{q+2})$, which happens when the analysis denotes that M_{12} is not identified. Define model M_1^* as the one with free parameters θ_1 up to θ_{q+1}, and with fixed parameter θ_{q+2} set at $\hat{\theta}_{q+2}$. Define similarly M_2^*, with free parameters θ_1 up to θ_q, and θ_{q+2}, and fixed parameter θ_{q+1} set at $\hat{\theta}_{q+1}$. Now, $M_1^*(\hat{\theta}_{12}) \sim_l M_2^*(\hat{\theta}_{12})$ with probability one because $\hat{\theta}_{12}$ is regular in M_{12} with probability one. It may be possible to prove that this local equivalence can be extended to a more global result, especially concerning local equivalence around $\overline{\theta}^{12}$, using the nice analytical characteristics of the (LISREL-) equations.

Appendix

Proof of Theorem 2

Sufficiency. According to Assumptions 2 and 3, and Theorem 1, there is an open $U_1 \subset \mathbb{R}^{q+1}$ with $\overline{\theta}^1 \in U_1$ so that for each $\theta^1 \in U_1$, the rank of $\Delta_1(\theta^1)$ is (full) $q + 1$. With the definition of σ_1 and the chain rule (see, e.g., Rudin, 1964, p. 190, Theorem 9.12) it follows that $\Delta_1(\theta^1) = \partial\sigma_{12}/\partial n_1\ (n_1(\theta^1))\ \partial n_1/\partial\theta\ (\theta^1)$. This implies that it may be assumed (possibly after permuting the m elements of σ_{12}) for the function $\sigma_{12}^I(\theta^{12}) = (\sigma_{12}^1(\theta^{12}), \ldots, \sigma_{12}^{q+1}(\theta^{12}))$, consisting of the first $q + 1$ elements of $\sigma_{12}(\theta^{12})$, the matrix $\partial\sigma_{12}^I/\partial\theta^{12}\ (\theta^{12})$ is of full rank $q + 1$ for all $\theta^{12} \in n_1(U_1)$. Consequently, define $\sigma_{12}^{II}(\theta^{12}) = (\sigma_{12}^{q+2}(\theta^{12}), \ldots, \sigma_{12}^m(\theta^{12}))$.

The rank of a matrix Δ is semicontinuous implying that if rank $(\Delta) = r$, there is a neighborhood W of Δ such that rank $\Delta' \geq r$ for all $\Delta' \in W$ (Bröcker & Jänich, 1973, p. 45). Thus, there is an open $U_{12} \subset \mathbb{R}^{q+2}$ with $\overline{\theta}^{12} \in U_{12}$, so that for all $\theta^{12} \in U_{12}$, the matrix $\partial\sigma_{12}^I/\partial\theta^{12}\ (\theta^{12})$ is of rank $q + 1$. Now, with the assumed deficiency of the rank of $\Delta_{12}(\overline{\theta}^{12})$ and the regularity of $\overline{\theta}^{12}$ in M_{12}, the conditions of the rank-theorem (see, e.g., Burkill & Burkill, 1970, p. 230, Theorem 7.63) are fulfilled. This theorem states that there is an open $U_{12}^* \subseteq U_{12}$ with $\overline{\theta}^{12} \in U_{12}^*$ and a function h: $\sigma_{12}^I(U_{12}^*) \rightarrow \mathbb{R}^{m-(q+1)}$ such that for all $\theta^{12} \in U_{12}^*$, $\sigma_{12}^{II}(\theta^{12}) = (h \circ \sigma_{12}^I)(\theta^{12})$. Now, define:

$$\sigma_1^I: U_1 \rightarrow \mathbb{R}^{q+1}, \quad \text{with} \quad \sigma_1^I(\theta^1) = (\sigma_{12}^I \circ n_1)(\theta^1), \quad \text{and}$$

$$\sigma_2^I: U_2 \rightarrow \mathbb{R}^{q+1}, \quad \text{with} \quad \sigma_2^I(\theta^2) = (\sigma_{12}^I \circ n_2)(\theta^2).$$

Define f:

$$U_1 \times U_2 \rightarrow \mathbb{R}^{q+1}, \quad \text{with} \quad f(\theta^1, \theta^2) = (\sigma_1^I(\theta^1) - \sigma_2^I(\theta^2)).$$

Then, f is differentiable, $f(\overline{\theta}^1, \overline{\theta}^2) = 0$, and $\partial f/\partial\theta^1\ (\overline{\theta}^1)$ is of full rank $(q + 1)$. With the implicit function theorem (see, e.g., Rudin, 1964, p. 196, Theorem 9.18) it follows that there is a differentiable function $g_{12}: U_2^* \rightarrow \mathbb{R}^{q+1}$ with $U_2^* \subset \mathbb{R}^{q+1}$ open and $\overline{\theta}^2 \in U_2^*$, and $g_{12}(\overline{\theta}^2) = \overline{\theta}^1$ and $f(g_{12}\theta^2), \theta^2) = 0$ for all $\theta^2 \in U_2^*$. Thus, $\sigma_2^I(\theta^2) =$

$\sigma_1^j(g_{12}(\theta^2))$ for all $\theta^2 \in U_2^*$. With the existence of the function h, it now follows that $\sigma_2(\theta^2) = \sigma_1(g_{12}(\theta^2))$ for all $\theta^2 \in U_2^*$.

Similar derivations can be made when one starts with M_2 instead of M_1, leading to the existence of a differentiable function $g_{21}: U_1^* \to \mathbb{R}^{q+1}$ with $U_1^* \subset \mathbb{R}^{q+1}$ open and $\bar{\theta}^1 \in U_1^*$, and $g_{21}(\bar{\theta}^1) = \bar{\theta}^2$ and $\sigma_1^j(\theta^1) = \sigma_2^j(g_{21}(\theta^1))$ for all $\theta^1 \in U_1^*$. With the existence of the function h, it now follows that $\sigma_1(\theta^1) = \sigma_2(g_{21}(\theta^1))$ for all $\theta^1 \in U_1^*$.

Necessity. Because M_1 and M_2 are locally equivalent at $(\bar{\theta}^1, \bar{\theta}^2)$ there exist sequences of vectors $\{\theta^{1(k)}\}$ and $\{\theta^{2(k)}\}$ such that for all k, $\sigma_1(\theta^{1(k)}) = \sigma_2(\theta^{2(k)})$ and $\lim_{k\to\infty} \theta^{1(k)} = \bar{\theta}^1 = (\bar{\theta}^0, 0)$. Moreover, all $\theta^{2(k)}$ are within a bounded U_2 so that there is a θ^{2^*} such that, passing to a subsequence if necessary, $\lim_{k\to\infty} \theta^{2(k)} = \theta^{2^*}$. Because of the continuity of σ_1 and σ_2, $\sigma_1(\bar{\theta}^1) = \sigma_2(\theta^{2^*})$, but also $\sigma_1(\bar{\theta}^1) = \sigma_2(\bar{\theta}^2)$; consequently, $\sigma_2(\theta^{2^*}) = \sigma_2(\bar{\theta}^2)$. This implies that $\theta^{2^*} = \bar{\theta}^2 = (\bar{\theta}^0, 0)$ because of the local identification of M_2 at $\bar{\theta}^2$.

The i-th components of the functions n_1 and n_2 are denoted n_{1i} and n_{2i}, respectively ($1 \le i \le q + 2$); thus, $n_{11} = \theta_1, n_{1q} = \theta_q, n_{1q+1} = \theta_{q+1}$, and $n_{1q+2} = 0$. Then, the multivariate mean-value theorem (see, e.g., Magnus & Neudecker, 1988, p. 98) can be applied for all j ($1 \le j \le m$) and all k: $0 = \sigma_1{}^j(\theta^{1(k)}) - \sigma_2{}^j(\theta^{2(k)}) = \sigma_{12}{}^j(n_1(\theta^{1(k)})) - \sigma_{12}{}^j(n_2(\theta^{2(k)})) = \Sigma_{i=1}^{q+2}\ \partial\sigma_{12}{}^j/\partial\theta_i\ (\theta_j^{12(k)})\ [n_{1i}(\theta^{1(k)}) - n_{2i}(\theta^{2(k)})]$, where $\theta_j^{12(k)}$ lies on the line between $n_1(\theta^{1(k)})$ and $n_2(\theta^{2(k)})$. This implies, for all j and k, that $\Sigma_{i=1}^{q+2}\ \partial\sigma_{12}{}^j/\partial\theta_i\ (\theta_j^{12(k)})\ d_i^{(k)} = 0$, where $d_i^{(k)} = [n_{1i}(\theta^{1(k)}) - n_{2i}(\theta^{2(k)})]/\|n_1(\theta^{1(k)}) - n_2(\theta^{2(k)})\|$.

The sequence $\{d^{(k)}\}$ is an infinite sequence within the closed unit sphere, and therefore, passing to a subsequence if necessary, it may be assumed that there exists a limit point $d = (d_1, \ldots, d_{q+2})$, where $d_i = \lim_{k\to\infty} [n_{1i}(\theta^{1(k)}) - n_{2i}(\theta^{2(k)})]/\|n_1(\theta^{1(k)}) - n_2(\theta^{2(k)})\|$. As, $\lim_{k\to\infty} d^{(k)} = d$ and $\lim_{k\to\infty} \theta_j^{12(k)} = n_1(\bar{\theta}^1) = n_2(\bar{\theta}^2) = (\bar{\theta}^0, 0, 0)$, for all j, and because $\partial\sigma_{12}{}^j/\partial\theta_i$ is continuous (Assumption 1), we have in the limit: $0 = \Sigma_{i=1}^{q+2}\ \partial\sigma_{12}{}^j/\partial\theta_i\ (\bar{\theta}^0, 0, 0)\ d_i$ for all j. This means that $\Delta_{12}(\bar{\theta}^{12})\ d = 0$, and $\Delta_{12}(\bar{\theta}^{12})$ is of rank $< q + 2$, and thus rank $q + 1$ (see Note 2). □

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