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THE AREA BETWEEN TWO ITEM CHARACTERISTIC CURVES

NAMBURY S. RAJU

ILLINOIS INSTITUTE OF TECHNOLOGY

Formulas for computing the exact signed and unsigned areas between two item characteristic curves (ICCs) are presented. It is further shown that when the c parameters are unequal, the area between two ICCs is infinite. The significance of the exact area measures for item bias research is discussed.

Key words: item response theory, item bias, item characteristic curves.

In item bias research, the area between two item characteristic curves (ICCs) for two different groups is sometimes used as a measure of item bias (Ironson & Subkoviak, 1979; Shepard, Camilli, & Averill, 1981; Shepard, Camilli, & Williams, 1984; Rudner, Geston, & Knight, 1980a, 1980b). At the present time, the area between two ICCs is only estimated, either by integrating the appropriate function between two finite points such as −4.00 and +4.00 (Shepard et al., 1981), or by adding successive rectangles of width 0.005 between two finite points (Rudner et al., 1980a, 1980b). One of the purposes of this paper is to offer formulas for computing the exact area between two ICCs for the one-, two-, and three-parameter IRT models. The other purpose is to discuss the significance of the exact area measures for item bias research, including a conceptual problem for the three-parameter model when the c parameters are not equal.

The Signed and Unsigned Area Formulas

Let $F_1(\theta)$ and $F_2(\theta)$ represent two ICCs which, in the three-parameter model, can be expressed as

$$F_1 = F_1(\theta) = c_1 + (1 - c_1)P_1 \tag{1}$$

$$F_2 = F_2(\theta) = c_2 + (1 - c_2)P_2 \tag{2}$$

where

$$P_1 = P_1(\theta) = \frac{\exp(Da_1(\theta - b_1))}{1 + \exp(Da_1(\theta - b_1))}, \tag{3}$$

$$P_2 = P_2(\theta) = \frac{\exp(Da_2(\theta - b_2))}{1 + \exp(Da_2(\theta - b_2))}, \tag{4}$$

a_i, b_i, and c_i are the three item parameters for the ith ICC, and D is a scaling constant, usually set to 1.7 to equate the logistic ogive to approximately the normal ogive (Lord,

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Requests for reprints should be sent to Nambury S. Raju, Department of Psychology, Illinois Institute of Technology, Chicago, IL 60616.

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1980). Furthermore, let the signed and unsigned areas be defined as

$$\text{Signed Area } (SA) = \int_{-\infty}^{\infty} (F_1 - F_2)\, d\theta, \tag{5}$$

$$\text{Unsigned Area } (UA) = \int_{-\infty}^{\infty} |F_1 - F_2|\, d\theta. \tag{6}$$

The SA is also referred to as the *difference* between two curves whereas the UA as the *distance*. Because of their popularity in the item bias literature, the SA and UA notations will be adopted here. In evaluating the integrals in Equations 5 and 6, two different cases will be considered—Case I: $c = c_1 = c_2$ and Case II: $c_1 \neq c_2$.

Theorem 1: Case I. Let F_1 and F_2 represent two ICCs with the stipulation that $a_1 \neq a_2$ and the restriction that $c = c_1 = c_2$. Then

$$SA = (1 - c)(b_2 - b_1) \tag{7}$$

$$UA = (1 - c)\left|\frac{2(a_2 - a_1)}{Da_1a_2} \ln\left(1 + \exp\left(\frac{Da_1a_2(b_2 - b_1)}{a_2 - a_1}\right)\right) - (b_2 - b_1)\right| \tag{8}$$

Proof. Since $c_1 = c_2$, the two ICCs must intersect at a point, θ_0, which can be expressed as

$$\theta_0 = \frac{a_2 b_2 - a_1 b_1}{a_2 - a_1}. \tag{9}$$

In fact, θ_0 is the *only* finite intersection point. Now, to evaluate SA and UA, the appropriate integrals can be written as

$$SA = \int_{-\infty}^{\infty} (F_1 - F_2)\, d\theta = (1 - c)\int_{-\infty}^{\infty} (P_1 - P_2)\, d\theta$$

$$= (1 - c)\left[\int_{-\infty}^{\theta_0} (P_1 - P_2)\, d\theta + \int_{\theta_0}^{\infty} (P_1 - P_2)\, d\theta\right] \tag{10}$$

or

$$SA = (1 - c)[I_1 + I_2] \tag{11}$$

where

$$I_1 = \int_{-\infty}^{\theta_0} (P_1 - P_2)\, d\theta \tag{12}$$

$$I_2 = \int_{\theta_0}^{\infty} (P_1 - P_2)\, d\theta \tag{13}$$

$$UA = \int_{-\infty}^{\infty} |F_1 - F_2|\, d\theta = (1 - c)\int_{-\infty}^{\infty} |P_1 - P_2|\, d\theta$$

$$= (1 - c)\left[\int_{-\infty}^{\theta_0} |P_1 - P_2|\, d\theta + \int_{\theta_0}^{\infty} |P_1 - P_2|\, d\theta\right] \tag{14}$$

Since F_1 and F_2 intersect at only one finite point and since this must also be true of P_1 and P_2, if $P_1 - P_2$ is positive in the open interval $(-\infty, \theta_0)$, then $P_1 - P_2$ must be negative in the open interval (θ_0, ∞). Therefore,

$$|P_1 - P_2| = P_1 - P_2 \qquad \text{when } \theta \in (-\infty, \theta_0)$$

$$|P_1 - P_2| = -(P_1 - P_2) \qquad \text{when } \theta \in (\theta_0, \infty)$$

Hence, the integral for UA for this case can be written as

$$(1-c)\left[\int_{-\infty}^{\theta_0} (P_1 - P_2)\, d\theta - \int_{\theta_0}^{\infty} (P_1 - P_2)\, d\theta\right] = (1-c)[I_1 - I_2]. \tag{15}$$

If, on the other hand, $P_1 - P_2$ is negative in the open interval $(-\infty, \theta_0)$ and positive in (θ_0, ∞), then

$$|P_1 - P_2| = -(P_1 - P_2) \qquad \text{for } \theta \in (-\infty, \theta_0),$$

$$|P_1 - P_2| = P_1 - P_2 \qquad \text{for } \theta \in (\theta_0, \infty).$$

Therefore, for this case, the integral for UA can be written as

$$(1-c)\left[-\int_{-\infty}^{\theta_0} (P_1 - P_2)\, d\theta + \int_{\theta_0}^{\infty} (P_1 - P_2)\, d\theta\right] = (1-c)(-1)[I_1 - I_2]. \tag{16}$$

Combining (15) and (16), a single, general expression for UA can be written as

$$UA = (1-c)|I_1 - I_2|. \tag{17}$$

Therefore, in order to prove Theorem 1, only I_1 and I_2 need to be evaluated. The integral I_1 can be rewritten as

$$I_1 = \lim_{x\to\infty} \int_{-x}^{\theta_0} (P_1 - P_2)\, d\theta$$

$$= \lim_{x\to\infty} \left[\frac{1}{Da_1} \ln\left(1 + \exp\left(Da_1(\theta - b_1)\right)\right) - \frac{1}{Da_2} \ln\left(1 + \exp\left(Da_2(\theta - b_2)\right)\right)\right]_{-x}^{\theta_0}$$

$$= \ln\left[\frac{(1 + \exp(Da_1(\theta_0 - b_1)))^{1/Da_1}}{(1 + \exp(Da_2(\theta_0 - b_2)))^{1/Da_2}}\right] - \lim_{x\to\infty} \ln\left[\frac{(1 + \exp(Da_1(-x - b_1)))^{1/Da_1}}{(1 + \exp(Da_2(-x - b_2)))^{1/Da_2}}\right]. \tag{18}$$

It should be noted that $a_1(\theta_0 - b_1) = a_2(\theta_0 - b_2)$ since θ_0 is the finite intersection point of P_1 and P_2. Therefore, the first term in (18), in view of (9), can be written as

$$B = \frac{a_2 - a_1}{Da_1 a_2} \ln\left(1 + \exp\left(\frac{Da_1 a_2 (b_2 - b_1)}{a_2 - a_1}\right)\right). \tag{19}$$

Rewriting the second term of (18) gives

$$\lim_{x\to\infty} \ln\left[\frac{\left(1 + \dfrac{\exp(1 - Da_1 b_1)}{\exp(Da_1 x)}\right)^{1/Da_1}}{\left(1 + \dfrac{\exp(-Da_2 b_2)}{\exp(Da_2 x)}\right)^{1/Da_2}}\right].$$

In order to evaluate this limit (as well as other limits that we will encounter later), certain theorems about limits are presented without proofs. Let f and g be any two functions and let $f(g(x))$ be the composite function. Furthermore, let $f(x) \to A$ as $x \to \infty$ and $g(x) \to B$ $(B \neq 0)$ as $x \to \infty$. Then, $f(x)/g(x) \to A/B$ as $x \to \infty$ and $f(g(x)) \to f(B)$ as $x \to \infty$. (Proofs of these theorems can be found in most calculus texts.) In view of these limit theorems, the limit of the above equation can be written as ln (1) = 0. Therefore, $I_1 = B - 0 = B$. To evaluate the second integral, I_2 can be written as

$$I_2 = \lim_{x \to \infty} \ln \left[\frac{(1 + \exp(Da_1(x - b_1)))^{1/Da_1}}{(1 + \exp(Da_2(x - b_2)))^{1/Da_2}}\right] - \ln \left[\frac{(1 + \exp(Da_1(\theta_0 - b_1)))^{1/Da_1}}{(1 + \exp(Da_2(\theta_0 - b_2)))^{1/Da_2}}\right]. \quad (20)$$

Based on the preceding discussion, the second term in (20) equals B. The first term in (20) can be rewritten as

$$\lim_{x \to \infty} \ln \left[\frac{(1 + \exp(Da_1(x - b_1)))^{1/Da_1} \cdot \dfrac{\exp(x - b_1)}{\exp(x - b_1)}}{(1 + \exp(Da_2(x - b_2)))^{1/Da_2} \cdot \dfrac{\exp(x - b_2)}{\exp(x - b_2)}}\right]$$

$$= \lim_{x \to \infty} \ln \frac{\left(\dfrac{1}{P_1}\right)^{1/Da_1}}{\left(\dfrac{1}{P_2}\right)^{1/Da_2}} + (b_2 - b_1).$$

Since $P_1 \to 1$ and $P_2 \to 1$ as $x \to \infty$, I_2 in (20) can be written as $I_2 = (b_2 - b_1) - B$. Based on (11), the signed area is

$$SA = (1 - c)[I_1 + I_2] = (1 - c)[B + (b_2 - b_1) - B] = (1 - c)[b_2 - b_1]. \quad (21)$$

The unsigned area can be written as

$$UA = (1 - c)|I_1 - I_2| = (1 - c)|2B - (b_2 - b_1)|$$

$$= (1 - c)\left|\frac{2(a_2 - a_1)}{Da_1 a_2} \ln\left(1 + \exp\left(\frac{Da_1 a_2(b_2 - b_1)}{a_2 - a_1}\right)\right) - (b_2 - b_1)\right|. \quad (22)$$

This completes the proof of Theorem 1. □

Lemma. Let F_1 and F_2 represent two ICCs with the restrictions that $a_1 = a_2$ and $c = c_1 = c_2$. Then

$$SA = (1 - c)[b_2 - b_1] \quad (23)$$

$$UA = (1 - c)|b_2 - b_1|. \quad (24)$$

Proof. The proof of this lemma is very similar to the proof of Theorem 1 with the expception that θ_0 is any arbitrary point between $-\infty$ and $+\infty$. Unlike in Theorem 1, F_1 and F_2 in this lemma do not have a finite intersection point because $a_1 = a_2$ and $b_1 \neq b_2$. Therefore, the expression for B in (19) will be different but still finite for any arbitrary point between $-\infty$ and $+\infty$. Furthermore, since P_1 and P_2 do not intersect at a finite point, integrals I_1 and I_2 (as given in Theorem 1) are both either positive or negative. This, in turn, implies that $UA = (1 - c)|I_1 + I_2|$. These special considerations, when incorporated into the proof for Theorem 1, establish the validity of (23) and (24). □

Corollary 1. Let P_1 and P_2 represent two ICCs in the two-parameter model. Then

$$SA = (b_2 - b_1) \tag{25}$$

$$UA = \left| \frac{2(a_2 - a_1)}{Da_1 a_2} \ln \left(1 + \exp\left(\frac{Da_1 a_2 (b_2 - b_1)}{a_2 - a_1}\right)\right) - (b_2 - b_1) \right| \quad \text{if } a_1 \neq a_2, \tag{26}$$

or

$$UA = |b_2 - b_1| \qquad \text{if } a_1 = a_2. \tag{27}$$

Corollary 2. For the Rasch model (Wright & Stone, 1979), the unsigned and signed areas between two ICCs are

$$SA = (b_2 - b_1) \tag{28}$$

$$UA = |b_2 - b_1|. \tag{29}$$

The proofs of Corollary 1 and Corollary 2 are straightforward and are therefore omitted.

Having presented the signed and unsigned areas for Case I and its special cases, let us now turn to Case II where $c_1 \neq c_2$.

Theorem 2: Case II. Let F_1 and F_2 represent two three-parameter ICCs with $c_1 \neq c_2$. Then the signed area is $+\infty$ or $-\infty$; the unsigned area is $+\infty$.

Proof. Let θ_0 be any arbitrary point between $-\infty$ and $+\infty$. Then the signed area between the two (three-parameter) ICCs can be written as

$$\begin{aligned} I &= \int_{-\infty}^{\infty} (F_1 - F_2)\, d\theta \\ &= \int_{-\infty}^{\theta_0} (F_1 - F_2)\, d\theta + \int_{\theta_0}^{\infty} (F_1 - F_2)\, d\theta \end{aligned} \tag{30}$$

$$I = I_1 + I_2. \tag{31}$$

The integral I_2 can be written as

$$\begin{aligned} I_2 &= \lim_{x \to \infty} \int_{\theta_0}^{x} (F_1 - F_2)\, d\theta \\ &= \lim_{x \to \infty} \int_{\theta_0}^{x} [(c_1 - c_2) + (1 - c_1)P_1 - (1 - c_2)P_2]\, d\theta \\ &= \lim_{x \to \infty} \Bigg[(c_1 - c_2)\theta + \frac{1 - c_1}{Da_1} \ln (1 + \exp (Da_1(\theta - b_1))) \\ &\qquad - \frac{1 - c_2}{Da_2} \ln (1 + \exp (Da_2(\theta - b_2))) \Bigg]_{\theta_0}^{x} \\ &= \lim_{x \to \infty} \left[(c_1 - c_2)\theta + \ln \frac{(1 + \exp (Da_1(\theta - b_1)))^{(1 - c_1)/Da_1}}{(1 + \exp (Da_2(\theta - b_2)))^{(1 - c_2)/Da_2}} \right]_{\theta_0}^{x} \end{aligned} \tag{32}$$

For derivational convenience, let

$$G(\theta) = \frac{(1 + \exp(Da_1(\theta - b_1)))^{(1-c_1)/Da_1}}{(1 + \exp(Da_2(\theta - b_2)))^{(1-c_2)/Da_2}}$$

$$= \frac{(1 + \exp(Da_1(\theta - b_1)))^{(1-c_1)/Da_1}\left(\frac{\exp((1-c_1)(\theta - b_1))}{\exp((1-c_1)(\theta - b_1))}\right)}{(1 + \exp(Da_2(\theta - b_2)))^{(1-c_2)/Da_2}\left(\frac{\exp((1-c_2)(\theta - b_2))}{\exp((1-c_2)(\theta - b_2))}\right)}$$

$$= \frac{\left(\frac{1}{P_1(\theta)}\right)^{(1-c_1)/Da_1}}{\left(\frac{1}{P_2(\theta)}\right)^{(1-c_2)/Da_2}} \exp[(1-c_1)(\theta - b_1) - (1-c_2)(\theta - b_2)]. \tag{33}$$

Using the definition of $G(\theta)$, I_2 can be rewritten as

$$I_2 = \lim_{x \to \infty} [(c_1 - c_2)x + \ln G(x)] - (c_1 - c_2)\theta_0 - \ln G(\theta_0). \tag{34}$$

Since $P_1 \to 1$ and $P_2 \to 1$ as $x \to \infty$,

$$\lim_{x \to \infty} [(c_1 - c_2)x + \ln G(x)] = b_2(1 - c_2) - b_1(1 - c_1).$$

Substituting this in (34) yields

$$I_2 = b_2(1 - c_2) - b_1(1 - c_1) - (c_1 - c_2)\theta_0 - \ln G(\theta_0),$$

which is finite. Turning now to I_1, it should be noted that

$$I_1 = \lim_{x \to \infty} \int_{-x}^{\theta_0} (F_1 - F_2)\, d\theta$$

$$= \lim_{x \to \infty} \left[(c_1 - c_2)\theta + \frac{1 - c_1}{Da_1} \ln(1 + \exp(Da_1(\theta - b_1))) - \frac{1 - c_2}{Da_2} \ln(1 + \exp(Da_2(\theta - b_1)))\right]_{-x}^{\theta_0}$$

$$= \left[(c_1 - c_2)\theta_0 + \frac{1 - c_1}{Da_1} \ln(1 + \exp(Da_1(\theta_0 - b_1))) - \frac{1 - c_2}{Da_2} \ln(1 + \exp(Da_2(\theta_0 - b_2)))\right]$$

$$- \lim_{x \to \infty} \left[(c_1 - c_2)(-x) + \frac{1 - c_1}{Da_1} \ln(1 + \exp(Da_1(-x - b_1))) - \frac{1 - c_2}{Da_2} \ln(1 + \exp(Da_2(-x - b_2)))\right].$$

In the above equation, the first part is finite whereas the limit of the second part equals $(c_1 - c_2)(-\infty)$ since

$$\lim_{x \to \infty} \exp(Da_i(-x - b_i)) = \lim_{x \to \infty} \frac{\exp(-Da_i b_i)}{\exp(xDa_i)} = 0.$$

Therefore,

$$I_1 = \left[(c_1 - c_2)\theta_0 + \frac{1 - c_1}{Da_1} \ln (1 + \exp (Da_1(\theta_0 - b_1))) - \frac{1 - c_2}{Da_2} \ln (1 + \exp (Da_2(\theta_0 - b_2))) \right] + (c_1 - c_2)\infty.$$

Unless $c_1 = c_2$, I_1 will be either $-\infty$ or ∞. Hence, in (31), I_2 is finite, whereas I_1 is infinite. This completes the proof for the signed area. Since by definition (see (5) and (6)) the unsigned area is always greater than or equal to the signed area, the unsigned area in the present context must be ∞. This completes the proof of Theorem 2. □

Discussion

All the area formulas assume that the item parameters for the two groups are on a common metric. Procedures for converting two sets of item parameters to a common metric are given in Hambleton and Swaminathan (1985).

Just as the estimated item parameters are likely to vary from one sample to the next, the signed and unsigned areas (when based on estimated item parameters) will also vary from one sample to the next due to sampling error. This variation could be severe at times, especially if the size of the sample used for estimating item parameters is small. Standard error formulas, which are not currently available, will be needed to assess the degree of variation to be expected in area estimates when such estimates are derived from estimated item parameters. The asymptotic procedures adopted by Oosterloo (1984) for establishing confidence intervals for test information function and relative efficiency may prove useful in developing formulas for standard errors for the area estimates.

When the lower asymptotes are equal, the area between two ICCs is finite, which can be evaluated readily with the formulas developed in this paper. The procedures of Rudner et al. (1980a, 1980b) and Shepard et al. (1981, 1984) also provide estimates of the area between two ICCs, but their estimates are obtained by integrating between two finite points on the theta scale. Their estimates will be smaller (and, at times, substantially smaller) than the area estimates given in this paper. For illustration, consider the example given by Linn, Levine, Hastings, and Wardrop (1981), where $a_1 = 1.8$, $b_1 = 3.5$, $c_1 = .2$ for Group 1 and $a_2 = .5$, $b_2 = 5.0$, and $c_2 = .2$ for Group 2. Based on (7) and (8), the signed and unsigned area are 1.200 and 1.415, respectively. If integrated between -3 and $+3$ on the theta scale, then the SA and UA are $-.106$ and $.107$, respectively. For this example, the areas based on (7) and (8) are substantial and are consistent with the fact that the differences in item difficulty and item discrimination are also substantial. On the other hand, the area estimates based on a finite interval $(-3, +3)$ of integration are quite small, which prompted Linn et al. (1981) to question the suggestion of item bias based on a large difference in estimated item difficulty or item discrimination. This example raises an important question about the estimation and use of the area between two ICCs as a measure of item bias: What is the appropriate score interval for computing the area between two ICCs?

On the one hand, the use of the entire theta scale range in computing the area between two ICCs removes the arbitrariness associated with defining a finite interval for integration. Also, the resulting areas will more accurately reflect the differences in the item parameters. Furthermore, given the item parameter invariance, any area measure used for identifying item bias must reflect the lack of invariance when it exists. The infinite interval approach would accomplish this; whereas, in the finite interval approach, it would depend upon the finite interval chosen for integration. On the other hand, as noted by I. W. Molenaar (personal communication, February 18, 1987) and two anonymous reviewers, the difference between two ICCs is more relevant in an ability region where there are more than just a few persons. This perspective argues in favor of a finite interval for integration but it also implies the need for an appropriate density function for $\theta(g(\theta))$. With $g(\theta)$ properly specified, one can compute the area between two ICCs by integrating the function: $(F_1 - F_2)g(\theta)$. Such a theoretically appealing approach will emphasize the ability region of interest while performing the integration on the entire theta scale. (It should be noted that the existing procedures for estimating the area between two ICCs implicitly assume that the density function for θ is uniform.) Both the specification of $g(\theta)$ and the integration of $(F_1 - F_2)g(\theta)$ are important questions for future research, and, as noted by Molenaar, the work of Bock and Aitkin (1981) may prove to be useful for solving the integral in this context.

Finally, Theorem 2 states that when the lower asymptotes are not equal, the area between two ICCs is infinite. This means that the current (finite interval) procedures for estimating the area between two ICCs with unequal lower asymptotes yield misleading results. For the area measure to be meaningful and valid, it must be finite and its estimate fairly accurate. Neither requirement is satisfied for these procedures when the lower asymptotes are unequal. One possible solution for this problem is to specify an appropriate density function for θ and use it to evaluate the area between two ICCs. As noted in the preceding paragraph, research is needed to assess the viability and practicality of this solution. In the meantime, practitioners may want to use Lord's chi-square test (Lord, 1980) for item bias analysis within the context of the three-parameter model.

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